J. Partial Diff. Eqs. **17**(2004), 283–288
©International Academic Publishers
Vol.17, No.3

ON THE INITIAL VALUE PROBLEM FOR THE BIPOLAR SCHRÖDINGER-POISSON SYSTEM

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(Received Feb. 16, 2004)

Abstract In this paper, we prove the existence and uniqueness of global solutions in $H^s(\mathbb{R}^3)$ ($s \in \mathbb{R}, s \geqslant 0$) for the initial value problem of the bipolar Schrödinger-Poisson systems.

Key Words Schrödinger-Poisson system; Strichartz' estimates; initial value problem; H^s-solution.

2000 MR Subject Classification 35Q55.

Chinese Library Classification O175.28.

1. Introduction

In the present paper, we study the global existence and uniqueness of solutions for the initial value problem to the (pure state) bipolar Schrödinger-Poisson systems

$$i\partial_t\psi = -\Delta\psi + V\psi, \tag{1.1a}$$

$$i\partial_t\phi = -\Delta\phi - V\phi, \tag{1.1b}$$

$$-\Delta V = |\psi|^2 - |\phi|^2, \tag{1.1c}$$

$$\psi(0,x) = \psi_0, \ \phi(0,x) = \phi_0, \tag{1.1d}$$

where $\psi = \psi(t,x)$ and $\phi = \phi(t,x) : \mathbb{R}^{1+3} \to \mathbb{C}$, Δ is the Laplacian operator on $\mathbb{R}^3$, and the electrostatic potential $V = V(\psi, \phi)$ is a real function. This system appears in

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quantum mechanics, semi-conductor and plasma physics. A large amount of interesting works has been devoted to the study for the Schrödinger-Poisson systems (see [1-4] and references therein). In [3], Castella proved the global existence and uniqueness of solutions in $H^m(m \in \mathbb{Z},\ m \geqslant 0)$ for the mixed-state unipolar Schrödinger-Poisson systems. And in [4], Jüngel and Wang discussed the combined semi-classical and quasineutral limit in the bipolar defocusing nonlinear Schrödinger-Poisson system in the whole space.

First, we introduce some notations. For any $p \in [2,\infty)$, we denote $\frac{1}{\gamma(p)} = \frac{3}{2}(\frac{1}{2}-\frac{1}{p})$. $S(t)$ denotes the unitary group generated by $i\Delta$ in $L^2(\mathbb{R}^3)$. For $p \in [1,\infty]$, we denote by p' the conjugate exponent of p, defined by $1/p+1/p'=1$. $\bar{z}$ denotes the conjugate of the complex number z. H^s_p or $\dot{H}^s_p$ (resp. $B^s_{p,2}$ or $\dot{B}^s_{p,2}$) denotes the inhomogeneous or homogeneous Sobolev (Besov) space respectively.

Now we state the main result of this paper as follows.

Theorem 1.1 *Let $s \in \mathbb{R}$, $s \geqslant 0$. Let $a \in [2, \frac{18}{7}]$. Assume that ψ_0, $\phi_0 \in H^s(\mathbb{R}^3)$. Then, there exists a unique solution of the IVP (1.1) such that (ψ,ϕ)*

$$\psi, \phi \in C(\mathbb{R}; H^s(\mathbb{R}^3)) \cap L^{\gamma(a)}_{loc}(\mathbb{R}; B^s_{a,2}(\mathbb{R}^3)). \tag{1.2}$$

Moreover, when s is an integer, the result in (1.2) also holds with the Besov space $B^s_{a,2}$ replaced by H^s_a.

Remark The result that we prove here for the single bipolar Schrödinger-Poisson system can be extended to the mixed-state bipolar Schrödinger-Poisson system within the same framework.

2. Global Existence

By (1.1c), we have the potential

$$V(t,x) = \frac{1}{4\pi}\cdot\frac{1}{r} * (|\psi|^2 - |\phi|^2), \tag{2.1}$$

where $r := |x|$. Now we recall the lemma needed to estimate $V(\psi,\phi)\psi$ and $V(\psi,\phi)\phi$.

Lemma 2.1([5, Lemma 1.1]) *Let $0 \leqslant s < \infty$, $1 \leqslant r' < \infty$. Assume that $l_k, m_k, p_k, q_k > 0$ satisfy*

$$\frac{1}{r'} = \frac{1}{l_k} + \frac{1}{m_k} = \frac{1}{p_k} + \frac{1}{q_k}, \quad k = 0, 1, ..., [s]. \tag{2.2}$$

Then there exists a constant $C > 0$ dependent only on r', n, s such that

$$\|uv\|_{\dot{B}^s_{r',2}} \leqslant C\sum_{k=0}^{[s]}(\|u\|_{\dot{H}^k_{p_k}}\|v\|_{\dot{B}^{s-k}_{q_k,2}} + \|u\|_{\dot{B}^{s-k}_{l_k,2}}\|v\|_{\dot{H}^k_{m_k}}), \tag{2.3}$$

where $[s]$ *denotes the maximal integer that is less than or equal to* s.

If s is an integer, we also have

$$\|uv\|_{\dot H^s_{r'}} \leqslant C\sum_{k=0}^{s}\|u\|_{\dot H^k_{p_k}}\|v\|_{\dot H^{s-k}_{q_k}}, \tag{2.4}$$

with

$$\frac{1}{r'}=\frac{1}{p_k}+\frac{1}{q_k},\quad k=0,1,\dots,s. \tag{2.5}$$

Notice that if $s>0$, we have $B^s_{r,2}=L^r\cap \dot B^s_{r,2}$ and $H^s_r=L^r\cap \dot H^s_r$, we can obtain the following

Remark 2.2 Both (2.3) and (2.4) hold true if we replace the homogeneous space by the corresponding inhomogeneous space.

By the equivalent norm of $\dot B^s_{r,2}$ (cf. [5, 6])

$$\|u\|_{\dot B^s_{r,2}}=\left(\int_0^\infty t^{-2(s-[s])}\sum_{|\alpha|=[s]}\sup_{|h|\leqslant t}\|\Delta_h D^\alpha u\|^2_{L^r}\frac{dt}{t}\right)^{1/2}, \tag{2.6}$$

where $\Delta_h u(\cdot)=u(\cdot+h)-u(\cdot)$ and the Hardy-Littlewood-Sobolev inequality, we get

Lemma 2.3 *Let* $r=|x|$, $0\leqslant s<\infty$. *Assume that* $q>0$ *satisfies*

$$\frac{1}{p}=\frac{1}{q}+\frac{1}{3}-1. \tag{2.7}$$

Then, we have

$$\text{(i)}\quad \|\frac{1}{r}*u\|_{B^s_{p,2}}\leqslant C\|u\|_{B^s_{q,2}}, \tag{2.8}$$

$$\text{(ii)}\quad \|\frac{1}{r}*u\|_{H^s_p}\leqslant C\|u\|_{H^s_q}. \tag{2.9}$$

We have the following estimates

Lemma 2.4 *Let* $\frac{1}{p'}=\frac{3}{a}+\frac{1}{3}-1$, $u,v,w\in L^{\gamma(a)}(0,T;B^s_{a,2}(\mathbb{R}^3))$. *Then we have*

$$\begin{aligned}\|(\frac{1}{r}*uv)w\|_{L^{\gamma(p)'}(0,T;B^s_{p',2})}\leqslant CT^{\frac12}\ &\|u\|_{L^{\gamma(a)}(0,T;B^s_{a,2})}\\ &\cdot\ \|v\|_{L^{\gamma(a)}(0,T;B^s_{a,2})}\|w\|_{L^{\gamma(a)}(0,T;B^s_{a,2})}.\end{aligned} \tag{2.10}$$

Proof By Remark 2.2 and Lemma 2.3, we have

$$\|(\frac{1}{r}*uv)w\|_{B^s_{p',2}}\leqslant C\sum_{k=0}^{[s]}\left(\|\frac{1}{r}*uv\|_{H^k_{p_k}}\|w\|_{B^{s-k}_{q_k,2}}+\|\frac{1}{r}*uv\|_{B^{s-k}_{p_k,2}}\|w\|_{H^k_{q_k}}\right)$$

$$\leqslant C\sum_{k=0}^{[s]}\left(\|uv\|_{H^k_{r_k}}\|w\|_{B^{s-k}_{q_k,2}}+\|uv\|_{B^{s-k}_{r_k,2}}\|w\|_{H^k_{q_k}}\right)$$

$$\leqslant C\sum_{k=0}^{[s]}\left\{\sum_{k_1=0}^{k}\|u\|_{H^{k_1}_{2r_k}}\|v\|_{H^{k-k_1}_{2r_k}}\|w\|_{B^{s-k}_{q_k,2}}\right.$$

$$\left.+\sum_{k_2=0}^{[s]-k}\left(\|u\|_{H^{k_2}_{2r_k}}\|v\|_{B^{s-k-k_2}_{2r_k,2}}+\|u\|_{B^{s-k-k_2}_{2r_k,2}}\|v\|_{H^{k_2}_{2r_k}}\right)\|w\|_{H^k_{q_k}}\right\}$$

$$\leqslant C\sum_{k=0}^{[s]}\left\{\sum_{k_1=0}^{k}\|u\|_{H^s_{2r_k}}\|v\|_{H^s_{2r_k}}\|w\|_{B^s_{q_k,2}}\right.$$

$$\left.+\sum_{k_2=0}^{[s]-k}\left(\|u\|_{H^s_{2r_k}}\|v\|_{B^s_{2r_k,2}}+\|u\|_{B^s_{2r_k,2}}\|v\|_{H^s_{2r_k}}\right)\|w\|_{H^s_{q_k}}\right\}$$

where $\frac{1}{p'}=\frac{1}{p_k}+\frac{1}{q_k}$, $\frac{1}{p_k}=\frac{1}{r_k}+\frac{1}{3}-1$.

Let $q_k=2r_k=a$, i.e. $\frac{1}{p'}=\frac{3}{a}+\frac{1}{3}-1$. By the Sobolev embedding, we obtain

$$\|(\frac{1}{r}*uv)w\|_{B^s_{p',2}}\leqslant C\|u\|_{B^s_{a,2}}\|v\|_{B^s_{a,2}}\|w\|_{B^s_{a,2}}.$$

Since

$$\frac{1}{\gamma(p)'}=1-\frac{3}{2}(\frac{1}{p'}-\frac{1}{2})=1-\frac{3}{2}(\frac{3}{a}+\frac{1}{3}-1-\frac{1}{2})=\frac{1}{2}+\frac{3}{\gamma(a)},$$

we have the desired result.

We will use the following Strichartz' estimates derived in [5, 7, 8].

Lemma 2.5 *Let $2\leqslant r,\ q\leqslant 6$, $S(t)=e^{i\Delta t}$. Then there exists a constant $C>0$ such that*

$$\|S(t)\varphi\|_{L^{\gamma(r)}(0,\infty;B^s_{r,2})}\leqslant C\|\varphi\|_{H^s},\tag{2.11}$$

$$\|\int_{\tau<t}S(t-\tau)f(\tau)d\tau\|_{L^{\gamma(r)}(0,T;B^s_{r,2})}\leqslant C\|f\|_{L^{\gamma(q)'}(0,T;B^s_{q',2})},\tag{2.12}$$

for all $\varphi\in H^s$, $f\in L^{\gamma(q)'}(0,T;B^s_{q',2})$ and any $0<T\leqslant\infty$, where $1/p+1/p'=1$. Both (2.11) and (2.12) hold true if we replace the homogeneous space with the corresponding inhomogeneous space when s is an integer. And the constant C in (2.11) and (2.12) is independent of $r,\ q\in[2,6]$ (cf. [5, 7, 8]).

Proof of Theorem 1.1 We first prove the local existence. It is sufficient to consider the integral equations

$$\psi(t)=S(t)\psi_0+i\int_0^t S(t-\tau)V(\psi(\tau),\phi(\tau))\psi(\tau)d\tau,\tag{2.13}$$

$$\phi(t) = S(t)\phi_0 - i\int_0^t S(t-\tau)V(\psi(\tau),\phi(\tau))\phi(\tau)d\tau. \tag{2.14}$$

Define the workspace $(\mathcal{L}, d)$

$$\mathcal{L} := \left\{(\psi,\phi) : \ \psi,\phi \in L^{\gamma(a)}(0,T;B^s_{a,2}),\ \|(\psi,\phi)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} \leqslant M\right\}$$

with the metric

$$d((\psi,\phi),\ (\psi_1,\phi_1)) = \|(\psi-\psi_1,\phi-\phi_1)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})},$$

which is obviously a Banach space. Consider the mapping $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2 : \mathcal{L} \to \mathcal{L}$ such that

$$\mathcal{T}_1 : \psi \longmapsto S(t)\psi_0 + i\int_0^t S(t-\tau)V(\psi,\phi)\psi d\tau, \tag{2.15}$$

$$\mathcal{T}_2 : \phi \longmapsto S(t)\phi_0 - i\int_0^t S(t-\tau)V(\psi,\phi)\phi d\tau. \tag{2.16}$$

By Lemmas 2.4–2.5, we have

$$\begin{aligned}
\|\mathcal{T}(\psi,\phi)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} &= \|(\mathcal{T}_1\psi,\mathcal{T}_2\phi)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} \\
&\leqslant \|\mathcal{T}_1\psi\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} + \|\mathcal{T}_2\phi\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} \\
&\leqslant C\|(\psi_0,\phi_0)\|_{H^s} + CT^{1/2}\Big(\|\psi\|^2_{L^{\gamma(a)}(0,T;B^s_{a,2})} \\
&\qquad + \|\phi\|^2_{L^{\gamma(a)}(0,T;B^s_{a,2})}\Big)\|(\psi,\phi)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} \\
&\leqslant M/2 + CT^{1/2}M^2\|(\psi,\phi)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} \\
&\leqslant M,
\end{aligned} \tag{2.17}$$

where $M := 2C\|(\psi_0,\phi_0)\|_{H^s}$ and $\|(\psi,\phi)\|_X := \|\psi\|_X + \|\phi\|_X$. Here we take T so small that $CT^{1/2}M^2 \leqslant \frac{1}{2}$. Furthermore, a straightforward computation shows that it holds

$$\|V(\psi,\phi)\psi - V(\psi_1,\phi_1)\psi_1\|_{L^{\gamma(p)'}(0,T;B^s_{p',2})} \leqslant CT^{1/2}M^2\|(\psi-\psi_1,\phi-\phi_1)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})}$$

and

$$\|V(\psi,\phi)\phi - V(\psi_1,\phi_1)\phi_1\|_{L^{\gamma(p)'}(0,T;B^s_{p',2})} \leqslant CT^{1/2}M^2\|(\psi-\psi_1,\phi-\phi_1)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})},$$

from which, we obtain

$$\begin{aligned}
\|\mathcal{T}(\psi,\phi) - \mathcal{T}(\psi_1,\phi_1)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} &= \|(\mathcal{T}_1\psi - \mathcal{T}_1\psi_1, \mathcal{T}_2\phi - \mathcal{T}_2\phi_1\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} \\
&\leqslant C\|V(\psi,\phi)\psi - V(\psi_1,\phi_1)\psi_1\|_{L^{\gamma(p)'}(0,T;B^s_{p',2})} \\
&\quad + C\|V(\psi,\phi)\phi - V(\psi_1,\phi_1)\phi_1\|_{L^{\gamma(p)'}(0,T;B^s_{p',2})} \\
&\leqslant CM^2T^{1/2}\|(\psi-\psi_1,\phi-\phi_1)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})} \\
&\leqslant \frac{1}{2}\|(\psi-\psi_1,\phi-\phi_1)\|_{L^{\gamma(a)}(0,T;B^s_{a,2})}.
\end{aligned} \tag{2.18}$$

Hence, $\mathcal{T}$ is a contracted mapping from the Banach space ($\mathcal{L}$, d) to itself. By the Banach contraction mapping principle, we know that there exists a unique solution $(\psi, \phi) \in L^{\gamma(a)}(0,T; B^s_{a,2}) \times L^{\gamma(a)}(0,T; B^s_{a,2})$ to (2.13) and (2.14). Once we obtain the local existence of solutions, we can use the standard argument to extend it to a global one satisfying

$$\psi(t,x), \phi(t,x) \in C(\mathbb{R};\ H^s(\mathbb{R}^3)) \cap L^{\gamma(a)}_{loc}(\mathbb{R};\ B^s_{a,2}(\mathbb{R}^3)),$$

and prove the uniqueness of the global solution.

Acknowledgement The authors would like to express their great thanks to Professor Ling Hsiao and Professor Peter Markowich for their valuable suggestions and frequent encouragement for this work. H.L. acknowledges the partial support from the Wittgenstein Award of P. Markowich, funded by the Austrian FWF, and from JSPS Post-doctor fellowship.

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