Global well posedness for the Gross-Pitaevskii equation with an angular momentum rotational term in three dimensions

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In this paper, we establish the global well posedness of the Cauchy problem for the Gross-Pitaevskii equation with an angular momentum rotational term in which the angular velocity is equal to the isotropic trapping frequency in the space $\mathbb{R}^3$. [DOI: 10.1063/1.2795218]

I. INTRODUCTION

The Gross-Pitaevskii equation (GPE), derived independently by Gross[9] and Pitaevskii,[18] arises in various models of nonlinear physical phenomena. This is a Schrödinger-type equation with an external field potential $V_{\text{ext}}(t,x)$ and a local cubic nonlinearity,

$$i\hbar\partial_t u + \frac{\hbar^2}{2m}\Delta u = V_{\text{ext}}u + \beta|u|^2 u. \tag{1.1}$$

The GPE (1.1) in physical dimensions (two and three dimensions) is used in the mean-field quantum theory of Bose-Einstein condensate (BEC) formed by ultracold bosonic coherent atomic ensembles. A rigorous derivation of the GPE for the dynamics of (nonrotating) BEC has been obtained by Erdös *et al.*[7,8] Recently, several research groups[10,15–17] have produced quantized vortices in trapped BECs, and a typical method they used is to impose a laser beam on the magnetic trap to create a harmonic anisotropic rotating trapping potential. Seiringer has discussed the stationary GPE in Ref. 20; Lieb and Seiringer have rigorously derived for the description of the ground state asymptotics of rotating Bose gases in Ref. 13. The properties of BEC in a rotational frame at temperature T being much smaller than the critical condensation temperature T_c (Ref. 12) are well described by the macroscopic wave function $u(t,x)$, whose evolution is governed by a self-consistent, mean-field nonlinear Schrödinger equation in a rotational frame, also known as the GPE with an angular momentum rotation term,

$$i\hbar\partial_t u + \frac{\hbar^2}{2m}\Delta u = V(x)u + NU_0|u|^2 u - \Omega L_z u, \quad x \in \mathbb{R}^3, \quad t \geqslant 0, \tag{1.2}$$

where the wave function $u(t,x)$ corresponds to a condensate state, m is the atomic mass, $\hbar$ is the Planck constant, N is the number of atoms in the condensate, Ω is the angular velocity of the rotating laser beam, and $V(x)$ is an external trapping potential. When a harmonic trap potential is concerned, $V(x) = \frac{m}{2}\Sigma_{j=1}^3 \omega_j^2 x_j^2$, with ω_1, ω_2, and ω_3 being the trap frequencies in the x_1, x_2, and x_3 directions, respectively. The local nonlinearity term $NU_0|u|^2 u$ arises from an assumption about the

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delta-shape interatomic potential, where $U_0=4\pi\hbar^2 a_s/m$ describes the interaction between atoms in the condensate with a_s (positive for repulsive interaction and negative for attractive interaction), the s-wave scattering length. $L_z=-i\hbar(x_1\partial_{x_2}-x_2\partial_{x_1})$ is the third component of the angular momentum $L=x\times P$ with the momentum operator $P=-i\hbar\nabla$.

After normalization, proper nondimensionalization and dimension reduction in certain limiting trapping frequency regime,[19] the system (1.2) becomes, to the dimensionless GPE in d dimensions ($d=2,3$),

$$iu_t+\tfrac{1}{2}\Delta u = V_d(x)u+\beta_d|u|^2u-\Omega L_z u, \quad x\in\mathbb{R}^d, \quad t>0, \tag{1.3}$$

where $L_z=i(x_1\partial_{x_2}-x_2\partial_{x_1})$ and

$$\beta_d=\begin{cases}\beta\sqrt{\gamma_3/2\pi},\\ \beta,\end{cases} \qquad V_d(x)=\begin{cases}(\gamma_1^2x_1^2+\gamma_2^2x_2^2)/2, & d=2,\\ (\gamma_1^2x_1^2+\gamma_2^2x_2^2+\gamma_3^2x_3^2)/2, & d=3,\end{cases} \tag{1.4}$$

with $\gamma_1>0$, $\gamma_2>0$, and $\gamma_3>0$ constants, $\beta=4\pi a_s N/a_0$, $a_0=\sqrt{\hbar/m\omega_m}$, and $\omega_m=\min\{\omega_1,\omega_2,\omega_3\}$.

In general, it is a rather complicated process about the dynamics of solutions (in particular, vortex) for GPE (1.2) under the interaction of trapping frequencies and angular rotating motion. The recent numerical simulation of GPE (1.2) for different choices of trap frequencies (γ_1,γ_2) can help us to understand the complicated dynamical phenomena caused by the angular rotating and spatial high frequency motion. The case of different frequencies $\gamma_1\neq\gamma_2$ gives much complicated behavior and thus is rather difficult to be studied rigorously.[2,3] To our knowledge, Eq. (1.2) has been only investigated for some specific cases by numerical simulation. Therefore, to develop methods for constructing analytical solutions to the GPE (1.1) or some specific cases is the first step in order to understand the dynamics caused by the trapping and rotation.

To begin with, we first consider the case $\gamma_1=\gamma_2=\gamma_3=\omega$ which means the spatial isotropic motion. In order to derive the exact analytic formula for the solution to the linear equation, we have to assume that the angular velocity is equal to the isotropic trapping frequency, i.e., $\Omega=\omega$. In the present paper, we focus on the Cauchy problem of the GPE with an angular momentum rotational term in three dimensions,

$$iu_t+\tfrac{1}{2}\Delta u=\tfrac{\omega^2}{2}|x|^2u+\beta|u|^2u-\omega L_z u, \quad x\in\mathbb{R}^3, \quad t\geqslant 0, \tag{1.5}$$

$$u(0,x)=u_0(x), \quad x\in\mathbb{R}^3, \tag{1.6}$$

where the wave function $u=u(t,x):[0,\infty)\times\mathbb{R}^3\to\mathbb{C}$ corresponds to a condensate state, Δ is the Laplace operator on $\mathbb{R}^3$, $\omega\geqslant 1$ and $\beta>0$ are constants, and $L_z=-i(x_1\partial_{x_2}-x_2\partial_{x_1})=i(x_2\partial_{x_1}-x_1\partial_{x_2})$ is the dimensionless angular momentum rotational term.

We assume that the initial value

$$u_0(x)\in\Sigma\coloneqq\{u\in H^1(\mathbb{R}^3):|x|u\in L^2(\mathbb{R}^3)\}, \tag{1.7}$$

with the norm

$$\|u\|_\Sigma=\|u\|_{H^1}+\||x|u\|_{L^2}.$$

Note that for multidimensional GPE (1.2), nothing is known about the exact integration except for the case $\gamma_1=\gamma_2$ ($=\gamma_3$ for three dimensions) considered in Refs. 4 and 5 without the angular momentum rotational term, namely, $\Omega=0$.

In the case of two dimensions,[11] the linear operator $i\partial_t+\frac{1}{2}(\Delta-\omega^2|x|^2+2\omega L_z)$ can be written as the form $i\partial_t+\frac{1}{2}(\nabla-ib)^2$, where $b=(-\omega x_2,\omega x_1)$ satisfies the Coulomb gauge condition $\nabla\cdot b=0$. However, in three dimensions, we cannot find such a potential b satisfying the Coulomb gauge condition $\nabla\cdot b=0$. So that it is impossible to write the linear operator as a similar operator as in the case of two dimensions, and this makes the problem be much more difficult than the two dimensional case.

In addition, there are three ingredients that play important roles in the proof of our result. The first involves the solution of the Cauchy problem to the linear equation

$$iu_t + \tfrac{1}{2}\Delta u = \tfrac{\omega^2}{2}|x|^2 u - \omega L_z u, \quad x \in \mathbb{R}^3, \quad t \geq 0,$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^3, \tag{1.8}$$

which is significant for investigating the properties of the evolution operator corresponding to the linear operator $i\partial_t + \frac{1}{2}\Delta - \frac{\omega^2}{2}|x|^2 + \omega L_z$. The second one is to obtain the Strichartz estimates for the foregoing linear operator. The last one is that there exist two Galilean operators $J(t)$ and $H(t)$ (as blow), which can commute approximatively with the linear operator and can be viewed as the substitute of ∇ and x, respectively, in the nonpotential case.

Now we state the main result of this paper.

Theorem 1.1: *Let $u_0 \in \Sigma$ and $\rho \in (2,6)$. Then, there exists a unique solution $u(t,x)$ to the Cauchy problem (1.5) and (1.6). And the solution satisfies, for any $T \in (0,\infty)$, that*

$$u(t,x), J(t)u(t,x), H(t)u(t,x) \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^3)) \cap L^{\gamma(\rho)}(0,T; L^\rho(\mathbb{R}^3)),$$

where $2/\gamma(\rho) = \frac{3}{2} - \frac{3}{\rho}$, $J(t)$ and $H(t)$ are defined as below as in (2.14) and (2.15), respectively.

Remark 1.2: Since the GPE (1.5) (or (1.3)) in a rotational frame is time reversible and time transverse invariant, the above result is also valid for the case when $t<0$.

The paper is organized as follows. In Sec. II, the evolution operator of the linear equation and the Strichartz estimates about the former operator are first established. Section III is devoted to the derivation of some conservation identities such as the mass, the energy, the angular momentum expectation, and the pseudoconformal conservation laws in the whole space $\mathbb{R}^3$ for (1.5) and (1.6). Finally, the nonlinear estimates and the proof of Theorem 1.1 are obtained in Sec. IV.

II. THE STRICHARTZ ESTIMATES AND SOME MAIN OPERATORS

On the analogy of the Schrödinger operators with magnetic fields in Refs. 1 and 6, we can also define the propagator associated with the self-adjoint operator,

$$H_0 \equiv \tfrac{1}{2}[-\Delta + \omega^2|x|^2 - 2\omega L_z],$$

i.e., $S(t) = e^{-iH_0 t}$, which can be explicitly expressed as

$$S(t)\phi := \left(\frac{\omega}{2\pi i \sin(\omega t)}\right)^{3/2} \int_{\mathbb{R}^3} e^{i\omega((|x-y|^2/2)\cot(\omega t) - \tilde{x}\cdot y)} \phi(y)\,dy \tag{2.1}$$

$$\equiv \left(\frac{\omega}{2\pi i \sin(\omega t)}\right)^{3/2} e^{i\omega(|x|^2/2)\cot(\omega t)} \int_{\mathbb{R}^3} e^{i\omega A(t)x\cdot y} e^{i\omega(|y|^2/2)\cot(\omega t)} \phi(y)\,dy, \tag{2.2}$$

$$\equiv \left(\frac{\omega}{2\pi i \sin(\omega t)}\right)^{3/2} e^{i\omega(|x|^2/2)\cot(\omega t)} \int_{\mathbb{R}^3} e^{i\omega x\cdot A^T(t) y} e^{i\omega(|y|^2/2)\cot(\omega t)} \phi(y)\,dy, \tag{2.3}$$

for $0 < t \leq \frac{\pi}{4\omega}$ through a complicated computation, where $\tilde{x} := (-x_2, x_1, (\csc(\omega t) - \cot(\omega t))x_3)$, the matrix $A(t)$ is defined by

$$A(t) = \begin{pmatrix} \cot(\omega t) & -1 & 0 \\ 1 & \cot(\omega t) & 0 \\ 0 & 0 & \csc(\omega t) \end{pmatrix},$$

and $A^T(t)$ is the transpose of the matrix $A(t)$. Note that this formula is valid only for small time due to the singularity formation for the fundamental solution.

Next, we derive the dual operator $S^*(t)$ of $S(t)$. From

$$\begin{aligned}\langle S(t)f,g\rangle &= \langle \widehat{S(t)f(x)},\overline{\widehat{g(x)}}\rangle = \langle \widehat{S(t)f}(\xi),\hat{g}(-\xi)\rangle \\ &= \left\langle \left(\frac{\sin(\omega t)}{i\omega}\right)^{3/2}\mathcal{F}(e^{i\omega(|x|^2/2)\cot(\omega t)})(\xi)(e^{i(|(A^T(t))^{-1}\xi|^2/2\omega)\cot(\omega t)}f((A^T(t))^{-1}\xi/\omega)),\hat{g}(-\xi)\right\rangle \\ &= \left(\frac{\sin(\omega t)}{i\omega}\right)^{3/2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{F}(e^{i\omega(|x|^2/2)\cot(\omega t)})(\eta)(e^{i(|(A^T(t))^{-1}(\xi-\eta)|^2/2\omega)\cot(\omega t)}f((A^T(t))^{-1} \\ &\quad\times(\xi-\eta)/\omega))\mathrm{d}\eta\hat{g}(-\xi)\mathrm{d}\xi \\ &= \left(\frac{\sin(\omega t)}{i\omega}\right)^{3/2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathcal{F}(e^{i\omega(|x|^2/2)\cot(\omega t)})(\eta)(e^{i(|(A^T(t))^{-1}\zeta|^2/2\omega)\cot(\omega t)} \\ &\quad\times f((A^T(t))^{-1}\zeta/\omega))\hat{g}(-\zeta-\eta)\mathrm{d}\zeta\mathrm{d}\eta \\ &= \left(\frac{\sin(\omega t)}{i\omega}\right)^{3/2}\int_{\mathbb{R}^3} e^{i(|(A^T(t))^{-1}\zeta|^2/2\omega)\cot(\omega t)}f((A^T(t))^{-1}\zeta/\omega)\mathcal{F}(e^{i\omega(|\cdot|^2/2)\cot(\omega t)}g(-\cdot))(\zeta)\mathrm{d}\zeta \\ &= \left(\frac{\omega}{i\sin(\omega t)}\right)^{3/2}\int_{\mathbb{R}^3} f(\xi)e^{i\omega(|\xi|^2/2)\cot(\omega t)}\mathcal{F}(e^{i\omega(|\cdot|^2/2)\cot(\omega t)}g(-\cdot))(\omega A^T(t)\xi)\mathrm{d}\xi = \langle f,S^*(t)g\rangle,\end{aligned}$$

we can define the dual operator as

$$\begin{aligned}S^*(t)g &:= \left(\frac{\omega}{2\pi i\sin(\omega t)}\right)^{3/2} e^{i\omega(|x|^2/2)\cot(\omega t)}\int_{\mathbb{R}^3} e^{-i\omega A^T(t)x\cdot y}e^{i\omega(|y|^2/2)\cot(\omega t)}g(-y)\mathrm{d}y \\ &= \left(\frac{\omega}{2\pi i\sin(\omega t)}\right)^{3/2} e^{i\omega(|x|^2/2)\cot(\omega t)}\int_{\mathbb{R}^3} e^{i\omega A^T(t)x\cdot y}e^{i\omega(|y|^2/2)\cot(\omega t)}g(y)\mathrm{d}y. \qquad (2.4)\end{aligned}$$

In order to obtain the Strichartz estimates, we have to estimate the norm $\|S(t)S^*(s)\|_{L^1\to L^\infty}$. Indeed, we have the following proposition:

Proposition 2.1: Let $u(t)=S(t)u_0$, then $u(t)$ satisfies the linear equation (1.8). And the operator $S(t)$ has following properties:

(1) $S(t)$ is unitary on L^2, i.e., $\|S(t)\|_{L^2\to L^2}=1$;

(2) $\|S(t)S^*(s)\|_{L^1\to L^\infty}\lesssim|t-s|^{-3/2}$ for $0<s<t\leqslant\frac{\pi}{4\omega}$.

Proof: By computation, it is easy to see that

$$\begin{aligned}iu_t &= \left(\frac{\omega}{2\pi i\sin(\omega t)}\right)^{3/2}\int_{\mathbb{R}^3} e^{i\omega((|x-y|^2/2)\cot(\omega t)-\tilde{x}\cdot y)}\left[-\frac{3}{2}i\omega\cot(\omega t)+\omega^2\csc^2(\omega t)\frac{|x-y|^2}{2}\right. \\ &\quad\left.+\,\omega^2(\csc(\omega t)-\cot(\omega t))\csc(\omega t)\ x_3y_3)\right]u_0(y)\mathrm{d}y, \qquad (2.5)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\Delta u &= \left(\frac{\omega}{2\pi i\sin(\omega t)}\right)^{3/2}\int_{\mathbb{R}^3} e^{i\omega((|x-y|^2/2)\cot(\omega t)-\tilde{x}\cdot y)}\left[\frac{3}{2}i\omega\cot(\omega t)-\omega^2\cot^2(\omega t)\frac{|x-y|^2}{2}\right. \\ &\quad\left.-\,\omega^2\frac{|y|^2}{2}+\omega^2\cot(\omega t)\tilde{x}\cdot y\right]u_0(y)\mathrm{d}y, \qquad (2.6)\end{aligned}$$

$$-\frac{\omega^2}{2}|x|^2u=\left(\frac{\omega}{2\pi i\sin(\omega t)}\right)^{3/2}\int_{\mathbb{R}^3}\mathrm{e}^{i\omega((|x-y|^2/2)\cot(\omega t)-\tilde{x}\cdot y)}\left(-\frac{\omega^2}{2}|x|^2\right)u_0(y)\mathrm{d}y, \tag{2.7}$$

and

$$\omega L_z u=\left(\frac{\omega}{2\pi i\sin(\omega t)}\right)^{3/2}\int_{\mathbb{R}^3}\mathrm{e}^{i\omega((|x-y|^2/2)\cot(\omega t)-\tilde{x}\cdot y)}\omega^2[(x_2y_1-x_1y_2)\cot(\omega t)+x_1y_1+x_2y_2]u_0(y)\mathrm{d}y. \tag{2.8}$$

Summing (2.5)–(2.8), we have

$$iu_t+\frac{1}{2}\Delta u-\frac{\omega^2}{2}|x|^2u+\omega L_z u=0,$$

which yields the desired result.

For the unitariness of $S(t)$ on L^2, it is trivial by using the Plancherel theorem. Thus, we omit the detailed proof.

Next, we prove the dispersive property of $S(t)$. Indeed, we have for $0<s<t\leqslant\pi/4\omega$

$$\begin{aligned}
S(t)S^*(s)f&=\left(\frac{\omega^2}{-(2\pi)^2\sin(\omega s)\sin(\omega t)}\right)^{3/2}\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega t)}\int_{\mathbb{R}^3}\mathrm{e}^{i\omega A(t)x\cdot y}\mathrm{e}^{i\omega(|y|^2/2)\cot(\omega t)}\\
&\quad\times\mathrm{e}^{i\omega(|y|^2/2)\cot(\omega s)}\int_{\mathbb{R}^3}\mathrm{e}^{i\omega A^T(s)y\cdot z}\mathrm{e}^{i\omega(|z|^2/2)\cot(\omega s)}f(z)\mathrm{d}z\mathrm{d}y\\
&=\left(\frac{\omega^2}{-2\pi\sin(\omega s)\sin(\omega t)}\right)^{3/2}\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega t)}\int_{\mathbb{R}^3}\mathrm{e}^{i\omega A(t)x\cdot y}\mathrm{e}^{i\omega(|y|^2/2)[\cot(\omega t)+\cot(\omega s)]}\\
&\quad\times\mathcal{F}^{-1}(\mathrm{e}^{i\omega(|\cdot|^2/2)\cot(\omega s)}f(\cdot))(\omega A^T(s)y)\mathrm{d}y\\
&=\left(\frac{\sin(\omega s)}{-2\pi\sin(\omega t)}\right)^{3/2}\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega t)}\int_{\mathbb{R}^3}\mathrm{e}^{iA(t)x\cdot(A^T(s))^{-1}\xi}\mathrm{e}^{i(|(A^T(s))^{-1}\xi|^2/2\omega)[\cot(\omega t)+\cot(\omega s)]}\\
&\quad\times\mathcal{F}^{-1}(\mathrm{e}^{i\omega(|\cdot|^2/2)\cot(\omega s)}f(\cdot))(\xi)\mathrm{d}\xi\\
&=\left(\frac{\sin(\omega s)}{-2\pi\sin(\omega t)}\right)^{3/2}\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega t)}\int_{\mathbb{R}^3}\mathrm{e}^{iB(t,s)x\cdot\xi}\mathrm{e}^{i(|(A^T(s))^{-1}\xi|^2/2\omega)[\cot(\omega t)+\cot(\omega s)]}\\
&\quad\times\mathcal{F}^{-1}(\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega s)}f(x))(\xi)\mathrm{d}\xi\\
&=\left(\frac{\sin(\omega s)}{-\sin(\omega t)}\right)^{3/2}\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega t)}\\
&\quad\times[(\mathcal{F}^{-1}\mathrm{e}^{i(|(A^T(s))^{-1}\xi|^2/2\omega)[\cot(\omega t)+\cot(\omega s)]})(\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega s)}f(-x))](B(t,s)x)\\
&=\left(\frac{\sin(\omega s)}{-\sin(\omega t)}\right)^{3/2}\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega t)}\int_{\mathbb{R}^3}(\mathcal{F}^{-1}\mathrm{e}^{i(|(A^T(s))^{-1}\xi|^2/2\omega)[\cot(\omega t)+\cot(\omega s)]})(B(t,s)x-y)\\
&\quad\times\mathrm{e}^{i\omega(|y|^2/2)\cot(\omega s)}f(-y)\mathrm{d}y,
\end{aligned}$$

where the matrix $B(t,s)$ is given by

$$B(t,x) = \begin{pmatrix} B_{11}(t,s) & B_{12}(t,s) & 0 \\ -B_{12}(t,s) & B_{11}(t,s) & 0 \\ 0 & 0 & \csc(\omega t)\sin(\omega s) \end{pmatrix},$$

with

$$B_{11}(t,s) = \sin^2(\omega s)(\cot(\omega t)\cot(\omega s) + 1), \quad B_{12}(t,s) = \sin^2(\omega s)(\cot(\omega t) - \cot(\omega s)).$$

Noticing that

$$\begin{aligned}
&|(\mathcal{F}^{-1} e^{i(|(A^T(s))^{-1}\xi|^2/2\omega)[\cot(\omega t)+\cot(\omega s)]})(B(t,s)x - y)| \\
&= (2\pi)^{-3/2} \left| \int_{\mathbb{R}^3} e^{i(B(t,s)x-y)\cdot\xi} e^{i(|(A^T(s))^{-1}\xi|^2/2\omega)[\cot(\omega t)+\cot(\omega s)]} d\xi \right| \\
&= \csc^3(\omega s) \left(\frac{\omega}{\pi[\cot(\omega t) + \cot(\omega s)]} \right)^{3/2} \left| \int_{\mathbb{R}^3} e^{(i\sqrt{2\omega}/\sqrt{\cot(\omega t)+\cot(\omega s)})(B(t,s)x-y)\cdot A^T(s)\eta} e^{i|\eta|^2} d\eta \right| \\
&\lesssim \left(\frac{\omega}{\pi \sin^2(\omega s)[\cot(\omega t) + \cot(\omega s)]} \right)^{3/2},
\end{aligned}$$

we can get, for $0 < s < t \leq \pi/4\omega$, that

$$\begin{aligned}
\|S(t)S^*(s)f\|_{L^\infty} &\lesssim \left(\frac{\omega}{\pi \sin(\omega t)\sin(\omega s)[\cot(\omega t) + \cot(\omega s)]} \right)^{3/2} \int_{\mathbb{R}^3} |f(y)| dy \\
&= \left(\frac{\omega}{\pi \sin(\omega(t+s))} \right)^{3/2} \|f\|_{L^1} \lesssim |t+s|^{-3/2} \|f\|_{L^1} \lesssim |t-s|^{-3/2} \|f\|_{L^1},
\end{aligned}$$

since $|\sin t| \geq \frac{2}{\pi}|t|$ for $|t| \leq \frac{\pi}{2}$. □

Thus, we can obtain similar Strichartz estimates to the linear Schrödinger operator $e^{i(t/2)\Delta}$ by the standard methods (cf. Ref. 14) provided that only finite time intervals are involved (cf. Ref. 5).

Proposition 2.2: Let I be an interval contained in $[0, \pi/4\omega]$. Then, it holds the following.

(1) *For any admissible pair* $(\gamma(p), p)$ *(that is,* $2/\gamma(p) = 3(1/2 - 1/p)$ *for* $2 \leq p < 6$*), there exists* C_p *such that for any* $\phi \in L^2$,

$$\|S(t)\phi\|_{L^{\gamma(p)}(I;L^p)} \leq C_p \|\phi\|_{L^2}. \tag{2.9}$$

(2) *For any admissible pairs* $(\gamma(p_1), p_1)$ *and* $(\gamma(p_2), p_2)$*, there exists* C_{p_1,p_2} *such that*

$$\left\| \int_{I \cap \{s<t\}} S(t)S^*(s)F(s) ds \right\|_{L^{\gamma(p_1)}(I;L^{p_1})} \leq C_{p_1,p_2} \|F\|_{L^{\gamma(p_2)'}(I;L^{p_2'})}. \tag{2.10}$$

The above constants are independent of $I \subset [0, \pi/4\omega]$.

The integral equation reads

$$u(t) = S(t)u_0 - i\beta \int_0^t S(t)S^*(s)|u|^2 u(s) ds. \tag{2.11}$$

Since the initial data belong to Σ, we naturally need the estimates of $\nabla S(t)\phi$ and $xS(t)\phi$. In fact, from (2.1), we can compute and obtain that

$$\nabla S(t)\phi = i\omega x \cot(\omega t)S(t)\phi - i\omega S(t)(x \cot(\omega t) - \check{x})\phi,$$

where $\check{x} = (-x_2, x_1, (\cot \omega t - \csc \omega t)x_3)$, and

$$S(t)\nabla\phi = i\omega(x\cot(\omega t)+\breve{x})S(t)\phi - i\omega\cot(\omega t)S(t)(x\phi),$$

which yield

$$\nabla S(t)\phi = \cos(\omega t)S(t)(\cos(\omega t)\nabla - \sin(\omega t)\breve{\nabla})\phi - i\omega\sin(\omega t)S(t)[(\cos(\omega t)x - \sin(\omega t)\breve{x})\phi], \tag{2.12}$$

$$xS(t)\phi = \cos(\omega t)S(t)[(\cos(\omega t)x - \sin(\omega t)\breve{x})\phi] - \tfrac{i}{\omega}\sin(\omega t)S(t)(\cos(\omega t)\nabla - \sin(\omega t)\breve{\nabla})\phi, \tag{2.13}$$

where $\breve{\nabla}=(-\partial_{x_2},\partial_{x_1},(\cot\omega t - \csc\omega t)\partial_{x_3})$.

Thus, we have

$$S(t)(-i\nabla)\phi = [\omega\sin(\omega t)(\cos(\omega t)x+\sin(\omega t)\breve{x}) - i\cos(\omega t)(\cos(\omega t)\nabla + \sin(\omega t)\breve{\nabla})]S(t)\phi,$$

and

$$S(t)\omega x\phi = [\omega\cos(\omega t)(\cos(\omega t)x+\sin(\omega t)\breve{x}) + i\sin(\omega t)(\cos(\omega t)\nabla + \sin(\omega t)\breve{\nabla})]S(t)\phi.$$

For convenience, we denote

$$J(t) = \omega\sin(\omega t)(\cos(\omega t)x+\sin(\omega t)\breve{x}) - i\cos(\omega t)(\cos(\omega t)\nabla + \sin(\omega t)\breve{\nabla}), \tag{2.14}$$

and the corresponding "orthogonal" operator

$$H(t) = \omega\cos(\omega t)(\cos(\omega t)x+\sin(\omega t)\breve{x}) + i\sin(\omega t)(\cos(\omega t)\nabla + \sin(\omega t)\breve{\nabla}), \tag{2.15}$$

which will appear in the pseudoconformal conservation law and play a crucial role in the nonlinear estimates.

By computation, we can obtain the following commutation relation:

$$\begin{aligned} \left[J(t), i\partial_t + \frac{1}{2}\Delta - \frac{\omega^2}{2}|x|^2 + \omega L_z\right] &= O_J(t), \\ \left[H(t), i\partial_t + \frac{1}{2}\Delta - \frac{\omega^2}{2}|x|^2 + \omega L_z\right] &= O_H(t), \end{aligned} \tag{2.16}$$

where

$$O_J(t) = (0,0,2i\omega\sin(\omega t)(\omega\sin(\omega t)x_3 - i\cos(\omega t)\partial_{x_3})),$$

and

$$O_H(t) = (0,0,2i\omega\sin(\omega t)(\omega\cos(\omega t)x_3 + i\sin(\omega t)\partial_{x_3})).$$

It is clear that for $0<t\leqslant \pi/4\omega$

$$|O_J(t)u| = \left|\left(0,0,\frac{2i\omega\sin(\omega t)}{2\cos(\omega t)-1}J_3(t)u\right)\right| = \left|\frac{2\omega\sin(\omega t)}{2\cos(\omega t)-1}J_3(t)u\right| \leqslant \frac{2\omega^2 t}{\sqrt{2}-1}|J_3(t)u| \lesssim \omega^2 t|J(t)u|, \tag{2.17}$$

and

$$|O_H(t)u| = \left|\left(0,0,\frac{2i\omega\sin(\omega t)}{2\cos(\omega t)-1}H_3(t)u\right)\right| \lesssim \omega^2 t|H(t)u|, \tag{2.18}$$

where $J_3(t)$ and $H_3(t)$ are the third components of the operators $J(t)$ and $H(t)$, respectively.

In addition, denote $M(t)=\mathrm{e}^{-i\omega(|x|^2/2)\tan(\omega t)}$ and $Q(t)=\mathrm{e}^{i\omega(|x|^2/2)\cot(\omega t)}$, then

$$\begin{aligned} J(t) &= -i\cos(\omega t)M(t)(\cos(\omega t)\nabla + \sin(\omega t)\breve{\nabla})M(-t), \\ H(t) &= i\sin(\omega t)Q(t)(\cos(\omega t)\nabla + \sin(\omega t)\breve{\nabla})Q(-t). \end{aligned} \tag{2.19}$$

III. THE CONSERVED QUANTITIES

Proposition 3.1: Let u be a solution of the equation (1.5) with the initial data $\phi \in \Sigma(\mathbb{R}^3)$. *Then, we have the following conserved quantities for all* $t \geqslant 0$:

(1) The L^2-norm:

$$\|u(t)\|_2 = \|u_0\|_2. \tag{3.1}$$

(2) The energy for the nonrotating part:

$$E_0(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{\omega^2}{2}\|xu\|_2^2 + \frac{\beta}{2}\|u\|_4^4 = E_0(u_0). \tag{3.2}$$

(3) The angular momentum expectation:

$$\langle L_z\rangle(t) = \int_{\mathbb{R}^2} \bar{u}L_z u\,\mathrm{d}x = \langle L_z\rangle(0). \tag{3.3}$$

(4) The pseudo-conformal conservation law:

$$\|J(t)u\|_2^2 + \|H(t)u\|_2^2 + 4\cos(\omega t)(1-\cos(\omega t))[\|\omega x_3 u\|_2^2 + \|\partial_{x_3}u\|_2^2] + \beta\|u\|_4^4 = 2E_0(u_0). \tag{3.4}$$

Proof: For convenience, we introduce

$$eq(u) \coloneqq iu_t + \frac{1}{2}\Delta u - \frac{\omega^2}{2}|x|^2 u - \beta|u|^2 u + \omega L_z u.$$

It is clear that (3.1) holds by applying the L^2-inner product between $eq(u)$ and $\bar{u}$, and then taking the imaginary part of the resulting equation.

Since we can use the identity (3.3) in the proof of (3.2), we derive (3.3) first. Differentiating $\langle L_z\rangle(t)$ with respect to t, and integrating by parts, we have

$$\begin{aligned} \frac{\mathrm{d}\langle L_z\rangle(t)}{\mathrm{d}t} &= i\int_{\mathbb{R}^3}[\bar{u}_t(x_2\partial_{x_1}u - x_1\partial_{x_2}u) + \bar{u}(x_2\partial_{x_1}u_t - x_1\partial_{x_2}u_t)]\mathrm{d}x \\ &= \int_{\mathbb{R}^3}[\overline{-iu_t}(x_2\partial_{x_1}u - x_1\partial_{x_2}u) - iu_t(x_2\partial_{x_1}\bar{u} - x_1\partial_{x_2}\bar{u})]\mathrm{d}x \\ &= \int_{\mathbb{R}^3}\left[\left(\frac{1}{2}\Delta\bar{u} - \frac{\omega^2}{2}|x|^2\bar{u} - \beta|u|^2\bar{u} + \omega L_z\bar{u}\right)(x_2\partial_{x_1}u - x_1\partial_{x_2}u)\right. \\ &\quad \left.+\left(\frac{1}{2}\Delta u - \frac{\omega^2}{2}|x|^2 u - \beta|u|^2 u + \omega L_z u\right)(x_2\partial_{x_1}\bar{u} - x_1\partial_{x_2}\bar{u})\right]\mathrm{d}x \end{aligned}$$

$$= \int_{\mathbb{R}^3} \mathrm{Re}(\Delta u(x_2\partial_{x_1}\bar{u} - x_1\partial_{x_2}\bar{u})) - \frac{\omega^2}{2}(|x|^2(x_2\partial_{x_1}|u|^2 - x_1\partial_{x_2}|u|^2))$$

$$- \beta\,\mathrm{Re}(|u|^2(x_2\partial_{x_1}|u|^2 - x_1\partial_{x_2}|u|^2))\mathrm{d}x$$

$$= \frac{\omega^2}{2}\int_{\mathbb{R}^3} (2x_1x_2|u|^2 - 2x_2x_1|u|^2))\mathrm{d}x = 0,$$

which yields the desired identity (3.3).

Next, we prove the energy conservation for the nonrotating part (3.2). We consider

$$\mathrm{Re}(eq(u),u_t) = 0,$$

where $(\cdot,\cdot)$ denotes the L^2-inner product. From the above, we can get

$$\int_{\mathbb{R}^3}\left[\frac{1}{2}\partial_t|\nabla u|^2 + \frac{\omega^2}{2}\partial_t|xu|^2 + \frac{\beta}{2}\partial_t|u|^4 + \frac{\omega}{2}\partial_t(\bar{u}L_z u)\right]\mathrm{d}x = 0,$$

which implies the identity (3.2) with the help of (3.3).

Finally, the pseudoconformal conservation law (3.4) can be easily derived from the definitions (2.14) and (2.15) of the operators $J(t)$ and $H(t)$ with the help of the energy conservation for the nonrotating part (3.2). We omit the details. □

IV. NONLINEAR ESTIMATES AND THE PROOF OF THEOREM 1.1

With the help of (2.19), we can get

$$J(t)|u|^2u = 2|u|^2J(t)u - u^2\overline{J(t)u},$$

which implies, in view of $\frac{1}{\rho'}+\varepsilon=\frac{2}{q}+\frac{1}{\rho}$ with $\max\{0,\frac{2}{\rho}-\frac{2}{3}\}<\varepsilon<\frac{1}{\rho}-\frac{1}{6}$, $\rho\in(2,6)$, and some $q\in$, that

$$\|J(t)|u|^2u\|_{L^{(\rho/(1-\rho\varepsilon))'}} \leqslant C\|u\|_{L^q}^2\|J(t)u\|_{L^\rho}.$$

From the Sobolev embedding theorem and the Hölder inequality, it yields

$$\|J(t)|u|^2u\|_{L^{\gamma(\rho/(1-\rho\varepsilon))'}(0,T;L^{(\rho/(1-\rho\varepsilon))'})} \leqslant CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}\|u\|_{L^\infty(0,T;H^1)}^2\|J(t)u\|_{L^{\gamma(\rho)}(0,T;L^\rho)}.$$

Similarly, we have

$$\|H(t)|u|^2u\|_{L^{\gamma(\rho/(1-\rho\varepsilon))'}(0,T;L^{(\rho(1-\rho\varepsilon))'})} \leqslant CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}\|u\|_{L^\infty(0,T;H^1)}^2\|H(t)u\|_{L^{\gamma(\rho)}(0,T;L^\rho)},$$

and

$$\||u|^2u\|_{L^{\gamma(\rho/(1-\rho\varepsilon))'}(0,T;L^{(\rho/(1-\rho\varepsilon))'})} \leqslant CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}\|u\|_{L^\infty(0,T;H^1)}^2\|u\|_{L^{\gamma(\rho)}(0,T;L^\rho)}.$$

For convenience, we denote

$$|||u|||_G := \|u\|_G + \|J(t)u\|_G + \|H(t)u\|_G,$$

where G denotes a normalized space. Thus, we have

$$|||\,|u|^2u\,|||_{L^{\gamma(\rho/(1-\rho\varepsilon))'}(0,T;L^{(\rho/(1-\rho\varepsilon))'})} \leqslant CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}\|u\|_{L^\infty(0,T;H^1)}^2|||u|||_{L^{\gamma(\rho)}(0,T;L^\rho)}. \tag{4.1}$$

For any $\rho\in(2,6)$ and $M\geqslant 2C\|u_0\|_\Sigma$, define the workspace $(\mathcal{D},d)$ as

$$\mathcal{D} := \{u: |||u|||_{L^\infty(0,T;L^2)\cap L^{\gamma(\rho)}(0,T;L^\rho)} \leqslant M\},$$

with the distance

$$d(u,v)=|||u-v|||_{L^{\gamma(\rho)}(0,T;L^\rho)}.$$

It is clear that $(\mathcal{D},d)$ is a Banach space. Let us consider the mapping $\mathcal{T}:(\mathcal{D},d)\to(\mathcal{D},d)$ defined by

$$\mathcal{T}:u(t)\mapsto S(t)u_0-i\beta\int_0^t S(t)S^*(s)|u|^2u(s)\mathrm{d}s.$$

For $u\in(\mathcal{D},d)$, by the commutation relations (2.16)–(2.18), Proposition 2.2, and the nonlinear estimate (4.1), we obtain

$$|||\mathcal{T}u|||_{L^{\gamma(\rho)}(0,T;L^\rho)}\leqslant C\|u_0\|_\Sigma+CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}\|u\|^2_{L^\infty(0,T;H^1)}|||u|||_{L^{\gamma(\rho)}(0,T;L^\rho)}+\|O_J(t)u\|_{L^{\gamma(\rho)}(0,T;L^\rho)}+\|O_H(t)u\|_{L^{\gamma(\rho)}(0,T;L^\rho)} \tag{4.2}$$

$$\leqslant M/2+(CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}M^2+2CT)M\leqslant M, \tag{4.3}$$

where we have taken $T\in(0,\pi/4\omega]$ so small that $CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}M^2+2CT\leqslant 1/2$. Similar to the above, a straightforward computation shows that it holds

$$\begin{aligned}d(\mathcal{T}u,\mathcal{T}v)&\leqslant[CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}(\|u\|^2_{L^\infty(0,T;H^1)}+\|v\|^2_{L^\infty(0,T;H^1)})+2CT]|||u-v|||_{L^{\gamma(\rho)}(0,T;L^\rho)}\\&\leqslant[CT^{1-(3/2)\varepsilon-2/\gamma(\rho)}M^2+2CT]d(u,v)\leqslant\tfrac{1}{2}d(u,v).\end{aligned} \tag{4.4}$$

Hence, $\mathcal{T}$ is a contracted mapping from the Banach space $(\mathcal{D},d)$ to itself. By the Banach contraction mapping principle, we know that there exists a unique solution $u\in(\mathcal{D},d)$ to (1.5) and (1.6). In view of the conservation laws stated in Proposition 3.1, we can use the standard argument to extend it uniquely to a solution at the interval $[0,\pi/4\omega]$ which satisfies, for any $t\in[0,\pi/4\omega]$ and $\rho\in(2,6)$, that

$$u(t,x),J(t)u(t,x),H(t)u(t,x)\in\mathcal{C}(0,\pi/4\omega;L^2(\mathbb{R}^3))\cap L^{\gamma(\rho)}(0,\pi/4\omega;L^\rho(\mathbb{R}^3)).$$

Then, we can extend the above solution to a global one by translation. In fact, in order to get the solution in the interval $(\pi/4\omega,\pi/2\omega]$, we can apply a translation transformation with respect to the time variable t such that the initial data $u(\pi/4\omega)$ are replaced by $\tilde{u}(0)$. Let $\tilde{u}(t,x):=u(t-\pi/4\omega,x)$, then we have from the original equation with initial data $u(\pi/4\omega,x)$

$$i\tilde{u}_t+\frac{1}{2}\Delta\tilde{u}=\frac{\omega^2}{2}|x|^2\tilde{u}+\beta|\tilde{u}|^2\tilde{u}-\omega L_z\tilde{u},\quad x\in\mathbb{R}^3,\quad t\geqslant 0, \tag{4.5}$$

$$\tilde{u}(0,x)=\tilde{u}_0(x):=u(\pi/4\omega,x),\quad x\in\mathbb{R}^3. \tag{4.6}$$

In the same way, we can get a solution $\tilde{u}(t,x)$ of (4.5) and (4.6) for $t\in[0,\pi/4\omega]$. It is also a solution $u(t,x)$ to (1.5) and (1.6) for $t\in[\pi/4\omega,\pi/2\omega]$ and it is unique. Thus, by an induction argument with the help of those conserved identities stated in Proposition 3.1, we can obtain a global solution $u(t,x)$ to (1.5) and (1.6) satisfying for any $T\in(0,\infty)$

$$u(t,x),J(t)u(t,x),H(t)u(t,x)\in\mathcal{C}(\mathbb{R};L^2(\mathbb{R}^3))\cap L^{\gamma(\rho)}(0,T;L^\rho(\mathbb{R}^3)).$$

Therefore, we have completed the proof of the main theorem.

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