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WELL-POSEDNESS FOR ONE-DIMENSIONAL DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we investigate the one-dimensional derivative nonlinear Schrödinger equations of the form $iu_t - u_{xx} + i\lambda |u|^k u_x = 0$ with non-zero $\lambda \in \mathbb{R}$ and any real number $k \geqslant 5$. We establish the local well-posedness of the Cauchy problem with any initial data in $H^{1/2}$ by using the gauge transformation and the Littlewood-Paley decomposition.

1. **Introduction.** In the present paper, we consider the following Cauchy problem for the derivative nonlinear Schrödinger equation

$$iu_t - u_{xx} + i\lambda |u|^k u_x = 0, \quad (t, x) \in \mathbb{R}^2, \tag{1.1}$$

$$u(0, x) = u_0(x), \tag{1.2}$$

where $u = u(t, x) : \mathbb{R}^2 \to \mathbb{C}$ is a complex-valued wave function, both $\lambda \neq 0$ and $k \geqslant 5$ are real numbers.

A great deal of interesting research has been devoted to the mathematical analysis for the derivative nonlinear Schrödinger equations [3, 4, 6, 7, 8, 9, 10, 11, 13, 18, 21]. In [13], C. E. Kenig, G. Ponce and L. Vega studied the local existence theory for the Cauchy problem of the derivative nonlinear Schrödinger equations

$$iu_t + u_{xx} + f(u, \bar{u}, u_x, \bar{u}_x) = 0, \quad (t, x) \in \mathbb{R}^2,$$

with small data $u(0, x) = u_0(x)$ in $H^{7/2}$ where f is a polynomial having no constant or linear terms with the lowest order term of degree being greater than or equal to 3. Subsequently, it was improved to H^3 by N. Hayashi and T. Ozawa [11].

If the nonlinearity consists mostly of the conjugate wave $\bar{u}$, then it can be done much better. In the case $f = (\bar{u}_x)^k$, A. Grüenrock, in [8], obtained local well-posedness when $s > s_c = 3/2 - 1/(k-1)$, $s \geqslant 1$, and $k \geqslant 2$ was an integer. In particular, the global well-posedness in H^1 is obtained when $f = i(\bar{u}_x)^2$ with the help of the Bourgain spaces (cf. [2, 23]).

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In [21], H. Takaoka discussed the derivative nonlinear Schrödinger equation of the form

$$u_t - iu_{xx} + |u|^2 u_x = 0, \quad (t,x) \in \mathbb{R}^2,$$

and obtained the local well-posedness in H^s for $s \geqslant 1/2$ by performing a fixed point argument in an adapted Bourgain space $X_{s,b}$ which yields a C^∞-solution map.

A very similar equation to (1.1) is the generalized Benjamin-Ono equation

$$u_t + \mathcal{H}u_{xx} \pm u^k u_x = 0, \quad (t,x) \in \mathbb{R}^2, \tag{1.3}$$

where u is a real-valued function, $\mathcal{H}$ is the Hilbert transformation defined by

$$\mathcal{H}f(x) = -i \int_{\mathbb{R}} e^{ix\xi} \mathrm{sgn}(\xi) \hat{f}(\xi) d\xi,$$

and $k \geqslant 2$ is an integer, the symbol $\hat{\cdot}$ (or $\mathscr{F}$) denotes the spatial Fourier transform. For this equation, L. Molinet and F. Ribaud [16, 17] obtained the local well-posedness in the Sobolev space H^s for $s > 1/2$ if $k = 2, 4$, $s \geqslant 3/4$ if $k = 3$ and $s \geqslant 1/2$ if $k \geqslant 5$ by using Tao's gauge transformation. In [14], C. E. Kenig and H. Takaoka have shown the global well-posedness for the case $k = 2$ in H^s for $s \geqslant 1/2$ by combining the gauge transformation with a Littlewood-Paley decomposition and following the compactness argument with a priori estimates with the help of the preservation of the Hamiltonian and the L^2-mass.

In the present paper, we shall generalize the above results to the derivative nonlinear Schrödinger equation with $k \geqslant 5$ by using some ideas in [14]. However, we have to reconstruct new and complicated estimates for the case $k \geqslant 5$ which is quite different from the case $k = 2$.

We first state the main result of this paper as follows, though we shall define later the function space X_T at the end of this section.

Theorem 1.1. *For any $u_0 \in H^{1/2}$, there exist a $T = T(\|u_0\|_{H^{1/2}})$ and a unique solution u of* (1.1)-(1.2) *satisfying*

$$u \in C([-T,T]; H^{1/2}) \cap X_T.$$

For convenience, we now introduce some notations. For nonnegative real numbers A, B, we use $A \lesssim B$ to denote $A \leqslant CB$ for some $C > 0$ which is independent of A and B. $A \sim B$ means $A \lesssim B \lesssim A$, and $A \ll B$ denotes $A \leqslant CB$ for some small $C > 0$ which is also independent of A and B.

To give the Littlewood-Paley decomposition, let ψ be a fixed even C^∞ function with a compact support, $\mathrm{supp}\, \psi \subset \{|\xi| < 2\}$, and $\psi(\xi) = 1$ for $|\xi| \leqslant 1$. Define $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$. Let N be a dyadic number of the form $N = 2^j$, $j \in \mathbb{N} \cup \{0\}$ or $N = 0$. Writing $\varphi_N(\xi) = \varphi(\xi/N)$ for $N \geqslant 1$, we define the convolution operator P_N by $P_N u = \check{\varphi}_N * u$, where the symbol $\check{\cdot}$ (or $\mathscr{F}^{-1}$) denotes the spatial Fourier inverse transform. We define the function φ_0 by $\varphi_0(\xi) = 1 - \sum_N \varphi_N(\xi)$ and denote $P_0 u = \check{\varphi}_0 * u$. Then we introduce a spatial Littlewood-Paley decomposition [20]

$$\sum_N P_N = I.$$

Throughout this paper, we often use the Littlewood-Paley theorem (cf. [20, 23])

$$\left\| \left(\sum_N |P_N \phi|^2 \right)^{1/2} \right\|_{L^p} \sim \|\phi\|_{L^p},$$

for $1 < p < \infty$. We also use more general operators $P_{\ll N}$ and $P_{\lesssim N}$ which are defined by

$$P_{\ll N} = \sum_{M \ll N} P_M, \quad P_{\lesssim N} = \sum_{M \lesssim N} P_M,$$

and $P_{\gg N}$, $P_{\gtrsim N}$ and $P_{\sim N}$ which can be defined in a similar way. The Littlewood-Paley operators commute with derivative operators (including $|\nabla|^s$ and $i\partial_t - \partial_{xx}$), the propagator $S(t) = e^{-it\partial_x^2}$, and conjugation operations, are self-adjoint, and are bounded on every Lebesgue space L^p and homogeneous Sobolev space $\dot{H}^s$ if $1 \leqslant p \leqslant \infty$. Furthermore, they obey the following Sobolev and Bernstein estimates for $\mathbb{R}$ with $s \geqslant 0$ and $1 \leqslant p \leqslant \infty$ (which is similar to those of three dimensions [5]):

$$\begin{aligned} \|P_{\geqslant N} f\|_{L^p} &\lesssim N^{-s} \||\nabla|^s P_{\geqslant N} f\|_{L^p}, \\ \|P_{\leqslant N} |\nabla|^s f\|_{L^p} &\lesssim N^s \|P_{\leqslant N} f\|_{L^p}, \\ \left\|P_N |\nabla|^{\pm s} f\right\|_{L^p} &\lesssim N^{\pm s} \|P_N f\|_{L^p}, \end{aligned}$$

which can be verified by combining the Bernstein multiplier theorem [1] and the interpolation theorem of Sobolev spaces.

We define the Lebesgue spaces $L_T^q L_x^p$ and $L_x^p L_T^q$ by the norms

$$\|f\|_{L_T^q L_x^p} = \left\| \|f\|_{L_x^p(\mathbb{R})} \right\|_{L_t^q([0,T])}, \quad \|f\|_{L_x^p L_T^q} = \left\| \|f\|_{L_t^q([0,T])} \right\|_{L_x^p(\mathbb{R})}.$$

In particular, we abbreviate $L_T^q L_x^p$ or $L_x^p L_T^q$ as $L_{x,T}^p$ in the case $p = q$.

We also use the elementary inequality [5]

$$\left\| \left(\sum_N |f_N|^2 \right)^{1/2} \right\|_{L_T^q L_x^p} \leqslant \left(\sum_N \|f_N\|_{L_T^q L_x^p}^2 \right)^{1/2},$$

for all $2 \leqslant q, p \leqslant \infty$ and arbitrary functions f_N, and the dual version

$$\left(\sum_N \|f_N\|_{L_T^{q'} L_x^{p'}}^2 \right)^{1/2} \leqslant \left\| \left(\sum_N |f_N|^2 \right)^{1/2} \right\|_{L_T^{q'} L_x^{p'}},$$

where p' is the conjugate number of p given by $1/p + 1/p' = 1$. It is easy to verify that they also hold if we replace the norm $L_T^q L_x^p$ by the norm $L_x^p L_T^q$ in both side of the above inequalities.

Let $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. We use the fractional differential operators D_x^s and $\langle D_x \rangle^s$ defined by

$$D_x^s f = \mathscr{F}^{-1} |\xi|^s \mathscr{F} f, \quad \langle D_x \rangle^s f = \mathscr{F}^{-1} \langle \xi \rangle^s \mathscr{F} f.$$

Thus, we can introduce the resolution space. For $T > 0$, we define the function space X_T in a similar way as in [14] by

$$X_T := \{u \in \mathscr{S}'((-T,T) \times \mathbb{R}) : \|u\|_{X_T} < \infty\},$$

where

$$\|u\|_{X_T} = \|u\|_{L_T^\infty H^{1/2}} + \left(\sum_N \|\partial_x P_N u\|_{L_x^\infty L_T^2}^2 \right)^{1/2}$$

$$+\left(\sum_N \|P_N u\|^2_{L^2_x L^\infty_T}\right)^{1/2} + \left(\sum_N \left\|\langle D_x\rangle^{\frac14} P_N u\right\|^2_{L^4_x L^\infty_T}\right)^{1/2}.$$

2. **Gauge transformation.** We transform the equation (1.1) by introducing the following complex-valued function $v_N : \mathbb{R}^2 \to \mathbb{C}$ for a dyadic number N given by

$$v_N(t,x) = e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u(t,y)|^k dy} P_N u. \tag{2.1}$$

By computation, we have

$$\begin{aligned} i\partial_t v_N - \partial_x^2 v_N =& -i\lambda e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}\Big[P_N(|u|^k u_x) - |P_{\ll N}u|^k P_N u_x\Big] \\ &- \frac{i\lambda}{2} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} P_N u (i\partial_t - \partial_x^2)\int_{-\infty}^x |P_{\ll N}u(t,y)|^k dy \\ &+ \frac{\lambda^2}{4} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} |P_{\ll N}u|^{2k} P_N u. \end{aligned} \tag{2.2}$$

For the second term, we integrate by parts and have

$$\begin{aligned} &(i\partial_t - \partial_x^2)\int_{-\infty}^x |P_{\ll N}u(t,y)|^k dy \\ =& i\int_{-\infty}^x \frac{k}{2}|P_{\ll N}u|^{k-2}\left(\partial_t P_{\ll N}u\overline{P_{\ll N}u} + P_{\ll N}u\partial_t\overline{P_{\ll N}u}\right)dy \\ &- \frac{k}{2}|P_{\ll N}u|^{k-2}\left(\partial_x P_{\ll N}u\overline{P_{\ll N}u} + P_{\ll N}u\partial_x\overline{P_{\ll N}u}\right) \\ =& \int_{-\infty}^x \frac{k}{2}|P_{\ll N}u|^{k-2}\Big(\overline{P_{\ll N}u}P_{\ll N}(u_{xx} - i\lambda|u|^k u_x) \\ &\qquad - P_{\ll N}uP_{\ll N}(\bar u_{xx} + i\lambda|u|^k\bar u_x)\Big)dy \\ &- \frac{k}{2}|P_{\ll N}u|^{k-2}\left(P_{\ll N}u_x\overline{P_{\ll N}u} + P_{\ll N}u\overline{P_{\ll N}u_x}\right) \\ =& \int_{-\infty}^x \frac{k(k-2)}{4}|P_{\ll N}u|^{k-4}\left[(\overline{P_{\ll N}u_y}P_{\ll N}u)^2 - (P_{\ll N}u_y\overline{P_{\ll N}u})^2\right]dy \\ &- \int_{-\infty}^x \frac{i\lambda k}{2}|P_{\ll N}u|^{k-2}P_{\ll N}|u|^k(u_x + \bar u_x)\,dy - k|P_{\ll N}u|^{k-2}P_{\ll N}u\overline{P_{\ll N}u_x} \end{aligned}$$

Thus, v_N obeys the following differential-integral equation

$$\begin{aligned} &i\partial_t v_N - \partial_x^2 v_N(t,x) \\ =& -i\lambda e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}\Big[P_N(|u|^k u_x) - |P_{\ll N}u|^k P_N u_x\Big] \\ &- \frac{i\lambda k(k-2)}{8} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} P_N u \int_{-\infty}^x |P_{\ll N}u|^{k-4}\cdot \\ &\qquad \left[(\overline{P_{\ll N}u_x}P_{\ll N}u)^2 - (P_{\ll N}u_x\overline{P_{\ll N}u})^2\right]dy \\ &- \frac{\lambda^2 k}{4} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} P_N u \int_{-\infty}^x |P_{\ll N}u|^{k-2}P_{\ll N}|u|^k(u_x + \bar u_x)\,dy \\ &+ \frac{i\lambda k}{2} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} |P_{\ll N}u|^{k-2} P_N u P_{\ll N}u\overline{P_{\ll N}u_x} \\ &+ \frac{\lambda^2}{4} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} |P_{\ll N}u|^{2k} P_N u \end{aligned}$$

$$\equiv I_{N,1}(t,x)+I_{N,2}(t,x)+I_{N,3}(t,x)+I_{N,4}(t,x)+I_{N,5}(t,x). \tag{2.3}$$

The equivalent integral equation reads

$$v_N(t)=S(t)e^{-\frac{i\lambda}{2}\int_{-\infty}^{x}|P_{\ll N}u_0(y)|^k dy}P_N u_0 \\ -i\int_0^t S(t-\tau)[I_{N,1}+I_{N,2}+I_{N,3}+I_{N,4}+I_{N,5}](\tau)d\tau. \tag{2.4}$$

3. **Preliminaries.** In order to prove the a priori estimate for the equation of v_N, we need the linear estimates associated with the one-dimensional Schrödinger equation. We first recall the Strichartz estimates, smoothing effects and maximal function estimates. For the proofs, one can see [13, 14].

Lemma 3.1. *For all $\phi\in\mathscr{S}(\mathbb{R})$, $\theta\in[0,1]$ and $T\in(0,1)$,*

$$\|S(t)\phi\|_{L_T^{\frac{4}{\theta}}L_x^{\frac{2}{1-\theta}}}\lesssim\|\phi\|_{L^2}, \tag{3.1}$$

$$\|S(t)P_N\phi\|_{L_x^{\frac{2}{1-\theta}}L_T^{\frac{2}{\theta}}}\lesssim\langle N\rangle^{\frac{1}{2}-\theta}\|\phi\|_{L^2}, \tag{3.2}$$

$$\|S(t)\phi\|_{L_x^4L_T^\infty}\lesssim\|\phi\|_{\dot H^{\frac14}}. \tag{3.3}$$

We also need the $L_T^qL_x^p$ and $L_x^pL_T^q$ estimates for the linear operator $f\mapsto\int_0^t S(t-\tau)f(\tau)d\tau$. For the proofs, one can see [14].

Lemma 3.2. *For $f\in\mathscr{S}(\mathbb{R}^2)$, $\theta\in[0,1]$ and $T\in(0,1)$,*

$$\left\|\int_0^t S(t-\tau)f(\tau)d\tau\right\|_{L_T^{\frac{4}{\theta}}L_x^{\frac{2}{1-\theta}}}\lesssim\|f\|_{L_T^{(\frac{4}{\theta})'}L_x^{(\frac{2}{1-\theta})'}}, \tag{3.4}$$

$$\left\|\langle D_x\rangle^{\frac{\theta}{2}}\int_0^t S(t-\tau)f(\tau)d\tau\right\|_{L_T^\infty L_x^2}\lesssim\|f\|_{L_x^{p(\theta)}L_T^{q(\theta)}}, \tag{3.5}$$

$$\left\|\langle D_x\rangle^{\frac{\theta}{2}}\int_0^t S(t-\tau)P_Nf(\tau)d\tau\right\|_{L_x^2L_T^\infty}\lesssim\langle N\rangle^{\frac12}\|f\|_{L_x^{p(\theta)}L_T^{q(\theta)}}, \tag{3.6}$$

$$\left\|\langle D_x\rangle^{\frac{\theta}{2}-\frac14}\int_0^t S(t-\tau)f(\tau)d\tau\right\|_{L_x^4L_T^\infty}\lesssim\|f\|_{L_x^{p(\theta)}L_T^{q(\theta)}}, \tag{3.7}$$

$$\left\|\langle D_x\rangle^{\frac12}\int_0^t S(t-\tau)f(\tau)d\tau\right\|_{L_x^\infty L_T^2}\lesssim\|f\|_{L_T^1L_x^2}, \tag{3.8}$$

$$\left\|\int_0^t S(t-\tau)P_Nf(\tau)d\tau\right\|_{L_x^{\frac{2}{\theta}}L_T^{\frac{2}{1-\theta}}}\lesssim\langle N\rangle^{\frac12-\theta}\|f\|_{L_T^1L_x^2}, \tag{3.9}$$

$$\left\|\int_0^t S(t-\tau)f(\tau)d\tau\right\|_{L_x^4L_T^\infty}\lesssim\|f\|_{L_T^1\dot H_x^{\frac14}}, \tag{3.10}$$

$$\left\|\partial_x\int_0^t S(t-\tau)f(\tau)d\tau\right\|_{L_x^\infty L_T^2}\lesssim\|f\|_{L_x^1L_T^2}, \tag{3.11}$$

where p' is the conjugate number of $p\in[1,\infty]$, i.e. $1/p+1/p'=1$, and

$$\frac{1}{p(\theta)}=\frac{3+\theta}{4},\qquad\frac{1}{q(\theta)}=\frac{3-\theta}{4}.$$

Next, we recall the Leibniz' rule for a product of the form $e^{iF}g$ where F is the spatial primitive of some function f. For the proof, we refer to [14, 17].

Lemma 3.3 ([14, Lemma 3.5]). *Let* $\alpha \in (0,1)$, p, p_1, p_2, q, $q_1 \in (1,\infty)$, $q_2 \in (0,\infty]$ *with* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, *and let* $F(t,x) = \int_{-\infty}^{x} f(t,y)dy$, *with real-valued function* f. *Then*

$$\left\| D_x^\alpha (e^{iF} g) \right\|_{L_x^p L_T^q} \lesssim \|f\|_{L_x^{p_1} L_T^{q_1}} \|g\|_{L_x^{p_2} L_T^{q_2}} + \|\langle D_x \rangle^\alpha g\|_{L_x^p L_T^q}.$$

4. **Bilinear estimates.** In this section, we prove the following space-time estimate which is crucial to the proof of the nonlinear estimates.

Proposition 4.1. *Let* $u \in H^\infty$ *and* $p \geqslant 4$ *be a real number. Then we have*

$$\|u\bar{u}_x\|_{L_x^p L_T^2} \lesssim T^{\frac{1}{2}} \|u\|_{X_T}^2 + (1 + T^{\frac{1}{4}} \|u\|_{X_T}) \|u\|_{X_T} \|P_{\gg 1} u\|_{X_T}. \tag{4.1}$$

Proof. By the Littlewood-Paley decomposition, we can write

$$\begin{aligned} \|u\bar{u}_x\|_{L_x^p L_T^2} &= \left\| \sum_{N_1,N_2} P_{N_1} u P_{N_2} \bar{u}_x \right\|_{L_x^p L_T^2} \\ &\lesssim \left\| \sum_{N_1 \sim N_2} P_{N_1} u P_{N_2} \bar{u}_x \right\|_{L_x^p L_T^2} + \left\| \sum_{N_1 \ll N_2} P_{N_1} u P_{N_2} \bar{u}_x \right\|_{L_x^p L_T^2} \qquad (4.2) \\ &\quad + \left\| \sum_{N_1 \gg N_2} P_{N_1} u P_{N_2} \bar{u}_x \right\|_{L_x^p L_T^2} \\ &=: I_1 + I_2 + I_3. \qquad (4.3) \end{aligned}$$

Now, we derive the estimates for I_1, I_2 and I_3, respectively.

From the Hölder inequality, the Bernstein type inequalities and the real interpolation theorem, we have

$$\begin{aligned} I_1 &\lesssim \sum_{N_1 \sim N_2} \|P_{N_1} u\|_{L_x^{2p} L_T^4} \|P_{N_2} u_x\|_{L_x^{2p} L_T^4} \lesssim \sum_{N_1 \sim N_2} \|P_{N_1} u\|_{L_x^{2p} L_T^4} N_2 \|P_{N_2} u\|_{L_x^{2p} L_T^4} \\ &\lesssim \sum_{N_1 \sim N_2} \left\| D_x^{1/2} P_{N_1} u \right\|_{L_x^{2p} L_T^4} \left\| D_x^{1/2} P_{N_2} u \right\|_{L_x^{2p} L_T^4} \lesssim \sum_{N} \left\| D_x^{1/2} P_N u \right\|_{L_x^{2p} L_T^4}^2 \\ &\lesssim \sum_{N} \left\| D_x^{1/2} P_N (P_{\lesssim 1} u + P_{\gg 1} u) \right\|_{L_x^{2p} L_T^4}^2 \\ &\lesssim \sum_{N} \left\| D_x^{1/2} P_N P_{\lesssim 1} u \right\|_{L_x^{2p} L_T^4}^2 + \sum_{N} \left\| D_x^{1/2} P_N P_{\gg 1} u \right\|_{L_x^{2p} L_T^4}^2 \\ &\lesssim \sum_{N} \|P_N P_{\lesssim 1} u\|_{L_x^{2p} L_T^4}^2 + \sum_{N} N \|P_N P_{\gg 1} u\|_{L_x^{2p} L_T^4}^2 \\ &\lesssim \sum_{N \lesssim 1} \|P_N u\|_{L_x^{2p} L_T^4}^2 + \sum_{N} N \|P_N P_{\gg 1} u\|_{L_x^p L_T^\infty} \|P_N P_{\gg 1} u\|_{L_x^\infty L_T^2}. \end{aligned}$$

Applying the Sobolev embedding theorem and the Hölder inequality to the first term, and Bernstein estimates to the second term, we can see that it is bounded by

$$\lesssim T^{1/2} \|u\|_{X_T}^2 + \sum_{N} \|P_N P_{\gg 1} u\|_{L_x^p L_T^\infty} \|\partial_x P_N P_{\gg 1} u\|_{L_x^\infty L_T^2}.$$

By the Cauchy-Schwartz inequality and the Sobolev embedding theorem(i.e. $H_4^{1/4} \subset H_4^{1/4-1/p} \subset L^p$ for the real number $p \geqslant 4$), we can bound it by

$$\lesssim T^{1/2} \|u\|_{X_T}^2 + \|P_{\gg 1} u\|_{X_T}^2 .$$

For I_2 or I_3, it is suffice to consider one of them, e.g. I_2, in view of symmetry. For $N_1 \ll N_2$, we have

$$P_{N_1} u P_{N_2} \bar{u}_x = \tilde{P}_{N_2}(P_{N_1} u P_{N_2} \bar{u}_x),$$

where $\tilde{P}_N = \sum_{j=-2}^{2} P_{2^j N}$. We split these into three cases, i.e. $N_1 \lesssim 1 \ll N_2$, $N_1 \ll N_2 \lesssim 1$ and $1 \ll N_1 \ll N_2$. For the case $N_1 \lesssim 1 \ll N_2$, from the Hölder inequality and the Littlewood-Paley theorem, we can get

$$\begin{aligned}
&\left\| \sum_{N_1 \lesssim 1 \ll N_2} P_{N_1} u P_{N_2} \bar{u}_x \right\|_{L_x^p L_T^2} \\
&= \left\| P_{\lesssim 1} u P_{N_2 \gg 1} \bar{u}_x \right\|_{L_x^p L_T^2} \lesssim \left\| P_{\lesssim 1} u \right\|_{L_x^p L_T^\infty} \|P_{\gg 1} u_x\|_{L_x^\infty L_T^2} \\
&\lesssim \left\| \left(\sum_M \left| P_M P_{\lesssim 1} u \right|^2 \right)^{1/2} \right\|_{L_x^p L_T^\infty} \left\| \left(\sum_M |P_M P_{\gg 1} u_x|^2 \right)^{1/2} \right\|_{L_x^\infty L_T^2} \\
&\lesssim \left(\sum_{M \lesssim 1} \|P_M u\|_{L_x^p L_T^\infty}^2 \right)^{1/2} \left(\sum_M \|P_M P_{\gg 1} u_x\|_{L_x^\infty L_T^2}^2 \right)^{1/2} \\
&\lesssim \|u\|_{X_T} \|P_{\geqslant 1} u\|_{X_T} .
\end{aligned} \tag{4.4}$$

For the case $N_1 \ll N_2 \lesssim 1$, we have, by the Hölder inequality and the Littlewood-Paley theorem, that

$$\begin{aligned}
&\left\| \sum_{N_1 \ll N_2 \lesssim 1} P_{N_1} u P_{N_2} \bar{u}_x \right\|_{L_x^p L_T^2} = \left\| \sum_{N_2 \lesssim 1} P_{\ll N_2} u P_{N_2} \bar{u}_x \right\|_{L_x^p L_T^2} \\
&\lesssim T^{1/2} \left\| \left(\sum_{N_2 \lesssim 1} |P_{\ll N_2} u|^2 \right)^{1/2} \right\|_{L_x^{2p} L_T^\infty} \left\| \left(\sum_{N_2 \lesssim 1} |P_{N_2} \bar{u}_x|^2 \right)^{1/2} \right\|_{L_x^{2p} L_T^\infty} \\
&\lesssim T^{1/2} \left(\sum_{N_2 \lesssim 1} \|P_{\ll N_2} u\|_{L_x^{2p} L_T^\infty}^2 \right)^{1/2} \|u\|_{X_T} .
\end{aligned} \tag{4.5}$$

For $N_2 \lesssim 1$, we have, by the Sobolev embedding theorem, that

$$\begin{aligned}
&\|P_{\ll N_2} u\|_{L_x^{2p} L_T^\infty} \sim \left\| \left(\sum_M |P_M P_{\ll N_2} u|^2 \right)^{1/2} \right\|_{L_x^{2p} L_T^\infty} \lesssim \left(\sum_{M \ll N_2} \|P_M u\|_{L_x^{2p} L_T^\infty}^2 \right)^{1/2} \\
&\lesssim \left(\sum_{M \ll N_2} N_2^{2\varepsilon} M^{-2\varepsilon} \|P_M u\|_{L_x^{2p} L_T^\infty}^2 \right)^{1/2} \lesssim \left(\sum_{M \ll N_2} N_2^{2\varepsilon} \left\| D_x^{-\varepsilon} P_M u \right\|_{L_x^{2p} L_T^\infty}^2 \right)^{1/2}
\end{aligned}$$

$$\lesssim N_2^{\varepsilon}\left(\sum_{M\lesssim 1}\|P_M u\|_{L_x^2L_T^\infty}^2\right)^{1/2} \lesssim N_2^{\varepsilon}\|u\|_{X_T},$$

where $\varepsilon=(p-1)/2p$. Thus, (4.5) can be bounded by

$$\lesssim T^{1/2}\left(\sum_{N_2\lesssim 1}N_2^{2\varepsilon}\right)^{1/2}\|u\|_{X_T}^2\lesssim T^{1/2}\|u\|_{X_T}^2.$$

Now, we turn to the case $1\ll N_1\ll N_2$. From the Littlewood-Paley theorem, we have for $p\geqslant 4$

$$\begin{aligned}\left\|\sum_{1\ll N_1\ll N_2}P_{N_1}uP_{N_2}\bar u_x\right\|_{L_x^pL_T^2}&=\left\|\sum_{N_1\gg 1}P_{N_1}uP_{\gg N_1}\bar u_x\right\|_{L_x^pL_T^2}\\ \lesssim\sum_{N_1\gg 1}\|P_{N_1}uP_{\gg N_1}\bar u_x\|_{L_x^pL_T^2}&\lesssim\sum_{N_1\gg 1}\|P_{N_1}u\|_{L_x^pL_T^\infty}\|P_{\gg N_1}u_x\|_{L_x^\infty L_T^2}.\end{aligned}\tag{4.6}$$

Noticing that

$$\begin{aligned}\|P_{\gg N_1}u_x\|_{L_x^\infty L_T^2}\sim\left\|\left(\sum_M|P_MP_{\gg N_1}u_x|^2\right)^{1/2}\right\|_{L_x^\infty L_T^2}&\lesssim\left(\sum_{M\gg N_1}\|P_Mu_x\|_{L_x^\infty L_T^2}^2\right)^{1/2}\\ \lesssim\|u\|_{X_T},&\end{aligned}$$

and for $N_1\gg 1$, $\varepsilon=1/p$ and $p\geqslant 4$

$$\begin{aligned}\|P_{N_1}u\|_{L_x^pL_T^\infty}&=\left\|P_{N_1}(P_{\lesssim 1}u+P_{\gg 1}u)\right\|_{L_x^pL_T^\infty}=\|P_{N_1}P_{\gg 1}u\|_{L_x^pL_T^\infty}\\ &=N_1^{-\varepsilon}N_1^{\varepsilon}\|P_{N_1}P_{\gg 1}u\|_{L_x^pL_T^\infty}\sim N_1^{-\varepsilon}\|D_x^{\varepsilon}P_{N_1}P_{\gg 1}u\|_{L_x^pL_T^\infty}\\ &\lesssim N_1^{-\varepsilon}\left\|\langle D_x\rangle^{\frac14}P_{N_1}P_{\gg 1}u\right\|_{L_x^4L_T^\infty},\end{aligned}\tag{4.7}$$

we can bound (4.6) by

$$\begin{aligned}&\lesssim\|u\|_{X_T}\sum_{N_1\gg 1}N_1^{-\varepsilon}\left\|\langle D_x\rangle^{\frac14}P_{N_1}P_{\gg 1}u\right\|_{L_x^4L_T^\infty}\\ &\lesssim\|u\|_{X_T}\left(\sum_{N_1\gg 1}N_1^{-2\varepsilon}\right)^{\frac12}\left(\sum_{N_1\gg 1}\left\|\langle D_x\rangle^{\frac14}P_{N_1}P_{\gg 1}u\right\|_{L_x^4L_T^\infty}^2\right)^{\frac12}\\ &\lesssim\|u\|_{X_T}\|P_{\gg 1}u\|_{X_T},\end{aligned}\tag{4.8}$$

in view of the Hölder inequality. Thus, we have obtained

$$\left\|\sum_{1\ll N_1\ll N_2}P_{N_1}uP_{N_2}\bar u_x\right\|_{L_x^pL_T^2}\lesssim\|u\|_{X_T}\|P_{\gg 1}u\|_{X_T},\quad\forall p\geqslant 4.\tag{4.9}$$

Therefore, we have the desired result (4.1) for any real number $p\geqslant 4$. □

5. **Nonlinear estimates.** To state the estimates for the nonlinearities $I_{N,j}$, we define the function space Y_T equipped with the following norm:

$$\|u\|_{Y_T}=\|u\|_{L_T^\infty H_x^{1/2}}+\|\partial_x u\|_{L_x^\infty L_T^2}+\|u\|_{L_x^2L_T^\infty}+\left\|\langle D_x\rangle^{\frac14}u\right\|_{L_x^4L_T^\infty}.$$

We have the following proposition for the nonlinearities.

Proposition 5.1. *Let u be a H^∞-solution to* (1.1)-(1.2)*. Then,*

$$\begin{aligned}
&\left(\sum_{N\gg 1}\left\|\int_0^t S(t-\tau)\sum_{j=1}^{5} I_{N,j}(\tau)d\tau\right\|_{Y_T}^2\right)^{1/2}\\
\lesssim&(1+\|u\|_{X_T}^k)\left[T^{\frac12}\|u\|_{X_T}^{\tilde k+1}\|P_{\gg1}u\|_{X_T}^{k-\tilde k}+(1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{\tilde k}\|P_{\gg1}u\|_{X_T}^{k+1-\tilde k}\right]\\
&+T^{\frac12}\left(\|u\|_{X_T}^{2k-1}+\|u\|_{X_T}^{(5k-2)/2}\right)\|P_{\gg1}u\|_{X_T}\\
&+T^{\frac14}(1+T^{\frac14}\|u\|_{X_T})^{\frac12}\|u\|_{X_T}^{2k-1}\|P_{\gg1}u\|_{X_T}^{\frac32}\\
&+(1+\|u\|_{X_T}^k)\left[T^{\frac12}\|u\|_{X_T}^k\|P_{\gg1}u\|_{X_T}+(1+T^{\frac14}\|u\|_{X_T})^2\|u\|_{X_T}^{k-1}\|P_{\gg1}u\|_{X_T}^2\right]\\
&+T^{\frac12}\|u\|_{X_T}^{3k}\|P_{\gg1}u\|_{X_T},
\end{aligned}$$

where $\tilde k$ denotes the maximal integer that is less than k (i.e. $\tilde k=[k]$ if k is not an integer and $\tilde k = k-1$ if k is an integer where $[k]$ denotes the maximal integer that is less than or equal to k).

We consider each nonlinearity separately.

5.1. **Nonlinear estimates of $I_{N,1}$.** Noting that the term $P_N(|P_{\ll N}u|^k u_x)$ has Fourier support in $|\xi|\sim N$, we have

$$\begin{aligned}
&P_N(|u|^k u_x)-|P_{\ll N}u|^k P_N u_x\\
=&P_N((|u|^k-|P_{\ll N}u|^k)u_x)+P_N(|P_{\ll N}u|^k u_x)-|P_{\ll N}u|^k P_N u_x\\
=&P_N((|u|^k-|P_{\ll N}u|^k)u_x)+P_N(|P_{\ll N}u|^k \tilde P_N u_x)-|P_{\ll N}u|^k P_N\tilde P_N u_x, \qquad (5.1)
\end{aligned}$$

where $\tilde P_N = P_{N/2}+P_N+P_{2N}$.

For the second term in (5.1), we have the following estimate.

Lemma 5.1. *Let u be a solution of* (1.1)-(1.2)*. Then, we have for any $k\geqslant 4$*

$$\begin{aligned}
&\left(\sum_{N\gg1}\left\|P_N(|P_{\ll N}u|^k\tilde P_N u_x)-|P_{\ll N}u|^k P_N\tilde P_N u_x\right\|_{L_x^1L_T^2}^2\right)^{1/2} \qquad (5.2)\\
\lesssim&T^{\frac12}\|u\|_{X_T}^k\|P_{\gg1}u\|_{X_T}+(1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{k-1}\|P_{\gg1}u\|_{X_T}^2.
\end{aligned}$$

Proof. To shift a derivative from the high-frequency function $P_N u_x$ to the low-frequency function $|P_{\ll N}u|^k$, we require the following Leibniz rule for P_N from [14]:

$$(P_N(fg)-fP_Ng)(x)=\int_0^1\left(\int\check\varphi_N(y)yf_x(x-\eta y)g(x-y)dy\right)d\eta. \qquad (5.3)$$

Thus, we have

$$\begin{aligned}
&\left\|P_N(|P_{\ll N}u|^k\tilde P_N u_x)-|P_{\ll N}u|^k P_N\tilde P_N u_x\right\|_{L_x^1L_T^2}\\
\lesssim&\|\check\varphi_N(y)y\|_{L_y^1}\left\|(|P_{\ll N}u|^k)_x\right\|_{L_{x,T}^2}\left\|\tilde P_N u_x\right\|_{L_x^2L_T^\infty}\\
\lesssim&N^{-1}\|\check\varphi_1(y)y\|_{L_y^1}\left\|(|P_{\ll N}u|^k)_x\right\|_{L_{x,T}^2}\left\|\tilde P_N u_x\right\|_{L_x^2L_T^\infty}\\
\lesssim&\left\|(|P_{\ll N}u|^k)_x\right\|_{L_{x,T}^2}\left\|\tilde P_N u\right\|_{L_x^2L_T^\infty}
\end{aligned}$$

$$\lesssim \|P_{\ll N}u\|_{L_x^{2k}L_T^\infty}^{k-2} \left\|P_{\ll N}u_x\overline{P_{\ll N}u}\right\|_{L_x^kL_T^2} \left\|\tilde{P}_Nu\right\|_{L_x^2L_T^\infty}. \tag{5.4}$$

Decomposing $P_{\ll N}u = P_{\leqslant 1}u + P_{1<\cdot\ll N}u$ for $N \gg 1$, we have

$$P_{\ll N}u_x\overline{P_{\ll N}u} = P_{\leqslant 1}u_x\overline{P_{\ll N}u} + P_{1<\cdot\ll N}u_x\overline{P_{\leqslant 1}u} + P_{1<\cdot\ll N}u_x\overline{P_{1<\cdot\ll N}u}. \tag{5.5}$$

For the first term in (5.5), we have

$$\begin{aligned}\left\|P_{\leqslant 1}u_x\overline{P_{\ll N}u}\right\|_{L_x^kL_T^2} &\lesssim \|P_{\leqslant 1}u_x\|_{L_x^{2k}L_T^2}\|P_{\ll N}u\|_{L_x^{2k}L_T^\infty}\\ &\lesssim T^{1/2}\|P_{\leqslant 1}u\|_{L_x^{2k}L_T^\infty}\|P_{\ll N}u\|_{L_x^{2k}L_T^\infty}.\end{aligned}$$

By the Littlewood-Paley theorem, we can obtain

$$\begin{aligned}\|P_{\ll N}u\|_{L_x^{2k}L_T^\infty} &\sim \left\|\left(\sum_M |P_MP_{\ll N}u|^2\right)^{1/2}\right\|_{L_x^{2k}L_T^\infty} \lesssim \left(\sum_M \|P_MP_{\ll N}u\|_{L_x^{2k}L_T^\infty}^2\right)^{1/2}\\ &\lesssim \left(\sum_{M\ll N}\|P_Mu\|_{L_x^{2k}L_T^\infty}^2\right)^{1/2} \lesssim \left(\sum_{M\gg 1}\|P_Mu\|_{L_x^{2k}L_T^\infty}^2\right)^{1/2}\\ &\lesssim \left(\sum_{M\gg 1}\left\|\langle D_x\rangle^{\frac14}P_Mu\right\|_{L_x^4L_T^\infty}^2\right)^{1/2} \lesssim \|P_{\gg 1}u\|_{X_T}.\end{aligned} \tag{5.6}$$

In the similar way, we have

$$\|P_{\leqslant 1}u\|_{L_x^{2k}L_T^\infty} \lesssim \|u\|_{X_T}.$$

Thus,

$$\left\|P_{\leqslant 1}u_x\overline{P_{\ll N}u}\right\|_{L_x^kL_T^2} \lesssim T^{1/2}\|u\|_{X_T}\|P_{\gg 1}u\|_{X_T}. \tag{5.7}$$

For the last two term in (5.5), in a similar way as in the proof of Proposition 4.1, we can obtain the following bound:

$$\|P_{\leqslant 1}\bar{u}P_{1\ll\cdot\ll N}u_x\|_{L_x^kL_T^2} \lesssim T^{\frac12}\|u\|_{X_T}^2 + \|u\|_{X_T}\|P_{\gg 1}u\|_{X_T}, \tag{5.8}$$

$$\|P_{1\ll\cdot\ll N}\bar{u}P_{1\ll\cdot\ll N}u_x\|_{L_x^kL_T^2} \lesssim T^{\frac12}\|u\|_{X_T}^2 + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}\|P_{\gg 1}u\|_{X_T}. \tag{5.9}$$

From the Sobolev embedding theorem and (5.7)-(5.9), we obtain that (5.4) can be bounded by

$$\lesssim \left(T^{\frac12}\|u\|_{X_T}^k + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{k-1}\|P_{\gg 1}u\|_{X_T}\right)\left\|\tilde{P}_Nu\right\|_{L_x^2L_T^\infty}.$$

Thus, we can bound (5.2) by

$$\begin{aligned}&\lesssim \left(T^{\frac12}\|u\|_{X_T}^k + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{k-1}\|P_{\gg 1}u\|_{X_T}\right)\left(\sum_{N\gg 1}\|P_Nu\|_{L_x^2L_T^\infty}^2\right)^{1/2}\\ &\lesssim T^{\frac12}\|u\|_{X_T}^k\|P_{\gg 1}u\|_{X_T} + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{k-1}\|P_{\gg 1}u\|_{X_T}^2,\end{aligned}$$

which yields the desired result. □

For the first term in (5.1), we have the following estimate:

Lemma 5.2. *Let u be a solution of* (1.1)-(1.2). *Then, we have for any $k \geqslant 4$*

$$\left(\sum_{N\gg 1}\left\|P_N((|u|^k-|P_{\ll N}u|^k)u_x)\right\|^2_{L^1_xL^2_T}\right)^{1/2}$$
$$\lesssim T^{\frac{1}{2}}\left\|u\right\|^{\tilde{k}+1}_{X_T}\left\|P_{\gg 1}u\right\|^{k-\tilde{k}}_{X_T}+(1+T^{\frac{1}{4}}\left\|u\right\|_{X_T})\left\|u\right\|^{\tilde{k}}_{X_T}\left\|P_{\gg 1}u\right\|^{k+1-\tilde{k}}_{X_T}. \tag{5.10}$$

Proof. We split (5.1) into several terms for $N \gg 1$ and $k \geqslant 4$

$$P_N((|u|^k-|P_{\ll N}u|^k)u_x) \tag{5.11}$$
$$=P_N(|u|^{k-2}\bar{u}u_xP_{\gtrsim N}u) \tag{5.12}$$
$$+P_N((|u|^{k-2}-|P_{\ll N}u|^{k-2})\bar{u}u_xP_{\ll N}u) \tag{5.13}$$
$$+P_N(|P_{\ll N}u|^{k-2}u_xP_{\ll N}uP_{\gtrsim N}\bar{u}). \tag{5.14}$$

Notice that

$$\begin{aligned}\left\|P_{\gtrsim N}u\right\|_{L^k_xL^\infty_T}\lesssim&\left(\sum_{M\gtrsim N}\left\|P_Mu\right\|^2_{L^k_xL^\infty_T}\right)^{1/2}\lesssim\left(\sum_{M\gtrsim N}N^{-2\varepsilon_k}M^{2\varepsilon_k}\left\|P_Mu\right\|^2_{L^k_xL^\infty_T}\right)^{1/2}\\ \lesssim&\left(\sum_{M\gtrsim N}N^{-2\varepsilon_k}\left\|D^{\varepsilon_k}_xP_Mu\right\|^2_{L^k_xL^\infty_T}\right)^{1/2}\\ \lesssim&N^{-\varepsilon_k}\left(\sum_{M\gtrsim N}\left\|\langle D_x\rangle^{\frac{1}{4}}P_Mu\right\|^2_{L^4_xL^\infty_T}\right)^{1/2}\\ \lesssim&N^{-\varepsilon_k}\left\|P_{\gg 1}u\right\|_{X_T},\quad\forall k\geqslant 4,\end{aligned}$$

where $\varepsilon_k > 0$ is defined by $\varepsilon_k = 1/k$.

Thus, for the first term (5.12), from the fact $\|\check{\varphi}_N\|_{L^1} \lesssim 1$ and Proposition 4.1, we have for $k \geqslant 4$

$$\begin{aligned}&\left\|P_N(|u|^{k-2}\bar{u}u_xP_{\gtrsim N}u)\right\|_{L^1_xL^2_T}\lesssim\left\||u|^{k-2}\bar{u}u_xP_{\gtrsim N}u\right\|_{L^1_xL^2_T}\\ \lesssim&\left\|u\right\|^{k-2}_{L^k_xL^\infty_T}\left\|\bar{u}u_x\right\|_{L^k_xL^2_T}\left\|P_{\gtrsim N}u\right\|_{L^k_xL^\infty_T}\\ \lesssim&N^{-\varepsilon_k}\left[T^{\frac{1}{2}}\left\|u\right\|^k_{X_T}\left\|P_{\gg 1}u\right\|_{X_T}+(1+T^{\frac{1}{4}}\left\|u\right\|_{X_T})\left\|u\right\|^k_{X_T}\left\|P_{\gg 1}u\right\|^2_{X_T}\right].\end{aligned}$$

Therefore, we obtain, for any $k \geqslant 4$, that

$$\begin{aligned}&\left(\sum_{N\gg 1}\left\|P_N(|u|^{k-2}\bar{u}u_xP_{\gtrsim N}u)\right\|^2_{L^1_xL^2_T}\right)^{1/2}\\ \lesssim&T^{\frac{1}{2}}\left\|u\right\|^k_{X_T}\left\|P_{\gg 1}u\right\|_{X_T}+(1+T^{\frac{1}{4}}\left\|u\right\|_{X_T})\left\|u\right\|^k_{X_T}\left\|P_{\gg 1}u\right\|^2_{X_T}.\end{aligned}$$

For (5.14), in the same way as the case (5.12), we have

$$\begin{aligned}&\left(\sum_{N\gg 1}\left\|P_N(|P_{\ll N}u|^{k-2}u_xP_{\ll N}uP_{\gtrsim N}\bar{u})\right\|^2_{L^1_xL^2_T}\right)^{1/2}\\ \lesssim&T^{\frac{1}{2}}\left\|u\right\|^k_{X_T}\left\|P_{\gg 1}u\right\|_{X_T}+(1+T^{\frac{1}{4}}\left\|u\right\|_{X_T})\left\|u\right\|^k_{X_T}\left\|P_{\gg 1}u\right\|^2_{X_T}.\end{aligned}$$

Now, we derive the estimate for (5.13) by using the induction argument in k.

For $k=4$, we have

$$\left||u|^{k-2}-|P_{\ll N}u|^{k-2}\right|\lesssim\left|P_{\gtrsim N}u\right|^2+\left|P_{\gtrsim N}u\overline{P_{\ll N}u}\right|.$$

From the Young inequality, the Hölder inequality, (5.6) and Proposition 4.1, we can get for $k=4$

$$\begin{aligned}
&\left\|P_N((|u|^{k-2}-|P_{\ll N}u|^{k-2})\bar{u}u_xP_{\ll N}u)\right\|_{L^1_xL^2_T}\\
\lesssim&\left\||u|^{k-2}-|P_{\ll N}u|^{k-2}\right\|_{L_x^{2k/(k-2)}L_T^\infty}\|\bar{u}u_x\|_{L^k_xL^2_T}\|P_{\ll N}u\|_{L^2_xL^\infty_T}\\
\lesssim&(\left\|P_{\gtrsim N}u\right\|^2_{L^8_xL^\infty_T}+\left\|P_{\gtrsim N}u\right\|_{L^8_xL^\infty_T}\|P_{\ll N}u\|_{L^8_xL^\infty_T})\|\bar{u}u_x\|_{L^k_xL^2_T}\|P_{\ll N}u\|_{L^2_xL^\infty_T}\\
\lesssim&N^{-1/8}\|P_{\gg 1}u\|^2_{X_T}\|\bar{u}u_x\|_{L^k_xL^2_T}\|P_{\ll N}u\|_{L^2_xL^\infty_T}\\
\lesssim&N^{-1/8}\left[T^{\frac12}\|u\|^3_{X_T}\|P_{\gg 1}u\|^{k-2}_{X_T}+(1+T^{\frac14}\|u\|_{X_T})\|u\|^2_{X_T}\|P_{\gg 1}u\|^{k-1}_{X_T}\right].
\end{aligned}$$

From the triangle inequality for complex number, i.e. $||z_1|-|z_2||\leqslant|z_1-z_2|$ for $z_1,z_2\in\mathbb{C}$, we can get $\left||z_1|^\theta-|z_2|^\theta\right|\leqslant|z_1-z_2|^\theta$ for any $\theta\in(0,1]$.

For $k\in(4,5]$, we have

$$\begin{aligned}
&\left||u|^{k-2}-|P_{\ll N}u|^{k-2}\right|\\
\lesssim&|u|^2\left||u|^{k-4}-|P_{\ll N}u|^{k-4}\right|+|P_{\ll N}u|^{k-4}\left||u|^2-|P_{\ll N}u|^2\right|\\
\lesssim&|u|^2\left|P_{\gtrsim N}u\right|^{k-4}+|P_{\ll N}u|^{k-4}\left|P_{\gtrsim N}u\right|^2+\left|P_{\gtrsim N}u\right||P_{\ll N}u|^{k-3}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left\|P_N((|u|^{k-2}-|P_{\ll N}u|^{k-2})\bar{u}u_xP_{\ll N}u)\right\|_{L^1_xL^2_T}\\
\lesssim&\left[\|u\|^2_{L^{2k}_xL^\infty_T}\left\|P_{\gtrsim N}u\right\|^{k-4}_{L^{2k}_xL^\infty_T}+\|P_{\ll N}u\|^{k-4}_{L^{2k}_xL^\infty_T}\left\|P_{\gtrsim N}u\right\|^2_{L^{2k}_xL^\infty_T}\right.\\
&\left.+\left\|P_{\gtrsim N}u\right\|_{L^{2k}_xL^\infty_T}\|P_{\ll N}u\|^{k-3}_{L^{2k}_xL^\infty_T}\right]\|\bar{u}u_x\|_{L^k_xL^2_T}\|P_{\ll N}u\|_{L^2_xL^\infty_T}\\
\lesssim&N^{-\varepsilon_k}\left[T^{\frac12}\|u\|^4_{X_T}\|P_{\gg 1}u\|^{k-3}_{X_T}+(1+T^{\frac14}\|u\|_{X_T})\|u\|^3_{X_T}\|P_{\gg 1}u\|^{k-2}_{X_T}\right],
\end{aligned}$$

where $\varepsilon_k=(k-4)/2k$ for $k\in(4,5]$. By the same procedure, we can obtain for any $k\geqslant 4$

$$\begin{aligned}
&\left\|P_N((|u|^{k-2}-|P_{\ll N}u|^{k-2})\bar{u}u_xP_{\ll N}u)\right\|_{L^1_xL^2_T}\\
\lesssim&N^{-\varepsilon_k}\left[T^{\frac12}\|u\|^{\tilde{k}+1}_{X_T}\|P_{\gg 1}u\|^{k-\tilde{k}}_{X_T}+(1+T^{\frac14}\|u\|_{X_T})\|u\|^{\tilde{k}}_{X_T}\|P_{\gg 1}u\|^{k+1-\tilde{k}}_{X_T}\right],
\end{aligned}$$

where $\varepsilon_k=(k-\tilde{k})/2k>0$. Therefore, we have for any $k\geqslant 4$

$$\begin{aligned}
&\left(\sum_{N\gg 1}\left\|P_N((|u|^{k-2}-|P_{\ll N}u|^{k-2})\bar{u}u_xP_{\ll N}u)\right\|^2_{L^1_xL^2_T}\right)^{1/2}\\
\lesssim&T^{\frac12}\|u\|^{\tilde{k}+1}_{X_T}\|P_{\gg 1}u\|^{k-\tilde{k}}_{X_T}+(1+T^{\frac14}\|u\|_{X_T})\|u\|^{\tilde{k}}_{X_T}\|P_{\gg 1}u\|^{k+1-\tilde{k}}_{X_T}.
\end{aligned}\tag{5.15}$$

Thus, we have proved the desired result. □

Remark 5.1. From the proof of Lemma 5.2, we can see that

$$\left(\sum_{N\gg 1}\left\|P_N((|u|^k-|P_{\ll N}u|^k)u_x)\right\|^2_{L_x^{\frac{1}{1-\varepsilon}}L^2_T}\right)^{1/2}$$

$$\lesssim T^{\frac{1}{2}} \|u\|_{X_T}^{\tilde{k}+1} \|P_{\gg 1}u\|_{X_T}^{k-\tilde{k}} + (1+T^{\frac{1}{4}}\|u\|_{X_T}) \|u\|_{X_T}^{\tilde{k}} \|P_{\gg 1}u\|_{X_T}^{k+1-\tilde{k}}, \tag{5.16}$$

holds for any $\varepsilon \in [0,1)$ in view of Proposition 4.1.

We turn to the proof of Proposition 5.1 for the nonlinearity $I_{N,1}$. We also consider the decomposition in (5.1). For convenience, we denote $B_N = P_N(|P_{\ll N}u|^k \tilde{P}_N u_x) - |P_{\ll N}u|^k P_N \tilde{P}_N u_x$. From (3.5), (3.11), (3.6) and (3.7), we have

$$\left(\sum_{N \gg 1} \left\| \int_0^t S(t-\tau) e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} B_N d\tau \right\|_{Y_T}^2 \right)^{1/2}$$

$$\lesssim \left(\sum_{N \gg 1} \|B_N\|_{L_x^1 L_T^2}^2 \right)^{1/2} + \left(\sum_{N \gg 1} \left(\sum_M \left\| P_M (e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} B_N) \right\|_{L_x^1 L_T^2} \right)^2 \right)^{1/2}. \tag{5.17}$$

By Lemma 5.1, the first term can be bounded by

$$\lesssim T^{\frac{1}{2}} \|u\|_{X_T}^k \|P_{\gg 1}u\|_{X_T} + (1+T^{\frac{1}{4}}\|u\|_{X_T}) \|u\|_{X_T}^{k-1} \|P_{\gg 1}u\|_{X_T}^2.$$

For the second term, we split the sum $\sum_M$ into three parts $\sum_{M\sim N} + \sum_{M \ll N} + \sum_{M \gg N}$ as in [14]. For the part of $M \sim N$, it is the same as Lemma 5.1 by summing in M such that $M \sim N$. For the part $M \ll N$, we can add the projection operator $P_{\sim N}$ to $e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}$ since B_N has Fourier support in $|\xi| \sim N$. Thus, by the Hölder inequality, we have

$$\left(\sum_{N \gg 1} \left(\sum_{M \ll N} \left\| P_M (e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} B_N) \right\|_{L_x^1 L_T^2} \right)^2 \right)^{1/2}$$

$$\lesssim \left(\sum_{N \gg 1} \left(\sum_{M \ll N} \left\| P_{\sim N} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} B_N \right\|_{L_x^1 L_T^2} \right)^2 \right)^{1/2}$$

$$\lesssim \left(\sum_{N \gg 1} (\ln N)^2 \left\| P_{\sim N} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} \right\|_{L_x^{1/\varepsilon} L_T^\infty}^2 \|B_N\|_{L_x^{1/(1-\varepsilon)} L_T^2}^2 \right)^{1/2}, \tag{5.18}$$

where $\varepsilon \in (0, 1/k)$.

By the Bernstein inequality, we have

$$N \left\| P_{\sim N} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} \right\|_{L_x^{1/\varepsilon} L_T^\infty} \lesssim \left\| \partial_x P_{\sim N} e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} \right\|_{L_x^{1/\varepsilon} L_T^\infty}$$

$$\lesssim \|P_{\ll N}u\|_{L_x^{k/\varepsilon} L_T^\infty}^k \lesssim \|u\|_{X_T}^k,$$

and from (5.3) and the Hölder inequality, we can get, as a similar way as in (5.4), that

$$\|B_N\|_{L_x^{1/(1-\varepsilon)} L_T^2} = \|(|P_{\ll N}u|^k)_x\|_{L_x^{2/(1-2\varepsilon)} L_T^2} \left\| \tilde{P}_N u \right\|_{L_x^2 L_T^\infty}$$

$$\lesssim \|P_{\ll N}u\|_{L_x^{\frac{2k(k-2)}{k-2-2k\varepsilon}} L_T^\infty}^{k-2} \|P_{\ll N}u_x P_{\ll N}\bar{u}\|_{L_x^k L_T^2} \left\| \tilde{P}_N u \right\|_{L_x^2 L_T^\infty}$$

$$\lesssim \left(T^{\frac{1}{2}} \|u\|_{X_T}^k + (1+T^{\frac{1}{4}}\|u\|_{X_T}) \|u\|_{X_T}^{k-1} \|P_{\gg 1}u\|_{X_T} \right) \|P_N u\|_{L_x^2 L_T^\infty}.$$

Thus, (5.18) can be bounded by

$$
\begin{aligned}
&\lesssim \|u\|_{X_T}^k \left(T^{\frac12}\|u\|_{X_T}^k + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{k-1}\|P_{\gg 1}u\|_{X_T}\right)\left(\sum_{N\gg 1}\|P_N u\|_{L_x^2 L_T^\infty}^2\right)^{1/2}\\
&\lesssim T^{\frac12}\|u\|_{X_T}^{2k}\|P_{\gg 1}u\|_{X_T} + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{2k-1}\|P_{\gg 1}u\|_{X_T}^2 .
\end{aligned}
$$

For the part $M \gg N$, we can add the projection operator P_M to $e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}$. In a similar way with the part $M \ll N$, we have

$$
\begin{aligned}
&\left(\sum_{N\gg 1}\left(\sum_{M\gg N}\left\|P_M(e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}B_N)\right\|_{L_x^1 L_T^2}\right)^2\right)^{1/2}\\
\lesssim &\left(\sum_{N\gg 1}\left(\sum_{M\gg N}\left\|P_M e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}\right\|_{L_x^{1/\varepsilon} L_T^\infty}\|B_N\|_{L_x^{1/(1-\varepsilon)} L_T^2}\right)^2\right)^{1/2}\\
\lesssim &\Bigg(\sum_{N\gg 1}\Big(\sum_{M\gg N}\frac{1}{M}\Big(T^{\frac12}\|u\|_{X_T}^{2k}\|P_N u\|_{L_x^2 L_T^\infty}\\
&\qquad + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{2k-1}\|P_{\gg 1}u\|_{X_T}\|P_N u\|_{L_x^2 L_T^\infty}\Big)\Big)^2\Bigg)^{1/2}\\
\lesssim &T^{\frac12}\|u\|_{X_T}^{2k}\|P_{\gg 1}u\|_{X_T} + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{2k-1}\|P_{\gg 1}u\|_{X_T}^2 .
\end{aligned}
$$

For the first term in (5.1), we denote it by A_N, i.e. $A_N = P_N((|u|^k - |P_{\ll N}u|^k)u_x)$. Similarly, from (3.5), (3.11), (3.6) and (3.7), we can get

$$
\begin{aligned}
&\left(\sum_{N\gg 1}\left\|\int_0^t S(t-\tau)e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}A_N d\tau\right\|_{Y_T}^2\right)^{1/2}\\
\lesssim &\left(\sum_{N\gg 1}\|A_N\|_{L_x^1 L_T^2}^2\right)^{1/2} + \left(\sum_{N\gg 1}\left(\sum_{M}\left\|P_M(e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}A_N)\right\|_{L_x^1 L_T^2}\right)^2\right)^{1/2}.
\end{aligned}
\tag{5.19}
$$

From Lemma 5.2, the first term is bounded by

$$
\lesssim T^{\frac12}\|u\|_{X_T}^{\tilde k+1}\|P_{\gg 1}u\|_{X_T}^{k-\tilde k} + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{\tilde k}\|P_{\gg 1}u\|_{X_T}^{k+1-\tilde k}.
$$

Noticing that (5.16), and in the same way as in dealing with the second term of (5.17), we can bound the second term of (5.19) by

$$
\lesssim T^{\frac12}\|u\|_{X_T}^{k+\tilde k+1}\|P_{\gg 1}u\|_{X_T}^{k-\tilde k} + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{k+\tilde k}\|P_{\gg 1}u\|_{X_T}^{k+1-\tilde k}.
$$

Therefore, we have obtained

$$
\begin{aligned}
&\left(\sum_{N\gg 1}\left\|\int_0^t S(t-\tau)I_{N,1}(\tau)d\tau\right\|_{Y_T}^2\right)^{1/2}\\
\lesssim &(1+\|u\|_{X_T}^k)\left[T^{\frac12}\|u\|_{X_T}^{\tilde k+1}\|P_{\gg 1}u\|_{X_T}^{k-\tilde k} + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{\tilde k}\|P_{\gg 1}u\|_{X_T}^{k+1-\tilde k}\right].
\end{aligned}
$$

5.2. **Nonlinear estimates of $I_{N,2}$.** From (3.4), (3.8), (3.9) and (3.10), we have

$$
\begin{aligned}
&\left(\sum_{N\gg 1}\left\|\int_0^t S(t-\tau)I_{N,2}(\tau)d\tau\right\|_{Y_T}^2\right)^{1/2}\\
\lesssim&\left(\sum_{N\gg 1}\left\|e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}P_N u B_{N,2}\right\|_{L_T^1 H_x^{1/2}}^2\right)^{1/2} &(5.20)\\
&+\left(\sum_{N\gg 1}\left(\sum_M\left\|P_M(e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}P_N u B_{N,2})\right\|_{L_T^1 H_x^{1/2}}\right)^2\right)^{1/2}, &(5.21)
\end{aligned}
$$

where $B_{N,2}=\int_{-\infty}^x |P_{\ll N}u|^{k-4}\left[(\overline{P_{\ll N}u_x}P_{\ll N}u)^2-(P_{\ll N}u_x\overline{P_{\ll N}u})^2\right]dy$.

For the first term (5.20), from Lemma 3.3 and the Hölder inequality, it can be bounded by

$$
\begin{aligned}
\lesssim&\left(\sum_{N\gg 1}\|P_N u B_{N,2}\|_{L_T^1 L_x^2}^2\right.\\
&\left.+\|P_N u B_{N,2}\|_{L_T^1 L_x^2}\left\|\partial_x(e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}P_N u B_{N,2})\right\|_{L_T^1 L_x^2}\right)^{1/2}\\
\lesssim&\Big(\sum_{N\gg 1}\|P_N u\|_{L_T^\infty L_x^2}^2\|B_{N,2}\|_{L_T^1 L_x^\infty}^2+\|P_N u\|_{L_T^\infty H_x^{1/2}}^2\|B_{N,2}\|_{L_T^1 L_x^\infty}^2\\
&+\|P_N u\|_{L_T^\infty L_x^2}\|B_{N,2}\|_{L_T^1 L_x^\infty}^2\|P_{\ll N}u\|_{L_x^{2(k+1)}L_T^\infty}^k\|P_N u\|_{L_x^{2(k+1)}L_T^\infty}\\
&+\|P_N u\|_{L_T^\infty L_x^2}\|B_{N,2}\|_{L_T^1 L_x^\infty}\|P_N u\|_{L_x^\infty L_T^2}\|\partial_x B_{N,2}\|_{L_{x,T}^2}\Big)^{1/2}. &(5.22)
\end{aligned}
$$

By the Hölder inequality, we have for $k\geqslant 5$

$$
\begin{aligned}
\|B_{N,2}\|_{L_T^1 L_x^\infty}\lesssim&\left\||P_{\ll N}u|^{k-4}\left[(\overline{P_{\ll N}u_x}P_{\ll N}u)^2-(P_{\ll N}u_x\overline{P_{\ll N}u})^2\right]\right\|_{L_{x,T}^1}\\
\lesssim&\left\|\overline{P_{\ll N}u}P_{\ll N}u_x\right\|_{L_x^4 L_T^2}^2\|P_{\ll N}u\|_{L_x^{2(k-4)}L_T^\infty}^{k-4}\\
\lesssim&T\|u\|_{X_T}^k+(1+T^{\frac14}\|u\|_{X_T})^2\|u\|_{X_T}^{k-2}\|P_{\gg 1}u\|_{X_T}^2, &(5.23)
\end{aligned}
$$

and from Proposition 4.1 and the proof of Lemma 5.1,

$$
\begin{aligned}
&\|P_N u\|_{L_x^\infty L_T^2}\|\partial_x B_{N,2}\|_{L_{x,T}^2}\\
=&\|P_N u\|_{L_x^\infty L_T^2}\left\||P_{\ll N}u|^{k-4}\left[(\overline{P_{\ll N}u_x}P_{\ll N}u)^2-(P_{\ll N}u_x\overline{P_{\ll N}u})^2\right]\right\|_{L_{x,T}^2}\\
\lesssim&\|P_N u_x\|_{L_x^\infty L_T^2}\|P_{\ll N}u_x P_{\ll N}\bar u\|_{L_x^{2(k-1)}L_T^2}\|P_{\ll N}u\|_{L_x^{2(k-1)}L_T^\infty}^{k-2}\\
\lesssim&\|P_N u_x\|_{L_x^\infty L_T^2}\left[T^{\frac12}\|u\|_{X_T}^k+(1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{k-1}\|P_{\gg 1}u\|_{X_T}\right].
\end{aligned}
$$

Thus, we can bound (5.20) by

$$
\lesssim(1+\|u\|_{X_T}^k)\left(T^{1/2}\|u\|_{X_T}^k\|P_{\gg 1}u\|_{X_T}+(1+T^{\frac14}\|u\|_{X_T})^2\|u\|_{X_T}^{k-2}\|P_{\gg 1}u\|_{X_T}^3\right).
$$

For (5.21), we split the sum $\sum_M$ into two parts $\sum_{M\lesssim N}+\sum_{M\gg N}$, which gives the bound by

$$\lesssim\left(\sum_{N\gg1}\left(\sum_{M\lesssim N}\langle M\rangle^{\frac{1}{2}}\left\|P_NuB_{N,2}\right\|_{L_T^1L_x^2}\right)^2\right)^{1/2} \tag{5.24}$$

$$+\left(\sum_{N\gg1}\left(\sum_{M\gg N}\left\|P_M\langle D_x\rangle^{\frac{1}{2}}e^{-\frac{i\lambda}{2}\int_{-\infty}^x|P_{\ll N}u|^kdy}P_NuB_{N,2}\right\|_{L_T^1L_x^2}\right)^2\right)^{1/2}. \tag{5.25}$$

For the first term (5.24), noticing that $\sum_{M\lesssim N}\langle M\rangle^{1/2}\lesssim N^{1/2}$ and (5.23), we can bound it by

$$\begin{aligned}&\lesssim\left(\sum_{N\gg1}\left(\left\|D_x^{\frac{1}{2}}P_Nu\right\|_{L_T^\infty L_x^2}\left\|B_{N,2}\right\|_{L_T^1L_x^\infty}\right)^2\right)^{1/2}\\&\lesssim\|u\|_{X_T}^k\left\|P_{\gg1}u\right\|_{X_T}.\end{aligned}$$

For the second term (5.25), in a similar way with (5.22), we bound it by

$$\begin{aligned}&\lesssim\left(\sum_{N\gg1}\left(\sum_{M\gg N}M^{-\frac{1}{2}}\left\|P_M\langle D_x\rangle e^{-\frac{i\lambda}{2}\int_{-\infty}^x|P_{\ll N}u|^kdy}P_NuB_{N,2}\right\|_{L_T^1L_x^2}\right)^2\right)^{1/2}\\&\lesssim\left(\sum_{N\gg1}\left\|P_NuB_{N,2}\right\|_{L_T^1L_x^2}^2+\left\|\partial_xe^{-\frac{i\lambda}{2}\int_{-\infty}^x|P_{\ll N}u|^kdy}P_NuB_{N,2}\right\|_{L_T^1L_x^2}^2\right)^{1/2}\\&\lesssim(1+\|u\|_{X_T}^k)\left(T^{1/2}\|u\|_{X_T}^k\left\|P_{\gg1}u\right\|_{X_T}+(1+T^{\frac{1}{4}}\|u\|_{X_T})^2\|u\|_{X_T}^{k-2}\left\|P_{\gg1}u\right\|_{X_T}^3\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\left(\sum_{N\gg1}\left\|\int_0^tS(t-\tau)I_{N,2}(\tau)d\tau\right\|_{Y_T}^2\right)^{1/2}\\&\lesssim(1+\|u\|_{X_T}^k)\left(T^{1/2}\|u\|_{X_T}^k\left\|P_{\gg1}u\right\|_{X_T}+(1+T^{\frac{1}{4}}\|u\|_{X_T})^2\|u\|_{X_T}^{k-2}\left\|P_{\gg1}u\right\|_{X_T}^3\right).\end{aligned}$$

5.3. **Nonlinear estimates of $I_{N,3}$.** From (3.4), (3.8), (3.9) and (3.10), we have

$$\left(\sum_{N\gg1}\left\|\int_0^tS(t-\tau)I_{N,2}(\tau)d\tau\right\|_{Y_T}^2\right)^{1/2} \tag{5.26}$$

$$\lesssim\left(\sum_{N\gg1}\left\|e^{-\frac{i\lambda}{2}\int_{-\infty}^x|P_{\ll N}u|^kdy}P_NuB_{N,3}\right\|_{L_T^1H_x^{1/2}}^2\right)^{1/2} \tag{5.27}$$

$$+\left(\sum_{N\gg1}\left(\sum_{M}\left\|P_M(e^{-\frac{i\lambda}{2}\int_{-\infty}^x|P_{\ll N}u|^kdy}P_NuB_{N,3})\right\|_{L_T^1H_x^{1/2}}\right)^2\right)^{1/2}, \tag{5.28}$$

where $B_{N,3}=\int_{-\infty}^x|P_{\ll N}u|^{k-2}P_{\ll N}|u|^k(u_x+\bar{u}_x)dy$.

By Hölder inequality, we get

$$\|B_{N,3}\|_{L_T^1L_x^\infty}\lesssim\left\||P_{\ll N}u|^{k-2}P_{\ll N}|u|^k(u_x+\bar{u}_x)\right\|_{L_{x,T}^1}$$

$$
\begin{aligned}
&\lesssim T^{\frac{1}{2}} \|P_{\ll N} u\|_{L_x^{2k-2} L_T^\infty}^{k-2} \left\| P_{\ll N} |u|^k (u_x + \bar{u}_x) \right\|_{L_x^{(2k-2)/k} L_T^2} \\
&\lesssim T^{\frac{1}{2}} \|P_{\ll N} u\|_{L_x^{2k-2} L_T^\infty}^{k-2} \|u\|_{L_x^{2k-2} L_T^\infty}^k \|u_x\|_{L_x^\infty L_T^2} \\
&\lesssim T^{\frac{1}{2}} \|u\|_{X_T}^{2k-1}.
\end{aligned}
$$

From the Hölder inequality and Proposition 4.1, we have

$$
\begin{aligned}
\|\partial_x B_{N,3}\|_{L_{x,T}^2} &= \left\| |P_{\ll N} u|^{k-2} P_{\ll N} |u|^k (u_x + \bar{u}_x) \right\|_{L_{x,T}^2} \\
&\lesssim \|P_{\ll N} u\|_{L_x^{4k-4} L_T^\infty}^{k-2} \|u\|_{L_x^{4k-4} L_T^\infty}^k \|\bar{u} u_x\|_{L_x^{4k-4} L_T^2} \\
&\lesssim T^{\frac{1}{2}} \|u\|_{X_T}^{2k} + (1 + T^{\frac{1}{4}} \|u\|_{X_T}) \|u\|_{X_T}^{2k-1} \|P_{\gg 1} u\|_{X_T}.
\end{aligned}
$$

In addition, for $N \gg 1$, we have $\|P_N u\|_{L_x^\infty L_T^2} \lesssim \|P_N u_x\|_{L_x^\infty L_T^2}$. Thus, in the same way as in the case $I_{N,2}$, we can bound (5.26) by

$$
\begin{aligned}
&\lesssim T^{\frac{1}{2}} \left(\|u\|_{X_T}^{2k-1} + \|u\|_{X_T}^{(5k-2)/2} \right) \|P_{\gg 1} u\|_{X_T} \\
&\quad + T^{\frac{1}{4}} (1 + T^{\frac{1}{4}} \|u\|_{X_T})^{\frac{1}{2}} \|u\|_{X_T}^{2k-1} \|P_{\gg 1} u\|_{X_T}^{\frac{3}{2}}.
\end{aligned}
$$

5.4. **Nonlinear estimates of $I_{N,4}$.** From (3.5), (3.11), (3.6) and (3.7), we have

$$
\begin{aligned}
&\left(\sum_{N \gg 1} \left\| \int_0^t S(t-\tau) e^{-\frac{i\lambda}{2} \int_{-\infty}^x |P_{\ll N} u|^k dy} B_{N,4} d\tau \right\|_{Y_T}^2 \right)^{1/2} \\
\lesssim &\left(\sum_{N \gg 1} \|B_{N,4}\|_{L_x^1 L_T^2}^2 \right)^{1/2} \\
&+ \left(\sum_{N \gg 1} \left(\sum_M \left\| P_M (e^{-\frac{i\lambda}{2} \int_{-\infty}^x |P_{\ll N} u|^k dy} B_{N,4}) \right\|_{L_x^1 L_T^2} \right)^2 \right)^{1/2},
\end{aligned} \tag{5.29}
$$

where $B_{N,4} = |P_{\ll N} u|^{k-2} P_N u P_{\ll N} u \overline{P_{\ll N} u_x}$. By the Hölder inequality, we have

$$
\begin{aligned}
\|B_{N,4}\|_{L_x^1 L_T^2} &\lesssim \|P_{\ll N} u\|_{L_x^k L_T^\infty}^{k-2} \|P_N u\|_{L_x^k L_T^\infty} \left\| P_{\ll N} u \overline{P_{\ll N} u_x} \right\|_{L_x^k L_T^2} \\
&\lesssim \left[T^{\frac{1}{2}} \|u\|_{X_T}^k + (1 + T^{\frac{1}{4}} \|u\|_{X_T}) \|u\|_{X_T}^{k-1} \|P_{\gg 1} u\|_{X_T} \right] \|P_N u\|_{L_x^k L_T^\infty}.
\end{aligned}
$$

Thus, the first term in (5.29) can be bounded by

$$
\lesssim T^{\frac{1}{2}} \|u\|_{X_T}^k \|P_{\gg 1} u\|_{X_T} + (1 + T^{\frac{1}{4}} \|u\|_{X_T}) \|u\|_{X_T}^{k-1} \|P_{\gg 1} u\|_{X_T}^2.
$$

By the Hölder inequality, we get

$$
\begin{aligned}
\|B_{N,4}\|_{L_x^{\frac{1}{1-\varepsilon}} L_T^2} &\lesssim \|P_{\ll N} u\|_{L_x^{\frac{k(k-2)}{k(1-\varepsilon)-2}} L_T^\infty}^{k-2} \|P_N u\|_{L_x^k L_T^\infty} \left\| P_{\ll N} u \overline{P_{\ll N} u_x} \right\|_{L_x^k L_T^2} \\
&\lesssim \left[T^{\frac{1}{2}} \|u\|_{X_T}^k + (1 + T^{\frac{1}{4}} \|u\|_{X_T}) \|u\|_{X_T}^{k-1} \|P_{\gg 1} u\|_{X_T} \right] \|P_N u\|_{L_x^k L_T^\infty}.
\end{aligned}
$$

Noticing that $B_{N,4}$ has Fourier support in $|\xi| \sim N$, we can repeat the procedure which we use to deal with the second term in (5.17), and obtain that the second term in (5.29) can be bounded by

$$
\lesssim T^{\frac{1}{2}} \|u\|_{X_T}^{2k} \|P_{\gg 1} u\|_{X_T} + (1 + T^{\frac{1}{4}} \|u\|_{X_T}) \|u\|_{X_T}^{2k-1} \|P_{\gg 1} u\|_{X_T}^2.
$$

Therefore, we obtain

$$\left(\sum_{N\gg 1}\left\|\int_0^t S(t-\tau)I_{N,4}(\tau)d\tau\right\|_{Y_T}^2\right)^{1/2}$$
$$\lesssim(1+\|u\|_{X_T}^k)\left[T^{\frac{1}{2}}\|u\|_{X_T}^k\|P_{\gg 1}u\|_{X_T}+(1+T^{\frac{1}{4}}\|u\|_{X_T})\|u\|_{X_T}^{k-1}\|P_{\gg 1}u\|_{X_T}^2\right].$$

5.5. **Nonlinear estimates of $I_{N,5}$.** From (3.5), (3.11), (3.6) and (3.7), we have

$$\left(\sum_{N\gg 1}\left\|\int_0^t S(t-\tau)e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}B_{N,5}d\tau\right\|_{Y_T}^2\right)^{1/2}$$
$$\lesssim\left(\sum_{N\gg 1}\|B_{N,5}\|_{L_x^1L_T^2}^2\right)^{1/2}$$
$$+\left(\sum_{N\gg 1}\left(\sum_M\left\|P_M(e^{-\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy}B_{N,5})\right\|_{L_x^1L_T^2}\right)^2\right)^{1/2}, \tag{5.30}$$

where $B_{N,5}=|P_{\ll N}u|^{2k}P_Nu$. By the Hölder inequality, we have

$$\begin{aligned}\|B_{N,5}\|_{L_x^1L_T^2}\lesssim&T^{\frac{1}{2}}\|P_{\ll N}u\|_{L_x^{4k}L_T^\infty}^{2k}\|P_Nu\|_{L_x^2L_T^\infty}\\ \lesssim&T^{\frac{1}{2}}\|u\|_{X_T}^{2k}\|P_Nu\|_{L_x^2L_T^\infty},\end{aligned}$$

and

$$\begin{aligned}\|B_{N,5}\|_{L_x^{\frac{1}{1-\varepsilon}}L_T^2}\lesssim&T^{\frac{1}{2}}\|P_{\ll N}u\|_{L_x^{\frac{4k}{1-2\varepsilon}}L_T^\infty}^{2k}\|P_Nu\|_{L_x^2L_T^\infty}\\ \lesssim&T^{\frac{1}{2}}\|u\|_{X_T}^{2k}\|P_Nu\|_{L_x^2L_T^\infty}.\end{aligned}$$

Thus, in a similar way as dealing with $I_{N,1}$ and $I_{N,4}$, and noticing that $B_{N,5}$ has Fourier support in $|\xi|\sim N$, we can bound (5.30) by

$$\lesssim T^{\frac{1}{2}}\|u\|_{X_T}^{3k}\|P_{\gg 1}u\|_{X_T}.$$

6. **A priori estimates for solutions.** By the scaling argument

$$u(t,x)\mapsto u_\gamma(t,x)=\frac{1}{\gamma^{1/k}}u(\frac{t}{\gamma^2},\frac{x}{\gamma}),$$

we have

$$\begin{aligned}\|u_{0,\gamma}\|_{L^2}=&\gamma^{\frac{1}{2}-\frac{1}{k}}\|u_0\|_{L^2},\\ \|u_{0,\gamma}\|_{\dot H^{\frac{1}{2}}}=&\frac{1}{\gamma^{1/k}}\|u_0\|_{\dot H^{\frac{1}{2}}}.\end{aligned}$$

Thus, we may rescale

$$\begin{aligned}\left\|P_{\lesssim 1}u_{0,\gamma}\right\|_{L^2}\leqslant&\gamma^{\frac{1}{2}-\frac{1}{k}}\|u_0\|_{L^2}=C_{low},\\ \|P_{\gg 1}u_{0,\gamma}\|_{H^{\frac{1}{2}}}\leqslant&\frac{1}{\gamma^{1/k}}\|u_0\|_{H^{\frac{1}{2}}}<C_{high}\ll 1,\end{aligned}$$

where we choose $\gamma=\gamma(\|u_0\|_{H^{1/2}})\gg 1$, and take the time interval T depending on γ later. We now drop the writing of the scaling parameter γ and assume

$$\left\|P_{\lesssim 1}u_0\right\|_{L^2}\leqslant C_{low},$$

$$\|P_{\gg 1}u_0\|_{H^{\frac{1}{2}}} \leqslant C_{high} \ll 1.$$

We now apply this to the norms X_T and $H^{1/2}$, and define new version of the norms of X_T and $H^{1/2}$, given by with the decomposition $I = P_{\lesssim 1} + P_{\gg 1}$,

$$\|u\|_{\tilde{X}_T} = \frac{1}{C_{low}} \left\|P_{\lesssim 1}u\right\|_{X_T} + \frac{1}{C_{high}} \|P_{\gg 1}u\|_{X_T},$$

and

$$\|\phi\|_{\tilde{H}^{1/2}} = \frac{1}{C_{low}} \left\|P_{\lesssim 1}\phi\right\|_{L^2} + \frac{1}{C_{high}} \|P_{\gg 1}\phi\|_{H^{1/2}},$$

which implies that $\|u_0\|_{\tilde{H}^{1/2}} \leqslant 2$.

For the low frequency part, we have the following estimates.

Lemma 6.1. *Let u be a solution of* (1.1)-(1.2). *Then*

$$\left\|P_{\lesssim 1}u\right\|_{X_T} \lesssim C_{low} + T^{1/2} \|u\|_{X_T}^{k+1}.$$

Proof. Using the integral equation of (1.1)

$$u(t) = S(t)u_0 - \lambda \int_0^t S(t-\tau)|u(\tau)|^k u_x(\tau) d\tau,$$

and by (3.1), (3.2), (3.3), (3.8), (3.9), (3.10) and the Hölder inequality, we have

$$\begin{aligned}
\left\|P_{\lesssim 1}u\right\|_{X_T} &\lesssim \left\|S(t)P_{\lesssim 1}u_0\right\|_{X_T} + \left\|\int_0^t S(t-\tau)P_{\lesssim 1}(|u|^k u_x)(\tau)d\tau\right\|_{X_T} \\
&\lesssim \left\|P_{\lesssim 1}u_0\right\|_{L^2} + \left\|P_{\lesssim 1}(|u|^k u_x)\right\|_{L_T^1 H_x^{1/2}} \lesssim C_{low} + \left\||u|^k u_x\right\|_{L_T^1 L_x^2} \\
&\lesssim C_{low} + T^{1/2} \|u\|_{L_x^{2k} L_T^\infty}^k \|u_x\|_{L_x^\infty L_T^2} \\
&\lesssim C_{low} + T^{1/2} \|u\|_{X_T}^{k+1},
\end{aligned}$$

which is the desired result. □

For the high frequency part, we have

Lemma 6.2. *Let u and v_N be given in* (2.1). *Then*

$$\|P_{\gg 1}u\|_{X_T} \lesssim (1 + \|u\|_{L_T^\infty H_x^{1/2}}^{2k}) \left(\sum_{N \gg 1} \|v_N\|_{Y_T}^2\right)^{1/2}.$$

Proof. By (2.1), we have

$$P_N u = e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k dy} v_N.$$

For $L_T^\infty H_x^{1/2}$-norm, by the interpolation theorem, we obtain for $N \gg 1$,

$$\begin{aligned}
\|P_N u\|_{H_x^{1/2}} &\lesssim \|P_N u\|_{L^2}^{\frac{1}{2}} \|P_N u\|_{H^1}^{\frac{1}{2}} \lesssim \|v_N\|_{L^2}^{\frac{1}{2}} (\|P_N u\|_{L^2} + \|\partial_x P_N u\|_{L^2})^{\frac{1}{2}} \\
&\lesssim \|v_N\|_{L^2}^{\frac{1}{2}} \left(\|v_N\|_{L^2} + \left\||P_{\ll N}u|^k v_N\right\|_{L^2} + \|\partial_x v_N\|_{L^2}\right)^{\frac{1}{2}} \\
&\lesssim \|v_N\|_{L^2}^{\frac{1}{2}} \left(\|P_{\ll N}u\|_{L_x^{4k}}^k \|v_N\|_{L_x^4} + \|v_N\|_{H^1}\right)^{\frac{1}{2}} \\
&\lesssim \left(1 + \|P_{\ll N}u\|_{H_x^{1/2}}^k\right)^{\frac{1}{2}} \|v_N\|_{H_x^{1/2}} \lesssim \left(1 + \|P_{\ll N}u\|_{H_x^{1/2}}^k\right) \|v_N\|_{H_x^{1/2}},
\end{aligned}$$

which yields the desired estimate by summing on l_N^2.

For the $L_x^\infty L_T^2$-norm, noticing that

$$\partial_x P_N u = e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy}(\partial_x v_N + \frac{i\lambda}{2}|P_{\ll N}u|^k v_N),$$

we have

$$\begin{aligned}\|\partial_x P_N u\|_{L_x^\infty L_T^2} \lesssim{}& \|\partial_x v_N\|_{L_x^\infty L_T^2}\\ &+ \left\|\tilde{P}_N\left(\sum_{N_1} P_{N_1}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy})|P_{\ll N}u|^k \sum_{N_2} P_{N_2} v_N\right)\right\|_{L_x^\infty L_T^2}. \end{aligned}\tag{6.1}$$

To estimate the second term (6.1), we split the sum $\sum_{N_2} = \sum_{N_2\sim N} + \sum_{N_2\nsim N}$. For $N_2 \sim N$, from the Bernstein inequality, we bound (6.1) by

$$\begin{aligned}&\lesssim \left\||P_{\ll N}u|^k \sum_{N_2\sim N} P_{N_2} v_N\right\|_{L_x^\infty L_T^2} \lesssim \|P_{\ll N}u\|_{L_{x,T}^\infty}^k \sum_{N_2\sim N} \|P_{N_2} v_N\|_{L_x^\infty L_T^2}\\ &\lesssim N \left\|D_x^{-1/k} P_{\ll N}u\right\|_{L_{x,T}^\infty}^k \sum_{N_2\sim N} \|P_{N_2} v_N\|_{L_x^\infty L_T^2}\\ &\lesssim \|P_{\ll N}u\|_{L_T^\infty H_x^{1/2}}^k \sum_{N_2\sim N} \|P_{N_2}\partial_x v_N\|_{L_x^\infty L_T^2}\\ &\lesssim \|u\|_{L_T^\infty H_x^{1/2}}^k \sum_{N_2\sim N} \|P_{N_2}\partial_x v_N\|_{L_x^\infty L_T^2}.\end{aligned}$$

For the part $N_2 \nsim N$, we split it as $\sum_{N_2\nsim N} = \sum_{N_2\ll N} + \sum_{N_2\gg N}$. Noticing that for $N_2 \ll N$,

$$\begin{aligned}&\tilde{P}_N\left(\sum_{N_1} P_{N_1}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy})|P_{\ll N}u|^k \sum_{N_2} P_{N_2} v_N\right) \qquad (6.2)\\ =&\tilde{P}_N\left(P_{\sim N}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy})|P_{\ll N}u|^k P_{\ll N} v_N\right),\end{aligned}$$

and for $N_2 \gg N$,

$$(6.2) = \tilde{P}_N\left(\sum_{N_1\sim N_2\gg N} P_{N_1}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy})|P_{\ll N}u|^k P_{N_2} v_N\right),$$

we can bound (6.1), in view of the Bernstein inequality and the Hölder inequality, by

$$\begin{aligned}\lesssim& \left\|P_{\sim N} e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy}\right\|_{L_{x,T}^\infty} \|P_{\ll N}u\|_{L_{x,T}^\infty}^k \|P_{\ll N} v_N\|_{L_{x,T}^\infty}\\ &+ \sum_{N_1\sim N_2\gg N} \left\|P_{N_1} e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy}\right\|_{L_{x,T}^\infty} \|P_{\ll N}u\|_{L_{x,T}^\infty}^k \|P_{N_2} v_N\|_{L_{x,T}^\infty}\\ \lesssim& N^{-\frac{k}{(2k+1)}} \left\|P_{\sim N}\partial_x e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy}\right\|_{L_{x,T}^\infty}\\ &\cdot \left\|D_x^{-\frac{1}{(2k+1)}} P_{\ll N}u\right\|_{L_{x,T}^\infty}^k \left\|D_x^{-\frac{1}{(2k+1)}} P_{\ll N} v_N\right\|_{L_{x,T}^\infty}\\ &+ \sum_{N_1\sim N_2\gg N} N_1^{-\frac{k}{(2k+1)}} \left\|P_{N_1}\partial_x e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N}u|^k\,dy}\right\|_{L_{x,T}^\infty}\end{aligned}$$

$$
\begin{aligned}
&\cdot \left\| D_x^{-\frac{1}{(2k+1)}} P_{\ll N} u \right\|_{L^\infty_{x,T}}^k \left\| D_x^{-\frac{1}{(2k+1)}} P_{N_2} v_N \right\|_{L^\infty_{x,T}} \\
\lesssim & N^{-\frac{k}{(2k+1)}} \|P_{\ll N} u\|^k_{L^\infty_{x,T}} \|u\|^k_{L^\infty_T H^{1/2}_x} \|v_N\|_{L^\infty_T H^{1/2}_x} \\
&+ \sum_{N_1 \sim N_2 \gg N} N_1^{-\frac{k}{3(2k+1)}} N_2^{-\frac{k}{3(2k+1)}} N^{-\frac{k}{3(2k+1)}} \|P_{\ll N} u\|^k_{L^\infty_{x,T}} \|u\|^k_{L^\infty_T H^{1/2}_x} \|P_{N_2} v_N\|_{L^\infty_T H^{1/2}_x} \\
\lesssim & \|u\|^{2k}_{L^\infty_T H^{1/2}_x} \|v_N\|_{L^\infty_T H^{1/2}_x} .
\end{aligned}
$$

Therefore, summing on l^2_N, we complete the proof for the $L^\infty_x L^2_T$-norm.

For the $L^2_x L^\infty_T$-norm, it is easy to obtain the desired result since $|P_N u| = |v_N|$.

We turn to estimate the $L^4_x L^\infty_T$-norm. It is similar with the proof for the $L^\infty_x L^2_T$-norm, since $\left\|\langle D_x\rangle^{1/4} P_N u\right\|_{L^4_x L^\infty_T} \sim N^{1/4} \|P_N u\|_{L^4_x L^\infty_T}$ for $N \gg 1$. In fact, we have

$$
\left\|\langle D_x\rangle^{1/4} P_N u\right\|_{L^4_x L^\infty_T} \sim N^{1/4} \left\| \sum_{N_1} P_{N_1}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N} u|^k dy}) \sum_{N_2} P_{N_2} v_N \right\|_{L^4_x L^\infty_T} . \quad (6.3)
$$

We also split $\sum_{N_2} = \sum_{N_2 \sim N} + \sum_{N_2 \nsim N}$. For $N_2 \sim N$, we bound (6.3) by

$$
\lesssim N^{1/4} \sum_{N_2 \sim N} \|P_{N_2} v_N\|_{L^4_x L^\infty_T} \lesssim \left\|\langle D_x\rangle^{1/4} v_N\right\|_{L^4_x L^\infty_T} .
$$

For the part $N_2 \nsim N$, we split it as $\sum_{N_2 \nsim N} = \sum_{N_2 \ll N} + \sum_{N_2 \gg N}$. Noticing that for $N_2 \ll N$,

$$
\begin{aligned}
&\tilde{P}_N \left(\sum_{N_1} P_{N_1}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N} u|^k dy}) \sum_{N_2} P_{N_2} v_N \right) \qquad (6.4)\\
=&\tilde{P}_N \left(P_{\sim N}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N} u|^k dy}) P_{\ll N} v_N \right),
\end{aligned}
$$

and for $N_2 \gg N$,

$$
(6.4) = \tilde{P}_N \left(\sum_{N_1 \sim N_2 \gg N} P_{N_1}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N} u|^k dy}) P_{N_2} v_N \right),
$$

we can bound (6.3), in view of the Bernstein inequality and the Hölder inequality, by

$$
\begin{aligned}
\lesssim & \left\|\langle D_x\rangle^{1/4} v_N\right\|_{L^4_x L^\infty_T} + N^{1/4} \left\| P_{\sim N}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N} u|^k dy}) \right\|_{L^\infty_{x,T}} \|P_{\ll N} v_N\|_{L^4_x L^\infty_T} \\
&+ N^{1/4} \sum_{N_1 \sim N_2 \gg N} \left\| P_{N_1}(e^{\frac{i\lambda}{2}\int_{-\infty}^x |P_{\ll N} u|^k dy}) \right\|_{L^\infty_{x,T}} \|P_{N_2} v_N\|_{L^4_x L^\infty_T} \\
\lesssim & \left\|\langle D_x\rangle^{1/4} v_N\right\|_{L^4_x L^\infty_T} + N^{-3/4} \|P_{\ll N} u\|^k_{L^\infty_{x,T}} \left\|\langle D_x\rangle^{1/4} P_{\ll N} v_N\right\|_{L^4_x L^\infty_T} \\
&+ \sum_{N_1 \sim N_2 \gg N} N_1^{-1/3} N_2^{-1/3} N^{-1/3} \|P_{\ll N} u\|^k_{L^\infty_{x,T}} \left\|\langle D_x\rangle^{1/4} P_{N_2} v_N\right\|_{L^4_x L^\infty_T} \\
\lesssim & (1 + \|u\|^k_{L^\infty_T H^{1/2}_x}) \left\|\langle D_x\rangle^{1/4} v_N\right\|_{L^4_x L^\infty_T} ,
\end{aligned}
$$

which yields the desired estimate by applying l^2_N-sum.

Thus, we complete the proof of this Lemma. □

Of course, we need the following estimate of the data.

Lemma 6.3. *For any $u_0 \in H^{1/2}$, we have*

$$\left(\sum_{N\gg 1}\left\|S(t)(e^{-\frac{i\lambda}{2}\int_{-\infty}^{x}|P_{\ll N}u_0|^k dy}P_N u_0)\right\|_{Y_T}^2\right)^{1/2} \lesssim (1+\|u_0\|_{H^{1/2}}^k)\,\|P_{\gg 1}u_0\|_{H^{1/2}}. \tag{6.5}$$

Proof. From (3.1), (3.2) and (3.3), we bound the left hand side of (6.5) by

$$\lesssim \left(\sum_{N\gg 1}\left\|e^{-\frac{i\lambda}{2}\int_{-\infty}^{x}|P_{\ll N}u_0|^k dy}P_N u_0\right\|_{H^{1/2}}^2\right)^{1/2} \tag{6.6}$$

$$+\left(\sum_{N\gg 1}\left(\sum_{M}\left\|P_M(e^{-\frac{i\lambda}{2}\int_{-\infty}^{x}|P_{\ll N}u_0|^k dy}P_N u_0)\right\|_{H^{1/2}}\right)^2\right)^{1/2}. \tag{6.7}$$

From Lemma 3.3, we have

$$\begin{aligned}(6.6) \lesssim &\left(\sum_{N\gg 1}[\|P_{\ll N}u_0\|_{L^{4k}}^{2k}\|P_N u_0\|_{L^4}^2+\|P_N u_0\|_{H^{1/2}}^2]\right)^{1/2}\\ \lesssim&(1+\|u_0\|_{H^{1/2}}^2)\,\|P_{\gg 1}u_0\|_{H^{1/2}}.\end{aligned}$$

For the second term (6.7), it is similar with (5.21). We split the sum $\sum_M = \sum_{M\lesssim N}+\sum_{M\gg N}$. By the Bernstein inequality, the Hölder inequality and the Sobolev embedding theorem, we bound (6.7) by

$$\begin{aligned}
\lesssim&\left(\sum_{N\gg 1}\left(\sum_{M\lesssim N}\langle M\rangle^{\frac12}\|P_N u_0\|_{L_x^2}\right)^2\right)^{1/2}\\
&+\left(\sum_{N\gg 1}\left(\sum_{M\gg N}\left\|\langle D_x\rangle^{\frac12}P_M(e^{-\frac{i\lambda}{2}\int_{-\infty}^{x}|P_{\ll N}u_0|^k dy}P_N u_0)\right\|_{L_x^2}\right)^2\right)^{1/2}\\
\lesssim&\left(\sum_{N\gg 1}\left(N^{1/2}\|P_N u_0\|_{L_x^2}\right)^2\right)^{1/2}\\
&+\left(\sum_{N\gg 1}\left(\sum_{M\gg N}M^{\frac12}\left\|(P_{\sim M}e^{-\frac{i\lambda}{2}\int_{-\infty}^{x}|P_{\ll N}u_0|^k dy})P_N u_0\right\|_{L_x^2}\right)^2\right)^{1/2}\\
\lesssim&\|P_{\gg 1}u_0\|_{H^{1/2}}\\
&+\left(\sum_{N\gg 1}\left(\sum_{M\gg N}M^{\frac12}\left\|P_{\sim M}e^{-\frac{i\lambda}{2}\int_{-\infty}^{x}|P_{\ll N}u_0|^k dy}\right\|_{L^4}\|P_N u_0\|_{L_x^4}\right)^2\right)^{1/2}\\
\lesssim&\|P_{\gg 1}u_0\|_{H^{1/2}}\\
&+\left(\sum_{N\gg 1}\left(\sum_{M\gg N}M^{-\frac12}\left\|P_{\sim M}\partial_x e^{-\frac{i\lambda}{2}\int_{-\infty}^{x}|P_{\ll N}u_0|^k dy}\right\|_{L^4}\|P_N u_0\|_{L_x^4}\right)^2\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|P_{\gg 1}u_0\|_{H^{1/2}} + \left(\sum_{N\gg 1}\left(\|P_{\ll N}u_0\|_{L^{4k}}^k \|P_N u_0\|_{L^4_x}\right)^2\right)^{1/2}\\
&\lesssim (1+\|u_0\|_{H^{1/2}}^k)\,\|P_{\gg 1}u_0\|_{H^{1/2}}\,,
\end{aligned}
$$

which yields the desired result. □

With the help of the above lemmas, we can prove the following proposition which yields the a priori estimate.

Proposition 6.1. *Let u be a smooth solution to* (1.1)-(1.2) *and* $0 < T \leqslant C_{high}^4$. *Then we have*

$$
\|u\|_{\tilde{X}_T} \leqslant C(C_{low}) + C(C_{low} + \|u\|_{\tilde{X}_T})^{3k}(T^{1/4} + C_{high})\,\|u\|_{\tilde{X}_T}\,.
$$

Proof. Noticing that

$$
\left\|P_{\lesssim 1}u\right\|_{X_T} \lesssim C_{low}\,\|u\|_{\tilde{X}_T}\,, \quad \|P_{\gg 1}u\|_{X_T} \lesssim C_{high}\,\|u\|_{\tilde{X}_T}\,.
$$

and from Lemmas 6.1, 6.2, 6.3 and Proposition 5.1, we obtain through a complicated computation

$$
\begin{aligned}
&\|u\|_{\tilde{X}_T} = \frac{1}{C_{low}}\left\|P_{\lesssim 1}u\right\|_{X_T} + \frac{1}{C_{high}}\|P_{\gg 1}u\|_{X_T}\\
&\lesssim 1 + \frac{1}{C_{low}}T^{\frac12}\|u\|_{X_T}^{k+1} + \frac{1}{C_{high}}(1+\|u\|_{L_T^\infty H_x^{1/2}}^{2k})\left(\sum_{N\gg 1}\|v_N\|_{Y_T}^2\right)^{1/2}\\
&\lesssim 1 + \frac{1}{C_{low}}T^{\frac12}\|u\|_{X_T}^{k+1} + \frac{1}{C_{high}}(1+\|u\|_{L_T^\infty H_x^{1/2}}^{2k})(1+\|u_0\|_{H^{1/2}}^k)\,\|P_{\gg 1}u_0\|_{H^{1/2}}\\
&\quad + \frac{1}{C_{high}}(1+\|u\|_{L_T^\infty H_x^{1/2}}^{2k})\Big\{T^{\frac12}\|u\|_{X_T}^{3k}\|P_{\gg 1}u\|_{X_T}\\
&\quad + (1+\|u\|_{X_T}^k)\left[T^{\frac12}\|u\|_{X_T}^{\tilde{k}+1}\|P_{\gg 1}u\|_{X_T}^{k-\tilde{k}} + (1+T^{\frac14}\|u\|_{X_T})\|u\|_{X_T}^{\tilde{k}}\|P_{\gg 1}u\|_{X_T}^{k+1-\tilde{k}}\right]\\
&\quad + T^{\frac12}\left(\|u\|_{X_T}^{2k-1} + \|u\|_{X_T}^{(5k-2)/2}\right)\|P_{\gg 1}u\|_{X_T}\\
&\quad + T^{\frac14}(1+T^{\frac14}\|u\|_{X_T})^{\frac12}\|u\|_{X_T}^{2k-1}\|P_{\gg 1}u\|_{X_T}^{\frac32}\\
&\quad + (1+\|u\|_{X_T}^k)\left[T^{\frac12}\|u\|_{X_T}^k\|P_{\gg 1}u\|_{X_T} + (1+T^{\frac14}\|u\|_{X_T})^2\|u\|_{X_T}^{k-1}\|P_{\gg 1}u\|_{X_T}^2\right]\Big\}\\
&\leqslant C + C(C_{low})T^{\frac12}\|u\|_{\tilde{X}_T}^{k+1} + (1+\|u\|_{L_T^\infty H_x^{1/2}}^{2k})(1+\|u_0\|_{H^{1/2}}^k)\\
&\quad + C(C_{low})(1+\|u\|_{L_T^\infty H_x^{1/2}}^{2k})\Big\{T^{\frac14}\|u\|_{\tilde{X}_T}^{3k+1} + T^{\frac14}(1+\|u\|_{\tilde{X}_T}^k)\|u\|_{\tilde{X}_T}^{k+1}\\
&\quad + T^{\frac12}(\|u\|_{\tilde{X}_T}^{k+1} + \|u\|_{\tilde{X}_T}^{5k/2}) + C_{high}^{k-\tilde{k}}(1+T^{\frac14}\|u\|_{\tilde{X}_T})(1+\|u\|_{\tilde{X}_T}^k)\|u\|_{\tilde{X}_T}^{k+1}\\
&\quad + C_{high}^{3/2}(1+T^{\frac14}\|u\|_{\tilde{X}_T})\|u\|_{\tilde{X}_T}^{2k} + C_{high}^2(1+T^{\frac14}\|u\|_{\tilde{X}_T})^2(1+\|u\|_{\tilde{X}_T}^k)\|u\|_{\tilde{X}_T}^{k+1}\Big\}.
\end{aligned}
$$

Notice that

$$
\|u(t)\|_{H^{1/2}} \lesssim \left\|P_{\lesssim 1}u(t)\right\|_{L^2} + C_{high}\|P_{\gg 1}\|_{\tilde{H}^{1/2}}\,.
$$

The high frequency part $C_{high}\|P_{\gg 1}\|_{\tilde{H}^{1/2}}$ can be absorbed into the $\tilde{X}_T$-norm. Then substituting Lemma 6.1 again in estimating the low frequency part of the norm $\left\|P_{\lesssim 1}u\right\|_{L_T^\infty H_x^{1/2}}$, we complete the proof of Proposition 6.1. □

From Proposition 6.1, we have the following a priori estimate for the solution of (1.1)-(1.2) if we take T and C_{high} small enough.

Corollary 6.1. *Let u be a smooth solution to* (1.1)-(1.2)*. we have*

$$\|u\|_{\tilde{X}_T} \lesssim C_{low} + C_{high},$$

for T and C_{high} small enough.

For the proof of Theorem 1.1, we can follow the compactness argument with the a priori estimate. Since the proof is standard, we omit the details and refer to the papers [14, 15, 16, 17, 19, 22].

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