Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 328 (2007) 58–83
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Well-posedness of Cauchy problem for the fourth order nonlinear Schrödinger equations in multi-dimensional spaces

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Received 1 March 2005
Available online 12 June 2006
Submitted by M.M. Peloso

Abstract

We study the well-posedness of Cauchy problem for the fourth order nonlinear Schrödinger equations

$$i\partial_t u = -\varepsilon\Delta u + \Delta^2 u + P\big((\partial_x^\alpha u)_{|\alpha|\leqslant 2}, (\partial_x^\alpha \bar{u})_{|\alpha|\leqslant 2}\big), \quad t\in\mathbb{R},\ x\in\mathbb{R}^n,$$

where $\varepsilon\in\{-1,0,1\}$, $n\geqslant 2$ denotes the spatial dimension and $P(\cdot)$ is a polynomial excluding constant and linear terms.

Keywords: Fourth order nonlinear Schrödinger equations; Cauchy problem; Well-posedness; Smoothing effects

1. Introduction

We consider the Cauchy problem for the fourth order nonlinear Schrödinger equations

$$i\partial_t u = -\varepsilon\Delta u + \Delta^2 u + P\big((\partial_x^\alpha u)_{|\alpha|\leqslant 2}, (\partial_x^\alpha \bar{u})_{|\alpha|\leqslant 2}\big), \quad t\in\mathbb{R},\ x\in\mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = u_0(x), \tag{1.2}$$

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1 Supported partially by the Innovation Funds of AMSS, CAS of China.
2 Supported partially by NSFC (Grant No. 10431060), NSAF (Grant No. 10276036).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.05.031

where $\varepsilon \in \{-1, 0, 1\}$, $\alpha \in \mathbb{Z}^n$, $u = u(t, x)$ is a complex valued wave function of $(t, x) \in \mathbb{R}^{1+n}$ and Δ is the Laplace operator on $\mathbb{R}^n$ with $n \geqslant 2$. $P(\cdot)$ is a complex valued polynomial defined in $\mathbb{C}^{n^2+3n+2}$ such that

$$P(\vec{z}) = P(z_1, z_2, \dots, z_{n^2+3n+2}) = \sum_{\substack{l \leqslant |\beta| \leqslant h \\ \beta \in \mathbb{Z}^{n^2+3n+2}}} a_\beta z^\beta \quad \text{for } l, h \in \mathbb{N}, \tag{1.3}$$

and there exists $a_{\beta_0} \neq 0$ for some $\beta_0 \in \mathbb{Z}^{n^2+3n+2}$ with $|\beta_0| = l \geqslant 2$.

This class of nonlinear Schrödinger equations has been widely applied in many branches in applied science such as deep water wave dynamics, plasma physics, optical communications and so on [5]. A large amount of interesting works has been devoted to the study of Cauchy problem to dispersive equations, such as [1–4,6–23,25] and references cited therein.

In order to study the influence of higher order dispersion on solitary waves, instability and the collapse phenomena, V.I. Karpman [11] introduced a class of nonlinear Schrödinger equations

$$i\Psi_t + \frac{1}{2}\Delta\Psi + \frac{\gamma}{2}\Delta^2\Psi + f\big(|\Psi|^2\big)\Psi = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

This system with different nonlinearities was discussed by lots of authors. In [21], by using the method of Fourier restriction norm, J. Segata studied a special fourth order nonlinear Schrödinger equation in one-dimensional space and considered the three-dimensional motion of an isolated vortex filament, which was introduced by Da Rios, embedded in inviscid incompressible fluid fulfilled in an infinite region, and the results have been improved in [10,22].

In [1], M. Ben-Artzi, H. Koch and J.C. Saut discussed the sharp space-time decay properties of fundamental solutions to the linear equation

$$i\Psi_t - \varepsilon\Delta\Psi + \Delta^2\Psi = 0, \quad \varepsilon \in \{-1, 0, 1\}.$$

In [7], B.L. Guo and B.X. Wang considered the existence and scattering theory for the Cauchy problem of nonlinear Schrödinger equations of the form

$$iu_t + (-\Delta)^m u + f(u) = 0, \tag{1.4}$$

$$u(0, x) = \varphi(x), \tag{1.5}$$

where $m \geqslant 1$ is an integer. H. Pecher and W. von Wahl in [20] proved the existence of classical global solutions of (1.4)–(1.5) for the space dimensions $n \leqslant 7m$ for the case $m \geqslant 1$. In [8], we discussed the local well-posedness of the Cauchy problem (1.4)–(1.5) without smallness of data for $m = 2$ and $f(u) = P((\partial_x^j u)_{j \leqslant 2}, (\partial_x^j \bar{u})_{j \leqslant 2})$ in one dimension.

In the present paper we deal with Eq. (1.1) in which the difficulty arises from the operator semigroup itself and the fact that the nonlinearity of P involves the first and the second derivatives. This could cause the so-called loss of derivatives so long as we make direct use of the standard methods, such as the energy estimates, the space-time estimates, etc. In addition, compared to the one-dimensional case, it is quite different in the multi-dimensional case, since we have to separate the spatial space $\mathbb{R}^n$ into a family of nonoverlapping cubes of bounded size.

In [16], C.E. Kenig, G. Ponce and L. Vega made a great progress on the nonlinear Schrödinger equation of the form

$$\partial_t u = i\Delta u + P(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

and proved that the corresponding Cauchy problem is locally well-posed for small data in the Sobolev space $H^s(\mathbb{R}^n)$ and in its weighted version by pushing forward the linear estimates associated with the Schrödinger group $\{e^{it\Delta}\}_{-\infty}^{\infty}$ and by introducing suitable function spaces where these estimates act naturally. They also studied generalized nonlinear Schrödinger equations in [17] and quasi-linear Schrödinger equations in [18]. In the one-dimensional case, $n = 1$, the smallness assumption on the size of the data was removed by N. Hayashi and T. Ozawa [9] by using a change of variable to obtain an equivalent system with a nonlinear term independent of $\partial_x u$, where the new system could be treated by the standard energy method. And H. Chihara [3], was able to remove the size restriction on the data in any dimension by using an invertible classical pseudo-differential operator of order zero.

T. Kato showed, in [12], that solutions of the Korteweg–de Vries equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad t, x \in \mathbb{R},$$

satisfy

$$\int_{-T}^{T}\int_{-R}^{R} \left|\partial_x u(t,x)\right|^2 \mathrm{d}x\,\mathrm{d}t \leqslant C\left(T, R, \left\|u(0,x)\right\|_2\right).$$

And in [19], S.N. Kruzhkov and A.V. Faminskiĭ, for $u(0,x) = u_0(x) \in L^2$ such that $x^\alpha u_0(x) \in L^2((0;+\infty))$, proved that the weak solution of the Korteweg–de Vries equation constructed there has l-continuous space derivatives for all $t > 0$ if $l < 2\alpha$. The corresponding version of the above estimate for the Schrödinger group $\{e^{it\Delta}\}_{-\infty}^{\infty}$

$$\int_{-T}^{T}\int_{-R}^{R} \left|(1-\Delta)^{\frac{1}{4}} e^{it\Delta} u_0\right|^2 \mathrm{d}x\,\mathrm{d}t \leqslant C\left(T, R, \|u_0\|_2\right)$$

was simultaneously established by P. Constantin and J.-C. Saut [4], P. Sjölin [23] and L. Vega [25] and others.

In the present paper, we will develop the arguments in [16] to study the Cauchy problem (1.1)–(1.2). We first discuss the local smoothing effects of the unit group $\{S(t)\}_{-\infty}^{\infty}$ ($S(t)$ is defined as below) in order to overcome the loss of derivatives in Section 2. To construct the work space, we have to study, in Section 3, the boundedness properties of the maximal function $\sup_{[0,T]} |S(t)\cdot|$. This idea is implicit in the splitting argument introduced by J. Ginibre and Y. Tsutsumi [6] to deal with uniqueness for the generalized KdV equation. Finally, we will consider some special cases in Section 4 to apply the estimates we have obtained in the previous sections.

For convenience, we introduce some notations. $S(t) := e^{it(\varepsilon\Delta - \Delta^2)}$ denotes the unitary group generated by $i(\varepsilon\Delta - \Delta^2)$ in $L^2(\mathbb{R}^n)$. $\bar{z}$ denotes the conjugate of the complex number z. $\mathcal{F}u$ or $\hat{u}$ ($\mathcal{F}^{-1}u$, respectively) denotes the Fourier (inverse, respectively) transform of u with respect to all variables except the special announcement. $\mathcal{S}$ denotes the space of Schwartz' functions. $\{Q_\alpha\}_{\alpha\in\mathbb{Z}^n}$ is the family of nonoverlapping cubes of size R such that $\mathbb{R}^n = \bigcup_{\alpha\in\mathbb{Z}^n} Q_\alpha$. $D_x^\gamma f(x) = \mathcal{F}^{-1}|\xi|^\gamma \mathcal{F} f(x)$ is the homogeneous derivative of order γ of f. H^s ($\dot{H}^s$, respectively) denotes the usual inhomogeneous (homogeneous, respectively) Sobolev space. We denote $\|\cdot\|_{s,2} = \|\cdot\|_{H^s}$, $\|\cdot\|_p = \|\cdot\|_{L^p}$ for $1 \leqslant p \leqslant \infty$ and

$$\|f\|_{l,2,j} = \|f\|_{H^l(\mathbb{R};|x|^j\,\mathrm{d}x)} = \sum_{|\gamma|\leqslant l}\left(\int_{\mathbb{R}^n}\left|\partial_x^\gamma f(x)\right|^2 |x|^j\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{1.6}$$

Throughout the paper, the constant C might be different from each other and $[\frac{n}{2}]$ denotes the greatest integer that is less than or equal to $\frac{n}{2}$.

Now we state the main results of this paper.

Theorem 1 *(Case $l \geqslant 3$). Let $n \geqslant 2$. Given any polynomial P as in* (1.3) *with $l \geqslant 3$, then, for any $u_0 \in H^s(\mathbb{R}^n)$ with $s \geqslant s_0 = n + 2 + \frac{1}{2}$, there exists $T = T(\|u_0\|_{s,2}) > 0$ such that the Cauchy problem* (1.1)–(1.2) *has a unique solution $u(t)$ defined in the time interval* $[0, T]$ *and satisfying*

$$u \in C\big([0, T]; H^s(\mathbb{R}^n)\big),$$

and

$$u \in L^2\big([0, T]; \dot{H}^{s+\frac{1}{2}}(Q)\big)$$

for any cube Q of unit size in $\mathbb{R}^n$.

Theorem 2 *(Case $l = 2$). Let $n \geqslant 2$. Given any polynomial P as in* (1.3) *with $l = 2$, then, for any $u_0 \in H^s(\mathbb{R}^n) \cap H^{n+4[\frac{n}{2}]+8}(\mathbb{R}^n; |x|^{2[\frac{n}{2}]+2}\,\mathrm{d}x)$ with $s \geqslant s_0 = 2n + 3[\frac{n}{2}] + 15 + \frac{1}{2}$, there exists $T = T(\|u_0\|_{s,2}, \|u_0\|_{n+4[\frac{n}{2}]+8,2,[\frac{n}{2}]+2}) > 0$ such that the Cauchy problem* (1.1)–(1.2) *has a unique solution $u(t)$ defined in the time interval* $[0, T]$ *and satisfying*

$$u \in C\big([0, T]; H^s\big(\mathbb{R}^n\big) \cap H^{n+[\frac{n}{2}]+8}\big(\mathbb{R}^n; |x|^{2[\frac{n}{2}]+2}\,\mathrm{d}x\big)\big),$$

and

$$u \in L^2\big([0, T]; \dot{H}^{s+\frac{1}{2}}(Q)\big)$$

for any cube Q of unit size in $\mathbb{R}^n$.

2. Local smoothing effects

We will prove the local smoothing effects exhibited by the semigroup $\{S(t)\}_{-\infty}^{\infty}$ in this section.

Lemma 3 *(Local smoothing effect: homogeneous case). We have the following estimate*

$$\sup_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_{-\infty}^{\infty} \big| D_x^{\frac{3}{2}} S(t) u_0(x) \big|^2 \,\mathrm{d}t\,\mathrm{d}x \right)^{\frac{1}{2}} \leqslant C R^{\frac{1}{2}} \|u_0\|_2 \tag{2.1}$$

and the corresponding dual version

$$\left\| D_x^{\frac{3}{2}} \int_0^T S(-\tau) f(\tau, \cdot)\,\mathrm{d}\tau \right\|_2 \leqslant C R^{\frac{1}{2}} \sum_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_0^T \big| f(t, x) \big|^2 \,\mathrm{d}t\,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.2}$$

Proof. (2.1) and (2.2) can be derived from [14, Theorem 4.1] for which we omit the details. □

More precisely, (2.2) yields, for $t \in [0, T]$, that

$$\left\| D_x^{\frac{3}{2}} \int_0^t S(t-\tau) f(\tau, \cdot)\,\mathrm{d}\tau \right\|_2 \leqslant C R^{\frac{1}{2}} \sum_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_0^T \big| f(t, x) \big|^2 \,\mathrm{d}t\,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.3}$$

Moreover, with the help of Fubini theorem, (2.1) implies that

$$\sup_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\left|S(t)u_0(x)\right|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\leqslant CR^{\frac{1}{2}}\|u_0\|_{\dot H^{-\frac{3}{2}}}. \tag{2.4}$$

In addition, we have

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\left|S(t)u_0(x)\right|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad\leqslant T^{\frac{1}{2}}\left(\sup_t\int_{\mathbb{R}^n}\left|S(t)u_0(x)\right|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad\leqslant T^{\frac{1}{2}}\|u_0\|_2.
\end{aligned} \tag{2.5}$$

Interpolating both (2.4) and (2.5), we can obtain that

$$\sup_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\left|S(t)u_0(x)\right|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\leqslant CR^{\frac{1}{6}}T^{\frac{1}{3}}\|u_0\|_{\dot H^{-\frac{1}{2}}}. \tag{2.6}$$

Now we turn to consider the inhomogeneous Cauchy problem

$$i\partial_t u=-\varepsilon\Delta u+\Delta^2u+F(t,x),\qquad t\in\mathbb{R},\ x\in\mathbb{R}^n, \tag{2.7}$$

$$u(0,x)=0, \tag{2.8}$$

with $F\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^n)$. We have the following estimate on the local smoothing effect in this inhomogeneous case.

Lemma 4 *(Local smoothing effect: inhomogeneous case). The solution $u(t,x)$ of Cauchy problem* (2.7)–(2.8) *satisfies*

$$\sup_{\alpha\in\mathbb{Z}^n}\left\|D_x^2u(t,x)\right\|_{L_x^2(Q_\alpha;L_t^2([0,T]))}\leqslant CRT^{\frac{1}{4n}}\sum_{\alpha\in\mathbb{Z}^n}\|F\|_{L_x^2(Q_\alpha;L_t^2([0,T]))}. \tag{2.9}$$

It should be 1/4.

Proof. Separating

$$F=\sum_{\alpha\in\mathbb{Z}^n}F\chi_{Q_\alpha}=\sum_{\alpha\in\mathbb{Z}^n}F_\alpha$$

and

$$u=\sum_{\alpha\in\mathbb{Z}^n}u_\alpha,$$

where $u_\alpha(t,x)$ is the corresponding solution of Cauchy problem

$$i\partial_t u_\alpha=-\varepsilon\Delta u_\alpha+\Delta^2u_\alpha+F_\alpha(t,x),\qquad t\in\mathbb{R},\ x\in\mathbb{R}^n, \tag{2.10}$$

$$u_\alpha(0,x)=0, \tag{2.11}$$

we formally take Fourier transform in both variables t and x in Eq. (2.10) and obtain

$$\hat{u_\alpha}(\tau,\xi)=\frac{\hat{F}_\alpha(\tau,\xi)}{\tau-\varepsilon|\xi|^2-|\xi|^4}\quad \text{for each } \alpha\in\mathbb{Z}^n.$$

By Plancherel theorem in the time variable, we can get

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{Z}^n}\big\|D_x^2u_\beta(t,x)\big\|_{L_x^2(Q_\alpha;L_t^2([0,T]))}\\
&\quad=\sup_{\alpha\in\mathbb{Z}^n}\big\|D_x^2\mathcal{F}_tu_\beta(\tau,x)\big\|_{L_x^2(Q_\alpha;L_\tau^2(\mathbb{R}))}\\
&\quad=\sup_{\alpha\in\mathbb{Z}^n}\left\|D_x^2\mathcal{F}_\xi^{-1}\left(\frac{\hat{F}_\beta(\tau,\xi)}{\tau-\varepsilon|\xi|^2-|\xi|^4}\right)\right\|_{L_x^2(Q_\alpha;L_\tau^2(\mathbb{R}))}\\
&\quad=\sup_{\alpha\in\mathbb{Z}^n}\left\|\mathcal{F}_\xi^{-1}\left(\frac{|\xi|^2}{\tau-\varepsilon|\xi|^2-|\xi|^4}\hat{F}_\beta(\tau,\xi)\right)\right\|_{L_x^2(Q_\alpha;L_\tau^2(\mathbb{R}))}\\
&\quad=\sup_{\alpha\in\mathbb{Z}^n}\left(\int\limits_{Q_\alpha}\int\limits_{-\infty}^{\infty}\left|\mathcal{F}_\xi^{-1}\left(\frac{|\xi|^2}{\tau-\varepsilon|\xi|^2-|\xi|^4}\hat{F}_\beta(\tau,\xi)\right)\right|^2\mathrm{d}\tau\,\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}\tag{2.12}$$

In order to continue the above estimate, we introduce the following estimate.

Lemma 5. *Let*

$$\mathcal{M}f=\mathcal{F}^{-1}m(\xi)\mathcal{F}f\quad \textit{and}\quad m(\xi)=\frac{|\xi|^2}{1-\varepsilon\tau^{-\frac{1}{2}}|\xi|^2-|\xi|^4}$$

for $\tau>0$ and $\varepsilon\in\{-1,0,1\}$, where $\mathcal{F}$ ($\mathcal{F}^{-1}$) denotes the Fourier (inverse, respectively) transform in x only. Then, we have

$$\sup_{\alpha\in\mathbb{Z}^n}\left(\int\limits_{Q_\alpha}\big|\mathcal{M}(g\chi_{Q_\beta})\big|^2\mathrm{d}x\right)^{\frac{1}{2}}\leqslant CR\left(\int\limits_{Q_\beta}|g|^2\mathrm{d}x\right)^{\frac{1}{2}}.\tag{2.13}$$

Proof. We have

$$\begin{aligned}
&1-\varepsilon\tau^{-\frac{1}{2}}|\xi|^2-|\xi|^4\\
&\quad=\left(1-\frac{\varepsilon}{2\sqrt{\tau}}|\xi|^2\right)^2-\left(\frac{\varepsilon^2}{4\tau}+1\right)|\xi|^4\\
&\quad=\left[1+\left(\sqrt{\frac{\varepsilon^2}{4\tau}+1}-\frac{\varepsilon}{2\sqrt{\tau}}\right)|\xi|^2\right]\left[1-\left(\sqrt{\frac{\varepsilon^2}{4\tau}+1}+\frac{\varepsilon}{2\sqrt{\tau}}\right)|\xi|^2\right]\\
&\quad=\left(\frac{2\sqrt{\tau}}{\sqrt{\varepsilon^2+4\tau}-\varepsilon}+|\xi|^2\right)\left(\frac{2\sqrt{\tau}}{\sqrt{\varepsilon^2+4\tau}+\varepsilon}-|\xi|^2\right).
\end{aligned}$$

Let $\varphi\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi\subset[-1,1]$, $\varphi\equiv1$ in $[-\frac{1}{2},\frac{1}{2}]$ and $0\leqslant\varphi\leqslant1$. Denote

$$a=\sqrt{\frac{2\sqrt{\tau}}{\sqrt{\varepsilon^2+4\tau}+\varepsilon}}.$$

Let $\varphi_1(\xi)=\varphi(2(|\xi|-a))$ and choose $\varphi_2(\xi)$ such that $\varphi_1(\xi)+\varphi_2(\xi)=1$.

Thus, we have $\operatorname{supp}\varphi_1\subset\{\xi\colon a-\frac{1}{2}\leqslant|\xi|\leqslant a+\frac{1}{2}\}$ and $\varphi_1(\xi)=1$ for $a-\frac{1}{4}\leqslant|\xi|\leqslant a+\frac{1}{4}$.

Define

$$\mathcal{M}_j f(x)=\mathcal{F}^{-1}m_j(\xi)\mathcal{F}f(x),\quad j=1,2,$$

where $m_j(\xi)=\varphi_j(\xi)m(\xi)$.

First, we shall establish (2.13) for the operator $\mathcal{M}_2$ whose symbol $m_2(\xi)$ has no singularities. For p, p' such that

$$\frac{1}{p}+\frac{1}{p'}=1\quad\text{and}\quad\frac{1}{p'}-\frac{1}{p}=\frac{1}{n},$$

it follows, by Hölder's inequality, Sobolev's embedding theorem and Mihlin theorem, that

$$\begin{aligned}\Big(\int_{Q_\alpha}\big|\mathcal{M}_2(g\chi_{Q_\beta})\big|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}&\leqslant C\|1\|_{L^{\frac{1}{\frac{1}{2}-\frac{1}{p}}}(Q_\alpha)}\big\|\mathcal{M}_2(g\chi_{Q_\beta})\big\|_{L^p(Q_\alpha)}\\&\leqslant CR^{n(\frac{1}{2}-\frac{1}{p})}\big\|\mathcal{M}_2(g\chi_{Q_\beta})\big\|_{L^p(Q_\alpha)}\\&\leqslant CR^{n(\frac{1}{2}-\frac{1}{p})}\big\|D_x\mathcal{M}_2(g\chi_{Q_\beta})\big\|_{L^{p'}(Q_\alpha)}\\&\leqslant CR^{n(\frac{1}{2}-\frac{1}{p})}\|g\chi_{Q_\beta}\|_{L^{p'}(Q_\alpha)}\\&\leqslant CR\|g\chi_{Q_\beta}\|_{L^2(Q_\alpha)},\end{aligned}$$

where we have used the fact

$$\|D_x\mathcal{M}_2 f\|_p\leqslant C\|f\|_p\quad\text{for }1<p<\infty.\tag{2.14}$$

In fact,

$$\begin{aligned}D_x\mathcal{M}_2 f&=\mathcal{F}^{-1}|\xi|m_2(\xi)\mathcal{F}f\\&=\mathcal{F}^{-1}\frac{|\xi|^3[1-\varphi(2(|\xi|-a))]}{\big(\frac{2\sqrt{\tau}}{\sqrt{\varepsilon^2+4\tau}-\varepsilon}+|\xi|^2\big)(a^2-|\xi|^2)}\mathcal{F}f.\end{aligned}$$

Denote

$$\rho(\xi)=\frac{|\xi|^3[1-\varphi(2(|\xi|-a))]}{\big(\frac{2\sqrt{\tau}}{\sqrt{\varepsilon^2+4\tau}-\varepsilon}+|\xi|^2\big)(a^2-|\xi|^2)}.$$

In the case $|\xi|\geqslant a+\frac{1}{4}$, we have

$$\big|\rho(\xi)\big|\leqslant\frac{1-\varphi(2(|\xi|-a))}{4}\leqslant\frac{1}{4}.$$

In the case $|\xi|\leqslant a-\frac{1}{4}$ where we assume that $a>\frac{1}{4}$, we have the same estimate. Thus, $|\rho(\xi)|\leqslant C$. By calculating the derivatives of ρ with respect to the variable ξ, we are able to get

$$\left|\frac{\partial^\alpha}{\partial\xi^\alpha}\rho(\xi)\right|\leqslant C|\xi|^{-|\alpha|},\quad|\alpha|\leqslant L,$$

for some integer $L > \frac{n}{2}$. Therefore, we have the desired estimate (2.14) in view of Mihlin multiplier theorem.

To estimate $\mathcal{M}_1$, we split its symbol $m_1(\xi)$ into a finite number of pieces (depending only on the dimension n). Let $\theta \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\theta \subset [-a, a]$ where a is the same as above, define

$$m_{1,1}(\xi) = m_1(\xi)\theta\big(4|\bar{\xi}|\big)$$

and

$$\mathcal{M}_{1,1} f = \mathcal{F}^{-1} m_{1,1}(\xi)\mathcal{F} f,$$

where $\xi = (\bar{\xi}, \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. By a rotation argument, $m_1(\cdot)$ can be expressed as a finite sum of $m_{1,1}$s. Notice that

$$\operatorname{supp} m_{1,1} \subset \left\{\xi = (\bar{\xi}, \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\colon |\bar{\xi}| \leqslant \frac{a}{4} \text{ and } a - \frac{1}{2} \leqslant |\xi| \leqslant a + \frac{1}{2}\right\},$$
$$Q_\alpha \subset \mathbb{R}^{n-1} \times [a_\alpha, a_\alpha + R] \quad \text{and} \quad Q_\beta \subset \mathbb{R}^{n-1} \times [a_\beta, a_\beta + R]$$

for appropriate constants a_α and a_β.

Thus, by Plancherel theorem in the $\bar{x}$ variable, we have

$$\begin{aligned}
&\int_{Q_\alpha} \big|\mathcal{M}_{1,1}(g\chi_{Q_\beta})\big|^2 \,\mathrm{d}x \\
&\quad \leqslant \int_{a_\alpha}^{a_\alpha+R} \int_{\mathbb{R}^{n-1}} \big|\mathcal{M}_{1,1}(g\chi_{Q_\beta})\big|^2 \,\mathrm{d}\bar{x}\,\mathrm{d}x_n \\
&\quad = \int_{a_\alpha}^{a_\alpha+R} \int_{\mathbb{R}^{n-1}} \left| \int_{-\infty}^{\infty} e^{ix_n\xi_n} m_{1,1}(\xi)\widehat{g\chi_{Q_\beta}}(\bar{\xi}, \xi_n)\,\mathrm{d}\xi_n \right|^2 \mathrm{d}\bar{\xi}\,\mathrm{d}x_n \\
&\quad = \int_{a_\alpha}^{a_\alpha+R} \int_{\mathbb{R}^{n-1}} \left| \int_{-\infty}^{\infty} e^{ix_n\xi_n} m_{1,1}(\xi) \int_{a_\beta}^{a_\beta+R} \int_{\mathbb{R}^{n-1}} e^{-iy\xi}(g\chi_{Q_\beta})(\bar{y}, y_n)\,\mathrm{d}\bar{y}\,\mathrm{d}y_n\,\mathrm{d}\xi_n \right|^2 \mathrm{d}\bar{\xi}\,\mathrm{d}x_n \\
&\quad = \int_{a_\alpha}^{a_\alpha+R} \int_{\mathbb{R}^{n-1}} \left| \int_{a_\beta}^{a_\beta+R} \int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}}(g\chi_{Q_\beta})(\bar{y}, y_n) \int_{-\infty}^{\infty} e^{i(x_n-y_n)\xi_n} m_{1,1}(\xi)\,\mathrm{d}\xi_n\,\mathrm{d}\bar{y}\,\mathrm{d}y_n \right|^2 \mathrm{d}\bar{\xi}\,\mathrm{d}x_n \\
&\quad = \int_{a_\alpha}^{a_\alpha+R} \int_{\mathbb{R}^{n-1}} \left| \int_{a_\beta}^{a_\beta+R} \int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}}(g\chi_{Q_\beta})(\bar{y}, y_n) b(x_n, y_n, \bar{\xi})\,\mathrm{d}\bar{y}\,\mathrm{d}y_n \right|^2 \mathrm{d}\bar{\xi}\,\mathrm{d}x_n \\
&\quad =: E,
\end{aligned}$$

where

$$
\begin{aligned}
b(x_n, y_n, \bar{\xi}) &= \int_{-\infty}^{\infty} e^{i(x_n-y_n)\xi_n} m_{1,1}(\xi)\,\mathrm{d}\xi_n \\
&= \int_{-\infty}^{\infty} e^{i(x_n-y_n)\xi_n} \frac{|\xi|^2\varphi(2(|\xi|-a))}{\big(\frac{2\sqrt{\tau}}{\sqrt{\varepsilon^2+4\tau}-\varepsilon}+|\xi|^2\big)(a^2-|\xi|^2)}\theta\big(4|\bar{\xi}|\big)\,\mathrm{d}\xi_n \\
&= \theta\big(4|\bar{\xi}|\big)\int_{-\infty}^{\infty} e^{i\lambda\xi_n} \frac{|\xi|^2}{\big(\frac{2\sqrt{\tau}}{\sqrt{\varepsilon^2+4\tau}-\varepsilon}+|\xi|^2\big)(a^2-|\xi|^2)}\varphi\big(2\big(|\xi|-a\big)\big)\,\mathrm{d}\xi_n \\
&= K(\lambda, \bar{\xi}),
\end{aligned}
$$

here $\lambda = x_n - y_n$ and the support of the integrand is contained in

$$
A = \left\{\xi = (\bar{\xi}, \xi_n) \in \mathbb{R}^{n-1}\times\mathbb{R}\colon\ |\bar{\xi}| \leqslant \frac{a}{4} \text{ and } a - \frac{1}{2} \leqslant |\xi_n| \leqslant a + \frac{1}{2}\right\}.
$$

For $(\bar{\xi}, \xi_n) \in A$, we have $|\xi|^2 - a^2 = \xi_n^2 + |\bar{\xi}|^2 - a^2 = \xi_n^2 - \mu^2$, where $\mu^2 = a^2 - |\bar{\xi}|^2 > \frac{15}{16}a^2$. Then, we separate K into two parts, i.e., $K = K_+ + K_-$ with

$$
\begin{aligned}
K_+(\lambda, \bar{\xi}) &= \theta\big(4|\bar{\xi}|\big)\int_{0}^{\infty} e^{i\lambda\xi_n}\frac{1}{\xi_n-\mu}\cdot\frac{|\xi|^2}{\xi_n+\mu}\cdot\frac{\varphi(2(|\xi|-a))}{\big(\frac{2\sqrt{\tau}}{\sqrt{\varepsilon^2+4\tau}-\varepsilon}+|\xi|^2\big)}\,\mathrm{d}\xi_n \\
&= \theta\big(4|\bar{\xi}|\big)\mathcal{F}_{\xi_n}^{-1}\left(\frac{1}{\xi_n-\mu}\psi(\bar{\xi}, \xi_n)\right)(\lambda) \\
&= \theta\big(4|\bar{\xi}|\big)\mathcal{F}_{\xi_n}^{-1}\left(\frac{1}{\xi_n-\mu}\right)(\lambda) * \mathcal{F}_{\xi_n}^{-1}\psi(\bar{\xi}, \xi_n)(\lambda) \\
&= \theta\big(4|\bar{\xi}|\big)e^{i\lambda\mu}\operatorname{sgn}(\lambda) * \mathcal{F}_{\xi_n}^{-1}\big(\psi(\bar{\xi}, \xi_n)\big)(\lambda) \\
&= \theta\big(4|\bar{\xi}|\big)\int_{-\infty}^{\infty} e^{iy_n\mu}\operatorname{sgn}(y_n)\mathcal{F}_{\xi_n}^{-1}\psi(\bar{\xi}, \xi_n)(\lambda-y_n)\,\mathrm{d}y_n,
\end{aligned}
$$

where $\psi(\bar{\xi}, \xi_n) \in C_0^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^+)$.

Thus, it follows that

$$
\big|K_+(\lambda, \bar{\xi})\big| \leqslant \int_{-\infty}^{\infty}\big|\mathcal{F}_{\xi_n}^{-1}\psi(\bar{\xi}, \xi_n)(y_n)\big|\,\mathrm{d}y_n \leqslant C. \tag{2.15}
$$

In the same way, we have

$$
\big|K_-(\lambda, \bar{\xi})\big| \leqslant C. \tag{2.16}
$$

(2.15) and (2.16) yield that $|b(x_n, y_n, \bar{\xi})| \leqslant C$. Now, we can turn to the estimate of $\int_{Q_\alpha}|\mathcal{M}_{1,1}(g\chi_{Q_\beta})|^2\,\mathrm{d}x$ and get, by Schwartz inequality in y_n variable and Plancherel theorem in $\bar{y}$ variable, that

$$
\begin{aligned}
E &\leqslant R \int_{a_\alpha}^{a_\alpha+R} \int_{\mathbb{R}^{n-1}} \int_{a_\beta}^{a_\beta+R} \left| \int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}} (g\chi_{Q_\beta})(\bar{y}, y_n)\,\mathrm{d}\bar{y} \right|^2 b^2(x_n, y_n, \bar{\xi})\,\mathrm{d}y_n\,\mathrm{d}\bar{\xi}\,\mathrm{d}x_n \\
&\leqslant CR \int_{a_\alpha}^{a_\alpha+R} \int_{\mathbb{R}^{n-1}} \int_{a_\beta}^{a_\beta+R} \left| \int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}} (g\chi_{Q_\beta})(\bar{y}, y_n)\,\mathrm{d}\bar{y} \right|^2 \mathrm{d}y_n\,\mathrm{d}\bar{\xi}\,\mathrm{d}x_n \\
&= CR \int_{a_\alpha}^{a_\alpha+R} \int_{a_\beta}^{a_\beta+R} \int_{\mathbb{R}^{n-1}} \left| \int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}} (g\chi_{Q_\beta})(\bar{y}, y_n)\,\mathrm{d}\bar{y} \right|^2 \mathrm{d}\bar{\xi}\,\mathrm{d}y_n\,\mathrm{d}x_n \\
&= CR \int_{a_\alpha}^{a_\alpha+R} \int_{a_\beta}^{a_\beta+R} \int_{\mathbb{R}^{n-1}} \left| (g\chi_{Q_\beta})(\bar{y}, y_n) \right|^2 \mathrm{d}\bar{y}\,\mathrm{d}y_n\,\mathrm{d}x_n \\
&= CR^2 \int_{\mathbb{R}^n} \left| (g\chi_{Q_\beta})(y) \right|^2 \mathrm{d}y.
\end{aligned}
$$

Therefore, we obtain

$$
\sup_{\alpha\in\mathbb{Z}^n} \left(\int_{Q_\alpha} \left| \mathcal{M}(g\chi_{Q_\beta}) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \leqslant CR \|g\chi_{Q_\beta}\|_{L^2(\mathbb{R}^n)} = CR\|g\|_{L^2(Q_\beta)}
$$

as desired. □

Now, we go back to the proof of Lemma 4. We first consider the part when $\tau > 0$ in (2.12), i.e.,

$$
\begin{aligned}
&\sup_{\alpha\in\mathbb{Z}^n} \left(\int_{Q_\alpha} \int_0^\infty \left| \mathcal{F}_\xi^{-1} \left(\frac{|\xi|^2}{\tau - \varepsilon|\xi|^2 - |\xi|^4} \hat{F}_\beta(\tau, \xi) \right) \right|^2 \mathrm{d}\tau\,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad = \left(\int_0^\infty \tau^{\frac{n}{4}-1} \sup_{\alpha\in\mathbb{Z}^n} \int_{\tau^{\frac{1}{4}} Q_\alpha} \left| \int_{\mathbb{R}^n} e^{iy\eta} \frac{|\eta|^2}{1 - \varepsilon\tau^{-\frac{1}{2}}|\eta|^2 - |\eta|^4} \hat{F}_\beta\big(\tau, \tau^{\frac{1}{4}}\eta\big)\,\mathrm{d}\eta \right|^2 \mathrm{d}y\,\mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\quad = \left(\int_0^\infty \tau^{-\frac{n}{4}-1} \sup_{\alpha\in\mathbb{Z}^n} \int_{\tau^{\frac{1}{4}} Q_\alpha} \left| \mathcal{M}\mathcal{F}_t F_\beta\big(\tau, \tau^{-\frac{1}{4}} y\big) \right|^2 \mathrm{d}y\,\mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\quad \leqslant CR \left(\int_0^\infty \tau^{-\frac{n}{4}-\frac{1}{2}} \int_{\tau^{\frac{1}{4}} Q_\beta} \left| \mathcal{F}_t F_\beta\big(\tau, \tau^{-\frac{1}{4}} y\big) \right|^2 \mathrm{d}y\,\mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\quad \leqslant CR \left(\int_0^\infty \tau^{-\frac{1}{2}} \int_{Q_\beta} \left| \mathcal{F}_t F(\tau, x) \right|^2 \mathrm{d}x\,\mathrm{d}\tau \right)^{\frac{1}{2}},
\end{aligned}
$$

where we have used the changes of variables, Lemma 5 and the identity

$$\left(\mathcal{F}_\eta^{-1}\hat{F}_\beta\left(\tau,\tau^{\frac{1}{4}}\eta\right)\right)(\tau,y)=\tau^{-\frac{n}{4}}\mathcal{F}_t F_\beta\left(\tau,\tau^{-\frac{1}{4}}y\right).$$

For the part when $\tau\in(-\infty,0)$ in (2.12), we are able to split it into two cases. First, it is easier to handle the case $\varepsilon\in\{0,1\}$ since this corresponds to the symbol $\frac{|\eta|^2}{1+\varepsilon|\tau|^{-\frac{1}{2}}|\eta|^2+|\eta|^4}$ which has no singularity. Next, for the case $\varepsilon=-1$, we have

$$\frac{|\eta|^2}{1+\varepsilon|\tau|^{-\frac{1}{2}}|\eta|^2+|\eta|^4}=\frac{|\eta|^2}{\left(1-\frac{1}{2|\tau|^{\frac{1}{2}}}|\eta|^2\right)^2+\left(1-\frac{1}{4|\tau|}\right)|\eta|^4}=:B.$$

It is obvious that B has no singularity when $|\tau|>\frac{1}{4}$. Consequently, we have to deal with the case $|\tau|<\frac{1}{4}$. In fact, we have

$$B=\frac{|\eta|^2}{\left(\frac{2|\tau|^{\frac{1}{2}}}{1-\sqrt{1-4|\tau|}}-|\eta|^2\right)\left(\frac{2|\tau|^{\frac{1}{2}}}{1+\sqrt{1-4|\tau|}}-|\eta|^2\right)}.$$

Thus, we can use the same argument as in the proof of Lemma 5 step by step for the cases $|\eta|^2\leqslant\frac{1}{2\sqrt{|\tau|}}$ and $|\eta|^2>\frac{1}{2\sqrt{|\tau|}}$, respectively, and get that

$$\sup_{\alpha\in\mathbb{Z}^n}\left(\int_{Q_\alpha}\left|\mathcal{F}_\eta^{-1}\frac{|\eta|^2}{1-|\tau|^{-\frac{1}{2}}|\eta|^2+|\eta|^4}\mathcal{F}_x(g\chi_{Q_\beta})\right|^2dx\right)^{\frac{1}{2}}\leqslant CR\left(\int_{Q_\beta}|g|^2\,dx\right)^{\frac{1}{2}}.$$

Therefore, we obtain, by Plancherel theorem, Sobolev embedding theorem and Hölder inequality, that

$$\begin{aligned}
\sup_{\alpha\in\mathbb{Z}^n}\left\|D_x^2u_\beta(t,x)\right\|_{L_x^2(Q_\alpha;L_t^2([0,T]))}&\leqslant CR\left(\int_{-\infty}^{\infty}|\tau|^{-\frac{1}{2}}\int_{Q_\beta}\left|\mathcal{F}_tF(\tau,x)\right|^2dx\,d\tau\right)^{\frac{1}{2}}\\
&\leqslant CR\left(\int_{Q_\beta}\int_{-\infty}^{\infty}\left||\tau|^{-\frac{1}{4}}\mathcal{F}_tF(\tau,x)\right|^2d\tau\,dx\right)^{\frac{1}{2}}\\
&\leqslant CR\left(\int_{Q_\beta}\left\|F(\cdot,x)\right\|^2_{\dot{H}^{-\frac{1}{4}}([0,T])}dx\right)^{\frac{1}{2}}\\
&\leqslant CR\left(\int_{Q_\beta}\left\|F(\cdot,x)\right\|^2_{L^{\frac{4n}{2n+1}}([0,T])}dx\right)^{\frac{1}{2}}\\
&\leqslant CR\left(\int_{Q_\beta}\|1\|^2_{L^{4n}([0,T])}\left\|F(\cdot,x)\right\|^2_{L^2([0,T])}dx\right)^{\frac{1}{2}}\\
&\leqslant CRT^{\frac{1}{4n}}\|F\|_{L_x^2(Q_\beta;L^2_{t\in[0,T]})},
\end{aligned}$$

It should be L^{4/3}.

It should be L^4.

It should be 1/4.

which implies the desired result (2.9). □

3. Estimates for the maximal function

For simplicity, let $\{Q_\alpha\}_{\alpha\in\mathbb{Z}^n}$ denote the mesh of dyadic cubes of unit size. We start with an L^2-continuity result for the maximal function $\sup_{[0,T]}|S(t)\cdot|$.

Lemma 6. *For any $s > n+\frac{1}{2}$ and $T\in(0,1]$,*

$$\left(\sum_{\alpha\in\mathbb{Z}^n}\sup_{t\in[0,T]}\sup_{\alpha\in\mathbb{Z}^n}\left|S(t)u_0(x)\right|^2\right)^{\frac{1}{2}} \leqslant C\|u_0\|_{s,2}. \tag{3.1}$$

In particular,

$$\left(\int_{\mathbb{R}^n}\sup_{t\in[0,T]}\left|S(t)u_0(x)\right|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \leqslant C\|u_0\|_{s,2}. \tag{3.2}$$

Proof. It is clear that it suffices to prove the case $t=1$. Let $\{\psi_k\}_{k=0}^{\infty}$ be a smooth partition of unity in $\mathbb{R}^n$ such that the ψ_k's are radial with $\operatorname{supp}\psi_0\subset\{\xi\colon |\xi|\leqslant 1\}$, $\operatorname{supp}\psi_k\subset\{\xi\colon 2^{k-1}\leqslant|\xi|\leqslant 2^{k+1}\}$, $\psi_k(x)\in[0,1]$ and $|\psi_k'(x)|\leqslant C2^{-k}$ for $k\in\mathbb{Z}^+$. For $k\geqslant 1$ and $t\in[0,2]$, define

$$I(t,r)=\int_{-\infty}^{\infty} e^{i\phi_r(s)}\psi_k(s)\,\mathrm{d}s,$$

where the phase function $\phi_r(s)=-\varepsilon ts^2-ts^4+rs$. Consequently, we have $\phi_r'(s)=-2\varepsilon ts-4ts^3+r$, $\phi_r''(s)=-2\varepsilon t-12ts^2$ and $\phi_r''''(s)=-24t$.

Denote

$$\Omega=\left\{s\in\mathbb{R}^+\colon\ \left|\phi_r'(s)\right|<\frac{r}{2}\right\}$$

and

$$I_k=\left[2^{k-1},2^{k+1}\right].$$

For $r\in(0,1)$, it is obvious that $|I(t,r)|\leqslant\frac{3}{2}\cdot 2^k$.

For $r>1$, we divide it into three cases:

(i) Ω is located to the left of I_k;
(ii) $\Omega\cap I_k\neq\emptyset$;
(iii) Ω is located to the right of I_k.

In the cases (i) and (ii), we have $t\geqslant C\frac{r}{2^{3k}}$ and $|\phi_r''''(s)|\geqslant C\frac{r}{2^{3k}}$. Thus, with the help of Van der Corput lemma, we have $|I(t,r)|\leqslant C(\frac{2^{3k}}{r})^{\frac{1}{4}}$, where we have used $|\psi_k'(x)|\leqslant C2^{-k}$.

In the case (iii), we have $|\phi_r'(s)|\leqslant\frac{r}{2}$ and $\frac{r}{t}\leqslant C2^{3k}$. Integration by parts gives that

$$\left|I(t,r)\right|\leqslant\int_0^{\infty}\left[\left|\frac{\phi_r''(s)}{(\phi_r'(s))^2}\right|\psi_k(s)+\frac{1}{|\phi_r'(s)|}\left|\psi_k'(s)\right|\right]\mathrm{d}s$$

$$
\begin{aligned}
&\leqslant \int_{I_k} \left[\frac{4(2t+12ts^2)}{r^2}\psi_k(s)+\frac{2}{r}\big|\psi_k'(s)\big|\right]\mathrm{d}s\\
&\leqslant C\int_{I_k} \left[\frac{4(2+12\cdot 2^{2(k+1)})}{r\cdot 2^{3k}}\psi_k(s)+\frac{2}{r}2^{-k}\right]\mathrm{d}s\\
&\leqslant \frac{C}{r}\leqslant C\left(\frac{2^{3k}}{r}\right)^{\frac14},
\end{aligned}
$$

where we have used the condition $r>1$ in the last step.

Finally, we always have the case in which Ω is located to the right of I_k when $r \geqslant C2^{3k}$. Hence, we can integrate by parts as in the case (iii) N-times and get that $|I(t,r)| \leqslant \frac{C_N}{r^N}$ for any $N \in \mathbb{Z}^+$.

Therefore, we have obtained

$$
\big|I(t,r)\big| \leqslant \begin{cases} C2^k & \text{for } r\in(0,1),\\ C\big(\frac{2^{3k}}{r}\big)^{\frac14} & \text{for } r\in(1,C2^{3k}),\\ \frac{C_N}{r^N} & \text{for } r\geqslant C2^{3k}. \end{cases}
$$

In order to continue the proof, we have to introduce some estimates. The Fourier transform of a radial function $f(|x|)=f(s)$ is given by the formula (cf. [24, p. 154, Theorem 3.3])

$$
\hat f(r)=\hat f\big(|\xi|\big)=r^{-\frac{n-2}{2}}\int_0^\infty f(s)J_{\frac{n-2}{2}}(rs)s^{\frac n2}\,\mathrm{d}s,
$$

and the Bessel function is defined as

$$
J_m(r)=\frac{(\frac r2)^m}{\Gamma(m+\frac12)\Gamma(\frac12)}\int_{-1}^{1} e^{irs}\big(1-s^2\big)^{\frac{2m-1}{2}}\,\mathrm{d}s,
$$

where $\Gamma(\cdot)$ is Gamma functions.

Lemma 7. [16, Lemma 3.6]

$$
\begin{aligned}
&J_m(r)=O\big(r^m\big)\quad \textit{as } r\to 0,\\
&J_m(r)=e^{-ir}\sum_{j=0}^{N}\alpha_{m,j}r^{-(j+\frac12)}+e^{ir}\sum_{j=0}^{N}\beta_{m,j}r^{-(j+\frac12)}\\
&\qquad\quad +O\big(r^{-(N+\frac32)}\big)\quad \textit{as } r\to\infty,
\end{aligned}
\tag{3.3}
$$

for each $N\in\mathbb{Z}^+$. □

We continue the proof of Lemma 6 now. Denote

$$
\begin{aligned}
\tilde I(t,x) &\equiv \left|\int_{\mathbb{R}^n} e^{-it(\varepsilon|\xi|^2+|\xi|^4)}e^{ix\xi}\psi_k\big(|\xi|\big)\,\mathrm{d}\xi\right|\\
&=\frac{1}{r^{\frac{n-2}{2}}}\left|\int_0^\infty e^{-it(\varepsilon s^2+s^4)}J_{\frac{n-2}{2}}(rs)\psi_k(s)s^{\frac n2}\,\mathrm{d}s\right| \equiv \tilde I(t,r),
\end{aligned}
$$

where $r = |x|$.

For $r \in (0, 1)$, we have

$$\tilde{I}(t,r) \leqslant \frac{1}{r^{\frac{n-2}{2}}} \int_0^\infty (rs)^{\frac{n-2}{2}} \psi_k(s) s^{\frac{n}{2}}\, \mathrm{d}s \leqslant C \int_{I_k} s^{n-1}\, \mathrm{d}s \leqslant C2^{nk}.$$

For $r > 1$, we first consider the remainder term in (3.3) to obtain the bound

$$C\frac{1}{r^{\frac{n-2}{2}}} \int_{I_k} \frac{1}{(rs)^{N+\frac{3}{2}}} s^{\frac{n}{2}}\, \mathrm{d}s \leqslant C\frac{2^{\frac{kn}{2}}}{r^n \cdot 2^{\frac{kn}{2}}} \leqslant \frac{C}{r^n},$$

where we have taken N such that $N > \frac{n-1}{2}$.

Next, we deal with the j-term in (3.3) for $0 \leqslant j \leqslant N$,

$$\begin{aligned}
&\frac{1}{r^{\frac{n-2}{2}}} \left| \int_0^\infty e^{-it(\varepsilon s^2+s^4)} e^{isr} \frac{1}{(sr)^{j+\frac{1}{2}}} s^{\frac{n}{2}} \psi_k(s)\, \mathrm{d}s \right| \\
&\quad \leqslant \frac{1}{r^{\frac{n-2}{2}}} \frac{1}{r^{j+\frac{1}{2}}} 2^{-(k+1)(j+\frac{1}{2})} 2^{\frac{nk}{2}} \left| \int_0^\infty e^{-it(\varepsilon s^2+s^4)} e^{isr} \psi_k(s)\, \mathrm{d}s \right| \\
&\quad \leqslant C\frac{2^{\frac{kn}{2}-(j+\frac{1}{2})k}}{r^{\frac{n}{2}+j-\frac{1}{2}}} \cdot \begin{cases} \left(\frac{2^{3k}}{r}\right)^{\frac{1}{4}}, & r \in [1, C2^{3k}], \\ C_M r^{-M}, & r > C2^{3k}. \end{cases}
\end{aligned}$$

Now, we consider the L^1-norm of $\tilde{I}(t,x)$ with respect to x variable. For $r \in [0, 1]$, it is clear that

$$\int_{|x|\leqslant 1} \left|\tilde{I}(t,x)\right| \mathrm{d}x \leqslant C \int_0^1 2^{nk} r^{n-1}\, \mathrm{d}r \leqslant C2^{nk}.$$

In the case $r \in [1, C2^{3k}]$, we have

$$\begin{aligned}
\int_{1\leqslant|x|\leqslant C2^{3k}} \left|\tilde{I}(t,x)\right| \mathrm{d}x &\leqslant C \int_1^{C2^{3k}} \frac{2^{\frac{nk}{2}-(j+\frac{1}{2})k}}{r^{\frac{n}{2}+j-\frac{1}{2}}} \left(\frac{2^{3k}}{r}\right)^{\frac{1}{4}} r^{n-1}\, \mathrm{d}r \\
&\leqslant C2^{\frac{nk}{2}-(j-\frac{1}{4})k} \int_1^{C2^{3k}} r^{\frac{n}{2}-j-\frac{3}{4}}\, \mathrm{d}r \leqslant C2^{k(2n+1)}.
\end{aligned}$$

For $r > C2^{3k}$, it follows that $\int_{|x|\geqslant C2^{3k}} |\tilde{I}(t,x)|\, \mathrm{d}x \leqslant C2^{kn}$. Thus, we obtain

$$\int_{\mathbb{R}^n} \left|\tilde{I}(t,x)\right| \mathrm{d}x \leqslant C2^{k(2n+1)}. \tag{3.4}$$

We decompose the operator $S(t)$ as follows according to the partition of unity $\{\psi_k\}$,

$$W_k(t)u_0 = \mathscr{F}^{-1} e^{-it(\varepsilon|\xi|^2+|\xi|^4)} \psi_k(\xi) \mathscr{F} u_0.$$

To prove (3.1), it suffices to show that

$$\left(\sum_{\gamma\in\mathbb{Z}^n}\sup_{|t|\leqslant 1}\sup_{x\in Q_\gamma}\left|\int_{-1}^{1} W_k(t-\tau)g(\tau,\cdot)\,\mathrm{d}\tau\right|^2\right)^{\frac{1}{2}}$$
$$\leqslant C2^{k(2n+1)}\left(\sum_{\gamma\in\mathbb{Z}^n}\left(\int_{-1}^{1}\int_{Q_\gamma}\left|g(t,x)\right|\mathrm{d}x\,\mathrm{d}t\right)^2\right)^{\frac{1}{2}}. \tag{3.5}$$

In fact, we have

$$\begin{aligned}
&\left|\int_{-1}^{1} W_k(t-\tau)g(\cdot,\tau)\,\mathrm{d}\tau\right|\\
&\quad=\left|\int_{-1}^{1}\mathcal{F}^{-1}e^{-i(t-\tau)(\varepsilon|\xi|^2+|\xi|^4)}\psi_k(\xi)\mathcal{F}g(\tau,\xi)\,\mathrm{d}\tau\right|\\
&\quad=\left|\int_{-1}^{1}\int_{\mathbb{R}^n}\left[\mathcal{F}^{-1}e^{-i(t-\tau)(\varepsilon|\xi|^2+|\xi|^4)}\psi_k(\xi)\right](y)\cdot g(\tau,x-y)\,\mathrm{d}y\,\mathrm{d}\tau\right|\\
&\quad=\left|\int_{\mathbb{R}^n}\int_{-1}^{1}\left[\mathcal{F}^{-1}e^{-i(t-\tau)(\varepsilon|\xi|^2+|\xi|^4)}\psi_k(\xi)\right](y)\cdot g(\tau,x-y)\,\mathrm{d}\tau\,\mathrm{d}y\right|\\
&\quad\leqslant\int_{\mathbb{R}^n}\int_{-1}^{1}\left|\left[\mathcal{F}^{-1}e^{-i(t-\tau)(\varepsilon|\xi|^2+|\xi|^4)}\psi_k(\xi)\right](y)\right|\left|g(\tau,x-y)\right|\mathrm{d}\tau\,\mathrm{d}y\\
&\quad\leqslant\int_{\mathbb{R}^n}\sup_{t\in[0,2]}\tilde{I}\big(t,|y|\big)\int_{-1}^{1}\left|g(\tau,x-y)\right|\mathrm{d}\tau\,\mathrm{d}y\\
&\quad\leqslant\sum_{\alpha\in\mathbb{Z}^n}\Big(\sup_{t\in[0,2]}\sup_{y\in Q_\alpha}\tilde{I}\big(t,|y|\big)\Big)\int_{-1}^{1}\int_{Q_\alpha}\left|g(\tau,x-y)\right|\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}$$

Thus, the left-hand side of (3.5) is bounded by

$$\left(\sum_{\gamma\in\mathbb{Z}^n}\left[\sum_{\alpha\in\mathbb{Z}^n}\Big(\sup_{t\in[0,2]}\sup_{y\in Q_\alpha}\tilde{I}\big(t,|y|\big)\Big)\sup_{x\in Q_\gamma}\int_{-1}^{1}\int_{Q_\alpha}\left|g(\tau,x-y)\right|\mathrm{d}y\,\mathrm{d}\tau\right]^2\right)^{\frac{1}{2}}. \tag{3.6}$$

Let $E_{\alpha,\gamma} := Q_\alpha - Q_\gamma = 2^n Q_\alpha - x_\gamma$, where x_γ denotes the center of Q_γ. Then, we can get

$$\sup_{x\in Q_\gamma}\int_{-1}^{1}\int_{Q_\alpha}\left|g(\tau,x-y)\right|\mathrm{d}y\,\mathrm{d}\tau\leqslant\int_{-1}^{1}\int_{E_{\alpha,\gamma}}\left|g(\tau,z)\right|\mathrm{d}z\,\mathrm{d}\tau,$$

and consequently (3.6) is bounded, with the help of Minkowski inequality, by

$$\begin{aligned}
&\left(\sum_{\gamma\in\mathbb{Z}^n}\left[\sum_{\alpha\in\mathbb{Z}^n}\Bigl(\sup_{t\in[0,2]}\sup_{y\in Q_\alpha}\tilde I\bigl(t,|y|\bigr)\Bigr)\int_{-1}^{1}\int_{E_{\alpha,\gamma}}\bigl|g(\tau,z)\bigr|\,\mathrm{d}z\,\mathrm{d}\tau\right]^2\right)^{\frac12}\\
&\quad\leqslant\sum_{\alpha\in\mathbb{Z}^n}\left[\Bigl(\sup_{t\in[0,2]}\sup_{y\in Q_\alpha}\tilde I\bigl(t,|y|\bigr)\Bigr)\left(\sum_{\gamma\in\mathbb{Z}^n}\left(\int_{-1}^{1}\int_{E_{\alpha,\gamma}}\bigl|g(\tau,z)\bigr|\,\mathrm{d}z\,\mathrm{d}\tau\right)^2\right)^{\frac12}\right]\\
&\quad\leqslant C\sum_{\alpha\in\mathbb{Z}^n}\left[\Bigl(\sup_{t\in[0,2]}\sup_{y\in Q_\alpha}\tilde I\bigl(t,|y|\bigr)\Bigr)\left(\sum_{\gamma\in\mathbb{Z}^n}\left(\int_{-1}^{1}\int_{Q_\gamma}\bigl|g(\tau,z)\bigr|\,\mathrm{d}z\,\mathrm{d}\tau\right)^2\right)^{\frac12}\right],
\end{aligned}$$

where we have used the fact that $E_{\alpha,\gamma}$ can be covered by a finite number (independent of α) of Q_γ in the last step.

From (3.4), we have the estimate

$$\sum_{\alpha\in\mathbb{Z}^n}\sup_{t\in[0,2]}\sup_{y\in Q_\alpha}\tilde I\bigl(t,|y|\bigr)=\sup_{t\in[0,2]}\int_{\mathbb{R}^n}\tilde I\bigl(t,|y|\bigr)\,\mathrm{d}y\leqslant C2^{(2n+1)k}.$$

Therefore, the proof is completed. □

When we deal with the Cauchy problem (1.1)–(1.2) in the case $l=2$, we shall use the l^1-estimate of the maximal function $\sup_{[0,T]}|S(t)u_0(x)|$ as the following inequality.

Lemma 8. *We have the estimate*

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}^n}\sup_{[0,T]}\sup_{x\in Q_\alpha}\bigl|S(t)u_0(x)\bigr|\\
&\quad\leqslant C(1+T)^{[\frac n2]+2}\bigl(\|u_0\|_{n+3[\frac n2]+7,2}+\|u_0\|_{n+3[\frac n2]+7,2,2[\frac n2]+2}\bigr),
\end{aligned}\tag{3.7}$$

where $\|\cdot\|_{l,2,j}$ *is defined as in* (1.6).

Proof. Taking $t_0\in[0,T]$ such that

$$\bigl|f(t_0)\bigr|\leqslant\frac{\int_0^T|f(t)|\,\mathrm{d}t}{T},$$

we have for any t

$$f(t)=f(t_0)+f(t)-f(t_0)=f(t_0)+\int_{t_0}^{t}f'(s)\,\mathrm{d}s.$$

Thus, we can get

$$\bigl|f(t)\bigr|\leqslant\frac1T\int_0^T\bigl|f(t)\bigr|\,\mathrm{d}t+\int_{t_0}^{t}\bigl|f'(s)\bigr|\,\mathrm{d}s\leqslant\frac1T\int_0^T\bigl|f(t)\bigr|\,\mathrm{d}t+\int_0^T\bigl|\partial_t f(t)\bigr|\,\mathrm{d}t,$$

namely,

$$\sup_{t\in[0,T]}\left|f(t)\right| \leqslant \frac{1}{T}\int_0^T \left|f(t)\right| \mathrm{d}t + \int_0^T \left|\partial_t f(t)\right| \mathrm{d}t. \tag{3.8}$$

Notice that

$$\|f\|_{L^1(\mathbb{R}^n)} \leqslant C\|f\|_{L^2(\mathbb{R}^n)} + C\left\||x|^{\bar{n}} f\right\|_{L^2(\mathbb{R}^n)}, \tag{3.9}$$

where $\bar{n} > \frac{n}{2}$, and

$$x_j S(t) f = S(t)(x_j f) + 4itS(t)(\partial_{x_j}\Delta f) - 2\varepsilon it S(t)(\partial_{x_j} f), \tag{3.10}$$

$$\begin{aligned} x_k x_j S(t) f = {} & \big((-2\varepsilon it)^2 + 8it\big) S(t)(\partial_{x_k}\partial_{x_j} f) + 4itS(t)(\delta_{kj}\Delta f) - 2\varepsilon it S(t)(\delta_{kj} f) \\ & + 16\varepsilon t^2 S(t)(\partial_{x_k}\partial_{x_j}\Delta f) + (4it)^2 S(t)\big(\partial_{x_k}\partial_{x_j}\Delta^2 f\big) \\ & - 2\varepsilon it S(t)(x_k\partial_{x_j} f + x_j\partial_{x_k} f) + 4itS(t)(x_k\partial_{x_j}\Delta f + x_j\partial_{x_k}\Delta f) \\ & + S(t)(x_k x_j f), \end{aligned} \tag{3.11}$$

where $\delta_{kj} = 1$ if $k = j$, $\delta_{kj} = 0$ if $k \neq j$.

We have, from (3.8)–(3.11), Sobolev embedding theorem and Fubini theorem, that

$$\begin{aligned} & \sum_{\alpha\in\mathbb{Z}^n} \sup_{[0,T]} \sup_{x\in Q_\alpha} \left|S(t)u_0(x)\right| \\ & \quad \leqslant \sum_{\alpha\in\mathbb{Z}^n} \sup_{[0,T]} \left(\int_{Q_\alpha} \left|S(t)u_0(x)\right| \mathrm{d}x + \sum_{|\beta|\leqslant n} \int_{Q_\alpha} \left|\partial_x^\beta S(t)u_0(x)\right| \mathrm{d}x \right) \\ & \quad \leqslant \sum_{\alpha\in\mathbb{Z}^n} \left(\int_{Q_\alpha} \sup_{[0,T]} \left|S(t)u_0(x)\right| \mathrm{d}x + \sum_{|\beta|\leqslant n} \int_{Q_\alpha} \sup_{[0,T]} \left|\partial_x^\beta S(t)u_0(x)\right| \mathrm{d}x \right) \\ & \quad \leqslant \sum_{\alpha\in\mathbb{Z}^n} \Bigg\{ \frac{C}{T} \int_{Q_\alpha} \int_0^T \left|S(t)u_0(x)\right| \mathrm{d}t\,\mathrm{d}x + \int_{Q_\alpha} \int_0^T \left|\partial_t S(t)u_0(x)\right| \mathrm{d}t\,\mathrm{d}x \\ & \qquad + \sum_{|\beta|\leqslant n} \int_{Q_\alpha} \left(\frac{C}{T} \int_0^T \left|S(t)\partial_x^\beta u_0(x)\right| \mathrm{d}t + \int_0^T \left|\partial_t S(t)\partial_x^\beta u_0(x)\right| \mathrm{d}t \right) \mathrm{d}x \Bigg\} \\ & \quad \leqslant \frac{C}{T} \int_{\mathbb{R}^n} \int_0^T \left|S(t)u_0(x)\right| \mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}^n} \int_0^T \left|S(t)\Delta u_0(x)\right| \mathrm{d}t\,\mathrm{d}x \\ & \qquad + \int_{\mathbb{R}^n} \int_0^T \left|S(t)\Delta^2 u_0(x)\right| \mathrm{d}t\,\mathrm{d}x + \sum_{|\beta|\leqslant n} \frac{C}{T} \int_{\mathbb{R}^n} \int_0^T \left|S(t)\partial_x^\beta u_0(x)\right| \mathrm{d}t\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\sum_{|\beta|\leqslant n}\int_{\mathbb{R}^n}\int_0^T\left|S(t)\partial_x^\beta \Delta u_0(x)\right|\mathrm{d}t\,\mathrm{d}x+\sum_{|\beta|\leqslant n}\int_{\mathbb{R}^n}\int_0^T\left|S(t)\partial_x^\beta \Delta^2 u_0(x)\right|\mathrm{d}t\,\mathrm{d}x\\
&\leqslant \frac{C}{T}\int_0^T\int_{\mathbb{R}^n}\left|S(t)u_0(x)\right|\mathrm{d}x\,\mathrm{d}t+\frac{C}{T}\sum_{|\beta|\leqslant n}\int_0^T\int_{\mathbb{R}^n}\left|S(t)\partial_x^\beta u_0(x)\right|\mathrm{d}x\,\mathrm{d}t\\
&\quad+C\sum_{|\beta|\leqslant n+4}\int_0^T\int_{\mathbb{R}^n}\left|S(t)\partial_x^\beta u_0(x)\right|\mathrm{d}x\,\mathrm{d}t\\
&\leqslant \frac{C}{T}\int_0^T\left[\left\|S(t)u_0\right\|_2+\left\||x|^{\bar n}S(t)u_0\right\|_2\right]\mathrm{d}t\\
&\quad+\frac{C}{T}\sum_{|\beta|\leqslant n}\int_0^T\left[\left\|S(t)\partial_x^\beta u_0\right\|_2+\left\||x|^{\bar n}S(t)\partial_x^\beta u_0\right\|_2\right]\mathrm{d}t\\
&\quad+C\sum_{|\beta|\leqslant n+4}\int_0^T\left[\left\|S(t)\partial_x^\beta u_0\right\|_2+\left\||x|^{\bar n}S(t)\partial_x^\beta u_0\right\|_2\right]\mathrm{d}t\\
&\leqslant C(1+T)^{\bar n+1}\left(\|u_0\|_{n+3\bar n+4,2}+\|u_0\|_{n+3\bar n+4,2,2\bar n}\right),
\end{aligned}
$$

which implies the desired result if we choose $\bar n=[\frac{n}{2}]+1$. $\square$

4. The well-posedness

We will give the proofs of the main theorems in this section.

Proof of Theorem 1. Similar to the proof in [16, Theorem 4.1], we shall only consider the most interesting case $s=s_0$. The general case follows by combining this result with the fact that the highest derivatives involved in that proof always appear linearly and with some commutator estimates (see [13]) for the cases where $s\neq k+\frac{1}{2}$, $k\in\mathbb{Z}^+$. For simplicity of the exposition, we shall assume that

$$P\left(\left(\partial_x^\alpha u\right)_{|\alpha|\leqslant 2},\left(\partial_x^\alpha \bar u\right)_{|\alpha|\leqslant 2}\right)=\partial_{x_j}^2 u\,\partial_{x_k}^2 u\,\partial_{x_m}^2 u.$$

For $u_0\in H^{s_0}(\mathbb{R}^n)$, we denote by $u=\mathcal{T}(v)=\mathcal{T}_{u_0}(v)$ the solution of the linear inhomogeneous Cauchy problem

$$i\partial_t u=-\varepsilon\Delta u+\Delta^2 u+\partial_{x_j}^2 v\,\partial_{x_k}^2 v\,\partial_{x_m}^2 v,\quad t\in\mathbb{R},\ x\in\mathbb{R}^n, \tag{4.1}$$

$$u(0,x)=u_0(x). \tag{4.2}$$

In order to construct $\mathcal{T}$ being a contraction mapping in some space, we use the integral equation

$$u(t)=\mathcal{T}(v)(t)=S(t)u_0-i\int_0^t S(t-\tau)\partial_{x_j}^2 v(\tau)\partial_{x_k}^2 v(\tau)\partial_{x_m}^2 v(\tau)\,\mathrm{d}\tau. \tag{4.3}$$

We introduce the following work space

$$Z_T^E = \Bigg\{ w : [0,T] \times \mathbb{R}^n \to \mathbb{C}\colon \sup_{t\in[0,T]} \big\| w(t) \big\|_{s_0,2} \leqslant E;$$

$$\sum_{|\beta|=s_0+\frac{1}{2}} \sup_{\alpha\in\mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} \big|\partial_x^\beta w(t,x)\big|^2 \, dx\, dt \right)^{\frac{1}{2}} \leqslant T^\delta;$$

$$\left(\sum_{\alpha\in\mathbb{Z}^n} \sup_{t\in[0,T]} \sup_{x\in Q_\alpha} \big| D_x^2 w(t,x) \big|^2 \right)^{\frac{1}{2}} \leqslant E \Bigg\},$$

where $\delta < \frac{1}{3}$ is a constant.

We notice, for any $\beta \in \mathbb{Z}^n$ with $|\beta| = s_0 - \frac{3}{2}$, that

$$\partial_x^\beta \big(\partial_{x_j}^2 v \partial_{x_k}^2 v \partial_{x_m}^2 v\big) = \partial_x^\beta \partial_{x_j}^2 v \partial_{x_k}^2 v \partial_{x_m}^2 v + \partial_{x_j}^2 v \partial_x^\beta \partial_{x_k}^2 v \partial_{x_m}^2 v + \partial_{x_j}^2 v \partial_{x_k}^2 v \partial_x^\beta \partial_{x_m}^2 v$$
$$+ R\big(\big(\partial_x^\gamma v\big)_{2\leqslant|\gamma|\leqslant s_0-\frac{1}{2}}\big),$$

where $|\beta'| = |\beta| - 1$ and $|\beta''| = 1$ with $\beta', \beta'' \in \mathbb{Z}^n$.

Thus, from the integral equation (4.3), (2.6), (2.1), Sobolev embedding theorem and the commutator estimates [13], we can get, for $v \in Z_T^D$, that

$$\sum_{|\beta|=s_0+\frac{1}{2}} \sup_{\alpha\in\mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} \big|\partial_x^\beta u(t,x)\big|^2 \, dx\, dt \right)^{\frac{1}{2}}$$
$$\leqslant \sum_{|\beta|=s_0+\frac{1}{2}} \sup_{\alpha\in\mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} \big|S(t)\partial_x^\beta u_0(x)\big|^2 \, dx\, dt \right)^{\frac{1}{2}}$$
$$+ \sum_{|\beta|=s_0+\frac{1}{2}} \sup_{\alpha\in\mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} \left| \int_0^t S(t-\tau)\partial_x^\beta \big(\partial_{x_j}^2 v \partial_{x_k}^2 v \partial_{x_m}^2 v\big) \right|^2 dx\, dt \right)^{\frac{1}{2}}$$
$$\leqslant CT^{\frac{1}{3}} \|u_0\|_{s_0,2} + CT^{\frac{1}{4n}} \sum_{|\beta|=s_0-\frac{3}{2}} \sum_{j,k,m=1}^n \sum_{\alpha\in\mathbb{Z}^n} \big\| \partial_x^\beta \partial_{x_j}^2 v \partial_{x_k}^2 v \partial_{x_m}^2 v \big\|_{L_x^2(Q_\alpha; L_t^2([0,T]))}$$

It should be 1/4.

$$+ C \int_0^T \big\| D_x^{\frac{1}{2}} R\big(\big(\partial_x^\gamma v\big)_{2\leqslant|\gamma|\leqslant s_0-\frac{1}{2}}\big) \big\|_2 \, dt$$
$$\leqslant CT^{\frac{1}{3}} \|u_0\|_{s_0,2} + CT^{\frac{1}{4n}} \sum_{|\beta|=s_0+\frac{1}{2}} \sup_{\alpha\in\mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} \big|\partial_x^\beta v(t,x)\big|^2 \, dx\, dt \right)^{\frac{1}{2}}$$

It should be 1/4.

$$\cdot \left(\sum_{\alpha\in\mathbb{Z}^n} \sup_{t\in[0,T]} \sup_{x\in Q_\alpha} \big| D_x^2 v(t,x) \big|^2 \right) + CT \sup_{t\in[0,T]} \|v\|_{s_0,2}^3$$
$$\leqslant CT^{\frac{1}{3}} \|u_0\|_{s_0,2} + CT^{\frac{1}{4n}} T^\delta E^2 + CTE^3$$
$$\leqslant T^\delta, \tag{4.4}$$

It should be 1/4.

where we have taken T so small that

$$CT^{\frac{1}{3}-\delta}\|u_0\|_{s_0,2} + CT^{\frac{1}{4n}}E^2 + CT^{1-\delta}E^3 \leqslant 1, \tag{4.5}$$

It should be 1/4.

in the last step.

By Sobolev embedding theorem, (2.2) and Hölder inequality, we have

$$\begin{aligned}
\sup_{t\in[0,T]}\big\|u(t)\big\|_{s_0,2} &\leqslant \|u_0\|_{s_0,2} + \sup_{t\in[0,T]}\int_0^t \big\|S(t-\tau)\partial_{x_j}^2 v(\tau)\partial_{x_k}^2 v(\tau)\partial_{x_m}^2 v(\tau)\big\|_2\,\mathrm{d}\tau \\
&\quad + \sup_{t\in[0,T]}\left\|D_x^{\frac{3}{2}}\int_0^t S(t-\tau)D_x^{s_0-\frac{3}{2}}\partial_{x_j}^2 v(\tau)\partial_{x_k}^2 v(\tau)\partial_{x_m}^2 v(\tau)\,\mathrm{d}\tau\right\|_2 \\
&\leqslant \|u_0\|_{s_0,2} + T\sup_{t\in[0,T]}\big\|\partial_{x_j}^2 v(t)\partial_{x_k}^2 v(t)\partial_{x_m}^2 v(t)\big\|_2 \\
&\quad + \sum_{\alpha\in\mathbb{Z}^n}\left(\int_{Q_\alpha}\int_0^T\Big|D_x^{s_0-\frac{3}{2}}\big(\partial_{x_j}^2 v(t)\partial_{x_k}^2 v(t)\partial_{x_m}^2 v(t)\big)\Big|^2\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leqslant \|u_0\|_{s_0,2} + T\sup_{t\in[0,T]}\|v\|_{\frac{n}{3}+2,2}^3 \\
&\quad + \sum_{j,k,m=1}^{n}\sum_{\alpha\in\mathbb{Z}^n}\left(\int_{Q_\alpha}\int_0^T\Big|D_x^{s_0-\frac{3}{2}}\partial_{x_j}^2 v(t)\partial_{x_k}^2 v(t)\partial_{x_m}^2 v(t)\Big|^2\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad + \sum_{\alpha\in\mathbb{Z}^n}\left(\int_{Q_\alpha}\int_0^T\Big|\big(D_x^\gamma v\big)_{2\leqslant|\gamma|\leqslant s_0-\frac{1}{2}}\Big|^2\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leqslant \|u_0\|_{s_0,2} + T\sup_{t\in[0,T]}\|v\|_{\frac{n}{3}+2,2}^3 \\
&\quad + \sum_{|\beta|=s_0+\frac{1}{2}}\sup_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\big|\partial_x^\beta v\big|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\cdot\sum_{\alpha\in\mathbb{Z}^n}\sup_{t\in[0,T]}\sup_{x\in Q_\alpha}\big|D_x^2 v\big|^2 \\
&\quad + T^{\frac{1}{2}}\sup_{t\in[0,T]}\|v\|_{s_0,2}^3 \\
&\leqslant \|u_0\|_{s_0,2} + \big(T+T^{\frac{1}{2}}\big)E^3 + T^\delta E^2 \\
&\leqslant E,
\end{aligned} \tag{4.6}$$

where in the last step, we have chosen T small enough such that

$$\big(T+T^{\frac{1}{2}}\big)E^2 + T^\delta E \leqslant \frac{1}{2}. \tag{4.7}$$

Similar to the derivation of (4.4) and (4.6), we obtain, by inserting (3.2) in (4.3), that

$$\begin{aligned}
&\Big(\sum_{\alpha\in\mathbb{Z}^n}\sup_{t\in[0,T]}\sup_{x\in Q_\alpha}\big|D_x^2u(t,x)\big|^2\Big)^{\frac{1}{2}}\\
&\quad\leqslant C\|u_0\|_{s_0,2}+C\Bigg(\sum_{\alpha\in\mathbb{Z}^n}\sup_{t\in[0,T]}\sup_{x\in Q_\alpha}\bigg|\int_0^t S(t-\tau)D_x^2\partial_{x_j}^2v(\tau)\partial_{x_k}^2v(\tau)\partial_{x_m}^2v(\tau)\bigg|^2\Bigg)^{\frac{1}{2}}\\
&\quad\leqslant C\|u_0\|_{s_0,2}+CT\sup_{t\in[0,T]}\|v\|_{s_0,2}^3\\
&\quad\leqslant \|u_0\|_{s_0,2}+CTE^3\\
&\quad\leqslant E,
\end{aligned}\tag{4.8}$$

where we have taken

$$E=2C\|u_0\|_{s_0,2}\tag{4.9}$$

and T sufficiently small such that

$$CTE^2\leqslant 1.\tag{4.10}$$

Therefore, choosing an E as in (4.9) and then taking T sufficiently small such that (4.5), (4.7) and (4.10) hold, we obtain that the mapping

$$\mathcal{T}=\mathcal{T}_{u_0}:Z_T^E\to Z_T^E$$

is well defined.

For convenience, we denote

$$\begin{aligned}
\Lambda_T(w)=\max\Bigg\{&ET^{-\delta}\sum_{|\beta|=s_0+1/2}\sup_{\alpha\in\mathbb{Z}^n}\bigg(\int_0^T\int_{Q_\alpha}\big|\partial_x^\beta w(t,x)\big|^2\,\mathrm{d}x\,\mathrm{d}t\bigg)^{\frac{1}{2}};\\
&\sup_{t\in[0,T]}\big\|w(t)\big\|_{s_0,2};\Big(\sum_{\alpha\in\mathbb{Z}^n}\sup_{t\in[0,T]}\sup_{x\in Q_\alpha}\big|D_x^2w(t,x)\big|^2\Big)^{\frac{1}{2}}\Bigg\}.
\end{aligned}$$

To show that $\mathcal{T}$ is a contraction mapping, we apply the estimates obtained in (4.4), (4.6) and (4.8) to the following integral equation

$$\begin{aligned}
\mathcal{T}v(t)-\mathcal{T}w(t)=\int_0^t S(t-\tau)\big[&\partial_{x_j}^2v(\tau)\partial_{x_k}^2v(\tau)\partial_{x_m}^2v(\tau)\\
&-\partial_{x_j}^2w(\tau)\partial_{x_k}^2w(\tau)\partial_{x_m}^2w(\tau)\big]\,\mathrm{d}\tau,
\end{aligned}$$

and obtain, for $v,w\in Z_T^E$, that

$$\begin{aligned}
\Lambda_T(\mathcal{T}v-\mathcal{T}w)&\leqslant CT^\delta\Lambda_T(v-w)\cdot\big[\Lambda_T^2(v)+\Lambda_T^2(w)\big]\\
&\leqslant 2CT^\delta E^2\Lambda_T(v-w),
\end{aligned}$$

where the constant C depends only on the form of $P(\cdot)$ and the linear estimates (2.1), (2.3), (2.9) and (3.1).

Thus, we can choose $0<T\ll 1$ satisfying (4.5), (4.7), (4.10) and

$$2CT^\delta E^2 \leqslant \frac{1}{2}. \tag{4.11}$$

Therefore, for those T, satisfying (4.5), (4.7), (4.10) and (4.11), the mapping $\mathcal{T}_{u_0}$ is a contraction mapping in Z_T^E. Consequently, by the Banach contraction mapping principle, there exists a unique function $u \in Z_T^E$ such that $\mathcal{T}_{u_0} u = u$ which solves the Cauchy problem.

By the method given in [16, Theorem 4.1], we can prove the persistence property of $u(t)$ in H^{s_0}, i.e.,

$$u(t,x) \in C\big([0,T]; H^{s_0}\big(\mathbb{R}^n\big)\big),$$

the uniqueness and the continuous dependence on the initial data of solution. For simplicity, we omit the rest of the proof. □

Proof of Theorem 2. For simplicity, we assume

$$P\big(\big(\partial_x^\alpha u\big)_{|\alpha|\leqslant 2}, \big(\partial_x^\alpha \bar{u}\big)_{|\alpha|\leqslant 2}\big) = |\Delta u|^2.$$

It will be clear, from the argument presented below, that this does not represent any loss of generality. And as in the proof of Theorem 1, we consider the case $s = s_0 = 2n + 3[\frac{n}{2}] + 15 + \frac{1}{2}$.

We introduce the following work space

$$X_T^E = \Bigg\{ w : [0,T] \times \mathbb{R}^n \to \mathbb{C}\colon \sup_t \big\| w(t)\big\|_{s_0,2} \leqslant E;\ \sup_t \big\|w(t)\big\|_{n+[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2} \leqslant E;$$

$$\sum_{|\beta|=s_0+\frac{1}{2}} \sup_{\alpha\in\mathbb{Z}^n} \Bigg(\int_0^T \int_{Q_\alpha} \big|\partial_x^\beta w(t,x)\big|^2 \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\frac{1}{2}} \leqslant T^\delta;$$

It should be 1/4.

$$(1+T)^{-[\frac{n}{2}]-2} \Bigg(\sum_{\alpha\in\mathbb{Z}^n} \sup_{t\in[0,T]} \sup_{x\in Q_\alpha} \big|D_x^2 w(t,x)\big|^2 \Bigg)^{\frac{1}{2}} \leqslant E \Bigg\},$$

where $0 < \delta < \frac{1}{4n}$ is a constant and the norm is defined as

$$\big\|v(t)\big\|_{X_T^E} = \max\Bigg\{ \sup_t \big\|v(t)\big\|_{\frac{23}{2}};\ \sup_t \big\|w(t)\big\|_{n+[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2};$$

$$ET^{-\delta} \sum_{|\beta|=s_0+\frac{1}{2}} \sup_{\alpha\in\mathbb{Z}^n} \Bigg(\int_0^T \int_{Q_\alpha} \big|\partial_x^\beta w(t,x)\big|^2 \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\frac{1}{2}};$$

$$(1+T)^{-[\frac{n}{2}]-2} \Bigg(\sum_{\alpha\in\mathbb{Z}^n} \sup_{t\in[0,T]} \sup_{x\in Q_\alpha} \big|D_x^2 w(t,x)\big|^2 \Bigg)^{\frac{1}{2}} \Bigg\}.$$

For $u_0 \in H^{s_0}(\mathbb{R}^n) \cap H^{n+4[\frac{n}{2}]+8}(\mathbb{R}^n;\ |x|^{2[\frac{n}{2}]+2}\,\mathrm{d}x)$ and $v \in X_T^E$, we denote by $u = \mathcal{T}(v) = \mathcal{T}_{u_0}(v)$ the solution of the linear inhomogeneous Cauchy problem

$$i\partial_t u = -\varepsilon\Delta u + \Delta^2 u + |\Delta v|^2, \tag{4.12}$$

$$u(0,x) = u_0(x), \tag{4.13}$$

and consider the corresponding integral equation

$$u(t)=\mathcal{T}(v)(t)=S(t)u_0-i\int_0^t S(t-\tau)|\Delta v|^2\,\mathrm{d}\tau. \tag{4.14}$$

We notice that

$$\partial_x^\beta(\Delta v\Delta\bar{v})=\partial_x^\beta\Delta v\Delta\bar{v}+\Delta v\partial_x^\beta\Delta\bar{v}+R_0\big(\big(\partial_x^{\gamma_1}v\partial_x^{\gamma_2}v\big)_{|\gamma_1|,\,|\gamma_2|\leqslant s_0-\frac{1}{2}}\big),$$

where $\beta,\gamma_1,\gamma_2\in\mathbb{Z}^n$.

From the integral equation (4.14), (2.6), Sobolev embedding theorem and (2.9), we can get, as in (4.4), that

$$\begin{aligned}
&\sum_{|\beta|=s_0+\frac{1}{2}}\sup_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\left|\partial_x^\beta u(t,x)\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad\leqslant\sum_{|\beta|=s_0+\frac{1}{2}}\sup_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\left|\partial_x^\beta S(t)u_0(x)\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\qquad+\sum_{|\beta|=s_0+\frac{1}{2}}\sup_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\left|\int_0^t S(t-\tau)\partial_x^\beta\left|\Delta v(\tau)\right|^2\mathrm{d}\tau\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad\leqslant CT^{\frac{1}{3}}\|u_0\|_{s_0,2}+CT^{\frac{1}{4n}}\sum_{|\beta|=s_0-\frac{3}{2}}\sum_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\left|\partial_x^\beta\Delta v\Delta\bar{v}\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}
\end{aligned}$$

It should be 1/4.

$$\begin{aligned}
&\qquad+C\int_0^T\left\|D_x^{\frac{1}{2}}R_0\right\|_2\mathrm{d}t\\
&\quad\leqslant CT^{\frac{1}{3}}\|u_0\|_{s_0,2}+CT^{\frac{1}{4n}}\sum_{|\beta|=s_0+\frac{1}{2}}\sum_{\alpha\in\mathbb{Z}^n}\left(\int_0^T\int_{Q_\alpha}\left|\partial_x^\beta v\right|^2\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}
\end{aligned}$$

It should be 1/4.

$$\begin{aligned}
&\qquad\cdot\left(\sum_{\alpha\in\mathbb{Z}^n}\sup_t\sup_{x\in Q_\alpha}\left|D_x^2v\right|\right)+CT\sup_t\|v\|_{s_0,2}^2\\
&\quad\leqslant CT^{\frac{1}{3}}\|u_0\|_{s_0,2}+CT^{\frac{1}{4n}}T^\delta E+CTE^2\\
&\quad\leqslant T^\delta,
\end{aligned}\tag{4.15}$$

It should be 1/4.

if we take T sufficiently small such that

$$CT^{\frac{1}{3}-\delta}\|u_0\|_{s_0,2}+CT^{\frac{1}{4n}}E+CT^{1-\delta}E^2\leqslant 1. \tag{4.16}$$

It should be 1/4.

We can rewrite (4.14) as

$$u(t)=S(t)\left(u_0-i\int_0^t S(-\tau)\left|\Delta v(\tau)\right|^2\mathrm{d}\tau\right). \tag{4.17}$$

From (3.7), we have

$$
\begin{aligned}
&(1+T)^{-[\frac{n}{2}]-2}\sum_{\alpha\in\mathbb{Z}^n}\sup_t\sup_{x\in Q_\alpha}\left|D_x^2u(t,x)\right|\\
&\quad\leqslant C\big(\|u_0\|_{n+3[\frac{n}{2}]+9,2}+\|u_0\|_{n+3[\frac{n}{2}]+9,2,2[\frac{n}{2}]+2}\big)\\
&\qquad+\sum_{\alpha\in\mathbb{Z}^n}\sup_t\sup_{x\in Q_\alpha}\left|D_x^2S(t)\int_0^t S(-\tau)\left|\Delta v(\tau)\right|^2\,\mathrm{d}\tau\right|,
\end{aligned}
$$

where the second term in the right-hand side can be bounded by

$$
\begin{aligned}
&C\sup_t\left\|\int_0^t S(-\tau)|\Delta v|^2\,\mathrm{d}\tau\right\|_{n+3[\frac{n}{2}]+9,2}+\sup_t\left\|\int_0^t S(-\tau)|\Delta v|^2\,\mathrm{d}\tau\right\|_{n+3[\frac{n}{2}]+9,2,2[\frac{n}{2}]+2}\\
&\quad\leqslant C\int_0^T\left\||\Delta v|^2\right\|_{n+3[\frac{n}{2}]+9,2}\mathrm{d}\tau+C\int_0^T\left\|S(-\tau)|\Delta v|^2\right\|_{n+3[\frac{n}{2}]+9,2,2[\frac{n}{2}]+2}\mathrm{d}\tau\\
&\quad\leqslant CT\sup_t\|v\|^2_{n+3[\frac{n}{2}]+11,2}+CT\sup_t\sum_{|\gamma|\leqslant n+3[\frac{n}{2}]+9}\left\||x|^{[\frac{n}{2}]+1}S(-\tau)\partial_x^\gamma|\Delta v|^2\right\|_2\\
&\quad\leqslant CT\sup_t\|v\|^2_{n+3[\frac{n}{2}]+11,2}+CT\Bigg\{\sup_t\sum_{|\gamma|\leqslant n+3[\frac{n}{2}]+9}\left\||x|^{[\frac{n}{2}]+1}\partial_x^\gamma|\Delta v|^2\right\|_2\\
&\qquad+\sum_{|\gamma|\leqslant n+6[\frac{n}{2}]+12}t^{[\frac{n}{2}]+1}\left\|\partial_x^\gamma|\Delta v|^2\right\|_2+C\sum_{\substack{|\beta|+|\gamma|\leqslant n+6[\frac{n}{2}]+10\\1\leqslant|\beta|\leqslant[\frac{n}{2}]}}t^{|\beta|}\left\|x^\beta\partial_x^\gamma|\Delta v|^2\right\|_2\Bigg\}\\
&\quad\leqslant CT\sup_t\|v\|^2_{n+3[\frac{n}{2}]+11,2}\\
&\qquad+CT\sup_t\sum_{|\gamma|\leqslant n+3[\frac{n}{2}]+9}\ \sum_{\substack{|\gamma_1|+|\gamma_2|=|\gamma|+4\\|\gamma_1|\leqslant|\gamma_2|}}\left\|\partial_x^{\gamma_2}v\right\|_\infty\left\||x|^{[\frac{n}{2}]+1}\partial_x^{\gamma_1}v\right\|_2\\
&\qquad+CT^{[\frac{n}{2}]+2}\sup_t\|v\|_{n+6[\frac{n}{2}]+14,2}\\
&\qquad+\sum_{\substack{|\beta|+|\gamma|\leqslant n+4[\frac{n}{2}]+12\\1\leqslant|\beta|\leqslant[\frac{n}{2}]}}T^{|\beta|+1}\sup_t\sum_{\substack{|\gamma_1|+|\gamma_2|=|\gamma|+4\\|\gamma_1|\leqslant|\gamma_2|}}\left\|\partial_x^{\gamma_2}v\right\|_\infty\left\|x^\beta\partial_x^{\gamma_1}v\right\|_2\\
&\quad\leqslant CT\sup_t\|v\|^2_{n+3[\frac{n}{2}]+11,2}+CT\sup_t\|v\|_{n+4[\frac{n}{2}]+14,2}\sup_t\|v\|_{n+[\frac{n}{4}]+7,2,2[\frac{n}{2}]+2}\\
&\qquad+CT^{[\frac{n}{2}]+2}\sup_t\|v\|_{n+6[\frac{n}{2}]+14,2}\\
&\qquad+C\sum_{1\leqslant|\beta|\leqslant[\frac{n}{2}]}T^{|\beta|+1}\sup_t\|v\|_{\frac{3n}{2}+4[\frac{n}{2}]+15+\frac{1}{2},2}\sup_t\|v\|_{n+[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2}\\
&\quad\leqslant CT\sup_t\|v\|^2_{s_0,2}+CT\sup_t\|v\|_{s_0,2}\sup_t\|v\|_{n+[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2}+CT^{[\frac{n}{2}]+2}\sup_t\|v\|^2_{s_0,2}\\
&\qquad+CT\big(T+T^{[\frac{n}{2}]}\big)\sup_t\|v\|_{s_0,2}\sup_t\|v\|_{n+[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2}
\end{aligned}
$$

$$\leqslant CT\big(1+T^{[\frac{n}{2}]+1}\big)E^2$$
$$\leqslant \frac{1}{2}E,$$

if we take

$$E \geqslant 2C\big(\|u_0\|_{n+3[\frac{n}{2}]+9,2} + \|u_0\|_{n+4[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2}\big)$$

and T so small that

$$CT\big(1+T^{[\frac{n}{2}]+1}\big)E \leqslant \frac{1}{2}. \tag{4.18}$$

Thus, we obtain

$$(1+T)^{-[\frac{n}{2}]-2}\sum_{\alpha\in\mathbb{Z}^n}\sup_t\sup_{x\in Q_\alpha}\big|D_x^2u(t,x)\big| \leqslant E. \tag{4.19}$$

Similar to (4.19), we have

$$\begin{aligned}
&\sup_t\big\|u(t)\big\|_{n+[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2}\\
&\quad=\sup_t\sum_{|\gamma|\leqslant n+[\frac{n}{2}]+8}\big\||x|^{[\frac{n}{2}]+1}\partial_x^\gamma u(t)\big\|_2\\
&\quad\leqslant\sum_{|\gamma|\leqslant n+[\frac{n}{2}]+8}\left[\sup_t\big\||x|^{[\frac{n}{2}]+1}S(t)\partial_x^\gamma u_0\big\|_2+\sup_t\left\|\int_0^t|x|^{[\frac{n}{2}]+1}S(t-\tau)\partial_x^\gamma|\Delta v|^2\,\mathrm{d}\tau\right\|_2\right]\\
&\quad\leqslant\sup_t\sum_{|\gamma|\leqslant n+[\frac{n}{2}]+8}\big\||x|^{[\frac{n}{2}]+1}\partial_x^\gamma u_0\big\|_2+C\sup_t\sum_{|\gamma|\leqslant n+4[\frac{n}{2}]+11}t^{[\frac{n}{2}]+1}\big\|\partial_x^\gamma u_0\big\|_2\\
&\qquad+C\sup_t\sum_{\substack{|\beta|+|\gamma|\leqslant n+4[\frac{n}{2}]+9\\1\leqslant|\beta|\leqslant[\frac{n}{2}]}}t^{|\beta|}\big\|x^\beta\partial_x^\gamma u_0\big\|_2+\sup_t\int_0^t\sum_{|\gamma|\leqslant n+[\frac{n}{2}]+8}\big\||x|^{[\frac{n}{2}]+1}\partial_x^\gamma|\Delta v|^2\big\|_2\\
&\qquad+C\sup_t\sum_{|\gamma|\leqslant n+4[\frac{n}{2}]+11}t^{[\frac{n}{2}]+1}\big\|\partial_x^\gamma|\Delta v|^2\big\|_2\\
&\qquad+C\sup_t\sum_{\substack{|\beta|+|\gamma|\leqslant n+4[\frac{n}{2}]+9\\1\leqslant|\beta|\leqslant[\frac{n}{2}]}}t^{|\beta|}\big\|x^\beta\partial_x^\gamma|\Delta v|^2\big\|_2\\
&\quad\leqslant C(1+T)^{[\frac{n}{2}]+1}\big(\|u_0\|_{n+4[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2}+\|u_0\|_{s_0,2}\big)\\
&\qquad+T\sup_t\|v\|_{s_0,2}\|v\|_{n+[\frac{n}{2}]+8,2,2[\frac{n}{2}]+2}+CT^{[\frac{n}{2}]+1}\|v\|_{s_0,2}^2\\
&\quad\leqslant E.
\end{aligned}\tag{4.20}$$

As in the derivation of (4.6), we can get

$$\sup_t\big\|u(t)\big\|_{s_0,2}\leqslant E. \tag{4.21}$$

The estimates (4.15), (4.19), (4.20) and (4.21) yield $u\in X_T^E$. Thus, we can fix E and T as above such that $\mathcal{T}$ is a contraction mapping from X_T^E to itself. By the standard argument used in the proof of Theorem 1, we can complete the proof for which we omit the details. □

Acknowledgments

The authors thank the referees for valuable comments and suggestions for the original manuscript.

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