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CAUCHY PROBLEM FOR VISCOUS SHALLOW WATER EQUATIONS WITH SURFACE TENSION

Chengchun Hao

Institute of Mathematics, Academy of Mathematics & Systems Science, CAS
and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences
Beijing 100190, China

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Abstract. We are concerned with the Cauchy problem for a viscous shallow water system with a third-order surface-tension term. The global existence and uniqueness of the solution in the space of Besov type is shown for the initial data close to a constant equilibrium state away from the vacuum by using the Friedrich's regularization and compactness arguments.

1. Introduction.

In the present paper, we consider the Cauchy problem for viscous shallow water equations with a third-order surface-tension term:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \nu\nabla\cdot(\rho\nabla\mathbf{u}) + \rho\nabla\rho = \rho\nabla\Delta\rho, \\ \rho(0) = \rho_0, \quad \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1}$$

where $\rho(t,x)$ is the height of the fluid surface, $\mathbf{u}(t,x) = (u^1(t,x), u^2(t,x))^\top$ is the horizontal velocity field, $x = (x_1, x_2) \in \mathbb{R}^2$ and $0 < \nu < 1$ is the viscous coefficient.

The nonlinear shallow water equation is used to model the motion of a shallow layer of homogeneous incompressible fluid in a three dimensional rotating subdomain and, in particular, to simulate the vertical average dynamics of the fluid in terms of the horizontal velocity and depth variation. The related systems with a third-order term have been considered by many people. For examples, F. Marche recently derived a complicated shallow water model involving a third-order surface tension term by considering second order approximation and parabolic correction in [9]. R. Danchin and B. Desjardins studied a compressible fluid model of Korteweg type in [6]:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \nabla P(\rho) = \kappa\rho\nabla\Delta\rho, \end{cases}$$

where the third-order term $\rho\nabla\Delta\rho$ stems from the capillary tensor.

For the shallow water system without a third-order surface tension term, there is a mount of work to deal with small initial data. The local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for the shallow water equations by using Lagrangian coordinates and Hölder space estimates with initial data in

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$C^{2+\alpha}$ was studied in [1]. Kloeden [8] and Sundbye [14] proved the global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem using Sobolev space estimates by following the energy method of Matsumura and Nishida [11, 12, 10]. Sundbye [15] proved also the existence and uniqueness of classical solutions to the Cauchy problem by using the method in [11, 12, 10]. Wang and Xu, in [16], obtained local solutions for general initial data and global solutions for small initial data $\rho_0 - \bar{\rho}_0$, $\mathbf{u}_0 \in H^{2+s}(\mathbb{R}^2)$ with $s > 0$. The result was improved by Haspot to get the global existence in time for small initial data $\rho_0 - \bar{\rho}_0 \in \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}$ and $\mathbf{u}_0 \in \dot{B}^0_{2,1}$ as a special case in [7].

In the present paper, we first separate the velocity field into a compressible part and a incompressible part by the standard div-curl decomposition of velocities. Unlike [6] where a linearized equation without convection terms was considered, we investigate a linearized system with a convection term to get a uniform estimate for the compressible part in a usual time-spatial space instead of hybrid Besov spaces or Besov-Chemin-Lerner spaces used in [5, 6, 7]. It is a heat equation for the incompressible part and so we can obtain a uniform estimate by the properties of the heat equation. We use a classical Friedrich's regularization to build approximate solutions and prove the existence of a solution by compactness arguments. For the uniqueness of solutions, due to the contribution of the third-order surface tension term, we can prove it in the same space as for the existence. For the initial data ρ_0, we suppose that it is a small perturbation of some positive constant $\bar{\rho}_0$. The main theorem of this paper reads as follows.

Theorem 1.1. *There exist two positive constants ε_0 and M such that if $\rho_0 - \bar{\rho}_0 \in \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}$, $\mathbf{u}_0 \in \dot{B}^0_{2,1}$ and*

$$\|\rho_0 - \bar{\rho}_0\|_{\dot{B}^0_{2,1}\cap\dot{B}^1_{2,1}} + \|\mathbf{u}_0\|_{\dot{B}^0_{2,1}} \leqslant \varepsilon_0,$$

then (1) has a unique global solution $(\rho, \mathbf{u})$ in E^1 which satisfies:

$$\|(\rho,\mathbf{u})\|_{E^1} \leqslant M(\|\rho_0 - \bar{\rho}_0\|_{\dot{B}^0_{2,1}\cap\dot{B}^1_{2,1}} + \|\mathbf{u}_0\|_{\dot{B}^0_{2,1}}),$$

for some M independent of the initial data where $\dot{B}^0_{2,1}$ and $\dot{B}^1_{2,1}$ are homogeneous Besov spaces (defined in next section), and

$$\begin{aligned}\|(\rho,\mathbf{u})\|_{E^1} =&\|\rho\|_{L^\infty(\dot{B}^0_{2,1}\cap\dot{B}^1_{2,1})} + \|\mathbf{u}\|_{L^\infty(\dot{B}^0_{2,1})}\\ &+ \|\rho\|_{L^1(\dot{B}^2_{2,1}\cap\dot{B}^3_{2,1})} + \|\mathbf{u}\|_{L^1(\dot{B}^2_{2,1})}.\end{aligned}$$

The paper is organized as follows. We recall the Littlewood-Paley theory for homogeneous Besov spaces in the second section. In section 3, we are dedicated into proving uniform *a priori* estimates. In section 4, we prove the global existence and uniqueness of solution for small initial data by using a classical iteration and compactness method.

2. **Littlewood-Paley theory and Besov spaces.** Let $\psi : \mathbb{R}^2 \to [0,1]$ be a radial smooth cut-off function valued in $[0,1]$ such that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leqslant 3/4,\\ \text{smooth}, & 3/4 < |\xi| < 4/3,\\ 0, & |\xi| \geqslant 4/3.\end{cases}$$

Let $\varphi(\xi)$ be the function

$$\varphi(\xi) := \psi(\xi/2) - \psi(\xi).$$

Thus, ψ is supported in the ball $\left\{\xi \in \mathbb{R}^2 : |\xi| \leqslant 4/3\right\}$, and φ is also a smooth cut-off function valued in $[0,1]$ and supported in the annulus $\{\xi : 3/4 \leqslant |\xi| \leqslant 8/3\}$. By construction, we have

$$\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1, \quad \forall \xi \neq 0.$$

One can define the dyadic blocks as follows. For $k \in \mathbb{Z}$, let

$$\triangle_k f := \mathscr{F}^{-1}\varphi(2^{-k}\xi)\mathscr{F} f.$$

The formal decomposition

$$f = \sum_{k\in\mathbb{Z}} \triangle_k f \tag{2}$$

is called homogeneous Littlewood-Paley decomposition. Actually, this decomposition works for just about any locally integrable function which has some decay at infinity, and one usually has all the convergence properties of the summation that one needs. Thus, the r.h.s. of (2) does not necessarily converge in $\mathscr{S}'(\mathbb{R}^2)$. Even if it does, the equality is not always true in $\mathscr{S}'(\mathbb{R}^2)$. For instance, if $f \equiv 1$, then all the projections $\triangle_k f$ vanish. Nevertheless, (2) is true modulo polynomials, in other words (cf.[4, 13]), if $f \in \mathscr{S}'(\mathbb{R}^2)$, then $\sum_{k\in\mathbb{Z}} \triangle_k f$ converges modulo $\mathscr{P}[\mathbb{R}^2]$ and (2) holds in $\mathscr{S}'(\mathbb{R}^2)/\mathscr{P}[\mathbb{R}^2]$.

Definition 2.1. Let $s \in \mathbb{R}$, $1 \leqslant p,\, q \leqslant \infty$. For $f \in \mathscr{S}'(\mathbb{R}^N)$, we write

$$\|f\|_{\dot{B}^s_{p,q}} = \Big(\sum_{k\in\mathbb{Z}} \big(2^{ks}\|\triangle_k f\|_{L^p}\big)^q\Big)^{\frac{1}{q}}.$$

A difficulty comes from the choice of homogeneous spaces at this point. Indeed, $\|\cdot\|_{\dot{B}^s_{p,q}}$ cannot be a norm on $\{f \in \mathscr{S}'(\mathbb{R}^2) : \|f\|_{\dot{B}^s_{p,q}} < \infty\}$ because $\|f\|_{\dot{B}^s_{p,q}} = 0$ means that f is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces (cf. [5]).

Definition 2.2. Let $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$. Denote $m = \left[s - \frac{2}{p}\right]$ if $s - \frac{2}{p} \notin \mathbb{Z}$ or $q > 1$ and $m = s - \frac{2}{p} - 1$ otherwise. If $m < 0$, then we define $\dot{B}^s_{p,q}(\mathbb{R}^2)$ as

$$\dot{B}^s_{p,q} = \Big\{f \in \mathscr{S}'(\mathbb{R}^2) : \|f\|_{\dot{B}^s_{p,q}} < \infty \text{ and } u = \sum_{k\in\mathbb{Z}} \triangle_k f \text{ in } \mathscr{S}'(\mathbb{R}^2)\Big\}.$$

If $m \geqslant 0$, we denote by $\mathscr{P}_m[\mathbb{R}^2]$ the set of polynomials of degree less than or equal to m and define

$$\begin{aligned}\dot{B}^s_{p,q} = \Big\{f \in \mathscr{S}'(\mathbb{R}^2)/\mathscr{P}_m[\mathbb{R}^2] : &\|f\|_{\dot{B}^s_{p,q}} < \infty \text{ and} \\ &u = \sum_{k\in\mathbb{Z}} \triangle_k f \text{ in } \mathscr{S}'(\mathbb{R}^2)/\mathscr{P}_m[\mathbb{R}^2]\Big\}.\end{aligned}$$

Let us now recall some useful estimates for homogeneous Besov spaces.

Lemma 2.3 ([5, Proposition 1.5])**.** *Let $s > 0$ and $f, g \in L^\infty \cap \dot{B}^s_{p,q}$. Then $fg \in L^\infty \cap \dot{B}^s_{p,q}$ and*

$$\|fg\|_{\dot{B}^s_{p,q}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,q}} + \|f\|_{\dot{B}^s_{p,q}}\|g\|_{L^\infty}.$$

Let $s_1, s_2 \leqslant 1$ such that $s_1+s_2>0$, $f \in \dot{B}^{s_1}_{2,1}$ and $g \in \dot{B}^{s_2}_{2,1}$. Then $fg \in \dot{B}^{s_1+s_2-1}_{2,1}$ and

$$\|fg\|_{\dot{B}^{s_1+s_2-1}_{2,1}} \lesssim \|f\|_{\dot{B}^{s_1}_{2,1}} \|g\|_{\dot{B}^{s_2}_{2,1}}.$$

Lemma 2.4 ([5, Lemma 1.6])**.** *Let $s>0$ and $u \in \dot{B}^s_{2,1} \cap L^\infty$.*

i) *Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^2)$ such that $F(0)=0$. Then $F(u) \in \dot{B}^s_{2,1}$. Moreover, there exists a function of one variable C_0 depending only on s and F, and such that*

$$\|F(u)\|_{\dot{B}^s_{2,1}} \leqslant C_0(\|u\|_{L^\infty})\|u\|_{\dot{B}^s_{2,1}}.$$

ii) *If $u, v \in \dot{B}^1_{2,1}$, $(v-u) \in \dot{B}^s_{2,1}$ for a $s \in (-1,1]$ and $G \in W^{4,\infty}_{loc}(\mathbb{R}^2)$ satisfies $G'(0)=0$, then $G(v)-G(u) \in \dot{B}^s_{2,1}$ and there exists a function of two variables C depending only on s and G, and such that*

$$\|G(v)-G(u)\|_{\dot{B}^s_{2,1}} \leqslant C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) \left(\|u\|_{\dot{B}^1_{2,1}} + \|v\|_{\dot{B}^1_{2,1}}\right) \|v-u\|_{\dot{B}^s_{2,1}}.$$

Now, we define the following work space as follows.

Definition 2.5. For $T>0$ and $s \in \mathbb{R}$, we denote

$$\begin{aligned} E^s_T =& \Big\{(\rho,\mathbf{u}) \in \mathcal{C}\left([0,T]; \dot{B}^{s-1}_{2,1} \cap \dot{B}^s_{2,1}\right) \cap L^1\left(0,T; \dot{B}^{s+1}_{2,1} \cap \dot{B}^{s+2}_{2,1}\right) \\ & \times \left(\mathcal{C}\left([0,T]; \dot{B}^{s-1}_{2,1}\right) \cap L^1\left(0,T; \dot{B}^{s+1}_{2,1}\right)\right)^2 \Big\} \end{aligned}$$

and

$$\begin{aligned} \|(\rho,\mathbf{u})\|_{E^s_T} =& \|\rho\|_{L^\infty_T(\dot{B}^{s-1}_{2,1} \cap \dot{B}^s_{2,1})} + \|\mathbf{u}\|_{L^\infty_T(\dot{B}^{s-1}_{2,1})} \\ & + \|\rho\|_{L^1_T(\dot{B}^{s+1}_{2,1} \cap \dot{B}^{s+2}_{2,1})} + \|\mathbf{u}\|_{L^1_T(\dot{B}^{s+1}_{2,1})}. \end{aligned}$$

We use the notation E^s if $T=+\infty$, changing $[0,T]$ into $[0,+\infty)$ in the definition above.

3. *A priori* estimates.

For convenience, we take $\bar{\rho}_0 = 1$. Substituting ρ by $\rho+1$ in (1), we have

$$\begin{cases} \rho_t + \mathbf{u}\cdot\nabla\rho + \operatorname{div}\mathbf{u} = -\rho \operatorname{div}\mathbf{u}, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\rho - \nabla\Delta\rho = \nu\dfrac{\nabla\rho\cdot\nabla\mathbf{u}}{1+\rho}, \\ \rho(0) = \rho_0 - 1, \quad \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{3}$$

For all $s \in \mathbb{R}$, we denote $\Lambda^s f = \mathscr{F}^{-1}(|\xi|^s \hat{f})$. Let $c = \Lambda^{-1}\operatorname{div}\mathbf{u}$ and $\mathbf{I} = \Lambda^{-1}\operatorname{curl}\mathbf{u}$ where c represents the compressible part of the velocity and $\mathbf{I}$ the incompressible part, and $\operatorname{curl}\mathbf{g} = \nabla\mathbf{g} - (\nabla\mathbf{g})^\top$ for the vector function $\mathbf{g}$. Then, we have

$$\mathbf{u} = -\Lambda^{-1}\nabla c - \Lambda^{-1}\operatorname{div}\mathbf{I},$$

since $\operatorname{div}\operatorname{div}\mathbf{I} = 0$. In fact,

$$\operatorname{div}\mathbf{I} = \Lambda^{-1}\operatorname{div}\operatorname{curl}\mathbf{u} = \Lambda^{-1}\begin{pmatrix} \partial_2^2 u^1 - \partial_2\partial_1 u^2 \\ \partial_1^2 u^2 - \partial_1\partial_2 u^1 \end{pmatrix} = \Lambda^{-1}(\Delta\mathbf{u} - \nabla\operatorname{div}\mathbf{u}),$$

which yields

$$\operatorname{div}\operatorname{div}\mathbf{I} = \Lambda^{-1}\operatorname{div}(\Delta\mathbf{u} - \nabla\operatorname{div}\mathbf{u}) = \Lambda^{-1}(\operatorname{div}\Delta\mathbf{u} - \Delta\operatorname{div}\mathbf{u}) = 0,$$

and

$$\begin{aligned}\operatorname{curl}\operatorname{div}\mathbf{I} =&\Lambda^{-1}\operatorname{curl}\begin{pmatrix}\partial_2^2 u^1-\partial_2\partial_1 u^2\\ \partial_1^2 u^2-\partial_1\partial_2 u^1\end{pmatrix}=\Lambda^{-1}\begin{pmatrix}0 & \Delta(\partial_2 u^1-\partial_1 u^2)\\ \Delta(\partial_1 u^2-\partial_2 u^1) & 0\end{pmatrix}\\ =&\Lambda^{-1}\Delta\operatorname{curl}\mathbf{u}=\Delta\mathbf{I}.\end{aligned}$$

Moreover, for any function f, we have

$$\operatorname{curl}\nabla f=\nabla\nabla f-(\nabla\nabla f)^\top=\mathbf{0}$$

because $\nabla\nabla f$ is a symmetric matrix.

We rewrite now the system (3) in terms of these notations as the following:

$$\begin{cases}\rho_t+\mathbf{u}\cdot\nabla\rho+\Lambda c=F,\\ c_t+\mathbf{u}\cdot\nabla c-\nu\Delta c-\Lambda\rho-\Lambda^3\rho=G,\\ \mathbf{I}_t-\nu\Delta\mathbf{I}=\Lambda^{-1}\operatorname{curl}H,\\ \mathbf{u}=-\Lambda^{-1}\nabla c-\Lambda^{-1}\operatorname{div}\mathbf{I},\\ \rho(0)=\rho_0-1,\quad \mathbf{u}(0)=\mathbf{u}_0,\end{cases}\tag{4}$$

where

$$F=-\rho\operatorname{div}\mathbf{u},\quad G=\mathbf{u}\cdot\nabla c+\Lambda^{-1}\operatorname{div}H,\quad H=-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\frac{\nabla\rho\cdot\nabla\mathbf{u}}{1+\rho}.$$

For the third equation of (4), we can use the estimates for the heat equation in [5].

Proposition 1. *Let $s\in\mathbb{R}$, $r\in[1,+\infty]$, and u solve*

$$\begin{cases}u_t-\nu\Delta u=f,\\ u(0)=u_0.\end{cases}$$

Then there exists $C>0$ depending only on ν and r such that, for all $0<T\leqslant+\infty$,

$$\|u\|_{L^r_T(\dot B^{s+2/r}_{2,1})}\leqslant C\left(\|u_0\|_{\dot B^s_{2,1}}+\|f\|_{L^1_T(\dot B^s_{2,1})}\right).$$

Moreover, $u\in\mathcal{C}([0,T];\dot B^s_{2,1})$.

For the first two equations, we study the following system:

$$\begin{cases}\rho_t+\mathbf{v}\cdot\nabla\rho+\Lambda c=F,\\ c_t+\mathbf{v}\cdot\nabla c-\nu\Delta c-\Lambda\rho-\Lambda^3\rho=G,\end{cases}\tag{5}$$

where $\mathbf{v}$ is a vector function and we will precise its regularity in the next proposition. System (5) has been studied in the case where $\mathbf{v}=0$ by Danchin and Desjardins in [6]. Here, we take into account the convection terms.

Proposition 2. *Let (ρ,c) be a solution of (5) on $[0,T)$, $0<s\leqslant 2$ and $V(t)=\int_0^t\|\mathbf{v}(\tau)\|_{\dot B^2_{2,1}}d\tau$. The following estimate holds on $[0,T)$:*

$$\begin{aligned}&\|\rho(t)\|_{\dot B^{s-1}_{2,1}\cap\dot B^s_{2,1}}+\|c(t)\|_{\dot B^{s-1}_{2,1}}+\nu\int_0^t(\|\rho(\tau)\|_{\dot B^{s+1}_{2,1}\cap\dot B^{s+2}_{2,1}}+\|c(\tau)\|_{\dot B^{s+1}_{2,1}})d\tau\\ \leqslant& Ce^{CV(t)}\Big(\|\rho(0)\|_{\dot B^{s-1}_{2,1}\cap\dot B^s_{2,1}}+\|c(0)\|_{\dot B^{s-1}_{2,1}}\\ &\qquad+\int_0^t e^{-CV(\tau)}\left(\|F(\tau)\|_{\dot B^{s-1}_{2,1}\cap\dot B^s_{2,1}}+\|G(\tau)\|_{\dot B^{s-1}_{2,1}}\right)d\tau\Big),\end{aligned}$$

where C depends only on s.

Proof. Let (ρ, c) be a solution of (5) and we set

$$(\tilde{\rho}, \tilde{c}, \tilde{F}, \tilde{G}) = e^{-KV(t)}(\rho, c, F, G).$$

Thus, (5) can be transformed into

$$\begin{cases} \tilde{\rho}_t + \mathbf{v} \cdot \nabla\tilde{\rho} + \Lambda\tilde{c} = \tilde{F} - KV'(t)\tilde{\rho}, \\ \tilde{c}_t + \mathbf{v} \cdot \nabla\tilde{c} - \nu\Delta\tilde{c} - \Lambda\tilde{\rho} - \Lambda^3\tilde{\rho} = \tilde{G} - KV'(t)\tilde{c}. \end{cases} \tag{6}$$

Applying the operator $\triangle_k$ to the system (6), and denoting $\tilde{\rho}_k = \triangle_k\tilde{\rho}$ and $\tilde{c}_k = \triangle_k\tilde{c}$, we obtain the following system:

$$\begin{cases} \partial_t\tilde{\rho}_k + \triangle_k(\mathbf{v} \cdot \nabla\tilde{\rho}) + \Lambda\tilde{c}_k = \tilde{F}_k - KV'(t)\tilde{\rho}_k, \\ \partial_t\tilde{c}_k + \triangle_k(\mathbf{v} \cdot \nabla\tilde{c}) - \nu\Delta\tilde{c}_k - \Lambda\tilde{\rho}_k - \Lambda^3\tilde{\rho}_k = \tilde{G}_k - KV'(t)\tilde{c}_k. \end{cases} \tag{7}$$

To begin with, we consider the case where $\mathbf{v} = \mathbf{0}$, $K = 0$ and $F = G = 0$ which implies that (7) takes the form

$$\begin{cases} \partial_t\tilde{\rho}_k + \Lambda\tilde{c}_k = 0, \\ \partial_t\tilde{c}_k - \nu\Delta\tilde{c}_k - \Lambda\tilde{\rho}_k - \Lambda^3\tilde{\rho}_k = 0. \end{cases} \tag{8}$$

Taking the L^2 scalar product of the first equation of (8) with $\tilde{\rho}_k$, and of the second equation with $\tilde{c}_k$, we get the following two identities:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|\tilde{\rho}_k\|_{L^2}^2 + (\Lambda\tilde{c}_k, \tilde{\rho}_k) = 0, \\ \dfrac{1}{2}\dfrac{d}{dt}\|\tilde{c}_k\|_{L^2}^2 + \nu\|\Lambda\tilde{c}_k\|_{L^2}^2 - (\Lambda\tilde{\rho}_k, \tilde{c}_k) - (\Lambda^2\tilde{\rho}_k, \Lambda\tilde{c}_k) = 0. \end{cases} \tag{9}$$

Now we want to get an equality involving $\Lambda\tilde{\rho}_k$. To achieve it, we apply Λ to the first equation of (8) and take L^2 scalar product with $\Lambda\tilde{\rho}_k$ and $\tilde{c}_k$ respectively, then take the L^2 scalar product of the second equation with $\Lambda\tilde{\rho}_k$ and sum both equalities, which yields

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|\Lambda\tilde{\rho}_k\|_{L^2}^2 + (\Lambda^2\tilde{c}_k, \Lambda\tilde{\rho}_k) = 0, \\ \dfrac{d}{dt}(\Lambda\tilde{\rho}_k, \tilde{c}_k) + \|\Lambda\tilde{c}_k\|_{L^2}^2 - \|\Lambda\tilde{\rho}_k\|_{L^2}^2 - \|\Lambda^2\tilde{\rho}_k\|_{L^2}^2 + \nu(\Lambda\tilde{c}_k, \Lambda^2\tilde{\rho}_k) = 0. \end{cases} \tag{10}$$

Let $K_1 > 0$ be a constant to be chosen later and denote

$$f_k^2 = \|\tilde{\rho}_k\|_{L^2}^2 + (1 + \nu K_1)\|\Lambda\tilde{\rho}_k\|_{L^2}^2 + \|\tilde{c}_k\|_{L^2}^2 - 2K_1(\Lambda\tilde{\rho}_k, \tilde{c}_k).$$

By a linear combination of (9) and (10), we can get

$$\frac{1}{2}\frac{d}{dt}f_k^2 + (\nu - K_1)\|\Lambda\tilde{c}_k\|_{L^2}^2 + K_1\|\Lambda\tilde{\rho}_k\|_{L^2}^2 + K_1\|\Lambda^2\tilde{\rho}_k\|_{L^2}^2 = 0. \tag{11}$$

Using Schwartz' inequality and Young's inequality, we find for any $M_1 > 0$

$$|(\Lambda\tilde{\rho}_k, \tilde{c}_k)| \leqslant \frac{M_1}{2}\|\Lambda\tilde{\rho}_k\|_{L^2}^2 + \frac{1}{2M_1}\|\tilde{c}_k\|_{L^2}^2.$$

Thus, we need to determine the values of K_1 and M_1 such that

$$1 + \nu K_1 - K_1M_1 > 0, \quad K_1 < M_1.$$

One can verify that it holds for

$$M_1 = \nu, \; K_1 = \nu/2.$$

Hence, we obtain

$$\frac{1}{2}f_k^2 \leqslant \|\tilde{c}_k\|_{L^2}^2 + \|\tilde{\rho}_k\|_{L^2}^2 + \|\Lambda\tilde{\rho}_k\|_{L^2}^2 \leqslant 2f_k^2. \tag{12}$$

Therefore, we obtain

$$f_k^2 = \|\tilde{\rho}_k\|_{L^2}^2 + (1+\frac{\nu^2}{2})\|\Lambda\tilde{\rho}_k\|_{L^2}^2 + \|\tilde{c}_k\|_{L^2}^2 - \nu(\Lambda\tilde{\rho}_k, \tilde{c}_k),$$

and

$$\frac{1}{2}\frac{d}{dt}f_k^2 + \frac{a\nu}{4}2^{2k}f_k^2 \leqslant 0,$$

where $a > 0$ is a constant from Bernstein's inequality

$$\|\triangle_k h\|_{L^2} \leqslant \frac{1}{\sqrt{a}}2^{-k}\|\Lambda\triangle_k h\|_{L^2}.$$

In the general case where F, G, K and $\mathbf{v}$ are not zero, we have, with the help of Lemma 5.1 in [5], that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}f_k^2 + \left(\frac{a\nu}{4}2^{2k} + KV'\right)f_k^2 \\
\leqslant&(\tilde{F}_k, \tilde{\rho}_k) + (\tilde{G}_k, \tilde{c}_k) + (1+\frac{\nu^2}{2})(\Lambda\tilde{F}_k, \Lambda\tilde{\rho}_k) - \frac{\nu}{2}(\Lambda\tilde{F}_k, \tilde{c}_k) - \frac{\nu}{2}(\tilde{G}_k, \Lambda\tilde{\rho}_k) \\
&- (\triangle_k(\mathbf{v}\cdot\nabla\tilde{\rho}), \tilde{\rho}_k) - (\triangle_k(\mathbf{v}\cdot\nabla\tilde{c}), \tilde{c}_k) - (1+\frac{\nu^2}{2})(\Lambda\triangle_k(\mathbf{v}\cdot\nabla\tilde{\rho}), \Lambda\tilde{\rho}_k) \\
&+ \frac{\nu}{2}(\Lambda\triangle_k(\mathbf{v}\cdot\nabla\tilde{\rho}), \tilde{c}_k) + \frac{\nu}{2}(\triangle_k(\mathbf{v}\cdot\nabla\tilde{c}), \Lambda\tilde{\rho}_k) \\
\lesssim&f_k\Big(\|\tilde{F}_k\|_{L^2} + \|\tilde{G}_k\|_{L^2} + \|\Lambda\tilde{F}_k\|_{L^2} + \alpha_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{\rho}\|_{\dot{B}^{s-1}_{2,1}} \\
&+ \alpha_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{c}\|_{\dot{B}^{s-1}_{2,1}} + \alpha_k 2^{-k(s-1)}\|\mathbf{v}\|_{\dot{B}^2_{2,1}}\|\tilde{\rho}\|_{\dot{B}^s_{2,1}} \\
&+ \alpha_k\|\mathbf{v}\|_{\dot{B}^2_{2,1}}(2^{-k(s-1)}\|\tilde{c}\|_{\dot{B}^{s-1}_{2,1}} + 2^{-k(s-1)}\min(2^k, \nu^{-1})\|\tilde{\rho}\|_{\dot{B}^{s-1}_{2,1}\cap\dot{B}^s_{2,1}})\Big),
\end{aligned}$$

where $\sum_k \alpha_k \leqslant 1$ and $s \in (0, 2]$.

Thus, we obtain

$$\begin{aligned}
&\frac{d}{dt}f_k + \left(\frac{a\nu}{4}2^{2k} + KV'\right)f_k \\
\lesssim&\|\tilde{F}_k\|_{L^2} + \|\tilde{G}_k\|_{L^2} + \|\Lambda\tilde{F}_k\|_{L^2} + \alpha_k 2^{-k(s-1)}V'\sum_l 2^{l(s-1)}f_l,
\end{aligned} \tag{13}$$

where we choose $V(t) = \int_0^t \|\mathbf{v}\|_{\dot{B}^2_{2,1}}$. Multiplying (13) with $2^{k(s-1)}$ and sum in k, we have

$$\begin{aligned}
&\frac{d}{dt}\sum_k 2^{k(s-1)}f_k + \frac{a\nu}{4}\sum_k 2^{k(s+1)}f_k + KV'\sum_k 2^{k(s-1)}f_k \\
\leqslant&C\|\tilde{F}\|_{\dot{B}^{s-1}_{2,1}\cap\dot{B}^s_{2,1}} + C\|\tilde{G}\|_{\dot{B}^{s-1}_{2,1}} + CV'\sum_k 2^{k(s-1)}f_k.
\end{aligned}$$

Taking $K > C$, we obtain

$$\sum_k 2^{k(s-1)} f_k(t) + \frac{a\nu}{4}\int_0^t \sum_k 2^{k(s+1)} f_k(\tau)d\tau$$
$$\lesssim \sum_k 2^{k(s-1)} f_k(0) + \int_0^t \left(\|\tilde{F}(\tau)\|_{\dot{B}^{s-1}_{2,1}\cap\dot{B}^{s}_{2,1}} + \|\tilde{G}(\tau)\|_{\dot{B}^{s-1}_{2,1}} \right) d\tau.$$

Hence, in view of (12), we obtain

$$\|\tilde{\rho}(t)\|_{\dot{B}^{s-1}_{2,1}\cap\dot{B}^{s}_{2,1}} + \|\tilde{c}(t)\|_{\dot{B}^{s-1}_{2,1}} + \nu\int_0^t (\|\tilde{\rho}(\tau)\|_{\dot{B}^{s+1}_{2,1}\cap\dot{B}^{s+2}_{2,1}} + \|\tilde{c}(\tau)\|_{\dot{B}^{s+1}_{2,1}})d\tau$$
$$\lesssim \|\tilde{\rho}(0)\|_{\dot{B}^{s-1}_{2,1}\cap\dot{B}^{s}_{2,1}} + \|\tilde{c}(0)\|_{\dot{B}^{s-1}_{2,1}} + \int_0^t \left(\|\tilde{F}(\tau)\|_{\dot{B}^{s-1}_{2,1}\cap\dot{B}^{s}_{2,1}} + \|\tilde{G}(\tau)\|_{\dot{B}^{s-1}_{2,1}} \right) d\tau.$$

Therefore, we complete the proof as long as we change the functions $(\tilde{\rho}, \tilde{c}, \tilde{F}, \tilde{G})$ into the original ones (ρ, c, F, G). □

4. **Existence and uniqueness results.** This section is devoted to the proof of Theorem 1.1. The principle of the proof is a very classical one. We shall use the classical Friedrichs' regularization method, which was used in [2, 3, 7] for examples, to construct the approximate solutions $(\rho^n, \mathbf{u}^n)_{n\in\mathbb{N}}$ to (3) which are solutions of (5) coupled with a heat equation, and then we will use Proposition 2 to get some uniform bounds on $(\rho^n, \mathbf{u}^n)_{n\in\mathbb{N}}$.

4.1. **Building of the sequence** $(\rho^n, \mathbf{u}^n)_{n\in\mathbb{N}}$**.** let us define the sequence of operators $(J_n)_{n\in\mathbb{N}}$ by

$$J_n f := \mathscr{F}^{-1}\mathbf{1}_{B(\frac{1}{n},n)}(\xi)\mathscr{F}f,$$

and consider the following approximate system:

$$\begin{cases} \rho^n_t + J_n(J_n\mathbf{u}^n\cdot\nabla J_n\rho^n) + \Lambda J_n c^n = F^n, \\ c^n_t + J_n(J_n\mathbf{u}^n\cdot\nabla J_n c^n) - \nu\Delta J_n c^n - \Lambda J_n\rho^n - \Lambda^3 J_n\rho^n = G^n, \\ \mathbf{I}^n_t - \nu\Delta J_n\mathbf{I}^n = J_n\Lambda^{-1}\,\mathrm{curl}H^n, \\ \mathbf{u}^n = -\Lambda^{-1}\nabla c^n - \Lambda^{-1}\,\mathrm{div}\mathbf{I}^n, \\ (\rho^n, c^n, \mathbf{I}^n)(0) = (\rho_n, \Lambda^{-1}\,\mathrm{div}\mathbf{u}_n, \Lambda^{-1}\,\mathrm{curl}\mathbf{u}_n), \end{cases} \tag{14}$$

where

$$\begin{aligned} &\rho_n = J_n(\rho_0 - 1), \quad \mathbf{u}_n = J_n\mathbf{u}_0, \\ &F^n = -J_n(J_n\rho^n\,\mathrm{div}J_n\mathbf{u}^n), \\ &G^n = J_n(J_n\mathbf{u}^n\cdot\nabla J_n c^n) + J_n\Lambda^{-1}\,\mathrm{div}H^n, \\ &H^n = -J_n(J_n\mathbf{u}^n\cdot\nabla J_n\mathbf{u}^n) + \nu J_n\frac{\nabla J_n\rho^n\cdot\nabla J_n\mathbf{u}^n}{\zeta(1+J_n\rho^n)}, \end{aligned}$$

with ζ a smooth function satisfying

$$\zeta(s) = \begin{cases} 1/4, & |s| \leqslant 1/4, \\ s, & 1/2 \leqslant |s| \leqslant 3/2, \\ 7/4, & |s| \geqslant 7/4. \\ \text{smooth}, & \text{otherwise}. \end{cases}$$

We want to show that (14) is only an ordinary differential equation in $L^2\times L^2\times L^2$. We can observe easily that all the source term in (14) turn out to be continuous in $L^2\times L^2\times L^2$. For example, we consider the term $J_n\Lambda^{-1}\operatorname{div}\frac{\nabla J_n\rho^n\cdot\nabla J_n\mathbf{u}^n}{\zeta(J_n\rho^n+1)}$. By Plancherel's theorem, Hausdorff-Young's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\|J_n\Lambda^{-1}\operatorname{div}\frac{\nabla J_n\rho^n\cdot\nabla J_n\mathbf{u}^n}{\zeta(J_n\rho^n+1)}\|_{L^2}=\|\mathbf{1}_{B(\frac{1}{n},n)}|\xi|^{-1}(\xi_1,\xi_2)\cdot\mathscr{F}\frac{\nabla J_n\rho^n\cdot\nabla J_n\mathbf{u}^n}{\zeta(J_n\rho^n+1)}\|_{L^2}\\
\leqslant&\|\frac{\nabla J_n\rho^n\cdot\nabla J_n\mathbf{u}^n}{\zeta(J_n\rho^n+1)}\|_{L^2}\leqslant\|\nabla J_n\rho^n\cdot\nabla J_n\mathbf{u}^n\|_{L^2}\|\frac{1}{\zeta(J_n\rho^n+1)}\|_{L^\infty}\\
\leqslant&4\|\nabla J_n\rho^n\|_{L^\infty}\|\nabla J_n\mathbf{u}^n\|_{L^2}\leqslant 4n\||\xi|\mathbf{1}_{B(\frac{1}{n},n)}\mathscr{F}\rho^n\|_{L^1}\|\mathbf{u}^n\|_{L^2}\\
\leqslant&4n^3\|\rho^n\|_{L^2}\|\mathbf{u}^n\|_{L^2}.
\end{aligned}$$

Thus, the usual Cauchy-Lipschitz theorem implies the existence of a strictly positive maximal time T_n such that a unique solution exists which is continuous in time with value in $L^2\times L^2\times L^2$. However, as $J_n^2=J_n$, we claim that $J_n(\rho^n,c^n,\mathbf{I}^n)$ is also a solution, so uniqueness implies that $J_n(\rho^n,c^n,\mathbf{I}^n)=(\rho^n,c^n,\mathbf{I}^n)$. So $(\rho^n,c^n,\mathbf{I}^n)$ is also a solution of the following system:

$$\begin{cases}
\rho_t^n+J_n(\mathbf{u}^n\cdot\nabla\rho^n)+\Lambda c^n=F_1^n,\\
c_t^n+J_n(\mathbf{u}^n\cdot\nabla c^n)-\nu\Delta c^n-\Lambda\rho^n-\Lambda^3\rho^n=G_1^n,\\
\mathbf{I}_t^n-\nu\Delta\mathbf{I}^n=J_n\Lambda^{-1}\operatorname{curl}H_1^n,\\
\mathbf{u}^n=-\Lambda^{-1}\nabla c^n-\Lambda^{-1}\operatorname{div}\mathbf{I}^n,\\
(\rho^n,c^n,\mathbf{I}^n)(0)=(\rho_n,\Lambda^{-1}\operatorname{div}\mathbf{u}_n,\Lambda^{-1}\operatorname{curl}\mathbf{u}_n),
\end{cases}\tag{15}$$

with

$$\begin{aligned}
F_1^n&=-J_n(\rho^n\operatorname{div}\mathbf{u}^n),\\
G_1^n&=J_n(\mathbf{u}^n\cdot\nabla c^n)+J_n\Lambda^{-1}\operatorname{div}H_1^n,\\
H_1^n&=-J_n(\mathbf{u}^n\cdot\nabla\mathbf{u}^n)+\nu J_n\frac{\nabla\rho^n\cdot\nabla\mathbf{u}^n}{\zeta(1+\rho^n)}.
\end{aligned}$$

The system (15) appears to be an ordinary differential equation in the space

$$L_n^2:=\left\{a\in L^2(\mathbb{R}^2):\,\operatorname{supp}\mathscr{F}a\subset B(\frac{1}{n},n)\right\}.$$

Due to the Cauchy-Lipschitz theorem again, a unique maximal solution exists on an interval $[0,T_n^*)$ which is continuous in time with value in $L_n^2\times L_n^2\times L_n^2$.

4.2. **Uniform estimates.** In this part, we prove uniform estimates independent of $T<T_n^*$ in E_T^1 for $(\rho^n,\mathbf{u}^n)$. We shall show that $T_n^*=+\infty$ by the Cauchy-Lipschitz theorem. Denote

$$\begin{aligned}
E(0):=&\|\rho_0-1\|_{\dot B^0_{2,1}\cap\dot B^1_{2,1}}+\|\mathbf{u}_0\|_{\dot B^0_{2,1}},\\
E(\rho,\mathbf{u},t):=&\|(\rho,\mathbf{u})\|_{E_t^1},\\
\tilde T_n:=&\sup\left\{t\in[0,T_n^*):E(\rho^n,\mathbf{u}^n,t)\leqslant A\tilde CE(0)\right\},
\end{aligned}$$

where $\tilde C$ corresponds to the constant in Proposition 2 and $A>\max(2,\tilde C^{-1})$ is a constant. Thus, by the continuity we have $\tilde T_n>0$.

We are going to prove that $\tilde T_n=T_n^*$ for all $n\in\mathbb{N}$ and we will conclude that $T_n^*=+\infty$ for any $n\in\mathbb{N}$.

According to Proposition 2 and Proposition 1, and to the definition of $(\rho_n, \mathbf{u}_n)$, the following inequality holds

$$\begin{aligned}\|(\rho^n, \mathbf{u}^n)\|_{E^1_T} \leqslant & \tilde{C} e^{\tilde{C}\|\mathbf{u}^n\|_{L^1_T(\dot{B}^2_{2,1})}} \Big(\|\rho_0 - 1\|_{\dot{B}^0_{2,1}\cap \dot{B}^1_{2,1}} + \|\mathbf{u}_0\|_{\dot{B}^0_{2,1}} \\ & + \|F^n_1\|_{L^1_T(\dot{B}^0_{2,1}\cap \dot{B}^1_{2,1})} + \|\mathbf{u}^n \cdot \nabla c^n\|_{L^1_T(\dot{B}^0_{2,1})} + \|H^n_1\|_{L^1_T(\dot{B}^0_{2,1})}\Big).\end{aligned}$$

Therefore, it is only a matter of proving appropriate estimates for F^n_1, H^n_1 and $\mathbf{u}^n \cdot \nabla c^n$. The estimate of F^n is straightforward. From Lemma 2.3, we have

$$\|F^n_1\|_{L^1(\dot{B}^0_{2,1}\cap \dot{B}^1_{2,1})} \leqslant C\|\rho^n\|_{L^\infty(\dot{B}^0_{2,1}\cap \dot{B}^1_{2,1})}\|\mathbf{u}^n\|_{L^1(\dot{B}^2_{2,1})} \leqslant CE^2(\rho^n, \mathbf{u}^n, T). \tag{16}$$

With the help of Lemma 2.3 and interpolation arguments, we have

$$\begin{aligned}\|\mathbf{u}^n \cdot \nabla c^n\|_{L^1(\dot{B}^0_{2,1})} \leqslant & C\|\mathbf{u}^n\|_{L^2(\dot{B}^1_{2,1})}\|\nabla c^n\|_{L^2(\dot{B}^0_{2,1})} \leqslant C\|\mathbf{u}^n\|^2_{L^2(\dot{B}^1_{2,1})} \\ \leqslant & C\|\mathbf{u}^n\|_{L^\infty(\dot{B}^0_{2,1})}\|\mathbf{u}^n\|_{L^1(\dot{B}^2_{2,1})} \\ \leqslant & CE^2(\rho^n, \mathbf{u}^n, T).\end{aligned} \tag{17}$$

In the same way, we can get

$$\|\mathbf{u}^n \cdot \nabla \mathbf{u}^n\|_{L^1(\dot{B}^0_{2,1})} \leqslant CE^2(\rho^n, \mathbf{u}^n, T). \tag{18}$$

To estimate other terms of H^n_1, we make the following assumption on $E(0)$:

$$2C_1 A\tilde{C}E(0) \leqslant 1,$$

where C_1 is the continuity modulus of $\dot{B}^1_{2,1} \subset L^\infty$. If $T < \tilde{T}_n$, it implies

$$\|\rho^n\|_{L^\infty} \leqslant C_1\|\rho^n\|_{\dot{B}^1_{2,1}} \leqslant C_1\|\rho^n\|_{\dot{B}^0_{2,1}\cap \dot{B}^1_{2,1}} \leqslant C_1 A\tilde{C}E(0) \leqslant \frac{1}{2}.$$

Thus, we have

$$\|\rho^n\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leqslant \frac{1}{2},$$

which yields

$$\rho^n + 1 \in [\frac{1}{2}, \frac{3}{2}] \text{ and } \zeta(\rho^n + 1) = \rho^n + 1.$$

From Lemma 2.3, Lemma 2.4 and interpolation arguments, we have

$$\begin{aligned}&\|\frac{\nabla\rho^n \cdot \nabla \mathbf{u}^n}{1+\rho^n}\|_{L^1(\dot{B}^0_{2,1})} \\ \leqslant & \|\nabla\rho^n \cdot \nabla\mathbf{u}^n\|_{L^1(\dot{B}^0_{2,1})} + \|\frac{\rho^n\nabla\rho^n \cdot \nabla\mathbf{u}^n}{1+\rho^n}\|_{L^1(\dot{B}^0_{2,1})} \\ \leqslant & C\|\nabla\rho^n\|_{L^2(\dot{B}^1_{2,1})}\|\nabla\mathbf{u}^n\|_{L^2(\dot{B}^0_{2,1})} + C\|\frac{\rho^n\nabla\rho^n}{1+\rho^n}\|_{L^2(\dot{B}^1_{2,1})}\|\nabla\mathbf{u}^n\|_{L^2(\dot{B}^0_{2,1})} \\ \leqslant & C\|\rho^n\|_{L^2(\dot{B}^2_{2,1})}\|\mathbf{u}^n\|_{L^2(\dot{B}^1_{2,1})}\Big(1 + \|\frac{\rho^n}{1+\rho^n}\|_{L^\infty(\dot{B}^1_{2,1})}\Big) \\ \leqslant & C\|\rho^n\|^{1/2}_{L^\infty(\dot{B}^1_{2,1})}\|\rho^n\|^{1/2}_{L^1(\dot{B}^3_{2,1})}\|\mathbf{u}^n\|^{1/2}_{L^\infty(\dot{B}^0_{2,1})}\|\mathbf{u}^n\|^{1/2}_{L^1(\dot{B}^2_{2,1})}\Big(1 + \|\rho^n\|_{L^\infty(\dot{B}^1_{2,1})}\Big) \\ \leqslant & CE^2(\rho^n, \mathbf{u}^n, T)(1 + E(\rho^n, \mathbf{u}^n, T)).\end{aligned} \tag{19}$$

Hence, from (17)-(19), we gather

$$\|\mathbf{u}^n \cdot \nabla c^n\|_{L^1_T(\dot{B}^0_{2,1})} + \|H^n_1\|_{L^1_T(\dot{B}^0_{2,1})} \leqslant C(1 + E(\rho^n, \mathbf{u}^n, T))E^2(\rho^n, \mathbf{u}^n, T), \tag{20}$$

whence

$$\|(\rho^n,\mathbf{u}^n)\|_{E^1_T} \leqslant \tilde{C}e^{A\tilde{C}^2E(0)}[1+CA^2\tilde{C}^2(1+A\tilde{C}E(0))E(0)]E(0).$$

So we can choose $E(0)$ so small that

$$\begin{aligned} &1+CA^2\tilde{C}^2(1+A\tilde{C}E(0))E(0) \leqslant \frac{A^2}{A+2},\\ &e^{A\tilde{C}^2E(0)} \leqslant \frac{A+1}{A} \ \text{ and } \ 2C_1A\tilde{C}E(0) \leqslant 1, \end{aligned} \tag{21}$$

which yields $\|(\rho^n,\mathbf{u}^n)\|_{E^1_T} \leqslant \frac{A+1}{A+2}A\tilde{C}E(0)$ for any $T<\tilde{T}_n$. It follows that $\tilde{T}_n = T^*_n$. In fact, if $\tilde{T}_n < T^*_n$, we have seen that $E(\rho^n,\mathbf{u}^n,\tilde{T}_n) \leqslant \frac{A+1}{A+2}A\tilde{C}E(0)$. So by continuity, for a sufficiently small constant $\sigma>0$ we can obtain $E(\rho^n,\mathbf{u}^n,\tilde{T}_n+\sigma) \leqslant A\tilde{C}E(0)$. This yields a contradiction with the definition of $\tilde{T}_n$.

Now, if $\tilde{T}_n = T^*_n < \infty$, then we have obtained $F(\rho^n,\mathbf{u}^n,T^*_n) \leqslant A\tilde{C}E(0)$. As $\|\rho^n\|_{L_{T^*_n}(\dot{B}^0_{2,1}\cap\dot{B}^1_{2,1})} < \infty$ and $\|\mathbf{u}^n\|_{L_{T^*_n}(\dot{B}^0_{2,1})} < \infty$, it implies that $\|\rho^n\|_{L_{T^*_n}(L^2_n)} < \infty$ and $\|\mathbf{u}^n\|_{L_{T^*_n}(L^2_n)} < \infty$. Thus, we may continue the solution beyond T^*_n by the Cauchy-Lipschitz theorem. This contradicts the definition od T^*_n. Therefore, the approximate solution $(\rho^n,\mathbf{u}^n)_{n\in\mathbb{N}}$ is global in time.

4.3. **Existence of a solution.** In this part, we shall show that, up to an extraction, the sequence $(\rho^n,\mathbf{u}^n)_{n\in\mathbb{N}}$ converges in $\mathscr{D}'(\mathbb{R}^+\times\mathbb{R}^2)$ to a solution $(\rho,\mathbf{u})$ of (3) which has the desired regularity properties. The proof lies on compactness arguments. To start with, we show that the time first derivative of $(\rho^n,\mathbf{u}^n)$ is uniformly bounded in appropriate spaces. This enables us to apply Ascoli's theorem and get the existence of a limit $(\rho,\mathbf{u})$ for a subsequence. Now, the uniform bounds of the previous part provides us with additional regularity and convergence properties so that we may pass to the limit in the system.

It is convenient to split $(\rho^n,\mathbf{u}^n)$ into the solution of a linear system with initial data $(\rho_n,\mathbf{u}_n)$, and the discrepancy to that solution. More precisely, we denote by $(\rho^n_L,\mathbf{u}^n_L)$ the solution to the linear system

$$\begin{cases} \partial_t\rho^n_L + \operatorname{div}\mathbf{u}^n_L = 0,\\ \partial_t\mathbf{u}^n_L - \nu\Delta\mathbf{u}^n_L + \nabla\rho^n_L - \nabla\Delta\rho^n_L = 0,\\ (\rho^n_L,\mathbf{u}^n_L)_{t=0} = (\rho_n,\mathbf{u}_n), \end{cases} \tag{22}$$

and $(\bar{\rho}^n,\bar{\mathbf{u}}^n) = (\rho^n-\rho^n_L,\mathbf{u}^n-\mathbf{u}^n_L)$.

Obviously, the definition of $(\rho_n,\mathbf{u}_n)$ entails

$$\rho_n \to \rho_0 - 1 \text{ in } \dot{B}^0_{2,1}\cap\dot{B}^1_{2,1}, \quad \mathbf{u}_n \to \mathbf{u}_0 \text{ in } \dot{B}^0_{2,1}.$$

The Propositions 2 and 1 insure us that

$$(\rho^n_L,\mathbf{u}^n_L) \to (\rho_L,\mathbf{u}_L) \text{ in } E^1, \tag{23}$$

where $(\rho_L,\mathbf{u}_L)$ is the solution of the linear system

$$\begin{cases} \partial_t\rho_L + \operatorname{div}\mathbf{u}_L = 0,\\ \partial_t\mathbf{u}_L - \nu\Delta\mathbf{u}_L + \nabla\rho_L - \nabla\Delta\rho_L = 0,\\ (\rho_L,\mathbf{u}_L)_{t=0} = (\rho_0-1,\mathbf{u}_0). \end{cases} \tag{24}$$

Now, we have to prove the convergence of $(\bar{\rho}^n,\bar{\mathbf{u}}^n)$. This is of course a trifle more difficult and requires compactness results. Let us first state the following lemma.

Lemma 4.1. $((\bar\rho^n, \bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ *is uniformly bounded in the space*

$$\mathcal{C}^{\frac12}_{loc}(\mathbb{R}^+; \dot B^0_{2,1}) \times (\mathcal{C}^{\frac14}_{loc}(\mathbb{R}^+; \dot B^{-\frac12}_{2,1}))^2.$$

Proof. Throughout the proof, we will note u.b. for uniformly bounded. We first prove that $\partial_t\bar\rho^n$ is u.b. in $L^2(\mathbb{R}^+, \dot B^0_{2,1})$, which yields the desired result for $\bar\rho$. Let us observe that $\bar\rho^n$ verifies the following equation

$$\partial_t\bar\rho^n = -J_n(\rho^n \operatorname{div}\mathbf{u}^n) - J_n(\mathbf{u}^n\cdot\nabla\rho^n) - \operatorname{div}\mathbf{u}^n + \operatorname{div}\mathbf{u}^n_L.$$

According to the previous part, $(\rho^n)_{n\in\mathbb{N}}$ is u.b. in $L^\infty(\dot B^1_{2,1})$ and $(\mathbf{u}^n)_{n\in\mathbb{N}}$ is u.b. in $L^2(\dot B^1_{2,1})$ in view of interpolation arguments. Thus, $-J_n(\rho^n \operatorname{div}\mathbf{u}^n) - J_n(\mathbf{u}^n\cdot\nabla\rho^n) - \operatorname{div}\mathbf{u}^n$ is u.b. in $L^2(\dot B^0_{2,1})$. The definition of $\mathbf{u}^n_L$ obviously provides us with uniform bounds for $\operatorname{div}\mathbf{u}^n_L$ in $L^2(\dot B^0_{2,1})$, so we can conclude that $\partial_t\bar\rho^n$ is u.b. in $L^2(\dot B^0_{2,1})$.

Denote $c^n_L = \Lambda^{-1}\operatorname{div}\mathbf{u}^n_L$, $\bar c^n = \Lambda^{-1}\operatorname{div}\bar{\mathbf{u}}^n$, $\mathbf{I}^n_L = \Lambda^{-1}\operatorname{curl}\mathbf{u}^n_L$ and $\bar{\mathbf{I}}^n = \Lambda^{-1}\operatorname{curl}\bar{\mathbf{u}}^n$. Let us prove now that $\partial_t\bar c^n$ is u.b. in $L^{\frac43}(\dot B^{-\frac12}_{2,1}) + L^4(\dot B^{-\frac12}_{2,1})$ and that $\partial_t\bar{\mathbf{I}}^n$ is u.b. in $L^{\frac43}(\dot B^{-\frac12}_{2,1})$ which give the required result for $\bar{\mathbf{u}}^n$ by using the relation $\mathbf{u}^n = -\Lambda^{-1}\nabla c^n - \Lambda^{-1}\operatorname{div}\mathbf{I}^n$.

Let us recall that

$$\begin{aligned}\partial_t\bar c^n =& \nu\Delta(c^n - c^n_L) + \Lambda(\rho^n-\rho^n_L) + \Lambda^3(\rho^n - \rho^n_L)\\ &- J_n\Lambda^{-1}\operatorname{div}\left(J_n(\mathbf{u}^n\cdot\nabla\mathbf{u}^n) - \nu J_n\frac{\nabla\rho^n\cdot\nabla\mathbf{u}^n}{1+\rho^n}\right),\\ \partial_t\bar{\mathbf{I}}^n =& \nu\Delta(\mathbf{I}^n - \mathbf{I}^n_L) - J_n\Lambda^{-1}\operatorname{curl}\left(J_n(\mathbf{u}^n\cdot\nabla\mathbf{u}^n) - \nu J_n\frac{\nabla\rho^n\cdot\nabla\mathbf{u}^n}{1+\rho^n}\right).\end{aligned}$$

Results of the previous part and an interpolation argument yield uniform bounds for $\mathbf{u}^n$ in $L^\infty(\dot B^0_{2,1})\cap L^{\frac43}(\dot B^{\frac32}_{2,1})$. Since ρ^n is u.b. in $L^\infty(\dot B^1_{2,1})$ and c^n_L is u.b. in $L^{\frac43}(\dot B^{\frac32}_{2,1})$, we easily verify that $\nu\Delta(c^n - c^n_L)$ and $J_n\Lambda^{-1}\operatorname{div}J_n\left(\mathbf{u}^n\cdot\nabla\mathbf{u}^n - \nu\frac{\nabla\rho^n\cdot\nabla\mathbf{u}^n}{1+\rho^n}\right)$ are u.b. in $L^{\frac43}(\dot B^{-\frac12}_{2,1})$. Because ρ^n is u.b. in $L^\infty(\dot B^1_{2,1})\cap L^1(\dot B^3_{2,1})$, we have ρ^n is u.b. in $L^{\frac43}(\dot B^{\frac52}_{2,1})$ in view of interpolation arguments. Thus, $\Lambda^3\rho^n$ is u.b. in $L^{\frac43}(\dot B^{-\frac12}_{2,1})$. We also have $\Lambda^3\rho^n_L$ u.b. in $L^{\frac43}(\dot B^{-\frac12}_{2,1})$. Using the bounds for ρ^n in $L^1(\dot B^2_{2,1})\cap L^\infty(\dot B^0_{2,1})$, we get ρ^n u.b. in $L^4(\dot B^{\frac12}_{2,1})$ and then $\Lambda\rho^n$ is u.b. in $L^4(\dot B^{-\frac12}_{2,1})$. So we finally get $\partial_t\bar c^n$ u.b. in $L^{\frac43}(\dot B^{-\frac12}_{2,1}) + L^4(\dot B^{-\frac12}_{2,1})$. The case of $\partial_t\bar{\mathbf{I}}^n$ goes along the same lines. As the terms corresponding to $\Lambda(\rho^n - \rho^n_L)$ do not appear, we simply get $\partial_t\bar{\mathbf{I}}^n$ u.b. in $L^{\frac43}(\dot B^{-\frac12}_{2,1})$. □

Now, we can turn to the proof of the existence of a solution and use Ascoli theorem to get strong convergence. We need to localize the spatial space because we have some results of compactness for the local Sobolev spaces. Let $(\chi_p)_{p\in\mathbb{N}}$ be a sequence of $\mathcal{C}^\infty_0(\mathbb{R}^2)$ cut-off functions supported in the ball $B(0,p+1)$ of $\mathbb{R}^2$ and equal to 1 in a neighborhood of $B(0,p)$.

For any $p\in\mathbb{N}$, Lemma 4.1 tells us that $((\chi_p\bar\rho^n, \chi_p\bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ is uniformly equicontinuous in $\mathcal{C}(\mathbb{R}^+; \dot B^0_{2,1}\times(\dot B^{-\frac12}_{2,1})^2)$.

Let us observe that the application $f\mapsto\chi_p f$ is compact from $\dot B^0_{2,1}\cap\dot B^1_{2,1}$ into L^2, and from $\dot B^0_{2,1}\cap\dot B^{-\frac12}_{2,1}$ into $\dot H^{-\frac12}$. This can be proved by noting that $f\mapsto\chi_p f$ is compact from $\dot H^s\cap\dot H^{s'}$ into $\dot H^s$ for $s<s'$ and that $\dot B^s_{2,1}\subset\dot H^s$. After we apply

Ascoli's theorem to the family $((\chi_p\bar\rho^n,\chi_p\bar{\mathbf{u}}^n))_{n\in\mathbb{N}}$ on the time interval $[0,p]$, we use Cantor's diagonal process. This finally provides us with a distribution $(\bar\rho,\bar{\mathbf{u}})$ belonging to $\mathcal{C}(\mathbb{R}^+;\dot H^0\times(\dot H^{-\frac12})^2)$ and a subsequence (which we still denote by $((\bar\rho^n,\bar{\mathbf{u}}^n)_{n\in\mathbb{N}})$ such that, for all $p\in\mathbb{N}$, we have

$$(\chi_p\bar\rho^n,\chi_p\bar{\mathbf{u}}^n)\to(\chi_p\bar\rho,\chi_p\bar{\mathbf{u}}) \text{ as } n\to+\infty,\ \text{ in } \mathcal{C}([0,p];\dot H^0\times(\dot H^{-\frac12})^2). \tag{25}$$

This obviously infers that $(\bar\rho^n,\bar{\mathbf{u}}^n)$ tends to $(\bar\rho,\bar{\mathbf{u}})$ in $\mathscr{D}'(\mathbb{R}^+\times\mathbb{R}^2)$.

Coming back to the uniform estimates of the previous part, we moreover get that $(\bar\rho,\bar{\mathbf{u}})$ belongs to

$$L^\infty\left(\mathbb{R}^+;(\dot B^0_{2,1}\cap\dot B^2_{2,1})\times(\dot B^0_{2,1})^2\right)\cap L^1\left(\mathbb{R}^+;(\dot B^2_{2,1}\cap\dot B^3_{2,1})\times(\dot B^2_{2,1})^2\right)$$

and to $\mathcal{C}^{1/2}(\mathbb{R}^+;\dot B^0_{2,1})\times(C^{1/4}(\mathbb{R}^+;\dot B^{-\frac12}_{2,1}))^2$.

Let us now prove that $(\rho,\mathbf{u}):=(\rho_L,\mathbf{u}_L)+(\bar\rho,\bar{\mathbf{u}})$ solves (3). We first observe that, according to (14),

$$\begin{cases}\rho^n_t+J_n(\mathbf{u}^n\cdot\nabla\rho^n)+\Lambda c^n=-J_n(\rho^n\,\mathrm{div}\mathbf{u}^n),\\ \mathbf{u}^n_t-\nu\Delta\mathbf{u}^n+\nabla\rho^n-\nabla\Delta\rho^n+J_n(\mathbf{u}^n\cdot\nabla\mathbf{u}^n)=\nu J_n\Lambda^{-1}\nabla\dfrac{\nabla\rho^n\cdot\nabla\mathbf{u}^n}{1+\rho^n}.\end{cases}$$

The only problem is to pass to the limit in $\mathscr{D}'(\mathbb{R}^+\times\mathbb{R}^2)$ in the nonlinear terms. This can be done by using the convergence results stemming from the uniform estimates and the convergence results (23) and (25).

As it is just a matter of doing tedious verifications, we show, as an example, the case of the term $\frac{\nabla\rho^n\cdot\nabla\mathbf{u}^n}{1+\rho^n}$. Denote $L(z)=z/(z+1)$. Let $\theta\in\mathcal{C}^\infty_0(\mathbb{R}^+\times\mathbb{R}^2)$ and $p\in\mathbb{N}$ be such that $\operatorname{supp}\theta\subset[0,p]\times B(0,p)$. We consider the decomposition

$$\begin{aligned}&J_n\frac{\theta\nabla\rho^n\cdot\nabla\mathbf{u}^n}{1+\rho^n}-\frac{\theta\nabla\rho\cdot\nabla\mathbf{u}}{1+\rho}\\ =&J_n[\theta(1-L(\rho^n))\chi_p\nabla\rho^n\cdot\chi_p\nabla(\mathbf{u}^n_L-\mathbf{u}_L)+\theta(1-L(\rho^n))\chi_p\nabla\rho^n\cdot\chi_p\nabla(\chi_p(\bar{\mathbf{u}}^n-\bar{\mathbf{u}}))\\ &+\theta(1-L(\rho^n))\chi_p\nabla\mathbf{u}\cdot\nabla(\chi_p(\rho^n-\rho))+\theta\nabla\rho\chi_p\nabla\mathbf{u}(L(\chi_p\rho)-L(\chi_p\rho^n))]\\ &+(J_n-I)\frac{\theta\nabla\rho\cdot\nabla\mathbf{u}}{1+\rho}.\end{aligned}$$

The last term tends to zero as $n\to+\infty$ due to the property of J_n. As $\theta L(\rho^n)$ and ρ^n are u.b. in $L^\infty(\dot B^1_{2,1})$ and $\mathbf{u}^n_L$ tends to $\mathbf{u}_L$ in $L^1(\dot B^2_{2,1})$, the first term tends to 0 in $L^1(\dot B^0_{2,1})$. According to (25), $\chi_p(\bar{\mathbf{u}}^n-\bar{\mathbf{u}})$ tends to zero in $L^1([0,p];\dot H^2)$ so that the second term tends to 0 in $L^1([0,p];L^2)$. Clearly, $\chi_p\rho^n\to\chi_p\rho$ in $L^\infty(\dot H^1)$ and $L(\chi_p\rho^n)\to L(\chi_p\rho)$ in $L^\infty(L^\infty\cap\dot H^1)$, so that the third and the last terms also tend to 0 in $L^1(L^2)$. The other nonlinear terms can be treated in the same way.

We still have to prove that ρ is continuous in $\dot B^0_{2,1}\cap\dot B^1_{2,1}$ and that $\mathbf{u}$ belongs to $C(\mathbb{R}^+;\dot B^0_{2,1})$. The continuity of $\mathbf{u}$ is straightforward. Indeed, $\mathbf{u}$ satisfies

$$\partial_t\mathbf{u}=-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\Delta\mathbf{u}-\nabla\rho+\nabla\Delta\rho+\nu\frac{\nabla\rho\cdot\nabla\mathbf{u}}{1+\rho}$$

and the r.h.s. belongs to $L^1(\dot B^0_{2,1})$ by noting that we also have $\rho\in L^1(\dot B^1_{2,1})\cap L^2(\dot B^2_{2,1})$ in view of the interpolation argument. We have already got that $\rho\in\mathcal{C}(\mathbb{R}^+;\dot B^0_{2,1})$. Indeed, $\rho_0-1\in\dot B^0_{2,1}$, $\mathbf{u}\in L^2(\mathbb{R}^+;\dot B^1_{2,1})$, $\rho\in L^\infty(\mathbb{R}^+;\dot B^1_{2,1})$ and then $\partial_t\rho\in L^2(\mathbb{R}^+;\dot B^0_{2,1})$ from the equation $\partial_t\rho=-\,\mathrm{div}\mathbf{u}-\,\mathrm{div}(\rho\mathbf{u})$. Thus, there remains to prove the continuity of ρ in $\dot B^1_{2,1}$.

Let us apply the operator $\triangle_k$ to the first equation of (3) to get

$$\partial_t \triangle_k \rho = -\triangle_k(\mathbf{u}\cdot\nabla\rho) - \triangle_k \operatorname{div}\mathbf{u} - \triangle_k(\rho \operatorname{div}\mathbf{u}). \tag{26}$$

Obviously, for fixed k the r.h.s belongs to $L^1_{loc}(\mathbb{R}^+;L^2)$ so that each $\triangle_k\rho$ is continuous in time with values in L^2.

Now, we apply an energy method to (26) to obtain, with the help of Lemma 5.1 in [5], that

$$\frac{1}{2}\frac{d}{dt}\|\triangle_k\rho\|_{L^2}^2 \leqslant C\|\triangle_k\rho\|_{L^2}\Big(\alpha_k 2^{-k}\|\rho\|_{\dot B^1_{2,1}}\|\mathbf{u}\|_{\dot B^2_{2,1}} + \|\triangle_k \operatorname{div}\mathbf{u}\|_{L^2} + \|\triangle_k(\rho \operatorname{div}\mathbf{u})\|_{L^2}\Big),$$

where $\sum_k \alpha_k \leqslant 1$. Integrating in time and multiplying 2^k, we get

$$2^k\|\triangle_k\rho(t)\|_{L^2} \leqslant 2^k\|\triangle_k(\rho_0-1)\|_{L^2} + C\int_0^t \Big(\alpha_k\|\rho(\tau)\|_{\dot B^1_{2,1}}\|\mathbf{u}(\tau)\|_{\dot B^2_{2,1}} + 2^{2k}\|\triangle_k\mathbf{u}(\tau)\|_{L^2} + 2^k\|\triangle_k(\rho \operatorname{div}\mathbf{u})(\tau)\|_{L^2}\Big)d\tau.$$

Since $\rho\in L^\infty(\dot B^1_{2,1})$, $\mathbf{u}\in L^1(\dot B^2_{2,1})$ and $\rho \operatorname{div}\mathbf{u}\in L^1(\dot B^1_{2,1})$, we can get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\sup_{t\geqslant 0} 2^k\|\triangle_k\rho(t)\|_{L^2}\\ \lesssim&\|\rho_0-1\|_{\dot B^1_{2,1}} + \Big(1+\|\rho\|_{L^\infty(\dot B^1_{2,1})}\Big)\|\mathbf{u}\|_{L^1(\dot B^2_{2,1})} + \|\rho \operatorname{div}\mathbf{u}\|_{L^1(\dot B^1_{2,1})} < \infty.\end{aligned}$$

Thus, $\sum_{|k|\leqslant N}\triangle_k\rho$ converges uniformly in $L^\infty(\mathbb{R}^+;\dot B^1_{2,1})$ and we can conclude that $\rho\in\mathcal{C}(\mathbb{R}^+;\dot B^1_{2,1})$.

4.4. **Uniqueness.** Let $(\rho_1,\mathbf{u}_1)$ and $(\rho_2,\mathbf{u}_2)$ be solutions of (3) in E^1_T with the same data $(\rho_0-1,\mathbf{u}_0)$ constructed in the previous parts on the time interval $[0,T]$. Denote $(\delta\rho,\delta\mathbf{u})=(\rho_2-\rho_1,\mathbf{u}_2-\mathbf{u}_1)$. From the (3), we can get

$$\begin{cases}\partial_t\delta\rho + \mathbf{u}_2\cdot\nabla\delta\rho + \operatorname{div}\delta\mathbf{u} = F_2,\\ \partial_t\delta\mathbf{u} + \mathbf{u}_2\cdot\nabla\delta\mathbf{u} - \nu\Delta\delta\mathbf{u} + \nabla\delta\rho - \nabla\Delta\delta\rho = G_2,\\ (\delta\rho,\delta\mathbf{u}) = (0,\mathbf{0}),\end{cases} \tag{27}$$

where

$$\begin{aligned}F_2 =& -\delta\mathbf{u}\cdot\nabla\rho_1 - \delta\rho \operatorname{div}\mathbf{u}_2 - \rho_1 \operatorname{div}\delta\mathbf{u},\\ G_2 =& -\delta\mathbf{u}\cdot\nabla\mathbf{u}_1 + \nu\frac{\nabla\delta\rho\cdot\nabla\mathbf{u}_2}{1+\rho_2} + \nu\frac{\nabla\rho_1\cdot\nabla\delta\mathbf{u}}{1+\rho_2} + \nu\left(\frac{1}{1+\rho_2}-\frac{1}{1+\rho_1}\right)\nabla\rho_1\cdot\nabla\mathbf{u}_1.\end{aligned}$$

Similar to (3), we can get

$$\|(\delta\rho,\delta\mathbf{u})\|_{E^1_T} \leqslant Ce^{C\|\mathbf{u}_2\|_{L^1_T(\dot B^2_{2,1})}}\Big(\|F_2\|_{L^1_T(\dot B^0_{2,1}\cap\dot B^1_{2,1})} + \|G_2\|_{L^1_T(\dot B^0_{2,1})}\Big).$$

Noticing that $\rho_1\in L^\infty_T(\dot B^0_{2,1})\cap L^1_T(\dot B^2_{2,1})$ and $\mathbf{u}_2\in L^1_T(\dot B^2_{2,1})$, we can get

$$\begin{aligned}\|F_2\|_{L^1_T(\dot B^0_{2,1})} \lesssim&\|\delta\mathbf{u}\|_{L^\infty_T(\dot B^0_{2,1})}\|\rho_1\|_{L^1_T(\dot B^2_{2,1})} + \|\delta\rho\|_{L^\infty_T(\dot B^0_{2,1})}\|\mathbf{u}_2\|_{L^1_T(\dot B^2_{2,1})}\\ &+ \|\rho_1\|_{L^\infty_T(\dot B^0_{2,1})}\|\delta\mathbf{u}\|_{L^1_T(\dot B^2_{2,1})}.\end{aligned}$$

Moreover, from $\rho_1 \in L^2_T(\dot B^2_{2,1}) \cap L^\infty_T(\dot B^1_{2,1})$ by interpolation, we have

$$\begin{aligned}\|F_2\|_{L^1_T(\dot B^1_{2,1})} \lesssim & \|\delta\mathbf{u}\|^{\frac12}_{L^\infty_T(\dot B^0_{2,1})}\|\delta\mathbf{u}\|^{\frac12}_{L^1_T(\dot B^2_{2,1})}\|\rho_1\|_{L^2_T(\dot B^1_{2,1})}\\ & + \|\delta\rho\|_{L^\infty_T(\dot B^1_{2,1})}\|\mathbf{u}_2\|_{L^1_T(\dot B^2_{2,1})} + \|\rho_1\|_{L^\infty_T(\dot B^1_{2,1})}\|\delta\mathbf{u}\|_{L^1_T(\dot B^2_{2,1})}.\end{aligned}$$

Noting that $\rho_1,\, \rho_2 \in L^\infty_T(\dot B^1_{2,1})$, $\mathbf{u}_1, \mathbf{u}_2 \in L^1_T(\dot B^2_{2,1})$ and $\|\rho_1\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leqslant \frac12$, $\|\rho_2\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leqslant \frac12$ by the construction of solutions, we have

$$\begin{aligned}&\|G_2\|_{L^1_T(\dot B^0_{2,1})}\\ \lesssim &\|\delta\mathbf{u}\|_{L^\infty_T(\dot B^0_{2,1})}\|\mathbf{u}_1\|_{L^1_T(\dot B^2_{2,1})} + \nu(1+\|\rho_2\|_{L^\infty_T(\dot B^1_{2,1})})\|\delta\rho\|_{L^\infty_T(\dot B^1_{2,1})}\|\mathbf{u}_2\|_{L^1_T(\dot B^2_{2,1})}\\ &+ \nu(1+\|\rho_2\|_{L^\infty_T(\dot B^1_{2,1})})\|\rho_1\|_{L^\infty_T(\dot B^1_{2,1})}\|\delta\mathbf{u}\|_{L^1_T(\dot B^2_{2,1})}\\ &+ \nu\|\delta\rho\|_{L^\infty_T(\dot B^1_{2,1})}\|\rho_1\|_{L^\infty_T(\dot B^1_{2,1})}\|\mathbf{u}_1\|_{L^1_T(\dot B^2_{2,1})}\\ &\quad\times(1+\|\rho_1\|_{L^\infty_T(\dot B^1_{2,1})} + \|\rho_2\|_{L^\infty_T(\dot B^1_{2,1})} + \|\rho_1\|_{L^\infty_T(\dot B^1_{2,1})}\|\rho_2\|_{L^\infty_T(\dot B^1_{2,1})}).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|(\delta\rho,\delta\mathbf{u})\|_{E^1_T} \leqslant Ce^{C\|\mathbf{u}_2\|_{L^1_T(\dot B^2_{2,1})}}\Big\{&\Big(1+\nu+\nu\|\rho_2\|_{L^\infty_T(\dot B^0_{2,1}\cap\dot B^1_{2,1})}\Big)\|\rho_1\|_{L^\infty_T(\dot B^0_{2,1}\cap\dot B^1_{2,1})}\\ &+Z(T)\Big\}\|(\delta\rho,\delta\mathbf{u})\|_{E^1_T},\end{aligned}$$

where $\limsup_{T\to 0^+} Z(T) = 0$.

Supposing that $C(1+\nu(1+(2C_1)^{-1}))A\tilde C E(0) < \frac14$ besides (21) for $E(0)$ and taking $0 < T \leqslant 1$ so small that $C\|\mathbf{u}_2\|_{L^1_T(\dot B^2_{2,1})} \leqslant \ln 2$ and $Z(T) < \frac12$, we obtain $\|(\delta\rho,\delta\mathbf{u})\|_{E^1_T} \equiv 0$. Hence, $(\rho_1,\mathbf{u}_1) \equiv (\rho_2,\mathbf{u}_2)$ on $[0,T]$.

Let T_m (supposedly finite) be the largest time such that the two solutions coincide on $[0,T_m]$. If we denote

$$(\tilde\rho_i(t),\tilde{\mathbf{u}}_i(t)) := (\rho_i(t+T_m),\mathbf{u}_i(t+T_m)),\quad i=1,2,$$

we can use the above arguments and the fact that

$$\|\tilde\rho_i\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2)} \leqslant \frac12 \;\text{ and }\; \|\tilde\rho_i\|_{L^\infty(\mathbb{R}^+;\dot B^0_{2,1}\cap\dot B^1_{2,1})} \leqslant A\tilde C E(0)$$

to prove that $(\tilde\rho_1,\tilde{\mathbf{u}}_1) = (\tilde\rho_2,\tilde{\mathbf{u}}_2)$ on the interval $[0,T_m]$ with the same T_m as in the previous. Therefore, we complete the proofs.

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E-mail address: hcc@amss.ac.cn