J. Partial Diff. Eqs. **16**(2003), 306–320
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Vol.16, No.4

QUANTUM EULER-POISSON SYSTEM: LOCAL EXISTENCE OF SOLUTIONS

Hao Chengchun and Jia Yueling
(Academy of Mathematics and System Sciences, CAS, Beijing, 100080, China
E-mail: hcc@mail.amss.ac.cn; E-mail: yljia@mail.amss.ac.cn)
Li Hailiang
(Institut für Mathematik, Universität Wien, A-1090, Austria
E-mail: Hailiang.li@univie.ac.at)
(Received Sep. 22, 2002; revised Dec. 6, 2002)

Abstract The one-dimensional transient quantum Euler-Poisson system for semiconductors is studied in a bounded interval. The quantum correction can be interpreted as a dispersive regularization of the classical hydrodynamic equations and mechanical effects.

The existence and uniqueness of local-in-time solutions are proved with lower regularity and without the restriction on the smallness of velocity, where the pressure-density is general (can be non-convex or non-monotone).

Key Words Quantum Euler-Poisson system; existence of local classical solutions; non-linear fourth-order wave equation.

2000 MR Subject Classification 35Q40, 35L05, 35G25.

Chinese Library Classification O471.1, O175.28.

1. Introduction

In 1927, Madelung gave a fluid-dynamical description of quantum systems governed by the Schrödinger equation for the wave function ψ:

$$\begin{aligned} i\varepsilon\partial_t\psi &= -\frac{\varepsilon^2}{2}\Delta\psi - V\psi \quad \text{in} \quad \mathbb{R}^d \times (0,\infty), \\ \psi(\cdot,0) &= \psi_0 \quad \text{in} \quad \mathbb{R}^d, \end{aligned}$$

where $d \geqslant 1$ is the space dimension, $\varepsilon > 0$ denotes the scaled Planck constant, and $V = V(x,t)$ is some (given) potential. By separating the amplitude and phase of $\psi = |\psi|\exp(iS/\varepsilon)$, the particle density $\rho = |\psi|^2$ and the particle current density $j = \rho\nabla S$ for irrotational flow satisfy the so-called *Madelung equations* [1]

$$\partial_t\rho + \text{div} j = 0, \tag{1.1}$$

$$\partial_t j + \text{div}\left(\frac{j\otimes j}{\rho}\right) - \rho\nabla\phi - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\rho}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty). \tag{1.2}$$

The equations (1.1)-(1.2) can be interpreted as the pressureless Euler equations including the quantum Bohm potential

$$\frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho}}{\rho}. \tag{1.3}$$

They have been used for the modelling of superfluids like Helium II [2, 3].

Recently, Madelung-type equations have been derived to model quantum phenomena in semiconductor devices, like resonant tunnelling diodes, starting from the Wigner-Boltzmann equation [4] or from a mixed-state Schrödinger-Poisson system [5, 6]. There are several advantages to the fluid-dynamical description of quantum semiconductors. First, kinetic equations, like the Wigner equation, or Schrödinger systems are computationally very expensive, whereas for Euler-type equations efficient numerical algorithms are available [7, 8]. Second, the macroscopic description allows for a coupling of classical and quantum models. Indeed, setting the Planck constant ε in (1.2) equal to zero, we obtain the classical pressureless equations. So in both pictures, the same (macroscopic) variables can be used. Finally, as semiconductor devices are modelled in bounded domains, it is easier to find physically relevant boundary conditions for the macroscopic variables than for the Wigner function or for the wave function.

The Madelung-type equations derived by Gardner [4] and Gasser et al. [5] also include a pressure term and a momentum relaxation term taking into account of interactions of the electrons with the semiconductor crystal, and are self-consistently coupled to the Poisson equation for the electrostatic potential ϕ:

$$\partial_t\rho + \operatorname{div} j = 0, \tag{1.4}$$

$$\partial_t j + \operatorname{div}\left(\frac{j\otimes j}{\rho}\right) + \nabla P(\rho) - \rho\nabla\phi - \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\rho}\right) = -\frac{j}{\tau}, \tag{1.5}$$

$$\lambda^2\Delta\phi = \rho - C(x) \quad \text{in } \Omega\times(0,\infty), \tag{1.6}$$

where $\Omega\subset\mathbb{R}^d$ is a bounded domain, $\tau>0$ is the (scaled) momentum relaxation time constant, $\lambda>0$ the (scaled) Debye length, and $C(x)$ is the doping profile modelling the semiconductor device under consideration [9, 10]. The pressure is assumed to depend only on the particle density and, like in classical fluid dynamics, often to have the expression

$$P(\rho) = \frac{T_0}{\gamma}\rho^\gamma, \quad \rho\geqslant 0, \quad \gamma\geqslant 1, \tag{1.7}$$

with the temperature constant $T_0>0$ employed [4, 11]. *Isothermal* fluids correspond to $\gamma=1$, *isentropic* fluids to $\gamma>1$. Notice that the particle temperature is $T(\rho)=T_0\rho^{\gamma-1}$.

The equations (1.4)-(1.6) are referred to as the *quantum Euler-Poisson system* or the *quantum hydrodynamic model.*

In this paper, we investigate the local existence of solutions of the following one-dimensional quantum Euler-Poisson problem:

$$\rho_t + (\rho u)_x = 0, \tag{1.8}$$

$$(\rho u)_t + (\rho u^2 + P(\rho))_x = \rho\phi_x + \frac{1}{2}\varepsilon^2\rho\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x - \frac{\rho u}{\tau}, \tag{1.9}$$

$$\phi_{xx} = \rho - \mathcal{C}(x), \tag{1.10}$$

$$(u,\rho)(x,t=0) = (u_0,\rho_0), \tag{1.11}$$

$$(u,\rho_x,\phi_x)(x=0,t) = (u,\rho_x,\phi_x)(x=1,t) = (0,0,0), \tag{1.12}$$

for $(x,t) \in (0,1) \times (0,\infty)$, where $\rho_0 > 0$.

So far, to our knowledge, the known results on the existence of the time-dependent system (1.4)-(1.6) with different boundary conditions have been obtained in [12] for smooth local-in-time solutions on bounded domains and in [13] for general pressure and non-constant doping profile with the smallness of the velocity.

In this paper, we consider the initial-boundary-value problem (IBVP) (1.8)-(1.12), with different boundary conditions compared to the one in [13] and [12] with general pressure and non-constant doping profile, and without the restriction on the smallness of the velocity. We will establish the local-in-time existence of classical solutions (ρ, u, ϕ) of the IBVP (1.8)-(1.12).

In dealing with the IBVP (1.8)-(1.12), we introduce a new variable to overcome the difficulties caused by the lack of smallness of initial data and by the avoidance avoid hyperbolicity with the equation for u. This will be explained later in Section 3.

Notations: Denote $\Omega := (0,1)$. Let $L^2 = L^2(\Omega)$ and $H^k = H^k(\Omega)$ stand for the Lebesgue space of square integrable functions and the Sobolev space of function with square integrable weak derivatives of up to order k, respectively. The norm of L^2 is denoted by $\|\cdot\|$, and the norm of H^k is $\|\cdot\|_k$. The space $H_0^k = H_0^k(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of H^k. Let $T > 0$ and let $\mathcal{B}$ be a Banach space. Then $C^k(0,T;\mathcal{B})$ ($C^k([0,T];\mathcal{B})$, respectively) denotes the space of $\mathcal{B}$-valued up to k-times continuously differentiable functions on $(0,T)$ ($[0,T]$ respectively), $L^2(0,T;\mathcal{B})$ is the space of $\mathcal{B}$-valued L^2-functions on $(0,T)$. C always denotes a generic positive constant.

It is convenient to make use of the variable transformation $\rho = w^2$, $e = \phi_x$ in (1.8)-(1.12) which yields the following IBVP for (w,u,e):

$$2ww_t + (w^2u)_x = 0, \tag{1.13}$$

$$(w^2u)_t + (w^2u^2 + P(w^2))_x = w^2 e + \frac{1}{2}\varepsilon^2 w^2\left(\frac{w_{xx}}{w}\right)_x - \frac{w^2 u}{\tau}, \tag{1.14}$$

$$e_x = w^2 - \mathcal{C}(x), \tag{1.15}$$

$$(u,w,e)(x,t=0) = (u_0,w_0,e_0), \tag{1.16}$$

$$(u,w_x,e)(x=0,t) = (u,w_x,e)(x=1,t) = (0,0,0), \tag{1.17}$$

where $w_0 = \sqrt{\rho_0}$, $e_0 = \displaystyle\int_0^x (w_0^2(y) - \mathcal{C}(y))dy$. From (1.14) and (1.17), it is clear that $w_{xxx}(0,t) = w_{xxx}(1,t) = 0$. This problem is equivalent to (1.8)-(1.12) for the classical solution with positive particle density.

Throughout this paper, we will assume compatibility conditions for the IBVP (1.13)-(1.17) in the sense that the time derivatives of the boundary values and the spatial derivatives of the initial data are compatible at $(x,t) = (0,0)$ and $(x,t) = (1,0)$ in (1.13)-(1.17).

We have the following local existence result for the IBVP (1.8)-(1.12):

Theorem 1.1 *Assume that*

$$P(\rho) \in C^4(0,+\infty), \quad C(x) \in H^1, \tag{1.18}$$

$(\rho_0, u_0) \in H^5 \times H^4$ *such that* $\rho_0(x) > 0$ *for* $x \in [0,1]$. *Then, there is a number* $T_* > 0$ *(determined by (3.57)), such that there exists a classical solution* (ρ, u, ϕ) *of the IBVP (1.8)-(1.12) in the time interval* $[0, T_*]$ *satisfying*

$$\|\rho(t)\|_5^2 + \|u(t)\|_4^2 + \|\phi(t)\|_3^2 < \infty \quad \textit{for} \quad t \leqslant T_*.$$

Theorem 1.1 is proven by an iteration method and compactness arguments. More precisely, we construct a sequence of approximate solutions which is uniformly bounded in a certain Sobolev space in a fixed time interval. Compactness arguments imply that there is a limiting solution which is just a local-in-time solution of (1.13)-(1.17).

Following [13], the idea of the local existence result is first to linearize the system (1.13)-(1.15), around the initial data (w_0, u_0, e_0), and then, to consider the equations for the perturbation $(\psi, \eta, E) = (w - w_0, u - u_0, e - e_0)$. The main step is to change the evolution equation for the perturbation of particle density into a semi-linear fourth-order wave equation, then, construct iterated equations of (ψ_p, η_p, E_p) $(p \in \mathbb{N})$ from a fixed-point procedure, and obtain the uniformly boundedness of their solutions. At last, by the standard compactness argument, the desired solutions (ψ, η, E) of the perturbation problem come out as p tend to infinity. A further analysis shows that $(w, u, E) = (\psi + w_0, \eta + u_0, E + e_0)$ with $w > 0$ is the expected local-in-time solution of the original problem (1.13)-(1.17).

2. A Semi-linear Fourth-order Wave Equation

Consider $V := \{u \in H^4 : u_x|_{x=0,1} = u_{xxx}|_{x=0,1} = 0\}$ and L^2, endowed the scalar products $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$, respectively. Furthermore, we consider the following initial-value problem on L^2:

$$u'' + \frac{1}{\tau} u' + \nu A u + u + \mathcal{L} u' = F(t), \quad t > 0, \tag{2.1}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{2.2}$$

where the primes denote derivatives with respect to time, $\tau, \nu > 0$ are constant, $A = \partial_x^4$ is an operator defined on

$$D(A) = \{u \in H^4 : u_x|_{x=0,1} = u_{xxx}|_{x=0,1} = 0\}, \tag{2.3}$$

and the operators $\mathcal{L}$ and F are given by

$$\langle \mathcal{L}u, v\rangle = \int_0^1 b(x,t)u_x v dx, \quad u, v \in V,$$

$$(F(t), v) = \int_0^1 f(x,t) v dx, \quad v \in L^2,$$

where $b, f : [0,1] \times [0,T] \to \mathbb{R}$ are measurable functions.

Related to the operator A, we introduce the coercive, continuous, symmetric bilinear form $a(u,v)$

$$a(u,v) = \nu \int_0^1 u_{xx} v_{xx} dx, \quad u, v \in V.$$

There exists a complete orthonormal family of eigenvectors $\{r_i\}_{i\in\mathbb{N}}$ of L^2 and a family of eigenvalues $\{\mu_i\}_{i\in\mathbb{N}}$ such that $0 < \mu_1 < \mu_2 < \cdots$ and $\mu_i \to \infty$ as $i \to \infty$. The family $\{r_i\}_{i\in\mathbb{N}}$ is also orthogonal for $a(u,v)$ on H, i.e.

$$(r_i, r_j) = \delta_{ij}, \quad a(r_i, r_j) = \nu\langle A\mu_i, \mu_j\rangle = \nu\delta_{ij} \quad \forall i, j.$$

By using the Faedo-Galerkin method [14, 15], it is possible to prove the existence of the solutions of (2.1)-(2.2). The result is summarized in the following theorem.

Theorem 2.1 *Let $T_0 > 0$. Assume that*

$$F \in C^1(0, T_0; L^2), \quad b \in C^1([0,T_0]; H^2) \cap W^{2,\infty}(0, T_0; H^1). \tag{2.4}$$

Then, if $u_0 \in D(A)$ and $u_1 \in H$, there exists a solution of (2.1)-(2.2) satisfying

$$u \in C([0,T_0]; D(A)) \cap C^1([0,T_0]; H) \cap C^2([0,T_0]; L^2). \tag{2.5}$$

Moreover, assume additionally that

$$F \in C^1([0,T_0]; H^1).$$

Then, if $u_0 \in H^5 \cap D(A)$ and $u_1 \in H$ satisfy

$$\nu A u_0 + \mathcal{L}(u_1) - F(0) \in H^1,$$

it holds

$$u \in C^l([0,T_0]; H^{5-2l}), \quad l = 0, 1, 2. \tag{2.6}$$

Proof The proof is standard, cf. [13, 16], the details are omitted.

3. Local Existence

In this section we prove Theorem 1.1. We linearize the equations (1.13)-(1.15) around the initial state (w_0, u_0, e_0) and prove the local-in-time existence for the perturbation $(\psi, \eta, E) = (w - w_0, u - u_0, e - e_0)$. For this, we reformulate the original initial-boundary value problem (1.13)-(1.15). For given $U^p = (\psi_p, \eta_p, E_p)$, we obtain the following linearized problems for $U^{p+1} = (\psi_{p+1}, \eta_{p+1}, E_{p+1})$, $p \in \mathbb{N}$, write "∂_x" for the spatial derivative and " $'$ " for the time derivative:

$$\begin{cases} \eta'_{p+1} + \dfrac{1}{\tau}\eta_{p+1} = g_0(x, U^p), \\ \eta_{p+1}(x, 0) = 0. \end{cases} \tag{3.1}$$

$$\begin{cases} \psi''_{p+1} + \dfrac{1}{\tau}\psi'_{p+1} + \psi_{p+1} + \nu\partial_x^4\psi_{p+1} + \kappa(x, U^p)\partial_x\psi'_{p+1} = g_1(x, U^p), \\ \psi_{p+1}(x, 0) = 0,\ \psi'_{p+1}(x, 0) = \theta_1(x) := -w_{0x}u_0 - \dfrac{1}{2}w_0u_{0x}, \\ \partial_x\psi_{p+1}(0, t) = \partial_x\psi_{p+1}(1, t) = 0, \\ \partial_x^3\psi_{p+1}(0, t) = \partial_x^3\psi_{p+1}(1, t) = 0. \end{cases} \tag{3.2}$$

$$\begin{cases} \partial_x E_{p+1} = (2w_0 + \psi_p)\psi_p, \\ E_{p+1}(0, t) = E_{p+1}(1, t) = 0. \end{cases} \tag{3.3}$$

where $\nu = \frac{1}{4}\varepsilon^2$ and

$$\begin{aligned} g_0(x, U^p) =& -\frac{1}{\tau}u_0 - \frac{2P'((\psi_p + w_0)^2)(\psi_p + w_0)_x}{\psi_p + w_0} + E + e_0 + \frac{\varepsilon^2(\psi_p + w_0)_{xxx}}{2(\psi_p + w_0)} \\ & - \frac{\varepsilon^2(\psi_p + w_0)_x(\psi_p + w_0)_{xx}}{2(\psi_p + w_0)^2} + \frac{2\psi'_p(\eta_p + u_0)}{\psi_p + w_0} + \frac{2(\eta_p + u_0)^2(\psi_p + w_0)_x}{\psi_p + w_0}, \\ \kappa(x, U^p) =& 2(\eta_p + u_0), \\ g_1(x, U^p) =& -\nu\partial_x^4 w_0 - \frac{\psi_p'^2}{\psi_p + w_0} + \frac{(\psi_p + w_0)_x^2(\eta_p + u_0)}{\psi_p + w_0} \\ & + (\psi_p + w_0)_x(\eta_p + u_0)(\eta_p + u_0)_x - (\eta_p + u_0)^2(\psi_p + w_0)_{xx} \\ & + (\psi_p + w_0)(\eta_p + u_0)_x^2 + \frac{\left[P\left((\psi_p + w_0)^2\right)\right]_{xx}}{2(\psi_p + w_0)} + \psi_p - (\psi_p + w_0)_x(E_p + e_0) \\ & - \frac{1}{2}(\psi_p + w_0)^3 + \frac{1}{2}(\psi_p + w_0)\mathcal{C}(x) + \nu\frac{(\psi_p + w_0)_{xx}^2}{\psi_p + w_0}. \end{aligned} \tag{3.4}$$

We apply an induction argument to prove the existence of solutions of (3.1)-(3.3).

Lemma 3.1 *Under the assumption of Theorem 1.1, i.e. $P(\rho) \in C^4(0, \infty)$, $\mathcal{C}(x) \in H^1$, $(w_0, u_0) \in H^5 \times H^2$ with $w_0(x) > 0$, $x \in (0, 1)$, there exists a sequence $\{U^p\}_{p=1}^\infty$ of solutions of (3.1)-(3.3) in the time interval $t \in [0, T_*]$ for some $T_* > 0$ which is independent of p, satisfying the regularity properties*

$$\begin{cases} \eta_p \in C^1\left([0, T_*]; H^2\right) \\ \psi_p \in C^l\left([0, T_*]; H^{5-2l}\right) \quad l = 0, 1, 2;\ p \in \mathbb{N} \\ E_p \in C^1\left([0, T_*]; H^2\right) \end{cases} \tag{3.5}$$

and the uniform boundedness

$$\begin{cases} \|\eta_p'(t)\|_2^2 + \|(E_p, E_p')(t)\|_2^2 \leqslant M_0, \\ \|(\psi_p, \psi_p', \psi_p'')(t)\|^2_{H^5\times H^3\times H^1} \leqslant M_0, \quad p\in\mathbb{N},\ t\in[0,T_*] \\ \|\eta_p(t)\|_2^2 \leqslant 1, \quad \|\psi_{pxx}(t)\|^2 \leqslant \frac{4}{\nu}\|\theta_1\|^2 \end{cases} \tag{3.6}$$

where $M_0>0$ is a constant independent of U^p $(p\in\mathbb{N})$ and T_*.

Proof Step1: solutions of (3.1)-(3.3). We introduce a new variable φ_p satisfying

$$\begin{cases} 2\varphi_p' + 2\varphi_{px}u_p + \varphi_p u_{px} = 0, \\ \varphi_p(0, x(0)) = w_0(x_0), \\ x'(t) = u_p(x(t), t). \end{cases} \tag{3.7}$$

where $x_0 = x(0)$. From (1.6), (1.9) and (3.7), for $(x,t)\in(0,1)\times(0,\infty)$ we have

$$[w_p - \varphi_p](x,t) = 0, \quad if\ [w_p-\varphi_p](x,0) = 0.$$

Along the characteristic

$$x'(t) = u_p(x(t),t), \quad x(0) = x_0 \in (0,1),$$

it follows from (3.7) that

$$\varphi_p(x(t),t) = \sqrt{\rho_0(x_0)}\exp\left\{\int_0^t -\frac{1}{2}u_{px}(x(s),s)ds\right\}$$

which implies

$$\varphi_p > 0 \quad if \quad \rho_0 > 0,$$

and

$$m \leqslant \varphi_p \leqslant M \quad if \quad u_p \in H^2$$

where m and M are constants independent of p. Moreover, we have

$$\left\|\frac{\varphi_p'}{\varphi_p}\right\|_{L^\infty(dt)} \leqslant C\|u_p\|_{C^1([0,T];H^2)} \leq CM.$$

Therefore, we shall replace $\dfrac{1}{w_p}$ or $\dfrac{1}{\psi_p+w_0}$ by $\dfrac{1}{\varphi_p}$ and reset $U^p := (\psi_p, \eta_p, E_p, \varphi_p)$.

Obviously, $U^1 = (0,0,0,w_0(x_0))$ satisfies (3.5)-(3.6). Starting with $U^1 = (0,0,0,w_0(x_0))$, we prove the existence of a solution $U^2 = (\psi_2, \eta_2, E_2, \varphi_2)$ to (3.1)-(3.3) satisfying (3.5)-(3.6). The functions $g_0(x,U^1)$, $g_1(x,U^1)$ and $\kappa(x,U^1)$ only depend on the initial value $(0,0,0,w_0(x_0))$ and the boundedness of φ_p such that

$$\begin{aligned} &g_0(x,U^1)\in H^2, \quad g_1(x,U^1)\in H^1, \quad \kappa(x,U^1)\in H^2, \\ &\|g_0(x,U^1)\|_2 + \|g_1(x,U^1)\|_1 + \|\kappa(x,U^1)\|_2 \leqslant C_0(\tau,\varepsilon,M)(I_0+1), \\ &\partial_t g_0(x,U^1) = \partial_t g_1(x,U^1) = \partial_t\kappa(x,U^1) = 0 \end{aligned} \tag{3.8}$$

where $C_0 > 0$ is some constant and

$$I_0 = \|w_0\|_5^4 + \|u_0\|_2^4 + \|C\|_1^2 + \|e_0\|_2^2. \tag{3.9}$$

For the linear system (3.1)-(3.3), the local existence of a solution $U^2 = (\psi_2, \eta_2, E_2, \varphi_2)$ follows from the theory of ordinary differential equations, applied to (3.1); Theorem 2.1 with $f(x,t) = g_0(x, U^1)$ and $b(x,t) = \kappa(x, U^1)$, applied to (3.2); and integral theory, applied to (3.3). The solution U^2 satisfies (3.5) with $T_* = T$ and the first two inequalities of (3.6) with $p = 2$.

We show in the following that U^2 satisfies the last two inequalities of (3.6) for $t \in [0, T_1]$, where $T_1 > 0$ is given by

$$T_1 = \min\left\{\frac{\|\theta_1\|^2}{C_0^2(I_0+1)^2}, \frac{\ln 2}{2(3+I_0)}, \frac{1}{C_0(I_0+1)}\right\}. \tag{3.10}$$

From (3.1) and by integrating, we obtain

$$\eta_2(t) = g_0(x, U^1)\int_0^t \exp\{-(t-s)\}ds, \quad t \in [0, T_1],$$

hence, in view of (3.10),

$$\|\eta_2(t)\|_2^2 \leqslant T_1^2\|g_0(x, U^1)\|_2^2 \leqslant 1, \quad t \in [0, T_1]. \tag{3.11}$$

Multiplying the differential equation in (3.2) by ψ_2', integrating the resulted equation over $(0,1) \times (0,t)$ for $t \in [0, T_1]$ and integrating by parts, we have

$$\begin{aligned}\|\psi_{2xx}(t)\|^2 &\leqslant \frac{1}{\nu}(\|\theta_1\|^2 + C_0(I_0+1)^2 T_1)e^{2(3+I_0)T_1} \\ &\leqslant \frac{4}{\nu}\|\theta_1\|^2.\end{aligned} \tag{3.12}$$

This proves the last two estimates in (3.6). Moreover, by the Poincaré inequality and the boundary conditions in (3.2), it follows from (3.12) that

$$\|\psi_2(t)\|_1^2 \leqslant (\frac{4}{\nu} + 4)\|\theta_1\|^2, \quad t \in [0, T_1]. \tag{3.13}$$

Now, assume that there exist solutions $\{U^i\}_{i=1}^p$ $(p \geqslant 2)$ of (3.1)-(3.3) on the time interval $[0, T_1]$ where T_1 is given by (3.10), satisfying (3.5)-(3.6). As the procedure above, for given U^p, there exists a solution $U^{p+1} = (\psi_{p+1}, \eta_{p+1}, E_{p+1}, \varphi_{p+1})$ of (3.1)-(3.3) in the interval $[0, T_1]$, satisfying

$$\begin{aligned}&\eta_{p+1} \in C^1([0, T_1]; H^2), \\ &\psi_{p+1} \in C^l([0, T_1]; H^{5-2l}), \quad l = 0, 1, 2. \\ &E_{p+1} \in C^1([0, T_1]; H^2).\end{aligned} \tag{3.14}$$

We will prove that there exist constants $T_* \in (0, T_1]$ and $K_i > C_0\,(i = 1, 2, 3, 4, 5)$ independent of $\{U^i\}_{i=1}^p$, such that if U^p satisfies on $[0, T^*]$ that

$$\|\partial_x^2 \psi_p(t)\|^2 \leqslant \frac{4}{\nu}\|\theta_1\|^2 =: K_0, \tag{3.15}$$

$$\|(\psi_p, \psi_p')(t)\|^2 + \|\psi_p''(t)\|_1^2 \leqslant K_1, \tag{3.16}$$

$$\|\partial_x^3 \psi_p(t)\|^2 \leqslant K_2, \quad \|\partial_x^4 \psi_p(t)\|_1^2 \leqslant K_3, \quad \|\partial_x^2 \psi_p'(t)\|_1^2 \leqslant K_4, \tag{3.17}$$

$$\|\eta_p(t)\|_2^2 \leqslant 1, \quad \|\partial_x \eta_p'(t)\|_1^2 \leqslant K_5, \tag{3.18}$$

then on $[0, T_*]$, U^{p+1} also satisfies the following inequalities

$$\|\partial_x^2 \psi_{p+1}(t)\|^2 \leqslant \frac{4}{\nu}\|\theta_1\|^2, \tag{3.19}$$

$$\|(\psi_{p+1}, \psi_{p+1}')(t)\|^2 + \|\psi_{p+1}''(t)\|_1^2 \leqslant K_1, \tag{3.20}$$

$$\|\partial_x^3 \psi_{p+1}(t)\|^2 \leqslant K_2, \quad \|\partial_x^4 \psi_{p+1}(t)\|_1^2 \leqslant K_3, \quad \|\partial_x^2 \psi_{p+1}'(t)\|_1^2 \leqslant K_4, \tag{3.21}$$

$$\|\eta_{p+1}(t)\|_2^2 \leqslant 1, \quad \|\partial_x \eta_{p+1}'(t)\|_1^2 \leqslant K_5. \tag{3.22}$$

Notice that it follows from (3.15) and (3.19), employing the boundary conditions in (3.2) and Poincaré's inequality,

$$\|\psi_p(t)\|_1^2 \leqslant (\frac{4}{\nu} + 4)\|\theta_1\|^2, \quad \|\psi_{p+1}(t)\|_1^2 \leqslant (\frac{4}{\nu} + 4)\|\theta_1\|^2, \quad t \in [0, T^*]. \tag{3.23}$$

Step 2: estimates for g_0, g_1 and κ. Let U^p satisfy (3.15)-(3.18). Then a direct computation shows the following estimates for $g_0(x, U^p)$ and $g_1(x, U^p)$, for $t \in [0, T_*]$,

$$\begin{aligned} \|g_0(\cdot, U^p)(t)\|^2 &\leqslant C(1 + I_0 + \|\partial_x^2 \psi_p(t)\|^2 + \|\partial_x^3 \psi_p(t)\|^2 + \|\psi_p'(t)\|^2)^4 \\ &\leqslant C(1 + I_0 + K_0 + K_1 + K_2)^4, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \|g_{0x}(\cdot, U^p)(t)\|^2 &\leqslant C(1 + I_0 + \|\partial_x^2 \psi_p(t)\|^2 + \|\psi_p'(t)\|^2 + \|\partial_x^3 \psi_p(t)\|^2 \\ &\quad + \|\partial_x^4 \psi_p(t)\|_1^2 + \|\partial_x^2 \psi_p'(t)\|_1^2)^4 \\ &\leqslant C(1 + I_0 + K_0 + K_1 + K_2 + K_3 + K_4)^4, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \|g_{0xx}(\cdot, U^p)(t)\|^2 &\leqslant C(1 + I_0 + \|\partial_x^2 \psi_p(t)\|^2 + \|(\psi_p, \psi_p')(t)\|^2 + \|\partial_x^3 \psi_p(t)\|^2 \\ &\quad + \|\partial_x^4 \psi_p(t)\|_1^2 + \|\partial_x^2 \psi_p'(t)\|_1^2)^6 \\ &\leqslant C(1 + I_0 + K_0 + K_1 + K_2 + K_3 + K_4)^6, \end{aligned} \tag{3.26}$$

$$\begin{aligned} \|g_1(\cdot, U^p)(t)\|^2 &\leqslant C(1 + I_0 + \|\psi_p'(t)\|_1^2 + \|\partial_x \psi_p(t)\|_1^2 + \|\partial_x^2 \psi_p(t)\|_1^2 \\ &\quad + \|\psi_p(t)\|_1^2 + \|\psi_p(t)\|^2)^6 \\ &\leqslant C(1 + I_0 + K_0 + K_1 + K_2 + K_4)^6, \end{aligned} \tag{3.27}$$

$$\begin{aligned} \|g_{1x}(\cdot, U^p)(t)\|^2 &\leqslant C(1 + I_0 + \|\psi_p(t)\|_1^2 + \|\psi_p'(t)\|_1^2 + \|\partial_x \psi_p(t)\|_1^2 \\ &\quad + \|\partial_x^2 \psi_p(t)\|^2 + \|\partial_x^3 \psi_p(t)\|^2)^6 \\ &\leqslant C(1 + I_0 + K_0 + K_1 + K_2 + K_4)^6, \end{aligned} \tag{3.28}$$

$$\begin{aligned}\|g_1'(\cdot,U^p)(t)\|^2 \leqslant & C(1+I_0+\|\psi_p'(t)\|_1^2+\|\psi_p''(t)\|^2+\|\partial_x\psi_p(t)\|_1^2+\|\eta_p'(t)\|^2 \\ &+\|\partial_x\psi_p'(t)\|_1^2+\|\partial_x\psi_p'(t)\|^2+\|\partial_x^2\psi_p(t)\|_1^2+\|\partial_x^2\psi_p'(t)\|^2)^8 \\ \leqslant & C(1+I_0+K_0+K_1+K_2+K_4+K_5)^8,\end{aligned} \tag{3.29}$$

$$\begin{aligned}\|g_{1x}'(\cdot,U^p)(t)\|^2 \leqslant & C(1+I_0+\|\partial_x^2\psi_p(t)\|_1^2+\|\partial_x\psi_p'(t)\|_1^2+\|\psi_p'(t)\|_1^2 \\ &+\|\partial_x\psi_p''(t)\|^2+\|\psi_p''(t)\|^2+\|\partial_x^2\psi_p'(t)\|^2+\|\eta_p'(t)\|_1^2 \\ &+\|\partial_x^2\eta_p'(t)\|^2+\|\partial_x^3\psi_p(t)\|_1^2+\|\partial_x^3\psi_p'(t)\|^2)^8 \\ \leqslant & C(1+I_0+K_0+K_1+K_2+K_3+K_4+K_5)^8,\end{aligned} \tag{3.30}$$

and estimates for $\kappa(x,U^p)$ and $E_p(x,t)$, for $t\in[0,T_*]$,

$$\begin{aligned}\|\kappa(\cdot,U^p)(t)\|_2^2 &\leqslant C(1+I_0+\|\eta_p(t)\|_2^2)^2 \\ &\leqslant C(1+I_0)^2,\end{aligned} \tag{3.31}$$

$$\begin{aligned}\|\kappa'(\cdot,U^p)(t)\|_2^2 &\leqslant C(1+I_0+\|\eta_p'(t)\|_2^2)^2 \\ &\leqslant C(1+I_0+K_5)^2,\end{aligned} \tag{3.32}$$

$$\begin{aligned}\|E_p(t)\|_2^2+\|E_p'(t)\|_1^2 &\leqslant C(1+I_0+\|\psi_p(t)\|_2^2+\|\psi_p(t)\|_1^2+\|\psi_p'(t)\|_1^2)^2 \\ &\leqslant C(1+I_0+K_0+K_1+K_4)^2,\end{aligned} \tag{3.33}$$

where $C>1$ is a constant independent of K_i $(i=0,1,2,3,4,5)$.

Step 3: estimates for η_{p+1}. Integrating (3.1), we have

$$\eta_{p+1}(x,t)=\int_0^t \exp\{-(t-s)\}g_0(x,U^p)(s)ds,\quad 0\leqslant t\leqslant T_*\leqslant T_1,\quad x\in[0,1]. \tag{3.34}$$

From (3.24)-(3.26) we obtain the following estimates

$$\begin{aligned}\|\eta_{p+1}(t)\|^2 &\leqslant \left(\int_0^t \|g_0(x,U^p)\|ds\right)^2 \\ &\leqslant CT_*^2(1+I_0+K_0+K_1+K_2)^4,\end{aligned} \tag{3.35}$$

$$\begin{aligned}\|\partial_x\eta_{p+1}(t)\|^2 &\leqslant \left(\int_0^t \|g_{0x}(x,U^p)\|ds\right)^2 \\ &\leqslant CT_*^2(1+I_0+K_0+K_1+K_2+K_3+K_4)^4,\end{aligned} \tag{3.36}$$

$$\begin{aligned}\|\partial_x^2\eta_{p+1}(t)\|^2 &\leqslant \left(\int_0^t \|g_{0xx}(x,U^p)\|ds\right)^2 \\ &\leqslant CT_*^2(1+I_0+K_0+K_1+K_2+K_3+K_4)^6.\end{aligned} \tag{3.37}$$

Moreover, from (3.1) we have

$$\begin{aligned}\|\eta_{p+1}'(t)\|^2 &\leqslant C(\|\eta_{p+1}(t)\|^2+\|g_0(x,U^p)\|^2) \\ &\leqslant C(1+T_1^2)(1+I_0+K_0+K_1+K_2)^4,\end{aligned} \tag{3.38}$$

$$\begin{aligned}\|\partial_x\eta_{p+1}'(t)\|^2 &\leqslant C(\|\partial_x\eta_{p+1}(t)\|^2+\|g_{0x}(x,U^p)\|^2) \\ &\leqslant C(1+T_1^2)(1+I_0+K_0+K_1+K_2+K_3+K_4)^4,\end{aligned} \tag{3.39}$$

$$\|\partial_x^2\eta'_{p+1}(t)\|^2 \leqslant C(\|\partial_x^2\eta_{p+1}(t)\|^2 + \|g_{0xx}(x,U^p)\|^2) \leqslant C(1+T_1^2)(1+I_0+K_0+K_1+K_2+K_3+K_4)^6. \tag{3.40}$$

Thus, η_{p+1} satisfies (3.22) if

$$K_5 = 2C(1+T_1^2)(1+I_0+K_0+K_1+K_2+K_3+K_4)^6 \tag{3.41}$$

and if T_* satisfies

$$T_* \leqslant \frac{1}{\sqrt{L_1}}, \tag{3.42}$$

where

$$L_1 = \min\{3C(1+I_0+K_0+K_1+K_2)^4,\ 3C(1+I_0+K_0+K_1+K_2+K_3+K_4)^6\}. \tag{3.43}$$

Step 4: estimates for ψ_{p+1}. We multiply the differential equation in (3.2) by ψ'_{p+1} and integrate the resulted equations over $(0,1)\times(0,T_*)$ and integrate by parts. In view of (3.27) and (3.22), we obtain

$$\begin{aligned}\|\psi_{p+1}(t)\|^2 &+ \|\psi'_{p+1}(t)\|^2 + \nu\|\partial_x^2\psi_{p+1}(t)\|^2 \\ &\leqslant \left(\|\theta_1\|^2 + \int_0^t \|g_1\|^2 ds\right)\exp\left\{\int_0^t (2+2\|\eta_{p+1}(t)\|_2^2 + 2\|u_0\|_2^2)ds\right\} \\ &\leqslant (1+I_0+T_*L_3)e^{T_*(6+2I_0)},\end{aligned} \tag{3.44}$$

where

$$L_3 = C(1+I_0+K_0+K_1+K_2+K_4)^6.$$

We have (3.16) if

$$K_1 = 4(1+I_0), \tag{3.45}$$

and

$$T_* \leqslant \min\left\{\frac{\ln 2}{3+I_0}, \frac{1}{L_3}\right\}.$$

By differentiating the evolution equation in (3.2) with respect to t, then multiplying the resulted equation by ψ'_{p+1} and ψ''_{p+1} and integrating over $(0,1)\times(0,T_*)$ respectively, and integrating by parts, we have from (3.29) and (3.22),

$$\begin{aligned}\|\psi''_{p+1}(t)\|^2 &+ 2\nu\|\partial_x^2\psi'_{p+1}(t)\|^2 \\ &\leqslant \left(3\|\psi''_{p+1}(0)\|^2 + 3\|\theta_1\|^2 + 2\nu\|\partial_x^2\theta\|^2 + 3\int_0^t \|g'_1(\cdot,U^p)(t)\|^2 ds\right) \\ &\quad \cdot \exp\left\{\max(2,\frac{2\tau}{\tau+1},\frac{1}{\tau})\int_0^t (\frac{11}{2} + 3\|\eta'_{p+1}\|^2 + 3\|\eta_{p+1}\|_2^2 + 3\|u_0\|_2^2)ds\right\} \\ &\leqslant C(1+I_0+T_*L_4)e^{T_*L_5},\end{aligned} \tag{3.46}$$

where

$$L_4 = C(1 + I_0 + K_0 + K_1 + K_2 + K_4 + K_5)^8, \quad L_5 = 3\max(2, \tfrac{1}{\tau})(\tfrac{23}{6} + K_5). \tag{3.47}$$

We have (3.21) if

$$K_4 = 2C(1 + I_0), \tag{3.48}$$

and

$$T_* \leqslant \min\left\{\frac{1}{L_4}, \frac{\ln 2}{L_5}\right\}.$$

Applying ∂_x to the differential equation in (3.2) with respect to x, then multiplying the resulted equation by $\partial_x\psi_{p+1}$ and $\partial_x\psi'_{p+1}$ and integrating over $(0,1)\times(0,T_*)$ respectively, and using integration by parts, we have, with the help of (3.28), (3.22) and (3.44), that

$$\begin{aligned}
&\|\partial_x\psi'_{p+1}(t)\|^2 + \left(\frac{1}{\tau}+1\right)\|\partial_x\psi_{p+1}(t)\|^2 + 2\nu\|\partial_x^3\psi_{p+1}(t)\|^2 \\
&\leqslant \left\{2\|\theta_{1x}\|^2 + \int_0^t \left[3\|g_{1x}(\cdot,U^p)(t)\|^2 + (3 + 6\|\partial_x\eta_{p+1}(t)\|^2 + 6\|u_{0x}\|^2)\|\psi'_{p+1}\|^2\right] ds\right\} \\
&\quad \cdot \exp\left\{\max(4, \frac{2\tau}{1+\tau}) \int_0^t (5 + 3\|\eta_{p+1}(t)\|_1^2 + 3\|u_0\|_1^2) ds\right\} \\
&\leqslant C(1 + I_0 + T_* L_6)e^{4(8+3I_0)T_*}
\end{aligned} \tag{3.49}$$

where

$$L_6 = 3C(1 + I_0 + K_0 + K_1 + K_2 + K_4)^6 + (15 + 6I_0)(1 + I_0 + T_1 L_3)e^{T_1(6+2I_0)}.$$

We have (3.21) if

$$K_2 = C(1 + I_0), \tag{3.50}$$

and

$$T_* \leqslant \min\left\{\frac{1}{L_6}, \frac{\ln 2}{4(8+3I_0)}\right\}.$$

Differentiating the differential equation in (3.2) with respect to x and t, then multiplying the resulted equation by $\partial_x\psi'_{p+1}$ and $\partial_x\psi''_{p+1}$ and integrating over $(0,1)\times(0,T_*)$ respectively, and using the integration by parts, we get, with the help of (3.30), (3.22) and (3.46), that

$$\begin{aligned}
\|\partial_x\psi''_{p+1}(t)\|^2 &\leqslant \Big(3\|\partial_x\psi''_{p+1}(0)\|^2 + (\frac{1}{\tau}+3)\|\theta_{1x}\|^2 + 2\nu\|\partial_x^3\theta_1\|^2 \\
&\quad + \int_0^t (3\|g'_{1x}(\cdot,U^p)(t)\|^2 + 3\|\eta'_{p+1}(t)\|^2 + 2\|\eta_{p+1}(t)\|^2 + 2\|u_0\|^2 + 4\|\partial_x^2\psi'_{p+1}(t)\|^2) ds\Big) \\
&\quad \cdot \exp\left\{2\int_0^t (3 + 2\|\eta_{p+1}(t)\|_1^2 + 2\|u_0\|_1^2 + \|\eta'_{p+1}(t)\|_1^2 + \frac{3}{2}\|\partial_x^2\psi'_{p+1}(t)\|^2) ds\right\} \\
&\leqslant C(1 + I_0 + T_*(1 + I_0 + K_5 + L_7))e^{T_* L_8}
\end{aligned} \tag{3.51}$$

where

$$L_7 = C(1 + I_0 + K_0 + K_1 + K_2 + K_3 + K_4 + K_5)^8, \tag{3.52}$$

$$L_8 = 2(7 + 2I_0 + K_5 + \frac{3C}{4\nu}(1 + I_0 + T_1 L_4)e^{T_1 L_5}). \tag{3.53}$$

We have (3.20) if

$$T_* \leqslant \min\{\frac{1}{K_5}, \frac{\ln 2}{L_8}\}.$$

From (3.2), (3.46), (3.44), (3.31), (3.49) and (3.27), we obtain

$$\begin{aligned}\|\partial_x^4 \psi_{p+1}(t)\|^2 \leqslant\, & C(\|\psi''_{p+1}(t)\|^2 + \|\psi'_{p+1}(t)\|^2 + \|\psi_{p+1}(t)\|^2 \\ & + \|\kappa(x, U^p)(t)\partial_x \psi'_{p+1}(t)\|^2 + \|g_1(x, U^p)(t)\|^2) \\ \leqslant\, & C(1 + I_0 + T_* L_4)e^{T_* L_5} + 2(1 + I_0 + T_* L_3)e^{T_*(6+2I_0)} + L_3 \\ & + C(1 + I_0)^2 C(1 + I_0 + T_* L_4)e^{T_* L_5} \\ & + C(1 + I_0 + T_* L_6)e^{4(8+3I_0)T_*}. \end{aligned} \tag{3.54}$$

Differentiating the differential equation in (3.2) with respect to x, integrating over (0,1), and employing (3.51), (3.49), (3.46) and (3.28), we can estimate $\partial_x^5 \psi_{p+1}$ as

$$\begin{aligned}\|\partial_x^5 \psi_{p+1}(t)\|^2 \leqslant\, & C(\|\partial_x \psi''_{p+1}(t)\|^2 + \|\partial_x \psi'_{p+1}(t)\|^2 + \|\partial_x \psi_{p+1}(t)\|^2 \\ & + \|\kappa_x(x, U^p)(t)\partial_x \psi'_{p+1}(t)\|^2 + \|\kappa(x, U^p)(t)\partial_x^2 \psi'_{p+1}(t)\|^2 + \|g_{1x}(x, U^p)(t)\|^2) \\ \leqslant\, & C(1 + I_0 + T_*(1 + I_0 + K_5 + L_7))e^{T_* L_8} + 2C(1 + I_0 + T_* L_6)e^{4(8+3I_0)T_*} \\ & + C(1 + I_0)^2(C(1 + I_0 + T_* L_4)e^{T_* L_5} \\ & + C(1 + I_0 + T_* L_4)e^{T_* L_5}) + L_3. \end{aligned} \tag{3.55}$$

We have (3.21) if

$$K_3 = L_3 + K_1 + C(1 + I_0 + L_3)^2 K_4, \tag{3.56}$$

and

$$T_* \leqslant \min\left\{\frac{1}{L_4}, \frac{\ln 2}{3\max(2, \frac{1}{\tau})(\frac{23}{6} + K_5)}\right\}.$$

Thus, we take

$$\begin{aligned} T_* = \min\Bigg\{ & \frac{\|\theta_1\|^2}{C_0^2(I_0 + 1)^2}, \frac{\ln 2}{2(3 + I_0)}, \frac{1}{C_0(I_0 + 1)}, \frac{1}{\sqrt{L_1}}, \frac{1}{L_3}, \frac{1}{L_4}, \frac{\ln 2}{L_5}, \frac{1}{L_6}, \\ & \frac{\ln 2}{4(8 + 3I_0)}, \frac{1}{K_5}, \frac{\ln 2}{L_8}, \frac{\ln 2}{3\max\left(2, \frac{1}{\tau}\right)\left(\frac{23}{6} + K_5\right)}\Bigg\}. \end{aligned} \tag{3.57}$$

Now we choose the constants K_i as follows. Let K_1 be given by (3.45), K_2 by (3.50), K_3 by (3.56), K_4 by (3.48), K_5 by (3.41). The constant T_* is determined by (3.57). This shows that (ψ_{p+1}, η_{p+1}) satisfies (3.19)-(3.22) for $t \in [0, T_*]$.

Step 5: the end of the proof. The uniform boundedness of $E_{p+1} \in C^1([0, T_*]; H^2)$ of (3.14) follows from similar computation as (3.33), where the index p is replaced by $p+1$.

By induction, we conclude that $\{U^p\}_{p=1}^{\infty}$ is uniformly bounded in $[0, T_*]$ with T_* given by (3.57) and satisfies (3.5)-(3.6) uniformly for

$$M_0 = \max\{K_0, K_1, K_2, K_3, K_4, K_5\}.$$

The proof of Lemma 3.1 is complete.

Proof of Theorem 1.1 By Lemma 3.1, the sequence $\{U^p\}_{p=1}^{\infty}$ satisfies (3.5)-(3.6) uniformly in $[0, T_*]$. By applying the Ascoli-Arzela theorem and the Aubin lemma to $\{U^p\}_{p=1}^{\infty}$, it follows that there exists $U = (\psi, \eta, E, \varphi)$ satisfying

$$\eta \in C^1([0, T_*]; H^2), \quad E \in C^1([0, T_*]; H^2),$$

$$\psi \in C^l([0, T_*]; H^{5-2l}), \quad l = 0, 1, 2, \tag{3.58}$$

and (maybe after extracting a subsequence)

$$\psi_p \xrightarrow{p \to \infty} \psi \quad \text{strongly in } C^l([0, T_*]; H^{5-2l-\sigma}), \quad l = 0, 1, 2, \tag{3.59}$$

$$\eta_p \xrightarrow{p \to \infty} \eta \quad \text{strongly in } C^1([0, T_*]; H^{2-\sigma}), \tag{3.60}$$

$$E_p \xrightarrow{p \to \infty} E \quad \text{strongly in } C^1([0, T_*]; H^{2-\sigma}), \tag{3.61}$$

for any $\sigma > 0$. Note here, as in [11], we can prove that the whole approximate solution sequence converges as $p \to \infty$.

It is not difficult to verify that U is a solution of (3.1)-(3.3) where U^p is replaced by U. Setting

$$w = w_0 + \psi > 0, \quad u = u_0 + \eta, \quad \phi_x = \phi_{0x} + E,$$

we see that $u \in C([0, T_*]; H^4)$ from (3.58) and (1.13), (w, u, ϕ) ($(\rho = w^2, u, \phi)$, respectively) is a local-in-time solution of the IBVP (1.13)-(1.17) ((1.8)-(1.12), respectively). The proof of Theorem 1.1 is complete.

Acknowledgement: The authors would like to express their great thanks to Professor Ling Hsiao for her valuable suggestions and frequent encouragement for this work. They are also grateful to the referee for helpful suggestions.

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