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THE INITIAL BOUNDARY VALUE PROBLEM FOR QUASI-LINEAR SCHRÖDINGER-POISSON EQUATIONS*

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Abstract In this article, the author studies the initial-(Dirichlet) boundary problem for a high-field version of the Schrödinger-Poisson equations, which include a nonlinear term in the Poisson equation corresponding to a field-dependent dielectric constant and an effective potential in the Schrödinger equations on the unit cube. A global existence and uniqueness is established for a solution to this problem.

Key words Quasi-linear Schrödinger-Poisson system, Dirichlet boundary conditions, global existence and uniqueness

2000 MR Subject Classification 35Q55, 82D10

1 Introduction

In the present article, we consider the self-consistent quasi-linear Schrödinger-Poisson equations (QSP) on the unit cube $\Omega := (0,1)^d$

$$\mathrm{i}\partial_t\psi_m = -\frac{1}{2}\Delta\psi_m + V\psi_m, \quad m \in \mathbb{N}, \tag{1.1}$$

$$-\nabla\cdot((\varepsilon_0+\varepsilon_1|\nabla V|^2)\nabla V) = n - n^*, \tag{1.2}$$

$$n(x,t) = \sum_{m=1}^{\infty}\lambda_m|\psi_m(x,t)|^2, \tag{1.3}$$

with the following initial data and boundary conditions

$$\psi_m(x,0) = \phi_m(x), \quad m \in \mathbb{N}, \tag{1.4}$$

$$\psi_m(x,t) = 0, \quad \text{on } \partial\Omega, \quad m \in \mathbb{N}, \tag{1.5}$$

$$V(x,t) = 0, \quad \text{on } \partial\Omega, \tag{1.6}$$

where $d \in \mathbb{N}$, $d \le 3$, $t \in \mathbb{R}$ and ε_0, $\varepsilon_1 > 0$. The wave functions $\{\psi_m(x,t)\}_{m\in\mathbb{N}}$ form a sequence of complex-valued functions. Δ is the Laplace operator on $\mathbb{R}^3$ and the electrostatic

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potential $V(x,t)$ is a real-valued function. $\{\lambda_m\}_{m\in\mathbb{N}}$ is a specified sequence of probabilities with $\sum_{m\in\mathbb{N}}\lambda_m = 1$. n^* is a given time-independent dopant density which may be represented as

$$n^* = n_D^+ - n_A^-, \tag{1.7}$$

where n_D^+ is the density of donors and n_A^- is the density of acceptors. We always look forward to seeking a solution satisfying the following charge neutrality:

$$\int_\Omega (n - n^*)\mathrm{d}x = 0. \tag{1.8}$$

This system (1.1)–(1.3) appears in the physical area of semi-conductor science and plasma in the simulation of transport of charged particle [12]. A great deal of interesting research has been devoted to the mathematical analysis for the Schrödinger-Poisson systems (see [1,2,3,5,6,7,8,9, 11,13] and references therein). In [2], Castella proved the global existence and the asymptotic behavior of solutions in the function space L^2 for the mixed-state unipolar Schrödinger-Poisson systems without the defocusing nonlinearity, and in [3], the initial boundary value problem for Schrödinger-Poisson system was considered where the system consists of a series of Schrödinger equations. In [5], by using the pseudo-conformal conservation law of the bipolar defocusing nonlinear Schrödinger-Poisson system and applying the time-space L^p-$L^{p'}$ estimate method, we established the global existence and uniqueness and large-time behavior of the solution to the bipolar defocusing nonlinear Schrödinger-Poisson system with initial data in $\Sigma := \{u \in L^2 : |x|u \in L^2\}$. In addition, the stationary solutions of (QSP) were discussed in [6]. And the boundary valued problem with periodic conditions to (QSP) was considered in [7]. In [8], by applying the estimates of a modulated energy functional and the Wigner measure method, Jüngel and Wang discussed the combined semi-classical and quasineutral limit of the bipolar nonlinear Schrödinger-Poisson in the whole space. However, to our knowledge, there was no previous result on Dirichlet boundary problem for (QSP). In this article, we study the initial-(Dirichlet) boundary problem for a high field version of the Schrödinger-Poisson system including nonlinear terms in the Poisson equation (corresponding to a field-dependent dielectric constant) and effective potentials in the Schrödinger equation on a unit cube. We prove the global existence and uniqueness of solution for this problem with the help of the Schauder fixed point theorem.

Now, let us introduce the following work spaces

$$X := \Big\{\Psi = (\psi_m)_{m\in\mathbb{N}} : \psi_m \in L^2(\Omega), \|\Psi\|_X = (\sum_{m\in\mathbb{N}} \lambda_m \|\psi_m\|^2_{L^2(\Omega)})^{1/2} < \infty\Big\}, \tag{1.9}$$

$$X^1 := \Big\{\Psi = (\psi_m)_{m\in\mathbb{N}} : \psi_m \in H_0^1(\Omega), \|\Psi\|_{X^1} = (\sum_{m\in\mathbb{N}} \lambda_m \|\psi_m\|^2_{H^1(\Omega)})^{1/2} < \infty\Big\} \tag{1.10}$$

and

$$X^2 := \Big\{\Psi = (\psi_m)_{m\in\mathbb{N}} : \psi_m \in H^2(\Omega)\cap H_0^1(\Omega), \|\Psi\|_{X^2} = (\sum_{m\in\mathbb{N}} \lambda_m \|\psi_m\|^2_{H^2(\Omega)})^{1/2} < \infty\Big\}. \tag{1.11}$$

We can state our result on the global existence as follows.

Theorem 1.1 Let $\Phi = (\phi_m)_{m\in\mathbb{N}} \in X^2$ and $n^* \in C^1(\overline{\Omega})$. Then, there is a unique solution (Ψ, V) of (1.1)–(1.6) such that

$$\Psi \in C^1(\mathbb{R}; X) \cap C(\mathbb{R}; X^2), \tag{1.12}$$

$$V \in C(\mathbb{R}; X^2), \tag{1.13}$$

with the conserved quantities

(i) $\|\Psi(\cdot,t)\|_X = \|\Phi(\cdot)\|_X$, (1.14)

(ii) $\|\nabla\Psi(\cdot,t)\|_X^2 + \varepsilon_0\|\nabla V(\cdot,t)\|_{L^2(\Omega)}^2 + \frac{3}{2}\varepsilon_1\|\nabla V(\cdot,t)\|_{L^4(\Omega)}^4 =$const. (1.15)

2 Basic Estimates on Quasi-linear Poisson Equation

For convenience, we first introduce some notations. Denote $\mathbf{A} = (A^i)_{i=1}^d$, $\mathbf{p} = (p_1, \cdots, p_d) \in \mathbb{R}^d$, $D_i = \partial/\partial x_i$, $D_{p_j} = \partial/\partial p_j$, $D_{\mathbf{p}} = (D_{p_1}, \cdots, D_{p_d}) = \nabla_{\mathbf{p}}$.

We now recall the following lemmas which will be used to prove our results.

Lemma 2.1(see [4, Theorem 15.19]) Let Ω be a bounded domain in $\mathbb{R}^d$ satisfying an exterior sphere condition at each point of the boundary $\partial\Omega$. Let $Q: u \mapsto \mathrm{div}\mathbf{A}(x,u,Du) + B(x,u,Du) = 0$ be a divergence structure operator with coefficients $A^i \in C^{1,\gamma}(\Omega\times\mathbb{R}\times\mathbb{R}^d)$, $i = 1,\cdots,d$, $B \in C^\gamma(\Omega\times\mathbb{R}\times\mathbb{R}^d)$, $0<\gamma<1$, satisfying the following hypotheses for all $\xi \in \mathbb{R}^d$ and $(x,z,\mathbf{p}) \in \Omega\times\mathbb{R}\times\mathbb{R}^d$,

$$\sum_{i,j} D_{p_j}A^i(x,z,\mathbf{p})\xi_i\xi_j \geqslant \nu(|z|)(1+|\mathbf{p}|)^\tau|\xi|^2 \tag{2.1}$$

where $\tau > -1$ is some real number and ν is a positive, non-increasing function on $\mathbb{R}$;

$$|D_{\mathbf{p}}\mathbf{A}(x,z,\mathbf{p})| \leqslant \mu(|z|)(1+|\mathbf{p}|)^\tau \tag{2.2}$$

where μ is a positive, non-decreasing function on $\mathbb{R}$;

$$(1+|\mathbf{p}|)|D_z\mathbf{A}| + |D_x| + |B| \le \mu(|z|)(1+|\mathbf{p}|)^{\tau+2}; \tag{2.3}$$

$$\mathbf{p}\cdot\mathbf{A}(x,z,\mathbf{p}) \geqslant |\mathbf{p}|^\alpha - |a_1 z|^\alpha - a_2^\alpha; \tag{2.4}$$

$$B(x,z,\mathbf{p})\mathrm{sign}z \le b_0|\mathbf{p}|^{\alpha-1} + |b_1 z|^{\alpha-1} + b_2^{\alpha-1} \tag{2.5}$$

where $\alpha = \tau+2$, a_1, a_2, b_0, b_1, b_2 are nonnegative constants. Then, for any function $\varphi \in C(\partial\Omega)$, there exists a solution $u \in C(\overline{\Omega}) \cap C^{2,\gamma}(\Omega)$ of the Dirichlet problem $Qu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$.

Lemma 2.2(cf. [10,Théorème I.1]) Let $\Omega \subset \mathbb{R}^d$ be an open domain with regular boundary $\partial\Omega$. Let $\mathbf{A}$ be an elliptic operator of second order

$$\mathbf{A}u = -\sum_{i,j} a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_i b_i\frac{\partial u}{\partial x_i} + cu$$

where

$$a_{ij} = a_{ji},\ b_i,\ c \in C^1(\overline{\Omega}),$$

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geqslant \nu|\xi|^2, \quad \forall\xi\in\mathbb{R}^d, \quad \text{for some}\ \ \nu > 0,$$

$$c \geqslant 0.$$

Assume that $f \in H^{1,\infty}(\Omega)$ and φ is a convex function in $C(\mathbb{R}^d)$. If there exists $v \in H^{2,p}(\Omega)(p > d)$ such that

$$\mathbf{A}v + \varphi(\nabla v) - f \le 0 \ \text{ in } \ \Omega,$$

$$v = 0 \ \text{ on } \ \partial\Omega.$$

Then, the problem

$$\mathbf{A}u + \varphi(\nabla u) - f = 0 \ \text{ in } \ \Omega,$$

$$u = 0 \ \text{ on } \ \partial\Omega.$$

admits a unique solution $u \in H_0^{1,\infty}(\Omega)$. Moreover, $u \in H^{3,p}(\Omega)$ for all $p < \infty$.

Let $A^i(x, z, \mathbf{p}) = (\varepsilon_0 + \varepsilon_1|\mathbf{p}|^2)p_i$, $B(x, z, \mathbf{p}) = n - n^*$ and $\varphi = 0$. By computing, we know that the structure conditions (2.1)–(2.5) hold. In fact,

$$\sum_{i,j} D_{p_j} A^i \xi_i \xi_j \geqslant (\varepsilon_0 + \varepsilon_1|\mathbf{p}|^2)|\xi|^2 \geqslant C(1 + |\mathbf{p}|)^2|\xi|^2,$$

$$|D_{\mathbf{p}}\mathbf{A}| = d(\varepsilon_0 + \varepsilon_1|\mathbf{p}|^2) + 2\varepsilon_1|\mathbf{p}|^2 = d\varepsilon_0 + \varepsilon_1(d+2)|\mathbf{p}|^2 \leq C(1 + |\mathbf{p}|)^2,$$

$$(1 + |\mathbf{p}|)|D_z\mathbf{A}| + |D_x\mathbf{A}| + |n - n^*| \leq \|n - n^*\|_{C(\overline{\Omega})} \leq \|n - n^*\|_{C(\overline{\Omega})}(1 + |\mathbf{p}|)^4,$$

$$\mathbf{p} \cdot \mathbf{A} = (\varepsilon_0 + \varepsilon_1|\mathbf{p}|^2)|\mathbf{p}|^2 \geqslant \varepsilon_1|\mathbf{p}|^4,$$

$$n - n^* \leqslant \|n - n^*\|_{C(\overline{\Omega})} = (\|n - n^*\|_{C(\overline{\Omega})}^{1/3})^{4-1}.$$

Then, for $n - n^* \in C^1(\overline{\Omega})$, we obtain from Lemma 2.1 that there exists a solution $V \in C(\overline{\Omega}) \cap C^{2,\gamma}(\overline{\Omega})$ of the Dirichlet problem (1.2) and (1.6).

Moreover, we have for $p > d$

$$\|\Delta V\|_{L^p(\overline{\Omega})}^p \leq C\|\Delta V\|_{C(\overline{\Omega})} \leq C\|\Delta V\|_{C^\gamma(\overline{\Omega})}$$

that is,

$$V \in H^{2,p}(\Omega).$$

By Lemma 2.2, and taking coefficients

$$a_{ij}(x) = (\varepsilon_0 + \varepsilon_1|\nabla V|^2)\delta_{ij}, \ b_i = 0, \ c = 0$$

satisfying

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geqslant (\varepsilon_0 + \varepsilon_1|\nabla V|^2)|\xi|^2 \geqslant \varepsilon_0|\xi|^2$$

and

$$\varphi = 0,$$

we obtain $V \in H^{3,p}$ $(p < \infty)$.

Lemma 2.3 Let V be a solution of equation (1.2) with the boundary condition (1.6) and $n - n^* \in H^1(\Omega)$. Then

(i) $\|\nabla V\|_{L^p(\Omega)} \leq \frac{C}{\varepsilon_0}\|n - n^*\|_{L^1(\Omega)}^{(2-4/p)/3}\|n - n^*\|_{L^2(\Omega)}^{4/p-1}$, for $2 \leq p \leq 4$, (2.6)

(ii) $\|\Delta V\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon_0^{1/2}}\|n - n^*\|_{L^1(\Omega)}^{1/6}\|n - n^*\|_{\dot{H}^1_{4/3}(\Omega)}^{1/2}$. (2.7)

Proof (i) Multiply (1.2) by V and integrate. We get

$$\begin{aligned}\int_\Omega (n - n^*)V\mathrm{d}x &= \int_\Omega -\nabla \cdot ((\varepsilon_0 + \varepsilon_1|\nabla V|^2)\nabla V)V\mathrm{d}x \\ &= \int_\Omega (\varepsilon_0 + \varepsilon_1|\nabla V|^2)|\nabla V|^2\mathrm{d}x = \varepsilon_0\|\nabla V\|_{L^2(\Omega)}^2 + \varepsilon_1\|\nabla V\|_{L^4(\Omega)}^4.\end{aligned}$$

Thus, by the Hölder and the Poincaré inequalities, we have

$$\varepsilon_0\|\nabla V\|^2_{L^2(\Omega)} \le \int_\Omega (n-n^*)V\mathrm{d}x \le \|n-n^*\|_{L^2(\Omega)}\|V\|_{L^2(\Omega)} \le C\|n-n^*\|_{L^2(\Omega)}\|\nabla V\|_{L^2(\Omega)}$$

that is,

$$\|\nabla V\|_{L^2(\Omega)} \le \frac{C}{\varepsilon_0}\|n-n^*\|_{L^2(\Omega)},$$

and

$$\varepsilon_1\|\nabla V\|^4_{L^4(\Omega)} \le \int_\Omega (n-n^*)V\mathrm{d}x \le \|n-n^*\|_{L^1(\Omega)}\|V\|_{L^\infty(\Omega)} \le C\|n-n^*\|_{L^1(\Omega)}\|\nabla V\|_{L^4(\Omega)}$$

that is,

$$\|\nabla V\|_{L^4(\Omega)} \le \frac{C}{\varepsilon_1}\|n-n^*\|^{1/3}_{L^1(\Omega)},$$

Thus, by interpolation, we get the desired result (2.6).

(ii) Multiply (1.2) by $-\Delta V$ and integrate. By applying the proceeding assertions and the boundary condition (1.6), we arrive at

$$\begin{aligned}
-\int_\Omega (n-n^*)\Delta V\mathrm{d}x &= \int_\Omega \nabla\cdot((\varepsilon_0+\varepsilon_1|\nabla V|^2)\nabla V)\Delta V\mathrm{d}x \\
&= \varepsilon_0\|\Delta V\|^2_{L^2(\Omega)} + \varepsilon_1\int_\Omega \nabla\cdot(|\nabla V|^2\nabla V)\Delta V\mathrm{d}x \\
&= \varepsilon_0\|\Delta V\|^2_{L^2(\Omega)} + \varepsilon_1\sum_{i,j}\int_\Omega \partial_{x_i}(|\nabla V|^2\partial_{x_i}V)\partial_{x_ix_j}V\mathrm{d}x \\
&= \varepsilon_0\|\Delta V\|^2_{L^2(\Omega)} + \varepsilon_1\sum_{i=j}\int_\Omega \partial_{x_j}(|\nabla V|^2\partial_{x_i}V)\partial_{x_jx_i}V\mathrm{d}x \\
&\quad -\varepsilon_1\sum_{i\neq j}\int_\Omega |\nabla V|^2\partial_{x_i}V\partial_{x_jx_jx_i}V\mathrm{d}x \\
&= \varepsilon_0\|\Delta V\|^2_{L^2(\Omega)} + \varepsilon_1\sum_{i,j}\int_\Omega \partial_{x_j}(|\nabla V|^2\partial_{x_i}V)\partial_{x_jx_i}V\mathrm{d}x \\
&= \varepsilon_0\|\Delta V\|^2_{L^2(\Omega)} + \varepsilon_1\sum_{i,j}\int_\Omega |\nabla V|^2(\partial_{x_ix_j}V)^2\mathrm{d}x \\
&\quad +\varepsilon_1\sum_{i,j}\int_\Omega \partial_{x_j}|\nabla V|^2\cdot\frac{1}{2}\partial_{x_j}|\partial_{x_i}V|^2\mathrm{d}x \\
&= \varepsilon_0\|\Delta V\|^2_{L^2(\Omega)} + \varepsilon_1\sum_{i,j}\int_\Omega |\nabla V|^2(\partial_{x_ix_j}V)^2\mathrm{d}x + \frac{\varepsilon_1}{2}\sum_j\int_\Omega(\partial_{x_j}|\nabla V|^2)^2\mathrm{d}x \\
&\geqslant \varepsilon_0\|\Delta V\|^2_{L^2(\Omega)}
\end{aligned}$$

This, with the help of the Hölder's inequality, yields

$$\begin{aligned}
\varepsilon_0\|\Delta V\|^2_{L^2(\Omega)} &\le -\int_\Omega (n-n^*)\Delta V\mathrm{d}x \le \|n-n^*\|_{\dot H^1_{4/3}(\Omega)}\|\Delta V\|_{\dot H^{-1}_4(\Omega)} \\
&\le \|\nabla V\|_{L^4(\Omega)}\|n-n^*\|_{\dot H^1_{4/3}(\Omega)} \le C\|n-n^*\|^{1/3}_{L^1(\Omega)}\|n-n^*\|_{\dot H^1_{4/3}(\Omega)}.
\end{aligned}$$

Hence, (2.7) follows.

3 Global Existence and Uniqueness

We give the proof of Theorem 1.1 in the following. Let us show the conserved quantities (1.14) and (1.15) first.

We use $\overline{\psi}_m$ as a multiplier to (1.1) and integrate the resulting equation over Ω. Then (1.14) follows by taking the imaginary part.

Multiplying (1.1) by $\partial_t\overline{\psi}_m$ and integrating it, we obtain

$$\int_\Omega \Big\{ -\mathrm{i}\psi_m\partial_t\overline{\psi}_m + \frac{1}{2}\nabla\psi_m\partial_t(\nabla\overline{\psi}_m) + V\psi_m\partial_t\overline{\psi}_m \Big\}\mathrm{d}x = 0.$$

Taking the real part, we get

$$\frac{1}{4}\partial_t\|\nabla\psi_m\|^2_{L^2(\Omega)} = -\frac{1}{2}\int_\Omega V\partial_t|\psi_m|^2\mathrm{d}x.$$

Thus, we have

$$\begin{aligned}\frac{1}{4}\partial_t\|\nabla\Psi\|^2_X &= -\frac{1}{2}\int_\Omega V\partial_t n\mathrm{d}x = \frac{1}{2}\int_\Omega \partial_t[\nabla\cdot((\varepsilon_0+\varepsilon_1|\nabla V|^2)\nabla V)]V\mathrm{d}x\\ &= \frac{1}{2}\int_\Omega \partial_t((\varepsilon_0+\varepsilon_1|\nabla V|^2)\nabla V)\cdot\nabla V\mathrm{d}x\\ &= -\frac{1}{4}\int_\Omega(\varepsilon_0\partial_t\|\nabla V\|^2_{L^2(\Omega)} + \frac{3}{2}\varepsilon_1\partial_t\|\nabla V\|^4_{L^4(\Omega)})\end{aligned}$$

which implies (1.15).

Denote for $R, T > 0$

$$\begin{aligned}Y_{R,T} := \{\ \Psi = (\psi_m)_{m\in\mathbb{N}} :\ &\Psi\in C^1(0,T;X)\cap L^\infty(0,T;X^2),\\ &\|\Psi\|_{C(0,T;X)} \leqslant \|\Phi\|_X,\ \|\Psi\|_{L^\infty(0,T;X^2)} \le R,\ \|\Psi\|_{C^1(0,T;X)} \le R\}.\end{aligned}$$

It is clear that $Y_{R,T}$ is a bounded closed convex set (cf.[3]).

Given $A = (a_m(t,x)) \in Y_{R,T}$, denote

$$n_A(t,x) = \sum_{m=1}^\infty \lambda_m|a_m|^2,$$

then, we have

$$\|n_A\|_{L^1} \leqslant \sum_{m=1}^\infty \lambda_m\|a_m\|^2_{L^2} \leqslant \|\Phi\|^2_X \le C,$$

$$\|n_A\|_{L^\infty} \leqslant \sum_{m=1}^\infty \lambda_m\|a_m\|^2_{L^\infty} \le \|\Phi\|^2_{X^2} \le CR^2.$$

By interpolation, we see

$$\|n_A\|_{L^p} \le CR^{2/p'},\ \forall p>1,\ 1/p+1/p'=1.$$

We replace n by n_A in the Poisson equation (1.2), then, we know that there exists a solution V of (1.2) by the statement of Section 2, and by Lemma 2.3, we have, for $2\le p\le 4$,

$$\begin{aligned}\|\nabla V\|_{L^p} &\le C\|n_A-n^*\|^{(2-4/p)/3}_{L^1}\|n_A-n^*\|^{4/p-1}_{L^2}\\ &\le C\|n_A-n^*\|^{4/p-1}_{L^2} \le CR^{4/p-1} + C\|n^*\|^{4/p-1}_{L^2}\\ &\le C+CR^{4/p-1}\end{aligned}$$

and

$$\|\Delta V\|_{L^2} \le C\|n_A - n^*\|^{1/2}_{\dot{H}^1_{4/3}} \le C(\|n_A\|^{1/2}_{\dot{H}^1_{4/3}} + \|n^*\|^{1/2}_{\dot{H}^1_{4/3}})$$
$$\le C(\|A\|^{1/2}_{X^2}\|A\|^{1/2}_{X} + C) \le C(R^{1/2}+1).$$

Let $S(t)$ be the L^2 isometry group generated by $\frac{1}{2}\mathrm{i}\Delta$. For any $t \in [0,T]$ we have, with the above V, that

$$\psi_m(t) = S(t)\phi_m - \mathrm{i}\int_0^t S(t-\tau)[V(\tau)\psi_m(\tau)]\mathrm{d}\tau, \quad \forall m \in \mathbb{N}.$$

$$\|\psi_m(t)\|_{H^2} \le \|S(t)\phi_m\|_{H^2} + \int_0^t \|S(t-\tau)[V(\tau)\psi_m(\tau)]\|_{H^2}\mathrm{d}\tau$$
$$\le \|\phi_m\|_{H^2} + \int_0^t \|V(\tau)\psi_m(\tau)\|_{H^2}\mathrm{d}\tau,$$

$$\|V(\tau)\psi_m(\tau)\|_{H^2} = \|V\psi_m\|_{L^2} + \|\nabla(V\psi_m)\|_{L^2} + \|\Delta(V\psi_m)\|_{L^2}$$
$$\le C(\|\nabla(V\psi_m)\|_{L^2} + \|\Delta V\psi_m\|_{L^2} + \|\nabla V\cdot\nabla\psi_m\|_{L^2} + \|V\Delta\psi_m\|_{L^2})$$
$$\le C(\|\nabla V\|_{L^4}\|\psi_m\|_{L^4} + \|V\|_{L^\infty}\|\nabla\psi_m\|_{L^2} + \|\Delta V\|_{L^2}\|\psi_m\|_{L^\infty}$$
$$+\|\nabla V\|_{L^4}\|\nabla\psi_m\|_{L^4} + \|V\|_{L^\infty}\|\Delta\psi_m\|_{L^2})$$
$$\le C(R^{1/2}+1)\|\psi_m\|_{H^2}$$

Thus, we have

$$\|\Psi\|_{X^2} \le \|\Phi\|_{X^2} + \int_0^t \|V(\tau)\psi_m(\tau)\|_{X^2}\mathrm{d}\tau \leqslant \|\Phi\|_{X^2} + \int_0^t C(R^{1/2}+1)\|\Psi\|_{X^2}\mathrm{d}\tau$$

$$\|\Psi_t\|_X \le \frac{1}{2}\|\Delta\Psi\|_X + \|V\Psi\|_X \le C\|\Phi\|_{X^2}\mathrm{e}^{C(1+R^{1/2})T}.$$

This implies, by the Gronwall's inequality, that

$$\max(\|\Psi_t(t)\|_X,\ \|\Psi(t)\|_{X^2}) \le C\|\Phi\|_{X^2}\mathrm{e}^{C(1+R^{1/2})T}, \quad t \in [0,T]. \tag{3.1}$$

Hence, taking $R \geqslant C\mathrm{e}\|\Phi\|_{X^2}$ and $T \leqslant \frac{C}{1+\|\Phi\|^{1/2}_{X^2}}$, we get

$$\Psi \in Y_{R,T}.$$

Now, we can define the mapping $\mathcal{T}$ from $Y_{R,T}$ to itself by

$$\mathcal{T} : A \mapsto \Psi^A.$$

In order to apply the Schauder fixed point theorem, we will subsequently prove that the mapping $\mathcal{T}$ is compact. Given any $A_k \in Y_{R,T}$, denote $\mathcal{T}A_k = \Psi^k$. By the definition of $Y_{R,T}$, we can assume that

$$A_k \rightharpoonup A \quad \text{weakly}^* \text{ in } \ L^\infty(0,T;X^2)$$

and $A \in L^\infty(0,T;X^2)$. By the compactness principle, we have

$$A_k \to A \quad \text{strongly in } \ C(0,T;X^1).$$

$$\|n_{A_k} - n_A\|_{L^2} \leq \|A_k - A\|_X \|A_k + A\|_{X^2} \leq C\|A_k - A\|_{X^1}.$$

Hence,

$$n_{A_k} \to n_A \quad \text{strongly in} \quad C(0,T;L^2(\Omega)).$$

From the Poisson equation (1.2), we have

$$-\nabla \cdot ((\varepsilon_0 + \varepsilon_1|\nabla V_{A_k}|^2)\nabla V_{A_k}) = n_{A_k} - n^*,$$

$$-\nabla \cdot ((\varepsilon_0 + \varepsilon_1|\nabla V_A|^2)\nabla V_A) = n_A - n^*,$$

that is,

$$-\nabla \cdot ((\varepsilon_0 + \varepsilon_1|\nabla V_{A_k}|^2)\nabla V_{A_k} - (\varepsilon_0 + \varepsilon_1|\nabla V_A|^2)\nabla V_A) = n_{A_k} - n_A.$$

By Lemma 3.1 as below, we get

$$\varepsilon_0\|\nabla(V_{A_k} - V_A)\|_{L^2}^2 + \frac{\varepsilon_1}{4}\|\nabla(V_{A_k} - V_A)\|_{L^4}^4 \leq \int_\Omega (n_{A_k} - n_A)(V_{A_k} - V_A)\mathrm{d}x,$$

Thus, we have

$$\|\nabla(V_{A_k} - V_A)\|_{L^2}^2 \leq C\|n_{A_k} - n_A\|_{L^2}.$$

Noticing Lemma 2.2 and the statement in Section 2, that is, V_{A_k}, $V_A \in H_p^3$, $(1 < p < \infty)$, by the interpolation theory, we have

$$\|V_{A_k} - V_A\|_{H^2} \leq C\|V_{A_k} - V_A\|_{H^1}^{1/2}\|V_{A_k} - V_A\|_{H^3}^{1/2} \leq C\|n_{A_k} - n_A\|_{L^2}^{1/2}.$$

that is,

$$V_{A_k} \to V_A \quad \text{strongly in} \quad C(0,T;H^2(\Omega) \cap H_0^1(\Omega)).$$

By the integral equation, we have

$$\psi_m^k(t) - \psi_m^A(t) = -\mathrm{i}\int_0^t S(t-\tau)[V_{A_k}(\tau)\psi_m^k(\tau) - V_A\psi_m^A(\tau)]\mathrm{d}\tau.$$

Similar to (3.1), it yields to

$$\|\Psi^k - \Psi^A\|_{X^2} \leqslant \|\Psi^k\|_{X^2}\|V_{A_k} - V_A\|_{H^2}e^{\|V_A\|_{H^2}},$$

that is,

$$\Psi^k \to \Psi^A \quad \text{strongly in} \quad L^\infty(0,T;X^2),$$

and $\Psi^A \in Y_{R,T}$.

From the above convergence and the Schrödinger equation (1.1), we know that $\mathcal{T}$ is a compact mapping. Therefore, by the Schauder fixed point theorem, there exists a fixed point Ψ such that Ψ, the corresponding V and n satisfy (QSP) with (1.4)−(1.6) for $t \in [0,T]$. By the standard argument, we can extend the local solution to a global one.

Finally, we give the uniqueness of (QSP) with (1.4)−(1.6). We first state the following lemma which will be used to prove the uniqueness.

Lemma 3.1 For any two functions u, $v \in H^{1,4}(\Omega)$, we have

$$\begin{aligned}&\int_\Omega \{(\varepsilon_0 + \varepsilon_1|\nabla u|^2)\nabla u - (\varepsilon_0 + \varepsilon_1|\nabla v|^2)\nabla v\} \cdot \nabla(u-v)\mathrm{d}x \\ &\geqslant \varepsilon_0\|\nabla(u-v)\|_{L^2(\Omega)}^2 + \frac{\varepsilon_1}{4}\|\nabla(u-v)\|_{L^4(\Omega)}^4.\end{aligned} \tag{3.2}$$

Proof Let $\mathbf{p} = \nabla u$, $\mathbf{q} = \nabla v$. By using the Young's inequality, we arrive at

$$\begin{aligned}\langle|\mathbf{p}-\mathbf{q}|^2, |\mathbf{p}-\mathbf{q}|^2\rangle &= \langle|\mathbf{p}|^2+|\mathbf{q}|^2-2\mathbf{p}\cdot\mathbf{q}, |\mathbf{p}|^2+|\mathbf{q}|^2-2\mathbf{p}\cdot\mathbf{q}\rangle \\ &= 4\langle|\mathbf{p}|^2\mathbf{p}-|\mathbf{q}|^2\mathbf{q}, \mathbf{p}-\mathbf{q}\rangle - 3\langle|\mathbf{p}|^2,|\mathbf{p}|^2\rangle - 3\langle|\mathbf{q}|^2,|\mathbf{q}|^2\rangle \\ &\quad +2\langle|\mathbf{p}|^2,|\mathbf{q}|^2\rangle + 4\langle\mathbf{p}\cdot\mathbf{q},\mathbf{p}\cdot\mathbf{q}\rangle \\ &\le 4\langle|\mathbf{p}|^2\mathbf{p}-|\mathbf{q}|^2\mathbf{q}, \mathbf{p}-\mathbf{q}\rangle\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Omega)$. Thus, (3.2) follows.

Next, we turn to the proof of the uniqueness. Let $(\Psi^{(1)}, V^{(1)})$ and $(\Psi^{(2)}, V^{(2)})$ be two solutions where $\Psi^{(1)} = (\psi_m^{(1)})_{m\in\mathbb{N}}$ and $\Psi^{(2)} = (\psi_m^{(2)})_{m\in\mathbb{N}}$. Let $\widetilde{\psi}_m = \psi_m^{(1)} - \psi_m^{(2)}$ and $\widetilde{\Psi} = (\widetilde{\psi}_m)_{m\in\mathbb{N}}$. Then we have

$$\mathrm{i}\partial_t\psi_m^{(1)} = -\frac{1}{2}\Delta\psi_m^{(1)} + V^{(1)}\psi_m^{(1)}, \tag{3.3}$$

$$\mathrm{i}\partial_t\psi_m^{(2)} = -\frac{1}{2}\Delta\psi_m^{(2)} + V^{(2)}\psi_m^{(2)}, \tag{3.4}$$

$$\mathrm{i}\partial_t\widetilde{\psi}_m = -\frac{1}{2}\Delta\widetilde{\psi}_m + (V^{(1)}\psi_m^{(1)} - V^{(2)}\psi_m^{(2)}). \tag{3.5}$$

Multiplying (3.5) by $\overline{\widetilde{\psi}}_m$, we get

$$\begin{aligned}\mathrm{i}\partial_t\widetilde{\psi}_m\overline{\widetilde{\psi}}_m &= -\frac{1}{2}\Delta\widetilde{\psi}_m\overline{\widetilde{\psi}}_m + (V^{(1)}\psi_m^{(1)} - V^{(2)}\psi_m^{(2)})(\overline{\psi}_m^{(1)} - \overline{\psi}_m^{(2)}) \\ &= -\frac{1}{2}\Delta\widetilde{\psi}_m\overline{\widetilde{\psi}}_m + V^{(1)}|\psi_m^{(1)}|^2 - V^{(1)}\psi_m^{(1)}\overline{\psi}_m^{(1)} - V^{(2)}\psi_m^{(2)}\overline{\psi}_m^{(1)} + V^{(2)}|\psi_m^{(2)}|^2\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}V^{(1)}\psi_m^{(1)}\overline{\psi}_m^{(2)} + V^{(2)}\psi_m^{(2)}\overline{\psi}_m^{(1)} &= (\mathrm{i}\partial_t\psi_m^{(1)} + \frac{1}{2}\Delta\psi_m^{(1)})\overline{\psi}_m^{(2)} + (\mathrm{i}\partial_t\psi_m^{(2)} + \frac{1}{2}\Delta\psi_m^{(2)})\overline{\psi}_m^{(1)} \\ &= \mathrm{i}(\partial_t\psi_m^{(1)}\overline{\psi}_m^{(2)} + \partial_t\psi_m^{(2)}\overline{\psi}_m^{(1)}) + \frac{1}{2}(\Delta\psi_m^{(1)}\overline{\psi}_m^{(2)} + \Delta\psi_m^{(2)}\overline{\psi}_m^{(1)}).\end{aligned}$$

Integrating (3.6) over Ω and taking the imaginary part, we get

$$\partial_t\|\widetilde{\psi}_m\|^2_{L^2(\Omega)} + \mathrm{Re}\int_\Omega(\partial_t\psi_{\mathrm{m}}^{(1)}\overline{\psi}_{\mathrm{m}}^{(2)} + \partial_t\psi_{\mathrm{m}}^{(2)}\overline{\psi}_{\mathrm{m}}^{(1)})\mathrm{dx} = 0. \tag{3.7}$$

By computing, we know

$$\mathrm{Re}(\partial_t\psi_{\mathrm{m}}^{(1)}\overline{\psi}_{\mathrm{m}}^{(2)} + \partial_t\psi_{\mathrm{m}}^{(2)}\overline{\psi}_{\mathrm{m}}^{(1)}) = \frac{1}{2}\partial_t[|\psi_{\mathrm{m}}^{(1)}|^2 + |\psi_{\mathrm{m}}^{(2)}|^2 - |\widetilde{\psi}_{\mathrm{m}}|^2].$$

Thus, we obtain

$$\partial_t\|\widetilde{\psi}_m\|^2_{L^2(\Omega)} = 0.$$

Noticing that $\widetilde{\psi}_m(x,0) = 0$, we have

$$\|\widetilde{\psi}_m\|_{L^2(\Omega)} = 0, \quad \forall m \in \mathbb{N}.$$

Hence, $\Psi^{(1)} = \Psi^{(2)}$ in the sense of $L^2(\Omega)$. We also have $n(\Psi^{(1)}) = n(\Psi^{(2)}) =: n$ and

$$-\nabla\cdot((\varepsilon_0 + \varepsilon_1|\nabla V^{(1)}|^2)\nabla V^{(1)}) = n - n^* \tag{3.8}$$

$$-\nabla\cdot((\varepsilon_0+\varepsilon_1|\nabla V^{(2)}|^2)\nabla V^{(2)})=n-n^*. \tag{3.9}$$

By (3.8) and (3.9), we have

$$-\nabla\cdot[(\varepsilon_0+\varepsilon_1|\nabla V^{(1)}|^2)\nabla V^{(1)}-(\varepsilon_0+\varepsilon_1|\nabla V^{(2)}|^2)\nabla V^{(2)}]=0. \tag{3.10}$$

Multiply (3.10) by $V^{(1)}-V^{(2)}$ and integrate. By Lemma 3.1, we have

$$\begin{aligned}0&=\int_\Omega[(\varepsilon_0+\varepsilon_1|\nabla V^{(1)}|^2)\nabla V^{(1)}-(\varepsilon_0+\varepsilon_1|\nabla V^{(2)}|^2)\nabla V^{(2)}]\cdot\nabla(V^{(1)}-V^{(2)})\mathrm{d}x\\&\geqslant\varepsilon_0\|\nabla(V^{(1)}-V^{(2)})\|^2_{L^2(\Omega)}+\frac{\varepsilon_1}{4}\|\nabla(V^{(1)}-V^{(2)})\|^4_{L^4(\Omega)}.\end{aligned}$$

Thus, by the Poincaré inequality, we obtain

$$\|V^{(1)}-V^{(2)}\|_{L^2(\Omega)}\le C\|\nabla(V^{(1)}-V^{(2)})\|_{L^2(\Omega)}\le 0.$$

Therefore, $V^{(1)}=V^{(2)}$ in the sense of $L^2(\Omega)$.

Remark 3.2 We do not need the condition

$$\int_\Omega V(x,t)\mathrm{d}x=0$$

which is needed to prove the uniqueness of solution to (QSP) with periodic boundary conditions in [7].

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