Global well posedness for the Gross–Pitaevskii equation with an angular momentum rotational term

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SUMMARY

In this paper, we establish the global well posedness of the Cauchy problem for the Gross–Pitaevskii equation with a rotational angular momentum term in the space $\mathbb{R}^2$. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: Gross–Pitaevskii equation; angular momentum rotation; harmonic trap potential; global well posedness

1. INTRODUCTION

The Gross–Pitaevskii equation (GPE), derived independently by Gross [1] and Pitaevskii [2], arises in various models of nonlinear physical phenomena. This is a Schrödinger-type equation with an external field potential $V_{\text{ext}}(t, x)$ and a local cubic nonlinearity:

$$i\hbar\partial_t u + \frac{\hbar^2}{2m}\Delta u = V_{\text{ext}}u + \beta|u|^2u \tag{1}$$

The GPE (1) in physical dimensions (two and three dimensions) is used in the meanfield quantum theory of Bose–Einstein condensate (BEC) formed by ultracold bosonic coherent atomic ensembles. Recently, several research groups [3–6] have produced quantized vortices in trapped BECs, and

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a typical method they used is to impose a laser beam on the magnetic trap to create a harmonic anisotropic rotating trapping potential. The properties of BEC in a rotational frame at temperature T being much smaller than the critical condensation temperature T_c [7] are well described by the macroscopic wave function $u(t, x)$, whose evolution is governed by a self-consistent, mean field nonlinear Schrödinger equation (NLS) in a rotational frame, also known as the GPE with an angular momentum rotation term:

$$i\hbar\partial_t u + \frac{\hbar^2}{2m}\Delta u = V(x)u + NU_0|u|^2 u - \Omega L_z u, \quad x \in \mathbb{R}^3, \ t \geqslant 0 \tag{2}$$

where the wave function $u(t, x)$ corresponds to a condensate state, m is the atomic mass, $\hbar$ is the Planck constant, N is the number of atoms in the condensate, Ω is the angular velocity of the rotating laser beam, and $V(x)$ is an external trapping potential. When a harmonic trap potential is considered, $V(x) = (m/2)(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2)$ with ω_1, ω_2 and ω_3 being the trap frequencies in the x_1-, x_2- and x_3-direction, respectively. The local nonlinearity term $NU_0|u|^2 u$ arises from an assumption about the delta-shape interatomic potential. $U_0 = 4\pi\hbar^2 a_s/m$ describes the interaction between atoms in the condensate with a_s (positive for repulsive interaction and negative for attractive interaction) the s-wave scattering length, and $L_z = -i\hbar(x_1\partial_{x_2} - x_2\partial_{x_1})$ is the third component of the angular momentum $L = x \times P$ with the momentum operator $P = -i\hbar\nabla$.

After normalization, proper nondimensionalization and dimension reduction in certain limiting trapping frequency regime [8], it turns to be the dimensionless GPE in d-dimensions ($d = 2, 3$):

$$iu_t + \tfrac{1}{2}\Delta u = V_d(x)u + \beta_d|u|^2 u - \Omega L_z u, \quad x \in \mathbb{R}^d, \ t > 0 \tag{3}$$

where $L_z = i(x_1\partial_{x_2} - x_2\partial_{x_1})$ and

$$\beta_d = \begin{cases} \beta\sqrt{\gamma_z/2\pi}, \\ \beta \end{cases}, \quad V_d(x) = \begin{cases} (\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2)/2, & d = 2 \\ (\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2 + \gamma_3^2)/2, & d = 3 \end{cases} \tag{4}$$

with $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 > 0$ constants, $\beta = 4\pi a_s N/a_0$, $a_0 = \sqrt{\hbar/m\omega_m}$ and $\omega_m = \min\{\omega_1, \omega_2, \omega_3\}$.

In general, it is a rather complicated process about the dynamics of solutions (in particular, vortex) for GPE (2) under the interaction of trapping frequencies and angular rotating motion. The recent numerical simulation of GPE (2) for different choice of trap frequencies (γ_1, γ_2) can help us to understand the complicated dynamical phenomena caused by the angular rotating and spatial high-frequency motion. The case of different frequency $\gamma_1 \neq \gamma_2$ gives much complicated behavior and thus is rather difficult to be studied rigorously [9, 10]. To our knowledge, Equation (2) has been only investigated for some specific cases by numerical simulation. Therefore, to develop methods for constructing analytical solutions of the GPE (1) or some specific cases is the first step in order to understand the dynamics caused by the trapping and rotation.

To begin with, we first consider the case $\gamma_1 = \gamma_2 = \omega$ that means the spatial isotropic motion and focus on the Cauchy problem of the GPE with an angular momentum rotational term in two dimensions:

$$iu_t + \tfrac{1}{2}\Delta u = \frac{\omega^2}{2}|x|^2 u + \beta|u|^{2\sigma}u - \omega L_z u, \quad x \in \mathbb{R}^2, \ t \geqslant 0 \tag{5}$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^2 \tag{6}$$

where the wave function $u = u(t, x) : [0, \infty) \times \mathbb{R}^2 \to \mathbb{C}$ corresponds to a condensate state, Δ is the Laplace operator on $\mathbb{R}^2$, $\omega > 0$, $\beta > 0$ and $\sigma \in [1/\omega, \infty)$ are constants, and $L_z = -\mathrm{i}(x_1 \partial_{x_2} - x_2 \partial_{x_1}) = \mathrm{i}(x_2 \partial_{x_1} - x_1 \partial_{x_2})$ is the dimensionless angular momentum rotational term [9, 10]. We assume that the initial value

$$u_0(x) \in \Sigma := \{u \in H^1(\mathbb{R}^2) : |x|u \in L^2(\mathbb{R}^2)\} \tag{7}$$

with the norm

$$\|u\|_\Sigma = \|u\|_{H^1} + \||x|u\|_{L^2}$$

It is clear that (5) is a special case of (3) with a spatial isotropic trapping frequency (i.e. $\gamma_1 = \gamma_2$) and $\Omega = \omega$ in two-dimensions. Note that for multi-dimensional GPE (2), nothing is known about the exact integration except for the case when $\gamma_1 = \gamma_2$ ($= \gamma_3$ for three dimensions) without the angular momentum rotational term, namely, $\Omega = 0$ considered in [11, 12].

There are three ingredients that play important roles in the proof of our result. The first involves the solution of the Cauchy problem to the linear equation:

$$\mathrm{i}u_t + \frac{1}{2}\Delta u = \frac{\omega^2}{2}|x|^2 u - \omega L_z u, \quad x \in \mathbb{R}^2, \ t \geqslant 0$$
$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^2 \tag{8}$$

which is significant for investigating the properties of the evolution operator corresponding to the linear operator:

$$\mathrm{i}\partial_t + \frac{1}{2}\Delta - \frac{\omega^2}{2}|x|^2 + \omega L_z$$

The second one is to obtain the Strichartz estimates for the foregoing linear operator. The last one is that there exist two Galilean operators $J(t)$ and $H(t)$ (as below) that can commute with the linear operator and can be viewed as the substitute of ∇ and x, respectively, in the nonpotential case.

Now we state the main result of this paper.

Theorem 1.1
Let $u_0 \in \Sigma$ and $\rho \in [2, \infty)$. Then, there exists a unique solution $u(t, x)$ to the Cauchy problem (5)–(6). And the solution satisfies, for any $T \in (0, \infty)$, that

$$u(t, x), J(t)u(t, x), H(t)u(t, x) \in \mathscr{C}(\mathbb{R}; L^2(\mathbb{R}^2)) \cap L^{\gamma(\rho)}(0, T; L^\rho(\mathbb{R}^2)) \quad \forall t \in [0, T]$$

where $1/\gamma(\rho) = \frac{1}{2} - 1/\rho$, $J(t)$ and $H(t)$ are defined as below as in (15) and (16), respectively.

Remark 1.2
Since the GPE (5) (or (3)) in a rotational frame is time reversible and time transverse invariant, the above result is also valid for the case when $t < 0$.

The paper is organized as follows. In Section 2, the evolution operator of the linear equation and the Strichartz estimates about the former operator are first established. Section 3 is devoted to the derivation of some conservation identities, such as the mass, the energy, the angular momentum expectation, the pseudo-conformal conservation laws in the whole space $\mathbb{R}^2$ for (5)–(6). Finally, the nonlinear estimates and the proof of Theorem 1.1 are obtained in Section 4.

2. THE STRICHARTZ ESTIMATES AND SOME MAIN OPERATORS

Let $u(t)$ be the solution of the linear equation (8), then by a computation, it can be expressed as

$$u(t) = S(t)u_0 = \frac{\omega}{2\pi i \sin(\omega t)} \int_{\mathbb{R}^2} e^{i\omega((|x-y|^2/2)\cot(\omega t) - x^\perp \cdot y)} u_0(y)\, dy \tag{9}$$

where $x^\perp := (-x_2, x_1)$ and $S(t)$ is the evolution operator that can be formally written as $S(t) := e^{i(t/2)(\nabla - i\omega x^\perp)^2}$. This formula, which can be deduced from the three-dimensional NSE with a magnetic field discussed in [13, Section 9.1, 14], defines a operator $S(t)$, unitary on L^2. Note that this formula is valid only for small time, due to the singularity formation for the fundamental solution. For this NSE, Strichartz estimates are available. These estimates, mixed time–space estimates, are exactly the same as for $S_0(t) = e^{(i/2)t\Delta}$. Recall the main properties from which such estimates stem. The operator $S(t)$ is unitary on L^2, $\|S(t)\|_{L^2 \to L^2} = 1$. In fact, by the Plancherel theorem, we have for any $\phi \in L^2$

$$
\begin{aligned}
\|S(t)\phi\|_2 &= \frac{\omega}{2\pi |\sin \omega t|} \left\| \int_{\mathbb{R}^2} e^{i\omega((|x-y|^2/2)\cot(\omega t) - x^\perp \cdot y)} \phi(y)\, dy \right\|_2 \\
&= \frac{\omega}{2\pi |\sin \omega t|} \left\| e^{i\omega |x|^2 \cot(\omega t)/2} \int_{\mathbb{R}^2} e^{i\omega(x\cot(\omega t) - x^\perp)\cdot y} e^{i\omega(|y|^2/2)\cot(\omega t)} \phi(y)\, dy \right\|_2 \\
&= \frac{\omega}{|\sin \omega t|} \left\| [\mathcal{F}^{-1}(e^{i\omega(|y|^2/2)\cot(\omega t)} \phi(y))](\omega(x\cot(\omega t) - x^\perp)) \right\|_2 \\
&= \|\mathcal{F}^{-1}(e^{i\omega(|y|^2/2)\cot(\omega t)} \phi(y))\|_2 = \|e^{i\omega(|y|^2/2)\cot(\omega t)} \phi(y)\|_2 \\
&= \|\phi\|_2
\end{aligned}
$$

And for $0 < t \leqslant \pi/2\omega$, the operator is dispersive, with $\|S(t)\|_{L^1 \to L^\infty} \leqslant (1/4)|t|^{-1}$, since $|\sin t| \geqslant (2/\pi)|t|$ for $|t| \leqslant \pi/2$. Thus, we can obtain similar Strichartz estimates to the linear Schrödinger operator $e^{i(t/2)\Delta}$ by the standard methods (cf. [15]) provided that only finite time intervals are involved (cf. [12]).

Proposition 2.1
Let I be an interval contained in $[0, \pi/2\omega]$. Then, it holds that

(1) For any admissible pair $(\gamma(p), p)$ (that is, $1/\gamma(p) = 1/2 - 1/p$ for $2 \leqslant p < \infty$), there exists C_p such that for any $\phi \in L^2$

$$\|S(t)\phi\|_{L^{\gamma(p)}(I; L^p)} \leqslant C_p \|\phi\|_{L^2} \tag{10}$$

(2) For any admissible pairs $(\gamma(p_1), p_1)$ and $(\gamma(p_2), p_2)$, there exists C_{p_1, p_2} such that

$$\left\| \int_{I \cap \{s \leqslant t\}} S(t-s) F(s)\, ds \right\|_{L^{\gamma(p_1)}(I; L^{p_1})} \leqslant C_{p_1, p_2} \|F\|_{L^{\gamma(p_2)'}(I; L^{p_2'})} \tag{11}$$

The above constants are independent of $I \subset [0, \pi/2\omega]$.

The integral equation reads

$$u(t) = S(t)u_0 - \mathrm{i}\beta \int_0^t S(t-s)|u|^{2\sigma}u(s)\,\mathrm{d}s \qquad (12)$$

Since the initial data belong to the space Σ, we naturally need the estimates of $\nabla S(t)\phi$ and $xS(t)\phi$. In fact, from (9), we can compute and obtain that

$$\nabla S(t)\phi = \mathrm{i}\omega x\cot(\omega t)S(t)\phi - \mathrm{i}\omega\cot(\omega t)S(t)(x\phi) + \mathrm{i}\omega S(t)(x^{\perp}\phi)$$

and

$$S(t)\nabla^{\perp}\phi = \mathrm{i}\omega x^{\perp}\cot(\omega t)S(t)\phi - \mathrm{i}\omega\cot(\omega t)S(t)(x^{\perp}\phi) - \mathrm{i}\omega x S(t)\phi$$

$$S(t)\nabla\phi = \mathrm{i}\omega x\cot(\omega t)S(t)\phi - \mathrm{i}\omega\cot(\omega t)S(t)(x\phi) + \mathrm{i}\omega x^{\perp}S(t)\phi$$

which yield to

$$\nabla S(t)\phi = \cos(\omega t)S(t)(\cos(\omega t)\nabla - \sin(\omega t)\nabla^{\perp})\phi$$

$$- \mathrm{i}\omega\sin(\omega t)S(t)[(\cos(\omega t)x - \sin(\omega t)x^{\perp})\phi] \qquad (13)$$

$$xS(t)\phi = \cos(\omega t)S(t)[(\cos(\omega t)x - \sin(\omega t)x^{\perp})\phi]$$

$$- \frac{\mathrm{i}}{\omega}\sin(\omega t)S(t)(\cos(\omega t)\nabla - \sin(\omega t)\nabla^{\perp})\phi \qquad (14)$$

Thus, we have

$$S(t)(-\mathrm{i}\nabla)\phi = [\omega\sin(\omega t)(\cos(\omega t)x + \sin(\omega t)x^{\perp}) - \mathrm{i}\cos(\omega t)(\cos(\omega t)\nabla + \sin(\omega t)\nabla^{\perp})]S(t)\phi$$

and

$$S(t)\omega x\phi = [\omega\cos(\omega t)(\cos(\omega t)x + \sin(\omega t)x^{\perp}) + \mathrm{i}\sin(\omega t)(\cos(\omega t)\nabla + \sin(\omega t)\nabla^{\perp})]S(t)\phi$$

For convenience of computations, we denote

$$J(t) = \omega\sin(\omega t)(\cos(\omega t)x + \sin(\omega t)x^{\perp}) - \mathrm{i}\cos(\omega t)(\cos(\omega t)\nabla + \sin(\omega t)\nabla^{\perp}) \qquad (15)$$

and the corresponding 'orthogonal' operator

$$H(t) = \omega\cos(\omega t)(\cos(\omega t)x + \sin(\omega t)x^{\perp}) + \mathrm{i}\sin(\omega t)(\cos(\omega t)\nabla + \sin(\omega t)\nabla^{\perp}) \qquad (16)$$

which will appear in the pseudo-conformal conservation law and play a crucial role in the nonlinear estimates.

Thus, we obtain

$$J(t) = S(t)(-\mathrm{i}\nabla)S(-t), \quad H(t) = S(t)\omega x S(-t)$$

By computation, we can obtain the following commutation relation

$$\left[J(t), \mathrm{i}\partial_t + \frac{1}{2}\Delta - \frac{\omega^2}{2}|x|^2 + \omega L_z \right] = 0$$

$$\left[H(t), \mathrm{i}\partial_t + \frac{1}{2}\Delta - \frac{\omega^2}{2}|x|^2 + \omega L_z \right] = 0$$

(17)

In addition, denote $M(t) = \mathrm{e}^{-\mathrm{i}\omega(|x|^2/2)\tan(\omega t)}$ and $Q(t) = \mathrm{e}^{\mathrm{i}\omega(|x|^2/2)\cot(\omega t)}$, then

$$J(t) = -\mathrm{i}\cos(\omega t)M(t)(\cos(\omega t)\nabla + \sin(\omega t)\nabla^\perp)M(-t)$$

$$H(t) = \mathrm{i}\sin(\omega t)Q(t)(\cos(\omega t)\nabla + \sin(\omega t)\nabla^\perp)Q(-t)$$

(18)

3. THE CONSERVED QUANTITIES

Proposition 3.1
Let u be a solution of Equation (5) with the initial data $\phi \in \Sigma(\mathbb{R}^2)$. Then, we have the following conserved quantities for all $t \geqslant 0$:

(1) The L^2-norm:

$$\|u(t)\|_2 = \|u_0\|_2 \tag{19}$$

(2) The energy for the nonrotating part:

$$E_0(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{\omega^2}{2}\|xu\|_2^2 + \frac{\beta}{\sigma+1}\|u\|_{2\sigma+2}^{2\sigma+2} = E_0(u_0) \tag{20}$$

(3) The angular momentum expectation:

$$\langle L_z \rangle(t) = \int_{\mathbb{R}^2} \bar{u} L_z u \, \mathrm{d}x = \langle L_z \rangle(0) \tag{21}$$

Proof
For convenience, we introduce

$$\mathrm{eq}(u) := \mathrm{i}u_t + \frac{1}{2}\Delta u - \frac{\omega^2}{2}|x|^2 u - \beta|u|^{2\sigma}u + \omega L_z u$$

It is clear that (19) holds by applying the L^2-inner product between $\mathrm{eq}(u)$ and $\bar{u}$, and then taking the imaginary part of the resulting equation.

Since we can use identity (21) in the proof of (20), we derive (21) first. Differentiating $\langle L_z \rangle(t)$ with respect to t, and integrating by parts, we have

$$\frac{\mathrm{d}\langle L_z \rangle(t)}{\mathrm{d}t} = \mathrm{i}\int_{\mathbb{R}^2}[\bar{u}_t(x_2\partial_{x_1}u - x_1\partial_{x_2}u) + \bar{u}(x_2\partial_{x_1}u_t - x_1\partial_{x_2}u_t)]\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^2}[\overline{-\mathrm{i}u_t}(x_2\partial_{x_1}u - x_1\partial_{x_2}u) - \mathrm{i}u_t(x_2\partial_{x_1}\bar{u} - x_1\partial_{x_2}\bar{u})]\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^2} \left[\left(\frac{1}{2} \Delta \bar{u} - \frac{\omega^2}{2} |x|^2 \bar{u} - \beta |u|^{2\sigma} \bar{u} + \omega L_z \bar{u} \right) (x_2 \partial_{x_1} u - x_1 \partial_{x_2} u) \right.$$

$$\left. + \left(\frac{1}{2} \Delta u - \frac{\omega^2}{2} |x|^2 u - \beta |u|^{2\sigma} u + \omega L_z u \right) (x_2 \partial_{x_1} \bar{u} - x_1 \partial_{x_2} \bar{u}) \right] dx$$

$$= \int_{\mathbb{R}^2} \operatorname{Re}(\Delta u (x_2 \partial_{x_1} \bar{u} - x_1 \partial_{x_2} \bar{u})) - \frac{\omega^2}{2} (|x|^2 (x_2 \partial_{x_1} |u|^2 - x_1 \partial_{x_2} |u|^2))$$

$$- \beta \operatorname{Re}(|u|^{2\sigma} (x_2 \partial_{x_1} |u|^2 - x_1 \partial_{x_2} |u|^2)) \, dx$$

$$= \frac{\omega^2}{2} \int_{\mathbb{R}^2} (2x_1 x_2 |u|^2 - 2x_2 x_1 |u|^2)) \, dx$$

$$= 0$$

which yields the desired identity (21).

Next, we prove the energy conservation law for the nonrotating part (20). We consider

$$\operatorname{Re}(eq(u), u_t) = 0$$

where $(\cdot, \cdot)$ denotes the L^2-inner product. From the above, we can get

$$\int_{\mathbb{R}^2} \left[\frac{1}{2} \partial_t |\nabla u|^2 + \frac{\omega^2}{2} \partial_t |xu|^2 + \frac{\beta}{\sigma + 1} \partial_t |u|^{2\sigma+2} + \frac{\omega}{2} \partial_t (\bar{u} L_z u) \right] dx = 0$$

which implies the identity (20) with the help of (21). $\qquad \square$

Remark 3.2
For Equation (5), the pseudo-conformal type conservation laws are also valid:

$$\|H(t)u\|_2^2 + \frac{2\beta \sin^2(\omega t)}{\sigma + 1} \|u\|_{2\sigma+2}^{2\sigma+2} + \frac{2\beta[\sigma\omega - 1]}{\sigma + 1} \int_0^t \sin 2\omega s \|u(s)\|_{2\sigma+2}^{2\sigma+2} \, ds = \omega^2 \|xu_0\|_2^2$$

and

$$\|J(t)u\|_2^2 + \frac{2\beta \cos^2(\omega t)}{\sigma + 1} \|u\|_{2\sigma+2}^{2\sigma+2} = \|\nabla u_0\|_2^2 + \frac{2\beta}{\sigma + 1} \|u_0\|_{2\sigma+2}^{2\sigma+2}$$

$$+ \frac{2\beta[\sigma\omega - 1]}{\sigma + 1} \int_0^t \sin 2\omega s \|u(s)\|_{2\sigma+2}^{2\sigma+2} \, ds$$

4. NONLINEAR ESTIMATES AND THE PROOF OF THEOREM 1.1

By computation, we can get, with the help of (18),

$$J(t)|u|^{2\sigma} u = (\sigma + 1)|u|^{2\sigma} J(t)u - \sigma |u|^{2\sigma-2} u^2 \overline{J(t)u}$$

which implies, in view of $1/\rho' + \varepsilon = 2\sigma/q + 1/\rho$ with $0 < \varepsilon < 1/\rho$, that

$$\| J(t)|u|^{2\sigma}u \|_{L^{(\rho/(1-\rho\varepsilon))'}} \leqslant C\|u\|_{L^q}^{2\sigma} \|J(t)u\|_{L^\rho}$$

From the Sobolev embedding theorem and the Hölder inequality, it yields

$$\| J(t)|u|^{2\sigma}u \|_{L^{\gamma(\rho/(1-\rho\varepsilon))'}(0,T;L^{(\rho/(1-\rho\varepsilon))'})}$$

$$\leqslant CT^{1-\varepsilon-2/\gamma(\rho)} \|u\|_{L^\infty(0,T;H^1)}^{2\sigma} \|J(t)u\|_{L^{\gamma(\rho)}(0,T;L^\rho)}$$

Similarly, we have

$$\| H(t)|u|^{2\sigma}u \|_{L^{\gamma(\rho/(1-\rho\varepsilon))'}(0,T;L^{(\rho/(1-\rho\varepsilon))'})}$$

$$\leqslant CT^{1-\varepsilon-2/\gamma(\rho)} \|u\|_{L^\infty(0,T;H^1)}^{2\sigma} \|H(t)u\|_{L^{\gamma(\rho)}(0,T;L^\rho)}$$

and

$$\| |u|^{2\sigma}u \|_{L^{\gamma(\rho/(1-\rho\varepsilon))'}(0,T;L^{(\rho/(1-\rho\varepsilon))'})} \leqslant CT^{1-\varepsilon-2/\gamma(\rho)} \|u\|_{L^\infty(0,T;H^1)}^{2\sigma} \|u\|_{L^{\gamma(\rho)}(0,T;L^\rho)}$$

For convenience, we denote

$$\|\|u\|\|_A := \|u\|_A + \|J(t)u\|_A + \|H(t)u\|_A$$

where A denotes a normalized space. Thus, we have

$$\|\| |u|^{2\sigma}u \|\|_{L^{\gamma(\rho/(1-\rho\varepsilon))'}(0,T;L^{(\rho/(1-\rho\varepsilon))'})}$$
$$\leqslant CT^{1-\varepsilon-2/\gamma(\rho)} \|u\|_{L^\infty(0,T;H^1)}^{2\sigma} \|\|u\|\|_{L^{\gamma(\rho)}(0,T;L^\rho)} \qquad (22)$$

For any $\rho \in [2, \infty)$ and $M \geqslant 2C\|u_0\|_\Sigma$, define the workspace $(\mathscr{D}, d)$ as

$$\mathscr{D} := \{u : \|\|u\|\|_{L^\infty(0,T;L^2) \cap L^{\gamma(\rho)}(0,T;L^\rho)} \leqslant M\}$$

with the distance

$$d(u, v) = \|\|u - v\|\|_{L^{\gamma(\rho)}(0,T;L^\rho)}$$

It is clear that $(\mathscr{D}, d)$ is a Banach space. Let us consider the mapping $\mathscr{T} : (\mathscr{D}, d) \to (\mathscr{D}, d)$ defined by

$$\mathscr{T} : u(t) \mapsto S(t)u_0 - \mathrm{i}\beta \int_0^t S(t-s)|u|^{2\sigma}u(s)\,\mathrm{d}s$$

For $u \in (\mathscr{D}, d)$, by the commutation relation (17), Proposition 2.1 and the nonlinear estimate (22), we obtain

$$\|\mathscr{T}u\|_{L^{\gamma(\rho)}(0,T;L^{\rho})} \leqslant C\|u_0\|_{\Sigma} + CT^{1-\varepsilon-2/\gamma(\rho)}\|u\|_{L^{\infty}(0,T;H^1)}^{2\sigma}\|u\|_{L^{\gamma(\rho)}(0,T;L^{\rho})}$$

$$\leqslant M/2 + CT^{1-\varepsilon-2/\gamma(\rho)}M^{2\sigma}M$$

$$\leqslant M \tag{23}$$

where we have taken $T \in (0, \pi/2\omega]$ so small that $CT^{1-\varepsilon-2/\gamma(\rho)}M^{2\sigma} \leqslant 1/2$. Similar to the above, a straightforward computation shows that it holds

$$d(\mathscr{T}u, \mathscr{T}v) \leqslant CT^{1-\varepsilon-2/\gamma(\rho)}(\|u\|_{L^{\infty}(0,T;H^1)}^{2\sigma} + \|v\|_{L^{\infty}(0,T;H^1)}^{2\sigma})\|u - v\|_{L^{\gamma(\rho)}(0,T;L^{\rho})}$$

$$\leqslant CT^{1-\varepsilon-2/\gamma(\rho)}M^{2\sigma}d(u,v)$$

$$\leqslant \frac{1}{2}d(u,v) \tag{24}$$

Hence, $\mathscr{T}$ is a contracted mapping from the Banach space $(\mathscr{D}, d)$ to itself. By the Banach contraction mapping principle, we know that there exists a unique solution $u \in (\mathscr{D}, d)$ to (5)–(6). In view of the conservation laws, we can use the standard argument to extend it uniquely to a solution at the interval $[0, \pi/2\omega]$ that satisfies for any $t \in [0, \pi/2\omega]$ and $\rho \in [2, \infty)$

$$u(t, x), J(t)u(t, x), H(t)u(t, x) \in \mathscr{C}(0, \pi/2\omega; L^2(\mathbb{R}^2)) \cap L^{\gamma(\rho)}(0, \pi/2\omega; L^{\rho}(\mathbb{R}^2))$$

Then, we can extend the above solution to a global one by translation. In fact, in order to get the solution in the interval $(\pi/2\omega, \pi/\omega]$, we can apply a translation transformation with respect to the time variable t such that the initial data $u(\pi/2\omega)$ are replaced by $\tilde{u}(0)$. Let $\tilde{u}(t, x) := u(t - \pi/2\omega, x)$, then we have from the original equation with initial data $u(\pi/2\omega, x)$

$$i\tilde{u}_t + \frac{1}{2}\Delta\tilde{u} = \frac{\omega^2}{2}|x|^2\tilde{u} + \beta|\tilde{u}|^{2\sigma}\tilde{u} - \omega L_z\tilde{u}, \quad x \in \mathbb{R}^2, \quad t \geqslant 0 \tag{25}$$

$$\tilde{u}(0, x) = \tilde{u}_0(x) := u(\pi/2\omega, x), \quad x \in \mathbb{R}^2 \tag{26}$$

In the same way, we can get a solution $\tilde{u}(t, x)$ of (25)–(26) for $t \in [0, \pi/2\omega]$. It is also a solution $u(t, x)$ of (5)–(6) for $t \in [\pi/2\omega, \pi/\omega]$ and it is unique. Thus, by an induction argument with the help of those conserved identities stated in Proposition 3.1, we can obtain a global solution $u(t, x)$ of (5)–(6) satisfying for any $T \in (0, \infty)$

$$u(t, x), J(t)u(t, x), H(t)u(t, x) \in \mathscr{C}(\mathbb{R}; L^2(\mathbb{R}^2)) \cap L^{\gamma(\rho)}(0, T; L^{\rho}(\mathbb{R}^2))$$

Therefore, we have completed the proof of the main theorem. $\qquad\square$

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