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Well-posedness to the compressible viscous magnetohydrodynamic system

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ABSTRACT

This paper is concerned with the Cauchy problem of the compressible viscous magnetohydrodynamic (MHD) system in whole spatial space $\mathbb{R}^d$ for $d \geqslant 3$. It is shown that the global solution exists uniquely in hybrid Besov spaces provided the initial data close to a constant equilibrium state away from the vacuum.

1. Introduction

Magnetohydrodynamic (MHD) studies the interaction between the flow of an electrically conducting fluid and magnetic fields. The dynamic motion of the fluid and the magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled. It involves such diverse topics as the evolution and dynamics of astrophysical objects, thermonuclear fusion, metallurgy and semiconductor crystal growth.

In the present paper, we study the Cauchy problem to the isentropic compressible viscous MHD system of the form (see, e.g., [1–4])

$$\begin{cases} \partial_t \tilde{\rho} + \operatorname{div}(\tilde{\rho}\mathbf{u}) = 0, \\ \partial_t(\tilde{\rho}\mathbf{u}) + \operatorname{div}(\tilde{\rho}\mathbf{u}\otimes\mathbf{u}) + \nabla P(\tilde{\rho}) = \tilde{\mathbf{H}}\cdot\nabla\tilde{\mathbf{H}} - \frac{1}{2}\nabla(|\tilde{\mathbf{H}}|^2) + \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}, \\ \partial_t\tilde{\mathbf{H}} + (\operatorname{div}\mathbf{u})\tilde{\mathbf{H}} + \mathbf{u}\cdot\nabla\tilde{\mathbf{H}} - \tilde{\mathbf{H}}\cdot\nabla\mathbf{u} = \nu\Delta\tilde{\mathbf{H}}, \qquad \operatorname{div}\tilde{\mathbf{H}} = 0, \\ (\tilde{\rho},\mathbf{u},\tilde{\mathbf{H}})|_{t=0} = (\tilde{\rho}_0,\mathbf{u}_0,\tilde{\mathbf{H}}_0), \qquad \operatorname{div}\tilde{\mathbf{H}}_0 = 0, \end{cases} \tag{1.1}$$

where $\tilde{\rho} = \tilde{\rho}(t,x)$ denotes the density, $x \in \mathbb{R}^d$, $d \geqslant 2$, $t > 0$, $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_d) \in \mathbb{R}^d$ ($\mathbf{u}_j = \mathbf{u}_j(t,x)$, $j = 1, \ldots, d$) is the velocity of the flow, and $\tilde{\mathbf{H}} = (\tilde{\mathbf{H}}_1, \tilde{\mathbf{H}}_2, \ldots, \tilde{\mathbf{H}}_d) \in \mathbb{R}^d$ ($\tilde{\mathbf{H}}_j = \tilde{\mathbf{H}}_j(t,x)$) stands for the magnetic field. The constants μ and λ are the shear and bulk viscosity coefficients of the flow, respectively, satisfying $\mu > 0$ and $2\mu + d\lambda > 0$, the constant $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, and all these kinetic coefficients and the magnetic diffusivity are independent of the magnitude and direction of the magnetic field. $P(\tilde{\rho})$ is the scalar pressure function satisfying $P'(\tilde{\rho}) > 0$.

Due to the physical importance and mathematical challenges, the study on MHD has attracted many physicists and mathematicians. Many results concerning the existence and uniqueness of (weak, strong or smooth) solutions in one dimension can be found in [5–8] and the references cited therein. But in this paper, we focus on the well-posedness of the Cauchy problem in the multi-dimensional cases (i.e., $d \geqslant 3$). The MHD problem, in contrast, presents serious difficulties due to the presence of the magnetic field and its interaction with the hydrodynamic motion in the MHD flow. When there is no electromagnetic field, system (1.1) reduces to the compressible Navier–Stokes equations. See [9–11] and their references

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for studies on the multi-dimensional Navier–Stokes equations. Motivated by [9,12–14], we can obtain, the global existence and uniqueness of the smooth solutions in multi-dimensional space under the assumption that the initial data are close to a constant equilibrium state in the sense of the norm of hybrid Besov spaces with lower regularities.

In the present paper, we use the div–curl decomposition to decompose the velocity vector field and magnetic field into three new unknowns. With the help of the Littlewood–Paley analysis technique, we can derive an *a priori* estimates for the linearized system in hybrid Besov spaces. It is crucial for the proof of the main result. Then, by the classical Friedrichs' regularization method, we are able to construct the approximate solutions sequence, and prove the global existence of the unique strong solution by the compactness arguments (see, e.g., [12]).

Now, we state the main result as follows.

Theorem 1.1. *Let $d \geqslant 3$, $\bar{\rho} > 0$, $\mu > 0$, $2\mu + d\lambda > 0$, $\nu > 0$, $P'(\cdot) > 0$ and $\mathbf{I}$ be an arbitrary nonzero constant vector. Assume that $\tilde{\rho}_0 - \bar{\rho} \in B^{d/2-1,d/2}$ and $\mathbf{u}_0, \tilde{\mathbf{H}}_0 - \mathbf{I} \in B^{d/2-1}$ with the condition $\mathrm{div}\tilde{\mathbf{H}}_0 = 0$. Then, there exist a small number ε and a constant M such that if $\|\tilde{\rho}_0 - \bar{\rho}\|_{B^{d/2-1,d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}} + \|\tilde{\mathbf{H}}_0 - \mathbf{I}\|_{B^{d/2-1}} \leqslant \varepsilon$, then the system (1.1) yields a unique global solution $(\tilde{\rho}, \mathbf{u}, \tilde{\mathbf{H}})$ such that $(\tilde{\rho} - \bar{\rho}, \mathbf{u}, \tilde{\mathbf{H}} - \mathbf{I})$ belongs to*

$$E := \mathcal{C}(\mathbb{R}^+; B^{d/2-1,d/2} \times (B^{d/2})^{d+d}) \cap L^1(\mathbb{R}^+; B^{d/2+1,d/2+2} \times (B^{d/2+1})^{d+d}),$$

and satisfies $\|(\tilde{\rho} - \bar{\rho}, \mathbf{u}, \tilde{\mathbf{H}} - \mathbf{I})\|_E \leqslant M(\|\tilde{\rho}_0 - \bar{\rho}\|_{B^{d/2-1,d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}} + \|\tilde{\mathbf{H}}_0 - \mathbf{I}\|_{B^{d/2-1}})$. Here M is independent of the initial data and $B^{s_1,s_2} = B^{s_1} \cap B^{s_2}$ for $s_1 \leqslant s_2$ a hybrid Besov space where B^s denotes the usual homogeneous Besov space (defined in the next section).

2. Hybrid Besov spaces and Besov–Chemin–Lerner type spaces

Let $\psi : \mathbb{R}^d \to [0, 1]$ be a radial smooth cut-off function valued in $[0, 1]$ such that $\psi(\xi) = 1$ for $|\xi| \leqslant 3/4$, $\psi(\xi) = 0$ for $|\xi| \geqslant 4/3$ and $\psi(\xi)$ is smooth otherwise. Let $\varphi(\xi)$ be the function $\varphi(\xi) := \psi(\xi/2) - \psi(\xi)$. Thus, ψ is supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leqslant 4/3\}$, and φ is also a smooth cut-off function valued in $[0, 1]$ and supported in the annulus $\{\xi \in \mathbb{R}^d : 3/4 \leqslant |\xi| \leqslant 8/3\}$. By construction, we have $\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1, \forall \xi \neq 0$. One can define the dyadic blocks as follows. For $k \in \mathbb{Z}$, let $\triangle_k f := \mathscr{F}^{-1}\varphi(2^{-k}\xi)\mathscr{F}f$.

The formal decomposition

$$f = \sum_{k\in\mathbb{Z}} \triangle_k f \tag{2.1}$$

is called homogeneous Littlewood–Paley decomposition. Actually, this decomposition works for just about any locally integrable function which has some decay at infinity, and one usually has all the convergence properties of the summation that one needs. Thus, the r.h.s. of (2.1) does not necessarily converge in $\mathscr{S}'(\mathbb{R}^d)$. Even if it does, the equality is not always true in $\mathscr{S}'(\mathbb{R}^d)$. For instance, if $f \equiv 1$, then all the projections $\triangle_k f$ vanish. Nevertheless, (2.1) is true modulo polynomials, in other words (cf. [15]), if $f \in \mathscr{S}'(\mathbb{R}^d)$, then $\sum_{k\in\mathbb{Z}} \triangle_k f$ converges modulo $\mathscr{P}[\mathbb{R}^d]$ and (2.1) holds in $\mathscr{S}'(\mathbb{R}^d)/\mathscr{P}[\mathbb{R}^d]$.

Definition 2.1. Let $s \in \mathbb{R}$. For $f \in \mathscr{S}'(\mathbb{R}^d)$, we write

$$\|f\|_{B^s} = \sum_{k\in\mathbb{Z}} 2^{ks}\| \triangle_k f\|_{L^2}.$$

A difficulty comes from the choice of homogeneous spaces at this point. Indeed, $\|\cdot\|_{B^s}$ cannot be a norm on $\{f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{B^s} < \infty\}$ because $\|f\|_{B^s} = 0$ means that f is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces (cf. [12]).

Definition 2.2. Let $s \in \mathbb{R}$ and $m = -[d/2 + 1 - s]$. If $m < 0$, then we define $B^s(\mathbb{R}^d)$ as

$$B^s = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{B^s} < \infty \text{ and } u = \sum_{k\in\mathbb{Z}} \triangle_k f \text{ in } \mathscr{S}'(\mathbb{R}^d) \right\}.$$

If $m \geqslant 0$, we denote by $\mathscr{P}_m$ the set of polynomials of degree less than or equal to m and define

$$B^s = \left\{ f \in \mathscr{S}'(\mathbb{R}^d)/\mathscr{P}_m : \|f\|_{B^s} < \infty \text{ and } u = \sum_{k\in\mathbb{Z}} \triangle_k f \text{ in } \mathscr{S}'(\mathbb{R}^d)/\mathscr{P}_m \right\}.$$

For the composition of functions, we have the following estimates.

Lemma 2.3 *([12, Lemma 2.7])*. *Let $s > 0$ and $u \in B^s \cap L^\infty$. Then, it holds*

(i) *Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^d)$ with $F(0) = 0$. Then $F(u) \in B^s$. Moreover, there exists a function of one variable C_0 depending only on s and F, and such that*

$$\|F(u)\|_{B^s} \leqslant C_0(\|u\|_{L^\infty})\|u\|_{B^s}.$$

(ii) *If* $u, v \in B^{d/2}$, $v - u \in B^s$ *for* $s \in (-d/2, d/2]$ *and* $G \in W^{[d/2]+3,\infty}_{loc}(\mathbb{R}^d)$ *satisfies* $G'(0) = 0$, *then* $G(v) - G(u) \in B^s$ *and there exists a function of two variables C depending only on s, N and G, and such that*

$$\|G(v) - G(u)\|_{B^s} \leqslant C(\|u\|_{L^\infty}, \|v\|_{L^\infty})(\|u\|_{B^{d/2}} + \|v\|_{B^{d/2}})\|v - u\|_{B^s}.$$

We also need hybrid Besov spaces for which regularity assumptions are different in low frequencies and high frequencies [12]. We are going to recall the definition of these new spaces and some of their main properties.

Definition 2.4. Let $s, t \in \mathbb{R}$. We define

$$\|f\|_{B^{s,t}} = \sum_{k\leqslant 0} 2^{ks}\| \triangle_k f\|_{L^2} + \sum_{k>0} 2^{kt}\| \triangle_k f\|_{L^2}.$$

Let $m = -[d/2 + 1 - s]$, we then define

$$B^{s,t}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{B^{s,t}} < \infty\}, \quad \text{if } m < 0,$$
$$B^{s,t}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d)/\mathscr{P}_m : \|f\|_{B^{s,t}} < \infty\}, \quad \text{if } m \geqslant 0.$$

Lemma 2.5. *We have the following inclusions for hybrid Besov spaces.*

(i) *We have* $B^{s,s} = B^s$.
(ii) *If* $s \leqslant t$ *then* $B^{s,t} = B^s \cap B^t$. *Otherwise*, $B^{s,t} = B^s + B^t$.
(iii) *The space* $B^{0,s}$ *coincides with the usual inhomogeneous Besov space* $B^s_{2,1}$.
(iv) *If* $s_1 \leqslant s_2$ *and* $t_1 \geqslant t_2$, *then* $B^{s_1,t_1} \hookrightarrow B^{s_2,t_2}$.

Let us now recall some useful estimates for the product in hybrid Besov spaces.

Lemma 2.6 (*[12, Proposition 2.10]*)**.** *Let* $s_1, s_2 > 0$ *and* $f, g \in L^\infty \cap B^{s_1,s_2}$. *Then* $fg \in B^{s_1,s_2}$ *and*

$$\|fg\|_{B^{s_1,s_2}} \lesssim \|f\|_{L^\infty}\|g\|_{B^{s_1,s_2}} + \|f\|_{B^{s_1,s_2}}\|g\|_{L^\infty}.$$

Let $s_1, s_2, t_1, t_2 \leqslant d/2$ *such that* $\min(s_1 + s_2, t_1 + t_2) > 0$, $f \in B^{s_1,t_1}$ *and* $g \in B^{s_2,t_2}$. *Then* $fg \in B^{s_1+s_2-1,t_1+t_2-1}$ *and*

$$\|fg\|_{B^{s_1+s_2-d/2,t_1+t_2-d/2}} \lesssim \|f\|_{B^{s_1,t_1}}\|g\|_{B^{s_2,t_2}}.$$

For $\alpha, \beta \in \mathbb{R}$, let us define the following characteristic function on $\mathbb{Z}$:

$$\tilde{\varphi}^{\alpha,\beta}(r) = \begin{cases} \alpha, & \text{if } r \leqslant 0, \\ \beta, & \text{if } r \geqslant 1. \end{cases}$$

Then, we can recall the following lemma.

Lemma 2.7 (*[12, Lemma 6.2]*)**.** *Let F be a homogeneous smooth function of degree m. Suppose that* $-N/2 < s_1, t_1, s_2, t_2 \leqslant 1 + N/2$. *The following two estimates hold:*

$$|(F(D)\triangle_k(\mathbf{v}\cdot\nabla a), F(D)\triangle_k a)| \lesssim \varepsilon_k 2^{-k(\tilde{\varphi}^{s_1,s_2}(k)-m)}\|\mathbf{v}\|_{B^{d/2+1}}\|a\|_{B^{s_1,s_2}}\|F(D)\triangle_k a\|_{L^2},$$
$$|(F(D)\triangle_k(\mathbf{v}\cdot\nabla a), \triangle_k b) + (\triangle_k(\mathbf{v}\cdot\nabla b), F(D)\triangle_k a)| \lesssim \varepsilon_k\|\mathbf{v}\|_{B^{d/2+1}} \times (2^{-k\tilde{\varphi}^{t_1,t_2}(k)}\|F(D)\triangle_k a\|_{L^2}\|b\|_{B^{t_1,t_2}} + 2^{-k(\tilde{\varphi}^{s_1,s_2}(k)-m)}\|a\|_{B^{s_1,s_2}}\|\triangle_k b\|_{L^2}),$$

where $(\cdot,\cdot)$ *denotes the* L^2*-inner product, the operator* $F(D)$ *is defined by* $F(D)f := \mathscr{F}^{-1}F(\xi)\mathscr{F}f$ *and* $\sum_{k\in\mathbb{Z}}\varepsilon_k \leqslant 1$.

In the context of this paper, we also need to use the interpolation spaces of hybrid Besov spaces together with a time space such as $L^p(0, T; B^{s,t})$. Thus, we have to introduce the Besov–Chemin–Lerner type space (cf. [16]) which is a refinement of the space $L^p(0, T; B^{s,t})$.

Definition 2.8. Let $p \in [1, \infty]$, $T \in (0, \infty]$ and $s_1, s_2 \in \mathbb{R}$. Then we define

$$\|f\|_{\tilde{L}^p_T(B^{s,t})} = \sum_{k\leqslant 0} 2^{ks}\|\triangle_k f\|_{L^p(0,T;L^2)} + \sum_{k>0} 2^{kt}\|\triangle_k f\|_{L^p(0,T;L^2)}.$$

Noting that Minkowski's inequality yields $\|f\|_{L^p_T(B^{s,t})} \leqslant \|f\|_{\tilde{L}^p_T(B^{s,t})}$, we define the space $\tilde{L}^p_T(B^{s,t})$ as

$$\tilde{L}^p_T(B^{s,t}) = \{f \in L^p_T(B^{s,t}) : \|f\|_{\tilde{L}^p_T(B^{s,t})} < \infty\}.$$

If $T = \infty$, then we omit the subscript T from the notation $\tilde{L}^p_T(B^{s,t})$, that is, $\tilde{L}^p(B^{s,t})$ for simplicity. We will denote by $\tilde{\mathcal{C}}([0, T]; B^{s,t})$ the subset of functions of $\tilde{L}^\infty_T(B^{s,t})$ which are continuous on $[0, T]$ with values in $B^{s,t}$. Let us observe that $L^1_T(B^{s,t}) = \tilde{L}^1_T(B^{s,t})$, but the embedding $\tilde{L}^p_T(B^{s,t}) \subset L^p_T(B^{s,t})$ is strict if $p > 1$.

We will use the following interpolation property which can be verified easily (cf. [17]).

Lemma 2.9. *Let $\theta \in [0,1]$, $s, t, s_1, t_1, s_2, t_2 \in \mathbb{R}$ and $p, p_1, p_2 \in [1, \infty]$. We have*

$$\|f\|_{\tilde{L}^p_T(B^{s,t})} \leqslant \|f\|^{\theta}_{\tilde{L}^{p_1}_T(B^{s_1,t_1})} \|f\|^{1-\theta}_{\tilde{L}^{p_2}_T(B^{s_2,t_2})},$$

where $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, $s = \theta s_1 + (1-\theta)s_2$ and $t = \theta t_1 + (1-\theta)t_2$.

3. Reformulation and *a priori* estimates

Since we study the case where the initial data are close to a constant equilibrium state, it is convenient to change unknown variables. Let $\tilde{\rho} = \bar{\rho}(\rho+1)$ for a constant $\bar{\rho} > 0$, $\bar{\mu} = \mu/\bar{\rho}$, $\bar{\lambda} = \lambda/\bar{\rho}$ and $\tilde{\mathbf{H}} = \mathbf{H} + \mathbf{I}$ for a nonzero constant vector $\mathbf{I}$.

$$\begin{cases} \partial_t \rho + \mathbf{u}\cdot\nabla\rho + \mathrm{div}\mathbf{u} = -\rho\,\mathrm{div}\mathbf{u}, \\ \partial_t \mathbf{u} - \bar{\mu}\Delta\mathbf{u} - (\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}\mathbf{u} + P'(\bar{\rho})\nabla\rho - \dfrac{1}{\bar{\rho}}\nabla(\mathbf{I}\cdot\mathbf{H}) - \dfrac{1}{\bar{\rho}}\mathbf{I}\cdot\nabla\mathbf{H} \\ \quad = -\dfrac{1}{2}\dfrac{1}{\bar{\rho}(\rho+1)}\nabla(|\mathbf{H}|^2) + \dfrac{\rho}{\bar{\rho}(\rho+1)}\nabla(\mathbf{I}\cdot\mathbf{H}) - \dfrac{\rho}{\bar{\rho}(\rho+1)}\mathbf{I}\cdot\nabla\mathbf{H} + \dfrac{1}{\bar{\rho}(\rho+1)}\mathbf{H}\cdot\nabla\mathbf{H} \\ \quad + \left(\dfrac{P'(\bar{\rho}(\rho+1))}{\rho+1} - P'(\bar{\rho})\right)\nabla\rho - \bar{\mu}\dfrac{\rho}{\rho+1}\Delta\mathbf{u} + (\bar{\mu}+\bar{\lambda})\dfrac{\rho}{\rho+1}\nabla\mathrm{div}\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}, \\ \partial_t\mathbf{H} - \nu\Delta\mathbf{H} + (\mathrm{div}\mathbf{u})\mathbf{I} - \mathbf{I}\cdot\nabla\mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{H} - (\mathrm{div}\mathbf{u})\mathbf{H} + \mathbf{H}\cdot\nabla\mathbf{u}, \qquad \mathrm{div}\mathbf{H} = 0. \end{cases} \tag{3.1}$$

For simplicity, we use the div–curl decomposition to decompose the velocity vector field and magnetic field into three new unknowns. Let $\Lambda^s f = \mathscr{F}^{-1}|\xi|^s\mathscr{F}f$, $\omega = \Lambda^{-1}\mathrm{div}\mathbf{u}$, $\Omega = \Lambda^{-1}\mathrm{curl}\mathbf{u}$ and $E = \Lambda^{-1}\mathrm{curl}\mathbf{H}$ where $\mathrm{curl}\mathbf{u} = (\partial_j\mathbf{u}_i - \partial_i\mathbf{u}_j)_{ij}$ is a $d\times d$ matrix.

$$\begin{cases} \partial_t\rho + \mathbf{u}\cdot\nabla\rho + \Lambda\omega = F, \\ \partial_t\omega + \mathbf{u}\cdot\nabla\omega - (2\bar{\mu}+\bar{\lambda})\Delta\omega - P'(\bar{\rho})\Lambda\rho - \bar{\rho}^{-1}(\mathbf{I}\cdot\mathrm{div}E) = G, \\ \partial_t\Omega - \bar{\mu}\Delta\Omega - \bar{\rho}^{-1}\mathbf{I}\cdot\nabla E = J, \\ \partial_t E - \nu\Delta E + \mathrm{curl}(\omega\mathbf{I}) - \mathbf{I}\cdot\nabla\Omega = K, \\ \mathbf{u} = -\Lambda^{-1}\nabla\omega - \Lambda^{-1}\mathrm{div}\Omega, \qquad \mathbf{H} = -\Lambda^{-1}\mathrm{div}E, \qquad \mathrm{div}\mathbf{H} = 0, \end{cases} \tag{3.2}$$

where $\mathrm{div}E = \sum_{i=1}^d \partial_i E_{ij}$ with entries E_{ij} of the matrix E, and

$$\begin{aligned} F &= -\rho\,\mathrm{div}\mathbf{u}, \\ G &= \Lambda^{-1}\mathrm{div}\left(-\frac{1}{2}\frac{1}{\bar{\rho}(\rho+1)}\nabla(|\mathbf{H}|^2) + \frac{\rho}{\bar{\rho}(\rho+1)}\nabla(\mathbf{I}\cdot\mathbf{H}) - \frac{\rho}{\bar{\rho}(\rho+1)}\mathbf{I}\cdot\nabla\mathbf{H} + \frac{1}{\bar{\rho}(\rho+1)}\mathbf{H}\cdot\nabla\mathbf{H}\right. \\ &\quad \left. + \left(\frac{P'(\bar{\rho}(\rho+1))}{\rho+1} - P'(\bar{\rho})\right)\nabla\rho - \bar{\mu}\frac{\rho}{\rho+1}\Delta\mathbf{u} + (\bar{\mu}+\bar{\lambda})\frac{\rho}{\rho+1}\nabla\mathrm{div}\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}\right) \\ J &= \Lambda^{-1}\mathrm{curl}\left(-\frac{1}{2}\frac{1}{\bar{\rho}(\rho+1)}\nabla(|\mathbf{H}|^2) + \frac{\rho}{\bar{\rho}(\rho+1)}\nabla(\mathbf{I}\cdot\mathbf{H}) - \frac{\rho}{\bar{\rho}(\rho+1)}\mathbf{I}\cdot\nabla\mathbf{H} + \frac{1}{\bar{\rho}(\rho+1)}\mathbf{H}\cdot\nabla\mathbf{H}\right. \\ &\quad \left. + \left(\frac{P'(\bar{\rho}(\rho+1))}{\rho+1} - P'(\bar{\rho})\right)\nabla\rho - \bar{\mu}\frac{\rho}{\rho+1}\Delta\mathbf{u} + (\bar{\mu}+\bar{\lambda})\frac{\rho}{\rho+1}\nabla\mathrm{div}\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}\right) \\ K &= \Lambda^{-1}\mathrm{curl}\left(-\mathbf{u}\cdot\nabla\mathbf{H} - (\mathrm{div}\mathbf{u})\mathbf{H} + \mathbf{H}\cdot\nabla\mathbf{u}\right). \end{aligned}$$

Next, we shall study the following linearized system with convection terms

$$\begin{cases} \partial_t\rho + \mathbf{v}\cdot\nabla\rho + \Lambda\omega = F, \\ \partial_t\omega + \mathbf{v}\cdot\nabla\omega - (2\bar{\mu}+\bar{\lambda})\Delta\omega - P'(\bar{\rho})\Lambda\rho - \bar{\rho}^{-1}(\mathbf{I}\cdot\mathrm{div}E) = G, \\ \partial_t\Omega - \bar{\mu}\Delta\Omega - \bar{\rho}^{-1}\mathbf{I}\cdot\nabla E = J, \\ \partial_t E - \nu\Delta E + \mathrm{curl}(\omega\mathbf{I}) - \mathbf{I}\cdot\nabla\Omega = K, \\ (\rho,\omega,\Omega,E)|_{t=0} = (\rho_0,\omega_0,\Omega_0,E_0), \end{cases} \tag{3.3}$$

to get the following proposition for describing their regularities precisely.

Proposition 3.1. *Let* $(\rho, \omega, \Omega, E)$ *be a solution of* (3.3) *on* $[0, T)$ *for* $T > 0$, $1 - d/2 < s \leqslant 1 + d/2$ *and* $V(t) = \int_0^t \|\mathbf{v}(\tau)\|_{B^{1+d/2}} d\tau$. *Then the following estimate holds on* $[0, T)$:

$$\begin{aligned}&\|\rho\|_{\tilde{L}^\infty_T(B^{s-1,s})} + \|\omega\|_{\tilde{L}^\infty_T(B^{s-1})} + \|\Omega\|_{\tilde{L}^\infty_T(B^{s-1})} + \|E\|_{\tilde{L}^\infty_T(B^{s-1})} + \|\rho\|_{L^1_T(B^{s+1,s+2})} + \|\omega\|_{L^1_T(B^{s+1})} + \|\Omega\|_{L^1_T(B^{s+1})} + \|E\|_{L^1_T(B^{s+1})}\\ &\leqslant Ce^{CV(t)}(\|\rho(0)\|_{B^{s-1,s}} + \|\omega(0)\|_{\tilde{L}^\infty_T(B^{s-1})} + \|\Omega(0)\|_{\tilde{L}^\infty_T(B^{s-1})} + \|E(0)\|_{\tilde{L}^\infty_T(B^{s-1})})\\ &\quad + Ce^{CV(t)} \int_0^t e^{-CV(\tau)}(\|F(\tau)\|_{B^{s-1,s}} + \|G(\tau)\|_{B^{s-1}} + \|J(\tau)\|_{B^{s-1}} + \|K(\tau)\|_{B^{s-1}})d\tau,\end{aligned}$$

where the constant $C > 0$ *depends only on s,* $\bar{\rho}$ *and the coefficients of the system.*

Proof. Let $(\rho, \omega, \Omega, E)$ be a solution of the system (3.3) and $\underline{f} := e^{-\gamma V(t)} f$ for $f = \rho, \omega, \Omega, E$. Thus, the system (3.3) can be transformed into

$$\begin{cases}\partial_t \underline{\rho} + \mathbf{v}\cdot\nabla\underline{\rho} + \Lambda\underline{\omega} = \underline{F} - \gamma V'(t)\underline{\rho},\\ \partial_t\underline{\omega} + \mathbf{v}\cdot\nabla\underline{\omega} - (2\bar{\mu}+\bar{\lambda})\Delta\underline{\omega} - P'(\bar{\rho})\Lambda\underline{\rho} - \bar{\rho}^{-1}(\mathbf{I}\cdot \mathrm{div}\underline{E}) = \underline{G} - \gamma V'(t)\underline{\omega},\\ \partial_t\underline{\Omega} - \bar{\mu}\Delta\underline{\Omega} - \bar{\rho}^{-1}\mathbf{I}\cdot\nabla\underline{E} = \underline{J} - \gamma V'(t)\underline{\Omega},\\ \partial_t\underline{E} - \nu\Delta\underline{E} + \mathrm{curl}(\underline{\omega}\mathbf{I}) - \mathbf{I}\cdot\nabla\underline{\Omega} = \underline{K} - \gamma V'(t)\underline{E}.\end{cases} \tag{3.4}$$

Applying the operator $\triangle_k$ to the system (3.4), we get the following system with the notation $f_k := \triangle_k \underline{f}$,

$$\begin{cases}\partial_t\rho_k + \triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}) + \Lambda\omega_k = F_k - \gamma V'(t)\rho_k,\\ \partial_t\omega_k + \triangle_k(\mathbf{v}\cdot\nabla\underline{\omega}) - (2\bar{\mu}+\bar{\lambda})\Delta\omega_k - P'(\bar{\rho})\Lambda\rho_k - \bar{\rho}^{-1}(\mathbf{I}\cdot\mathrm{div}E_k) = G_k - \gamma V'(t)\omega_k,\\ \partial_t\Omega_k - \bar{\mu}\Delta\Omega_k - \bar{\rho}^{-1}\mathbf{I}\cdot\nabla E_k = J_k - \gamma V'(t)\Omega_k,\\ \partial_t E_k - \nu\Delta E_k + \mathrm{curl}(\omega_k\mathbf{I}) - \mathbf{I}\cdot\nabla\Omega_k = K_k - \gamma V'(t)E_k,\end{cases} \tag{3.5}$$

Taking the L^2 scalar product of the first equation of (3.5) with ρ_k, of the second equation with ω_k, the third one with Ω_k and the fourth one with E_k, we get the following four identities

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\rho_k\|_2^2 + (\Lambda\omega_k, \rho_k) = (F_k, \rho_k) - \gamma V'(t)\|\rho_k\|_2^2 - (\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \rho_k),\\ &\frac{1}{2}\frac{d}{dt}\|\omega_k\|_2^2 + (2\bar{\mu}+\bar{\lambda})\|\Lambda\omega_k\|_2^2 - P'(\bar{\rho})(\Lambda\rho_k, \omega_k) - \bar{\rho}^{-1}(\mathbf{I}\cdot\mathrm{div}E_k, \omega_k)\\ &\quad = (G_k, \omega_k) - \gamma V'(t)\|\omega_k\|_2^2 - (\triangle_k(\mathbf{v}\cdot\nabla\underline{\omega}), \omega_k),\\ &\frac{1}{2}\frac{d}{dt}\|\Omega_k\|_2^2 + \bar{\mu}\|\Lambda\Omega_k\|_2^2 - \bar{\rho}^{-1}(\mathbf{I}\cdot\nabla E_k, \Omega_k) = (J_k, \Omega_k) - \gamma V'(t)\|\Omega_k\|_2^2,\\ &\frac{1}{2}\frac{d}{dt}\|E_k\|_2^2 + \nu\|\Lambda E_k\|_2^2 + (\mathrm{curl}(\omega_k\mathbf{I}), E_k) - (\mathbf{I}\cdot\nabla\Omega_k, E_k) = (K_k, E_k) - \gamma V'(t)\|E_k\|_2^2.\end{aligned}$$

Noticing that

$$\begin{aligned}&(\Lambda\omega_k, \rho_k) = (\omega_k, \Lambda\rho_k),\\ &(\mathbf{I}\cdot\nabla E_k, \Omega_k) = \left(\sum_{i=1}^d \mathbf{I}_i\partial_i E_k, \Omega_k\right) = -\left(E_k, \sum_{i=1}^d \mathbf{I}_i\partial_i\Omega_k\right) = -(E_k, \mathbf{I}\cdot\nabla\Omega_k),\\ &(\mathrm{curl}(\omega_k\mathbf{I}), E_k) = \sum_{i,j=1}^d \int_{\mathbb{R}^d} [\partial_j(\omega_k\mathbf{I}_i) - \partial_i(\omega_k\mathbf{I}_j)]E_{kij}dx = \sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_i(\omega_k\mathbf{I}_j)E_{kji} - \partial_i(\omega_k\mathbf{I}_j)E_{kij}dx\\ &\qquad = -2\sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_i(\omega_k\mathbf{I}_j)E_{kij}dx = -2\sum_{i,j=1}^d \int_{\mathbb{R}^d} \omega_k\mathbf{I}_j\partial_i E_{kij}dx = -2(\mathbf{I}\cdot\mathrm{div}E_k, \omega_k),\end{aligned}$$

since $E_{kij} = -E_{kji}$.

Combining these identities, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[P'(\bar{\rho})\|\rho_k\|_2^2 + \|\omega_k\|_2^2 + \frac{1}{2}\|\Omega_k\|_2^2 + \frac{1}{2\bar{\rho}}\|E_k\|_2^2\right] + (2\bar{\mu}+\bar{\lambda})\|\Lambda\omega_k\|_2^2 + \frac{\bar{\mu}}{2}\|\Lambda\Omega_k\|_2^2 + \frac{\nu}{2\bar{\rho}}\|\Lambda E_k\|_2^2\\ &= P'(\bar{\rho})(F_k, \rho_k) + (G_k, \omega_k) + \frac{1}{2}(J_k, \Omega_k) + \frac{1}{2\bar{\rho}}(K_k, E_k)\\ &\quad - \gamma V'(t)\left[P'(\bar{\rho})\|\rho_k\|_2^2 + \|\omega_k\|_2^2 + \frac{1}{2}\|\Omega_k\|_2^2 + \frac{1}{2\bar{\rho}}\|E_k\|_2^2\right] - P'(\bar{\rho})(\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \rho_k) - (\triangle_k(\mathbf{v}\cdot\nabla\underline{\omega}), \omega_k).\end{aligned} \tag{3.6}$$

In order to get an L^1_t-estimate of ρ_k, we consider the L^2 scalar product of the first equation of (3.5) with $\Lambda\omega_k$, the second one with $\Lambda\rho_k$ and the first one with $\Lambda^2\rho_k$ to obtain

$$
\begin{aligned}
&(\partial_t\rho_k, \Lambda\omega_k) + \|\Lambda\omega_k\|_2^2 = (F_k, \Lambda\omega_k) - \gamma V'(t)(\rho_k, \Lambda\omega_k) - (\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \Lambda\omega_k),\\
&(\partial_t\omega_k, \Lambda\rho_k) + (2\bar\mu+\bar\lambda)(\Lambda^2\omega_k, \Lambda\rho_k) - P'(\bar\rho)\|\Lambda\rho_k\|_2^2 - \bar\rho^{-1}(\mathbf{I}\cdot \mathrm{div}E_k, \Lambda\rho_k)\\
&\quad = (G_k, \Lambda\rho_k) - \gamma V'(t)(\rho_k, \Lambda\omega_k) - (\triangle_k(\mathbf{v}\cdot\nabla\underline{\omega}), \Lambda\rho_k),\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda\rho_k\|_2^2 + (\Lambda\omega_k, \Lambda^2\rho_k) = (\Lambda F_k, \Lambda\rho_k) - \gamma V'(t)\|\Lambda\rho_k\|_2^2 - (\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \Lambda^2\rho_k).
\end{aligned}
$$

From these three equalities, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[(2\bar\mu+\bar\lambda)\|\Lambda\rho_k\|_2^2 - 2(\rho_k, \Lambda\omega_k)] + P'(\bar\rho)\|\Lambda\rho_k\|_2^2 - \|\Lambda\omega_k\|_2^2 + \bar\rho^{-1}(\mathbf{I}\cdot\mathrm{div}E_k, \Lambda\rho_k)\\
&\quad = (2\bar\mu+\bar\lambda)(\Lambda F_k, \Lambda\rho_k) - \gamma V'(t)(2\bar\mu+\bar\lambda)\|\Lambda\rho_k\|_2^2 - (2\bar\mu+\bar\lambda)(\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \Lambda^2\rho_k)\\
&\qquad - (F_k, \Lambda\omega_k) + 2\gamma V'(t)(\rho_k, \Lambda\omega_k) + (\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \Lambda\omega_k) - (G_k, \Lambda\rho_k) + (\triangle_k(\mathbf{v}\cdot\nabla\underline{\omega}), \Lambda\rho_k).
\end{aligned}
\tag{3.7}
$$

Since

$$
|(\rho_k, \Lambda\omega_k)| \leqslant \frac{a}{2}\|\omega_k\|_2^2 + \frac{1}{2a}\|\Lambda\rho_k\|_2^2, \qquad |(\mathbf{I}\cdot\mathrm{div}E_k, \Lambda\rho_k)| \leqslant |\mathbf{I}|\left(\frac{b}{2}\|\Lambda E_k\|_2^2 + \frac{1}{2b}\|\Lambda\rho_k\|_2^2\right),
$$

we expect to find some a, b and β such that

$$
\beta < 2\bar\mu+\bar\lambda, \qquad 1/a < 2\bar\mu+\bar\lambda, \qquad \beta a < 1, \quad b\beta|\mathbf{I}| < \nu, \qquad |\mathbf{I}|/(2b\bar\rho) < P'(\bar\rho).
$$

In fact, we can choose

$$
a = 3/(4\bar\mu+2\bar\lambda), \qquad b = |\mathbf{I}|/(\bar\rho P'(\bar\rho)), \qquad 0 < \beta < \min((2\bar\mu+\bar\lambda)/2, \nu\bar\rho P'(\bar\rho)/|\mathbf{I}|^2).
$$

Denote

$$
\alpha_k^2 := P'(\bar\rho)\|\rho_k\|_2^2 + \|\omega_k\|_2^2 + \frac{1}{2}\|\Omega_k\|_2^2 + \frac{1}{2\bar\rho}\|E_k\|_2^2 + \beta(2\bar\mu+\bar\lambda)\|\Lambda\rho_k\|_2^2 - 2\beta(\rho_k, \Lambda\omega_k),
$$

then

$$
\alpha_k^2 \sim \|\rho_k\|_2^2 + \|\Lambda\rho_k\|_2^2 + \|\omega_k\|_2^2 + \|\Omega_k\|_2^2 + \|E_k\|_2^2.
$$

Thus, from (3.6) and (3.7), there exists a constant $\delta > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\alpha_k^2 + (\delta 2^{2k} + \gamma V'(t))\alpha_k^2 \leqslant P'(\bar\rho)(F_k, \rho_k) + \beta(2\bar\mu+\bar\lambda)(\Lambda F_k, \Lambda\rho_k) - \beta(F_k, \Lambda\omega_k)\\
&\qquad + (G_k, \omega_k) - \beta(G_k, \Lambda\rho_k) + \frac{1}{2}(J_k, \Omega_k) + \frac{1}{2\bar\rho}(K_k, E_k)\\
&\qquad - P'(\bar\rho)(\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \rho_k) - (\triangle_k(\mathbf{v}\cdot\nabla\underline{\omega}), \omega_k) - \beta(2\bar\mu+\bar\lambda)(\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \Lambda^2\rho_k)\\
&\qquad + \beta(\triangle_k(\mathbf{v}\cdot\nabla\underline{\rho}), \Lambda\omega_k) + \beta(\triangle_k(\mathbf{v}\cdot\nabla\underline{\omega}), \Lambda\rho_k)\\
&\quad \lesssim \alpha_k(\|F_k\|_2 + \|\Lambda F_k\|_2 + \|G_k\|_2 + \|J_k\|_2 + \|K_k\|_2 + \varepsilon_k 2^{-k(s-1)}\|\mathbf{v}\|_{B^{1+d/2}}\|\underline{\rho}\|_{B^{s-1}}\\
&\qquad + \varepsilon_k 2^{-k(s-1)}\|\mathbf{v}\|_{B^{1+d/2}}\|\underline{\omega}\|_{B^{s-1}} + \varepsilon_k 2^{-k(s-1)}\|\mathbf{v}\|_{B^{1+d/2}}\|\underline{\rho}\|_{B^s}\\
&\qquad + \varepsilon_k\|\mathbf{v}\|_{B^{1+d/2}}2^{-k(s-1)}(\|\underline{\omega}\|_{B^{s-1}} + \|\underline{\rho}\|_{B^s}))\\
&\quad \lesssim \alpha_k(\|F_k\|_2 + \|\Lambda F_k\|_2 + \|G_k\|_2 + \|J_k\|_2 + \|K_k\|_2 + \varepsilon_k 2^{-k(s-1)}V'(t)\|(\underline{\rho}, \underline{\omega})\|_{B^{s-1,s}\times B^{s-1}}).
\end{aligned}
\tag{3.8}
$$

where we have used Lemma 2.7 with $\sum_k \varepsilon_k \leqslant 1$ and $s \in (1-d/2, 1+d/2]$.

Dividing (3.8) by α_k, we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\alpha_k(t) + (\delta 2^{2k} + \gamma V')\alpha_k \lesssim \|F_k\|_2 + \|\Lambda F_k\|_2 + \|G_k\|_2 + \|J_k\|_2 + \|K_k\|_2 + \varepsilon_k 2^{-k(s-1)}V'(t)\|(\underline{\rho}, \underline{\omega})\|_{B^{s-1,s}\times B^{s-1}}.
$$

Integrating over $[0, t]$, we have

$$\begin{aligned}\alpha_k(t) + \delta 2^{2k}\int_0^t \alpha_k(\tau)\mathrm{d}\tau &\leqslant \alpha_k(0) + C\int_0^t (\|F_k(\tau)\|_2 + \|\Lambda F_k(\tau)\|_2 + \|G_k(\tau)\|_2 + \|J_k(\tau)\|_2 + \|K_k(\tau)\|_2)\mathrm{d}\tau \\ &\quad + C\int_0^t V'(\tau)[\varepsilon_k(\tau)2^{-k(s-1)}\|(\underline{\rho}, \underline{\omega})\|_{B^{s-1,s}\times B^{s-1}} - \gamma\alpha_k(\tau)]\mathrm{d}\tau \\ &\leqslant \alpha_k(0) + C\int_0^t (\|F_k(\tau)\|_2 + \|\Lambda F_k(\tau)\|_2 + \|G_k(\tau)\|_2 + \|J_k(\tau)\|_2 + \|K_k(\tau)\|_2)\mathrm{d}\tau, \end{aligned} \tag{3.9}$$

by taking γ so large that

$$\sum_{k\in\mathbb{Z}}[\varepsilon_k(\tau)2^{-k(s-1)}\|(\underline{\rho}, \underline{\omega})\|_{B^{s-1,s}\times B^{s-1}} - \gamma\alpha_k(\tau)] \leqslant 0.$$

Changing the functions $\underline{\rho}, \ldots$ into the original ones $\rho, \ldots$ and multiplying both side of (3.9) by $2^{k(s-1)}$, we conclude, after summation on k in $\mathbb{Z}$, that

$$\begin{aligned}&\|\rho\|_{\tilde{L}^\infty_T(B^{s-1,s})} + \|(\omega, \Omega, E)\|_{\tilde{L}^\infty_T(B^{s-1})} + \|\rho\|_{L^1_T(B^{s+1,s+2})} + \|(\omega, \Omega, E)\|_{L^1_T(B^{s+1})} \\ &\lesssim \mathrm{e}^{CV(t)}\left(\|\rho(0)\|_{B^{s-1,s}} + \|(\omega, \Omega, E)(0)\|_{B^{s-1}} + \int_0^t \mathrm{e}^{-CV(\tau)}(\|F(\tau)\|_{B^{s-1,s}} + \|(G, J, K)(\tau)\|_{B^{s-1}})\mathrm{d}\tau\right).\end{aligned}$$

Thus, we complete the proof of the proposition. □

4. Construction of the approximate solutions sequence

The following sections are devoted to the proof of the main theorem. We use the classical Friedrichs' regularization method to construct the approximate solutions.

Let us define the sequence of operators $(\mathbb{J}_n)_{n\in\mathbb{N}}$ by restricting the frequency of the function within the annuli $A(1/n, n)$, i.e. $B(O, n)\setminus B(O, 1/n)$ where $B(O, r)$ denotes the ball centered at the origin point O with the radius r,

$$\mathbb{J}_n f := \mathscr{F}^{-1}(\mathbf{1}_{A(1/n,n)}(\xi)\mathscr{F}f(\xi)).$$

Formally, we see that $\mathbb{J}_n f \to f$ as $n\to+\infty$ in the sense of L^2-norms.

With the help of this operator, we consider the following approximate system

$$\begin{cases}\partial_t\rho^n + \mathbb{J}_n(\mathbb{J}_n\mathbf{u}^n\cdot\nabla\mathbb{J}_n\rho^n) + \Lambda\mathbb{J}_n\omega^n = F^n,\\ \partial_t\omega^n + \mathbb{J}_n(\mathbb{J}_n\mathbf{u}^n\cdot\nabla\mathbb{J}_n\omega^n) - (2\bar{\mu}+\bar{\lambda})\Delta\mathbb{J}_n\omega^n - P'(\bar{\rho})\Lambda\mathbb{J}_n\rho^n - \bar{\rho}^{-1}(\mathbf{I}\cdot\mathrm{div}\mathbb{J}_nE^n) = G^n,\\ \partial_t\Omega^n - \bar{\mu}\Delta\mathbb{J}_n\Omega^n - \bar{\rho}^{-1}\mathbf{I}\cdot\nabla\mathbb{J}_nE^n = J^n,\\ \partial_tE^n - \nu\Delta\mathbb{J}_nE^n + \mathrm{curl}(\mathbb{J}_n\omega^n\mathbf{I}) - \mathbf{I}\cdot\nabla\mathbb{J}_n\Omega^n = K^n,\\ \mathbf{u}^n = -\Lambda^{-1}\nabla\omega^n - \Lambda^{-1}\mathrm{div}\Omega^n, \qquad \mathbf{H}^n = -\Lambda^{-1}\mathrm{div}E^n, \qquad \mathrm{div}\mathbf{H}^n = 0,\\ (\rho^n, \omega^n, \Omega^n, E^n)|_{t=0} = (\rho_n, \Lambda^{-1}\mathrm{div}\mathbf{u}_n, \Lambda^{-1}\mathrm{curl}\mathbf{u}_n, \Lambda^{-1}\mathrm{curl}\mathbf{H}_n),\end{cases} \tag{4.1}$$

where

$$\begin{aligned}&\rho_n = \mathbb{J}_n(\bar{\rho}^{-1}\tilde{\rho}_0 - 1), \qquad \mathbf{u}_n = \mathbb{J}_n\tilde{\mathbf{u}}_0, \qquad \mathbf{H}_n = \mathbb{J}_n\tilde{\mathbf{H}}_0, \qquad F^n = -\mathbb{J}_n(\mathbb{J}_n\rho^n\mathrm{div}\mathbb{J}_n\mathbf{u}^n),\\ &G^n = \mathbb{J}_n(\mathbb{J}_n\mathbf{u}^n\cdot\nabla\mathbb{J}_n\omega^n) + \mathbb{J}_n\Lambda^{-1}\mathrm{div}\tilde{G}^n, \qquad J^n = \mathbb{J}_n\Lambda^{-1}\mathrm{curl}\tilde{G}^n,\\ &\tilde{G}^n = -\frac{1}{2}\frac{1}{\bar{\rho}\zeta(\mathbb{J}_n\rho^n+1)}\mathbb{J}_n\nabla(|\mathbb{J}_n\mathbf{H}^n|^2) + \frac{\mathbb{J}_n\rho^n}{\bar{\rho}\zeta(\mathbb{J}_n\rho^n+1)}\nabla(\mathbf{I}\cdot\mathbb{J}_n\mathbf{H}^n) - \frac{\mathbb{J}_n\rho^n}{\bar{\rho}\zeta(\mathbb{J}_n\rho^n+1)}\mathbf{I}\cdot\nabla\mathbb{J}_n\mathbf{H}^n\\ &\qquad + \frac{1}{\bar{\rho}\zeta(\mathbb{J}_n\rho^n+1)}\mathbb{J}_n(\mathbb{J}_n\mathbf{H}^n\cdot\nabla\mathbb{J}_n\mathbf{H}^n) + \mathbb{J}_n\left[\frac{P'(\bar{\rho}\zeta(\mathbb{J}_n\rho^n+1))}{\zeta(\mathbb{J}_n\rho^n+1)} - P'(\bar{\rho})\right]\nabla\mathbb{J}_n\rho^n\\ &\qquad - \bar{\mu}\frac{\mathbb{J}_n\rho^n}{\zeta(\mathbb{J}_n\rho^n+1)}\Delta\mathbb{J}_n\mathbf{u}^n + (\bar{\mu}+\bar{\lambda})\frac{\mathbb{J}_n\rho^n}{\zeta(\mathbb{J}_n\rho^n+1)}\nabla\mathrm{div}\mathbb{J}_n\mathbf{u}^n - \mathbb{J}_n\mathbf{u}^n\cdot\nabla\mathbb{J}_n\mathbf{u}^n,\\ &K^n = \mathbb{J}_n\Lambda^{-1}\mathrm{curl}(-\mathbb{J}_n\mathbf{u}^n\cdot\nabla\mathbb{J}_n\mathbf{H}^n - (\mathrm{div}\mathbb{J}_n\mathbf{u}^n)\mathbb{J}_n\mathbf{H}^n + \mathbb{J}_n\mathbf{H}^n\cdot\nabla\mathbb{J}_n\mathbf{u}^n).\end{aligned}$$

Here $\zeta(\cdot)$ is a smooth function satisfying

$$\zeta(f) = \begin{cases}1/4, & |f|\leqslant 1/4,\\ f, & 1/2\leqslant|f|\leqslant 3/2,\\ 7/4, & |f|\geqslant 7/4,\\ \text{smooth}, & \text{otherwise}.\end{cases}$$

Now, we show that (4.1) is only an ordinary differential equation in $(L^2)^{1+1+d\times d+d\times d}$ w.r.t. the time variable t. First, we can observe easily that all the source terms in (4.1) turn out to be continuous in $(L^2)^{1+1+d\times d+d\times d}$ for each fixed n. For instance, we show the term $\mathbb{J}_n\Lambda^{-1}\mathrm{div}(\frac{1}{\zeta(\mathbb{J}_n\rho^n+1)}\mathbb{J}_n\nabla(|\mathbb{J}_n\mathbf{H}^n|^2))$ and the other terms can be dealt with in similar ways. By Plancherel's theorem, Hölder's inequality and Hausdorff–Young's inequality, we have

$$\begin{aligned}\left\|\mathbb{J}_n\Lambda^{-1}\mathrm{div}\left(\frac{1}{\zeta(\mathbb{J}_n\rho^n+1)}\mathbb{J}_n\nabla(|\mathbb{J}_n\mathbf{H}^n|^2)\right)\right\|_2 &= \left\|\mathbf{1}_{A(1/n,n)}(\xi)|\xi|^{-1}\xi\cdot\mathscr{F}\left(\frac{1}{\zeta(\mathbb{J}_n\rho^n+1)}\mathbb{J}_n\nabla(|\mathbb{J}_n\mathbf{H}^n|^2)\right)\right\|_2\\ &\leqslant \left\|\frac{1}{\zeta(\mathbb{J}_n\rho^n+1)}\mathbb{J}_n\nabla(|\mathbb{J}_n\mathbf{H}^n|^2)\right\|_2 \leqslant \|\zeta(\mathbb{J}_n\rho^n+1)\|_\infty\|\mathbb{J}_n\nabla(|\mathbb{J}_n\mathbf{H}^n|^2)\|_2\\ &\leqslant 4\|\mathbf{1}_{A(1/n,n)}(\xi)\xi\mathscr{F}|\mathbb{J}_n\mathbf{H}^n|^2\|_2 \leqslant 4n\|\,|\mathbb{J}_n\mathbf{H}^n|^2\|_2 \leqslant 4n\|\mathbb{J}_n\mathbf{H}^n\|_\infty\|\mathbb{J}_n\mathbf{H}^n\|_2\\ &\leqslant 4n\|\mathbf{1}_{A(1/n,n)}(\xi)\mathscr{F}\mathbf{H}^n\|_1\|\mathbf{H}^n\|_2 \leqslant 4n\|\mathbf{1}_{A(1/n,n)}(\xi)\|_2\|\mathbf{H}^n\|_2^2\\ &\lesssim n^{1+d/2}\|\mathbf{H}^n\|_2^2.\end{aligned}$$

Thus, the usual Cauchy–Lipschitz theorem implies the existence of a strictly positive maximal time T_n such that there is a unique solution which is continuous in time with value in $(L^2)^{1+1+d\times d+d\times d}$. Moreover, due to the fact $\mathbb{J}_n^2=\mathbb{J}_n$, we obtain that $\mathbb{J}_n(\rho^n,\omega^n,\Omega^n,E^n)$ is also a solution, thus the uniqueness implies that $\mathbb{J}_n(\rho^n,\omega^n,\Omega^n,E^n)=(\rho^n,\omega^n,\Omega^n,E^n)$. Hence, $(\rho^n,\omega^n,\Omega^n,E^n)$ is also a solution of the following system

$$\begin{cases}\partial_t\rho^n+\mathbb{J}_n(\mathbf{u}^n\cdot\nabla\rho^n)+\Lambda\omega^n=F_1^n,\\ \partial_t\omega^n+\mathbb{J}_n(\mathbf{u}^n\cdot\nabla\omega^n)-(2\bar{\mu}+\bar{\lambda})\Delta\omega^n-P'(\bar{\rho})\Lambda\rho^n-\bar{\rho}^{-1}(\mathbf{I}\cdot\mathrm{div}E^n)=G_1^n,\\ \partial_t\Omega^n-\bar{\mu}\Delta\Omega^n-\bar{\rho}^{-1}\mathbf{I}\cdot\nabla E^n=J_1^n,\\ \partial_tE^n-\nu\Delta E^n+\mathrm{curl}(\omega^n\mathbf{I})-\mathbf{I}\cdot\nabla\Omega^n=K_1^n,\\ \mathbf{u}^n=-\Lambda^{-1}\nabla\omega^n-\Lambda^{-1}\mathrm{div}\Omega^n,\qquad \mathbf{H}^n=-\Lambda^{-1}\mathrm{div}E^n,\quad \mathrm{div}\mathbf{H}^n=0,\\ (\rho^n,\omega^n,\Omega^n,E^n)|_{t=0}=(\rho_n,\Lambda^{-1}\mathrm{div}\mathbf{u}_n,\Lambda^{-1}\mathrm{curl}\mathbf{u}_n,\Lambda^{-1}\mathrm{curl}\mathbf{H}_n),\end{cases}\tag{4.2}$$

where

$$\begin{aligned}&F_1^n=-\mathbb{J}_n(\rho^n\mathrm{div}\mathbf{u}^n),\qquad G_1^n=\mathbb{J}_n(\mathbf{u}^n\cdot\nabla\omega^n)+\mathbb{J}_n\Lambda^{-1}\mathrm{div}\tilde{G}_1^n,\qquad J_1^n=\mathbb{J}_n\Lambda^{-1}\mathrm{curl}\tilde{G}_1^n,\\ &\tilde{G}_1^n = -\frac{1}{2}\frac{1}{\bar{\rho}\zeta(\rho^n+1)}\mathbb{J}_n\nabla(|\mathbf{H}^n|^2)+\frac{\rho^n}{\bar{\rho}\zeta(\rho^n+1)}\nabla(\mathbf{I}\cdot\mathbf{H}^n)-\frac{\rho^n}{\bar{\rho}\zeta(\rho^n+1)}\mathbf{I}\cdot\nabla\mathbf{H}^n\\ &\qquad+\frac{1}{\bar{\rho}\zeta(\rho^n+1)}\mathbb{J}_n(\mathbf{H}^n\cdot\nabla\mathbf{H}^n)+\mathbb{J}_n\left[\frac{P'(\bar{\rho}\zeta(\rho^n+1))}{\zeta(\rho^n+1)}-P'(\bar{\rho})\right]\nabla\rho^n\\ &\qquad-\bar{\mu}\frac{\rho^n}{\zeta(\rho^n+1)}\Delta\mathbf{u}^n+(\bar{\mu}+\bar{\lambda})\frac{\rho^n}{\zeta(\rho^n+1)}\nabla\mathrm{div}\mathbf{u}^n-\mathbf{u}^n\cdot\nabla\mathbf{u}^n,\\ &K_1^n=\mathbb{J}_n\Lambda^{-1}\mathrm{curl}(-\mathbf{u}^n\cdot\nabla\mathbf{H}^n-(\mathrm{div}\mathbf{u}^n)\mathbf{H}^n+\mathbf{H}^n\cdot\nabla\mathbf{u}^n).\end{aligned}$$

The system (4.2) appears to be an ordinary differential equation in the space

$$L_n^2:=\{f\in L^2(\mathbb{R}^d):\ \mathrm{supp}\mathscr{F}f\subset A(1/n,n)\}.$$

Due to the Cauchy–Lipschitz theorem again, there is a unique maximal solution on an interval $[0,T_n^*)$ which is continuous in time with value in $(L_n^2)^{1+1+d\times d+d\times d}$.

5. Uniform bounds

In this section, we prove uniform estimates which is independent of $T<T_n^*$ in the space $E^{d/2}$ for $(\rho^n,\mathbf{u}^n,\mathbf{H}^n)$. We will show that $T_n^*=+\infty$ by the Cauchy–Lipschitz theorem. Firstly, we define the functional space for $1-d/2<s\leqslant 1+d/2$

$$\begin{aligned}&E^s=\mathcal{C}(\mathbb{R}^+;B^{s-1,s}\times(B^{s-1})^{d+d})\cap L^1(\mathbb{R}^+;B^{s+1,s+2}\times(B^{s+1})^{d+d}),\\ &\|(\rho,\mathbf{u},\mathbf{H})\|_{E^s}=\|\rho\|_{\tilde{L}^\infty(B^{s-1,s})\cap L^1(B^{s+1,s+2})}+\|\mathbf{u}\|_{\tilde{L}^\infty(B^{s-1})\cap L^1(B^{s+1})}+\|\mathbf{H}\|_{\tilde{L}^\infty(B^{s-1})\cap L^1(B^{s+1})}.\end{aligned}$$

For the case of a finite interval $[0,T]$, we denote by E_T^s and $\|\cdot\|_{E_T^s}$ the corresponding spaces and norms. Now, we denote

$$\begin{aligned}&E(0):=\bar{\rho}^{-1}\|\tilde{\rho}_0-1\|_{B^{d/2-1,d/2}}+\|\mathbf{u}_0\|_{B^{d/2-1}}+\|\mathbf{H}_0\|_{B^{d/2-1}},\\ &E(\rho,\mathbf{u},\mathbf{H},t):=\|(\rho,\mathbf{u},\mathbf{H})\|_{E_t^{d/2}},\\ &\tilde{T}_n:=\sup\{t\in[0,T_n^*):\ E(\rho^n,\mathbf{u}^n,\mathbf{H}^n,t)\leqslant abE(0)\},\end{aligned}$$

where b corresponds to the constant in Proposition 3.1 and $a>\max(2,1/b)$ is a constant. Thus, by the continuity we get $\tilde{T}_n>0$.

We are going to prove that $\tilde{T}_n=T_n^*$ for all $n\in\mathbb{N}$ and we will conclude that $T_n^*=+\infty$ for any $n\in\mathbb{N}$.

According to Proposition 3.1 and to the definition of $(\rho_n, \mathbf{u}_n, \mathbf{H}_n)$, we have

$$\|(\rho^n, \mathbf{u}^n, \mathbf{H}^n)\|_{E_T^{d/2}} \leqslant b e^{b\|\mathbf{u}^n\|_{L_T^1(B^{1+d/2})}} (\bar{\rho}^{-1}\|\tilde{\rho}_0 - 1\|_{B^{d/2-1,d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}} + \|\mathbf{H}_0\|_{B^{d/2-1}} + \|F_1^n\|_{L_T^1(B^{d/2-1,d/2})} + \|\mathbf{u}^n \cdot \nabla\omega^n\|_{L_T^1(B^{d/2-1})} + \|\tilde{G}_1^n\|_{L_T^1(B^{d/2-1})} + \|K_1^n\|_{L_T^1(B^{d/2-1})}).$$

Thus, we only need to prove the appropriate estimates for F_1^n, $\tilde{G}_1^n$ and the convection term. Denote the continuity modulus of the embedding relation $B^{d/2}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ by the constant c, then we assume that $E(0)$ satisfies the condition

$$2abcE(0) \leqslant 1. \tag{5.1}$$

If $T < \tilde{T}_n$, it implies

$$\|\rho^n\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant c\|\rho^n\|_{L_T^\infty(B^{d/2})} \leqslant c\|\rho^n\|_{\tilde{L}_T^\infty(B^{d/2-1,d/2})} \leqslant abcE(0) \leqslant 1/2. \tag{5.2}$$

Thus, $\rho^n + 1 \in [1/2, 3/2]$ and then $\zeta(\rho^n + 1) = \rho^n + 1$. By Lemma 2.6, we get

$$\|F_1^n\|_{L_T^1(B^{d/2-1,d/2})} = \|\rho^n \mathrm{div}\mathbf{u}^n\|_{L_T^1(B^{d/2-1,d/2})} \lesssim \|\rho^n\|_{\tilde{L}_T^\infty(B^{d/2-1,d/2})}\|\mathrm{div}\mathbf{u}^n\|_{L_T^1(B^{d/2})} \lesssim \|\rho^n\|_{\tilde{L}_T^\infty(B^{d/2-1,d/2})}\|\mathbf{u}^n\|_{L_T^1(B^{d/2+1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T), \tag{5.3}$$

and

$$\|\mathbf{u}^n \cdot \nabla\omega^n\|_{L_T^1(B^{d/2-1})} \lesssim \|\mathbf{u}^n\|_{\tilde{L}_T^\infty(B^{d/2-1})}\|\nabla\omega^n\|_{L_T^1(B^{d/2})} \lesssim \|\mathbf{u}^n\|_{\tilde{L}_T^\infty(B^{d/2-1})}\|\mathbf{u}^n\|_{L_T^1(B^{d/2+1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T), \tag{5.4}$$

which also yields, at the same way, the estimates $\|\mathbf{u}^n \cdot \nabla\mathbf{u}^n\|_{L_T^1(B^{d/2-1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T)$ and $\|\frac{1}{\bar{\rho}\zeta(\rho^n+1)}\mathbb{J}_n(\mathbf{H}^n \cdot \nabla\mathbf{H}^n)\|_{L_T^1(B^{d/2-1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T)$ for the last term and $\frac{1}{\bar{\rho}\zeta(\rho^n+1)}\mathbb{J}_n(\mathbf{H}^n \cdot \nabla\mathbf{H}^n)$ in $\tilde{G}_1^n$, and $\|K_1^n\|_{L_T^1(B^{d/2-1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T)$ for K_1^n. By Lemma 2.6 and Sobolev's embedding theorem, we have

$$\left\|\frac{1}{2}\frac{1}{\bar{\rho}\zeta(\rho^n+1)}\mathbb{J}_n\nabla(|\mathbf{H}^n|^2)\right\|_{L_T^1(B^{d/2-1})} \lesssim \|\,|\mathbf{H}^n|^2\|_{L_T^1(B^{d/2})} \lesssim \|\mathbf{H}^n\|^2_{\tilde{L}_T^2(B^{d/2})} \lesssim \|\mathbf{H}^n\|_{\tilde{L}_T^\infty(B^{d/2-1})}\|\mathbf{H}^n\|_{L_T^1(B^{d/2+1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T). \tag{5.5}$$

From Lemma 2.6, we get

$$\left\|\frac{\rho^n}{\bar{\rho}\zeta(\rho^n+1)}\nabla(\mathbf{I}\cdot\mathbf{H}^n)\right\|_{L_T^1(B^{d/2-1})} \lesssim \|\rho^n\|_{\tilde{L}_T^\infty(B^{d/2-1})}\|\mathbf{I}\cdot\mathbf{H}^n\|_{L_T^1(B^{d/2+1})} \lesssim \|\rho^n\|_{\tilde{L}_T^\infty(B^{d/2-1,d/2})}\|\mathbf{H}^n\|_{L_T^1(B^{d/2+1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T). \tag{5.6}$$

In a similar way, we can also obtain $\|\frac{\rho^n}{\bar{\rho}\zeta(\rho^n+1)}\mathbf{I}\cdot\nabla\mathbf{H}^n\|_{L_T^1(B^{d/2-1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T)$. With the help of Lemma 2.3, 2.6 and the Sobolev embedding theorem, we have

$$\left\|\mathbb{J}_n\left[\frac{P'(\bar{\rho}\zeta(\rho^n+1))}{\zeta(\rho^n+1)} - P'(\bar{\rho})\right]\nabla\rho^n\right\|_{L_T^1(B^{d/2-1})} \lesssim \|\rho\|^2_{\tilde{L}_T^2(B^{d/2})} \lesssim \|\rho\|_{\tilde{L}_T^\infty(B^{d/2-1})}\|\rho\|_{L_T^1(B^{d/2+1})} \lesssim \|\rho\|_{\tilde{L}_T^\infty(B^{d/2-1,d/2})}\|\rho\|_{L_T^1(B^{d/2+1,d/2+2})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T). \tag{5.7}$$

Due to Lemma 2.6, it yields

$$\left\|-\bar{\mu}\frac{\rho^n}{\zeta(\rho^n+1)}\Delta\mathbf{u}^n + (\bar{\mu}+\bar{\lambda})\frac{\rho^n}{\zeta(\rho^n+1)}\nabla\mathrm{div}\mathbf{u}^n\right\|_{L_T^1(B^{d/2-1})} \lesssim \|\rho^n\|_{\tilde{L}_T^\infty(B^{d/2})}\|\mathbf{u}^n\|_{L_T^1(B^{d/2+1})} \lesssim E^2(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T). \tag{5.8}$$

Thus, we have obtained

$$\|(\rho^n, \mathbf{u}^n, \mathbf{H}^n)\|_{E_T^{d/2}} \leqslant b e^{b\|\mathbf{u}^n\|_{L_T^1(B^{1+d/2})}}(1 + Ca^2b^2E(0))E(0). \tag{5.9}$$

If we choose $E(0)$ small enough such that

$$1 + Ca^2b^2E(0) \leqslant a^2/(a+2), \qquad e^{ab^2E(0)} \leqslant 1 + 1/a, \qquad 2abcE(0) \leqslant 1, \tag{5.10}$$

then $\|(\rho^n, \mathbf{u}^n, \mathbf{H}^n)\|_{E_T^{d/2}} \leqslant ab(a+1)E(0)/(a+2)$ for any $T < \tilde{T}_n$. It follows that $\tilde{T}_n = T_n^*$. Indeed, if $\tilde{T}_n < T_n^*$, we have shown that $E(\rho^n, \mathbf{u}^n, \mathbf{H}^n, \tilde{T}_n) \leqslant ab(a+1)E(0)/(a+2)$. Thus, by the continuity, for a sufficiently small constant $\sigma > 0$, we can obtain $E(\rho^n, \mathbf{u}^n, \mathbf{H}^n, \tilde{T}_n + \sigma) \leqslant abE(0)$. It yields a contradiction with the definition of $\tilde{T}_n$.

Now, we show the solution $(\rho^n, \mathbf{u}^n, \mathbf{H}^n)_{n\in\mathbb{N}}$ of the approximate system is global in time. In fact, if $T_n^* < \infty$, then we have $E(\rho^n, \mathbf{u}^n, \mathbf{H}^n, T_n^*) \leqslant abE(0)$ as derived above. Thus, it implies $\|\rho^n\|_{L^\infty_{T_n^*}(B^{d/2-1,d/2})} < \infty$ and $\|(\mathbf{u}^n, \mathbf{H}^n)\|_{L^\infty_{T_n^*}(B^{d/2-1})} < \infty$ and then $\|(\rho^n, \mathbf{u}^n, \mathbf{H}^n)\|_{L^\infty_{T_n^*}(L^2_n)} < \infty$. Hence, we can extend the solution beyond T_n^* by the Cauchy–Lipschitz theorem. This contradicts the definition of T_n^*. Therefore, we conclude that $T_n^* = +\infty$.

6. Existence and uniqueness of the solution

In this section, we shall show the existence part of the main theorem. In other words, we prove that the sequence $(\rho^n, \mathbf{u}^n, \mathbf{H}^n)_{n\in\mathbb{N}}$ converges, up to an extraction, in $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$ to a solution $(\rho, \mathbf{u}, \mathbf{H})$ of (3.2) which has the desired regularity properties.

Roughly speaking, the proof is based on the standard compactness arguments (e.g., [9,12,14]), but we have to show some subtle and lengthy details. Firstly, we need to show that the first order derivative of $(\rho^n, \mathbf{u}^n, \mathbf{H}^n)$ w.r.t. the time variable is uniformly bounded in appropriate spaces. It enables us to apply Arzelà–Ascoli's theorem and get the existence of a limit $(\rho, \mathbf{u}, \mathbf{H})$ for a subsequence. Secondly, the uniform bounds proved in the previous section provides us with additional regularity and convergence properties such that we can pass to the limit in the system.

We first split the approximate solutions sequence $(\rho^n, \mathbf{u}^n, \mathbf{H}^n)$ into the solutions sequence of the corresponding linear system with initial data $(\rho_n, \mathbf{u}_n, \mathbf{H}_n)$, and the discrepancy to that solutions sequence. More precisely, we denote by $(\rho_L^n, \mathbf{u}_L^n, \mathbf{H}_L^n)$ the solution to the linear system

$$\begin{cases} \partial_t \rho_L^n + \mathrm{div}\mathbf{u}_L^n = 0, \\ \partial_t \mathbf{u}_L^n - \bar{\mu}\Delta \mathbf{u}_L^n - (\bar{\mu}+\bar{\lambda})\nabla \mathrm{div}\mathbf{u}_L^n + P'(\bar{\rho})\nabla\rho_L^n - \bar{\rho}^{-1}\nabla(\mathbf{I}\cdot\mathbf{H}_L^n) - \bar{\rho}^{-1}\mathbf{I}\cdot\nabla\mathbf{H}_L^n = 0, \\ \partial_t \mathbf{H}_L^n - \nu\Delta\mathbf{H}_L^n + (\mathrm{div}\mathbf{u}_L^n)\mathbf{I} - \mathbf{I}\cdot\nabla\mathbf{u}_L^n = 0, \qquad \mathrm{div}\mathbf{H}_L^n = 0, \\ (\rho_L^n, \mathbf{u}_L^n, \mathbf{H}_L^n)|_{t=0} = (\rho_n, \mathbf{u}_n, \mathbf{H}_n), \end{cases} \tag{6.1}$$

and $(\rho_D^n, \mathbf{u}_D^n, \mathbf{H}_D^n) = (\rho^n - \rho_L^n, \mathbf{u}^n - \mathbf{u}_L^n, \mathbf{H}^n - \mathbf{H}_L^n)$.

Obviously, the definition od $(\rho_n, \mathbf{u}_n, \mathbf{H}_n)$ entails

$$\rho_n \to \bar{\rho}^{-1}(\tilde{\rho}_0 - 1) \text{ in } B^{d/2-1,d/2}, \qquad \mathbf{u}_n \to \mathbf{u}_0 \quad \text{and} \quad \mathbf{H}_n \to \mathbf{H}_0 \text{ in } B^{d/2-1}, \quad \text{as } n \to \infty.$$

Proposition 3.1 insure us that

$$(\rho_L^n, \mathbf{u}_L^n, \mathbf{H}_L^n) \to (\rho_L, \mathbf{u}_L, \mathbf{H}_L) \text{ in } E^{d/2}, \quad \text{as } n \to \infty, \tag{6.2}$$

where $(\rho_L, \mathbf{u}_L, \mathbf{H}_L)$ satisfies the following linear system

$$\begin{cases} \partial_t \rho_L + \mathrm{div}\mathbf{u}_L = 0, \\ \partial_t \mathbf{u}_L - \bar{\mu}\Delta \mathbf{u}_L - (\bar{\mu}+\bar{\lambda})\nabla \mathrm{div}\mathbf{u}_L + P'(\bar{\rho})\nabla\rho_L - \bar{\rho}^{-1}\nabla(\mathbf{I}\cdot\mathbf{H}_L) - \bar{\rho}^{-1}\mathbf{I}\cdot\nabla\mathbf{H}_L = 0, \\ \partial_t \mathbf{H}_L - \nu\Delta\mathbf{H}_L + (\mathrm{div}\mathbf{u}_L)\mathbf{I} - \mathbf{I}\cdot\nabla\mathbf{u}_L = 0, \qquad \mathrm{div}\mathbf{H}_L = 0, \\ (\rho_L, \mathbf{u}_L, \mathbf{H}_L)|_{t=0} = (\bar{\rho}^{-1}(\tilde{\rho}_0 - 1), \mathbf{u}_0, \mathbf{H}_0). \end{cases} \tag{6.3}$$

Now, we have to prove the convergence of the discrepancy sequence $(\rho_D^n, \mathbf{u}_D^n, \mathbf{H}_D^n)$.

Lemma 6.1. *Let $d \geqslant 3$, then $(\rho_D^n, \mathbf{u}_D^n, \mathbf{H}_D^n)_{n\in\mathbb{N}}$ is uniformly bounded in*

$$\mathcal{C}^{1/2}(\mathbb{R}^+; B^{d/2-1}) \times (\mathcal{C}^{1/5}(\mathbb{R}^+; B^{d/2-7/5}))^{d+d}.$$

Proof. Since $\partial_t\rho_D^n = -\mathbb{J}_n(\mathbf{u}^n\cdot\nabla\rho^n) - \mathbb{J}_n(\rho^n \mathrm{div}\mathbf{u}^n) - \mathrm{div}(\mathbf{u}^n - \mathbf{u}_L^n)$, we can obtain $\partial_t\rho_D^n$ is uniformly bounded in $\tilde{L}^2(B^{d/2-1})$ by the fact $\rho^n \in \tilde{L}^\infty(B^{d/2})$ and $\mathbf{u}^n, \mathbf{u}_L^n \in \tilde{L}^2(B^{d/2})$.

Let $\omega_L^n = \Lambda^{-1}\mathrm{div}\mathbf{u}_L^n$, $\Omega_L^n = \Lambda^{-1}\mathrm{curl}\mathbf{u}_L^n$ and $E_L^n = \Lambda^{-1}\mathrm{curl}\mathbf{H}_L^n$, then $\partial_t\omega_D^n = (2\bar{\mu}+\bar{\lambda})\Delta(\omega^n - \omega_L^n) + P'(\bar{\rho})\Lambda(\rho^n - \rho_L^n) + \bar{\rho}^{-1}(\mathbf{I}\cdot\mathrm{div}(E^n - E_L^n)) + \mathbb{J}_n\Lambda^{-1}\mathrm{div}\tilde{G}_1^n$, $\partial_t\Omega^n = \bar{\mu}\Delta(\Omega^n - \Omega_L^n) + \bar{\rho}^{-1}\mathbf{I}\cdot\nabla(E^n - E_L^n) + J_1^n$ and $\partial_t E^n = \nu\Delta(E^n - E_L^n) - \mathrm{curl}((\omega^n - \omega_L^n)\mathbf{I}) + \mathbf{I}\cdot\nabla(\Omega^n - \Omega_L^n) + K_1^n$. By interpolation arguments and Lemma 2.6, we have $\partial_t\omega_D^n$, $\partial_t\Omega_D^n$ and $\partial_t E_D^n$ uniformly bounded in $(\tilde{L}^{5/4} + \tilde{L}^{10/3})(B^{d/2-7/5})$ via a tedious and lengthy computation. So is $\partial_t\mathbf{u}_D^n$ and $\partial_t E_D^n$ in view of the relations $\mathbf{u}_D^n = -\Lambda^{-1}\nabla\omega_D^n - \Lambda^{-1}\mathrm{div}\Omega_D^n$, and $\mathbf{H}_D^n = -\Lambda^{-1}\mathrm{div}E_D^n$.

Thus, it is easy to obtain the desired results. □

Next, we can turn to the proof of the existence of a solution and use Arzelà–Ascoli theorem to get strong convergence. We need to localize the spatial space because we have some results of compactness for the local Sobolev spaces. Let $(\chi_p)_{p\in\mathbb{N}}$ be a sequence of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ cut-off functions supported in the ball $B(O, p+1)$ of $\mathbb{R}^d$ and equal to 1 in a neighborhood of $B(O, p)$.

From Lemma 6.1, it implies that $(\chi_p \rho_D^n, \chi_p \mathbf{u}_D^n, \chi_p \mathbf{H}_D^n)_{n\in\mathbb{N}}$ is uniformly equicontinuous in $\mathcal{C}(\mathbb{R}^+; B^{d/2-1} \times (B^{d/2-7/5})^{d+d})$ for any $p \in \mathbb{N}$. It is easy to see that the mapping $f \mapsto \chi_p f$ is compact from $B^{d/2-1,d/2}$ into $B^{d/2-1}$ and from $B^{d/2-1}$ into $B^{d/2-7/5}$. On applying Arzelà–Ascoli theorem to the family $(\chi_p \rho_D^n, \chi_p \mathbf{u}_D^n, \chi_p \mathbf{H}_D^n)_{n\in\mathbb{N}}$ on the time interval $[0, p]$ and using Cantor's diagonal process, it yields that a subsequence (which we still denote by $(\chi_p \rho_D^n, \chi_p \mathbf{u}_D^n, \chi_p \mathbf{H}_D^n)_{n\in\mathbb{N}}$) converges to a distribution $(\chi_p \rho_D, \chi_p \mathbf{u}_D, \chi_p \mathbf{H}_D)$ in $\mathcal{C}(\mathbb{R}^+; B^{d/2-1} \times (B^{d/2-7/5})^{d+d})$ for all $p \in \mathbb{N}$. It infers that $(\rho_D^n, \mathbf{u}_D^n, \mathbf{H}_D^n)$ converges to $(\rho_D, \mathbf{u}_D, \mathbf{H}_D)$ in the sense of $\mathscr{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$.

Going back to the uniform estimates of the previous sections, we can get that $(\rho_D, \mathbf{u}_D, \mathbf{H}_D)$ belongs to $\tilde{L}^\infty(\mathbb{R}^+; B^{d/2-1,d/2} \times (B^{d/2-1})^{d+d}) \cap L^1(\mathbb{R}^+; B^{d/2+1,d/2+2} \times (B^{d/2+1})^{d+d})$ and also to $\mathcal{C}^{1/2}(\mathbb{R}^+; B^{d/2-1}) \times (\mathcal{C}^{1/5}(\mathbb{R}^+; B^{d/2-7/5}))^{d+d}$. In a standard way (cf. [9,13,14]), we can show that $(\rho, \mathbf{u}, \mathbf{H}) := (\rho_L, \mathbf{u}_L, \mathbf{H}_L) + (\rho_D, \mathbf{u}_D, \mathbf{H}_D)$ solves the system (3.1). The continuities of ρ, $\mathbf{u}$ and $\mathbf{H}$ in $B^{d/2-1}$ are straightforward from (3.1) with the help of the interpolation theory of homogeneous Besov spaces.

There remains to prove the uniqueness of the solution. By the standard method, we can prove the uniqueness over a small time interval $[0, T]$. Since $\|\rho\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} \leqslant 1/2$ and $\|\rho\|_{\tilde{L}^\infty(\mathbb{R}^+; B^{d/2-1,d/2})} \leqslant abE(0)$ are uniform for all time, we can prove the uniqueness in $[T, 2T], [2T, 3T], \ldots$, and so on. Then, the solution is unique in $[0, \infty)$.

Therefore, we complete the proof of the main theorem.

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