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REMARKS ON THE FREE BOUNDARY PROBLEM OF COMPRESSIBLE EULER EQUATIONS IN PHYSICAL VACUUM WITH GENERAL INITIAL DENSITIES

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Abstract. In this paper, we establish a priori estimates for three-dimensional compressible Euler equations with the moving physical vacuum boundary, the γ-gas law equation of state for $\gamma = 2$ and the general initial density $\rho_0 \in H^5$. Because of the degeneracy of the initial density, we investigate the estimates of the horizontal spatial and time derivatives and then obtain the estimates of the normal or full derivatives through the elliptic-type estimates. We derive a mixed space-time interpolation inequality which plays a vital role in our energy estimates and obtain some extra estimates for the space-time derivatives of the velocity in L^3.

1. **Introduction.** In recent years, the motion of physical vacuum in compressible fluids has been received much attention due to its great physical importance and mathematical challenges (cf. [14–18, 21]). Some significant progresses have been made particularly on the Euler equations (cf. [4,6,7,9,10]). Physical vacuum problems arise in many physical situations naturally, for example, in the study of the evolution and structure of gaseous stars (cf. [1,8]) for which vacuum boundaries are natural boundaries.

In the present paper, we consider the following compressible Euler equations

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad \text{in } \Omega(t)\times(0,T], \tag{1.1a}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0 \qquad \text{in } \Omega(t)\times(0,T], \tag{1.1b}$$

$$p = 0 \qquad \text{on } \Gamma_1(t)\times(0,T], \tag{1.1c}$$

$$u_3 = 0 \qquad \text{on } \Gamma_0\times(0,T], \tag{1.1d}$$

$$\partial_t \Gamma_1(t) = \mathcal{V}(\Gamma_1(t)) = u\cdot\mathcal{N} \qquad \text{in } (0,T], \tag{1.1e}$$

$$(\rho, u) = (\rho_0, u_0) \qquad \text{in } \Omega\times\{t=0\}, \tag{1.1f}$$

$$\Omega(0) = \Omega,\ \Gamma_1(0) = \Gamma_1, \tag{1.1g}$$

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where ρ denotes the density, the vector field $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ denotes the Eulerian velocity field, and p denotes the pressure function. $\nabla = (\partial_1, \partial_2, \partial_3)$ and div are the usual gradient operator and spatial divergence in the three-dimensional space where $\partial_i = \partial/\partial x_i$. The open, bounded domain $\Omega(t) \subset \mathbb{R}^3$ denotes the changing volume occupied by the fluid, $\Gamma_1(t)$ denotes the moving vacuum boundary, $\mathcal{V}(\Gamma_1(t))$ denotes the normal velocity of $\Gamma_1(t)$, $\mathcal{N}$ denotes the outward unit normal vector to the boundary $\Gamma_1(t)$, and Γ_0 is a fixed boundary. The equation of the pressure $p(\rho)$ is given by

$$p(x,t) = C_\gamma \rho^\gamma(x,t), \tag{1.2}$$

where γ is the adiabatic index, C_γ is the adiabatic constant which we set to unity, i.e., $C_\gamma = 1$; and

$$\rho > 0 \quad \text{in } \Omega(t) \quad \text{and} \quad \rho = 0 \text{ on } \Gamma_1(t). \tag{1.3}$$

For simplicity, we will only consider the case $\gamma = 2$ in this paper, which can supply some helpful information and a better understanding for the discussion of general cases of $\gamma > 1$ (e.g. Remark 2).

Equation (1.1a) is the conservation of the mass, (1.1b) is the conservation of the momentum. The boundary condition (1.1c) states that the pressure (and hence the density) vanishes along the vacuum boundary, (1.1d) describes that the normal component of the velocity vanishes on the fixed boundary Γ_0, (1.1e) indicates that the vacuum boundary is moving with the normal component of the fluid velocity, (1.1f) and (1.1g) are the initial conditions for the density, velocity, domain and boundary.

To avoid the use of local coordinate charts necessary for arbitrary geometries, for simplicity, we assume that the initial domain $\Omega \subset \mathbb{R}^3$ at time $t = 0$ is given by

$$\Omega = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{T}^2,\ x_3 \in (0,1)\right\},$$

where $\mathbb{T}^2$ denotes the 2-torus, which can be thought of as the unit square with periodic boundary conditions. This permits the use of one global Cartesian coordinate system. At $t = 0$, the reference vacuum boundary is the top boundary

$$\Gamma_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{T}^2,\ x_3 = 1\},$$

while the bottom boundary

$$\Gamma_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{T}^2,\ x_3 = 0\}$$

is fixed with the boundary condition (1.1d).

We set the unit normal vectors $N = (0,0,1)$ on Γ_1 and $N = (0,0,-1)$ on Γ_0. We use the standard basis on $\mathbb{R}^3$: $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Similarly, the unit tangent vectors on $\Gamma = \Gamma_0 \cup \Gamma_1$ are given by

$$T_1 = (1,0,0) \quad \text{and} \quad T_2 = (0,1,0).$$

Throughout the paper, the repeated indices will stand for the summation with respect to them in a monomial expression. The k^{th}-partial derivative of F will be denoted by $F_{,k} = \frac{\partial F}{\partial x^k}$. Then we have

$$(\operatorname{div}(\rho u \otimes u))^j = (\rho u^i u^j)_{,i} = \operatorname{div}(\rho u) u^j + \rho u \cdot \nabla u^j,$$

which yields that (1.1b) can be rewritten, in view of (1.1a), as

$$\rho(\partial_t u + u \cdot \nabla u) + \nabla p = 0.$$

Thus, the system (1.1) can be rewritten as

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad \text{in } \Omega(t)\times(0,T], \tag{1.4a}$$

$$\rho(\partial_t u + u\cdot\nabla u) + \nabla p = 0 \qquad \text{in } \Omega(t)\times(0,T], \tag{1.4b}$$

$$p = 0 \qquad \text{on } \Gamma_1(t)\times(0,T], \tag{1.4c}$$

$$u_3 = 0 \qquad \text{on } \Gamma_0\times(0,T], \tag{1.4d}$$

$$\partial_t \Gamma_1(t) = u\cdot\mathcal{N} \qquad \text{in } (0,T], \tag{1.4e}$$

$$(\rho,u) = (\rho_0,u_0) \qquad \text{in } \Omega\times\{t=0\}, \tag{1.4f}$$

$$\Omega(0) = \Omega,\ \Gamma_1(0) = \Gamma_1. \tag{1.4g}$$

With the sound speed given by $c := \sqrt{\partial p/\partial\rho}$ and N denoting the outward unit normal to Γ_1, the satisfaction of the condition

$$-\infty < \frac{\partial c_0^2}{\partial N} \leqslant -\varepsilon_0 < 0 \tag{1.5}$$

in a small neighborhood of the boundary defines a physical vacuum boundary (cf. [16]), where $c_0 = c|_{t=0}$ denotes the initial sound speed of the gas and $\varepsilon_0 > 0$ is a constant. In other words, the pressure accelerates the boundary in the normal direction. It is the physical vacuum that makes the study of free boundary problems of compressible fluids challenging and very interesting, because standard methods of symmetric hyperbolic systems (cf. [11]) can not be applied directly.

The physical vacuum condition (1.5) for $\gamma = 2$ is equivalent to the requirement

$$\frac{\partial\rho_0}{\partial N} \leqslant -\frac{\varepsilon_0}{2} < 0 \quad \text{ on } \Gamma_1. \tag{1.6}$$

Since $\rho_0 > 0$ in Ω, (1.6) implies that for some positive constant C and $x\in\Omega$ near the vacuum boundary Γ_1,

$$\rho_0(x) \geqslant C\operatorname{dist}(x,\Gamma_1), \tag{1.7}$$

where $\operatorname{dist}(x,\Gamma_1)$ denotes the distance of x away from Γ_1.

The moving boundary is characteristic because of the evolution law (1.1e), and the system of conservation laws is degenerate because of the appearance of the density function as a coefficient in the nonlinear wave equation which governs the dynamics of the divergence of the velocity of the gas. In turn, weighted estimates show that this wave equation indeed loses derivatives with respect to the uniformly hyperbolic non-degenerate case of a compressible liquid, wherein the density takes the value of a strictly positive constant on the moving boundary [3]. The condition (1.7) violates the uniform Kreiss-Lopatinskii condition [11] because of resonant wave speeds at the vacuum boundary for the linearized problem. The methods developed for symmetric hyperbolic conservation laws would be extremely difficult to implement for this problem, wherein the degeneracy of the vacuum creates further difficulties for the linearized estimates.

Now, we transform the system (1.4) in terms of Lagrangian variables. Let $\eta(x,t)$ denote the "position" of the gas particle x at time t. Thus,

$$\partial_t\eta = u\circ\eta \quad \text{for } t>0, \quad \text{and} \quad \eta(x,0) = x, \tag{1.8}$$

where $\circ$ denotes the composition, i.e., $(u\circ\eta)(x,t) := u(\eta(x,t),t)$.

We let

$$v = u\circ\eta, \quad f = \rho\circ\eta, \quad A = (\nabla\eta)^{-1}, \quad J = \det\nabla\eta, \quad a = JA,$$

where A is the inverse of the deformation tensor

$$\nabla\eta = (\eta^i{}_{,j}) = \begin{pmatrix} \eta^1{}_{,1} & \eta^1{}_{,2} & \eta^1{}_{,3} \\ \eta^2{}_{,1} & \eta^2{}_{,2} & \eta^2{}_{,3} \\ \eta^3{}_{,1} & \eta^3{}_{,2} & \eta^3{}_{,3} \end{pmatrix},$$

J is the Jacobian determinant and a is the classical adjoint of $\nabla\eta$, i.e., the transpose of the cofactor matrix of $\nabla\eta$, explicitly,

$$a = \begin{pmatrix} \eta^2{}_{,2}\,\eta^3{}_{,3} - \eta^2{}_{,3}\,\eta^3{}_{,2} & \eta^1{}_{,3}\,\eta^3{}_{,2} - \eta^1{}_{,2}\,\eta^3{}_{,3} & \eta^1{}_{,2}\,\eta^2{}_{,3} - \eta^1{}_{,3}\,\eta^2{}_{,2} \\ \eta^2{}_{,3}\,\eta^3{}_{,1} - \eta^2{}_{,1}\,\eta^3{}_{,3} & \eta^1{}_{,1}\,\eta^3{}_{,3} - \eta^1{}_{,3}\,\eta^3{}_{,1} & \eta^1{}_{,3}\,\eta^2{}_{,1} - \eta^1{}_{,1}\,\eta^2{}_{,3} \\ \eta^2{}_{,1}\,\eta^3{}_{,2} - \eta^2{}_{,2}\,\eta^3{}_{,1} & \eta^1{}_{,2}\,\eta^3{}_{,1} - \eta^1{}_{,1}\,\eta^3{}_{,2} & \eta^1{}_{,1}\,\eta^2{}_{,2} - \eta^1{}_{,2}\,\eta^2{}_{,1} \end{pmatrix}. \tag{1.9}$$

Since $\eta^3 = x_3 = 0$ on the fixed boundary Γ_0, according to (1.9), the components $a_1^3 = a_2^3 = 0$ on Γ_0, and $v_3 = 0$ on Γ_0 due to $v \cdot (0,0,-1) = 0$ where $(0,0,-1)$ is the outward unit normal vector to Γ_0, then the Lagrangian version of (1.4) can be written in the fixed reference domain Ω as

$$f_t + fA_i^j v^i{}_{,j} = 0 \qquad \text{in } \Omega \times (0,T], \tag{1.10a}$$

$$fv_t^i + A_i^j f^2{}_{,j} = 0 \qquad \text{in } \Omega \times (0,T], \tag{1.10b}$$

$$f = 0 \qquad \text{on } \Gamma_1 \times (0,T], \tag{1.10c}$$

$$a_1^3 = a_2^3 = 0,\ v_3 = 0 \qquad \text{on } \Gamma_0 \times (0,T], \tag{1.10d}$$

$$(\rho, v, \eta) = (\rho_0, u_0, e) \qquad \text{in } \Omega \times \{t = 0\}, \tag{1.10e}$$

where $e(x) = x$ denotes the identity map on Ω.

From the derivative formula of determinants, we have

$$J_t = JA_l^j v^l{}_{,j} = a_l^j v^l{}_{,j}\,. \tag{1.11}$$

It follows from (1.10a) and (1.11) that

$$f_t + fJ^{-1}J_t = 0,\ \text{or } \partial_t(fJ) = 0,\ \text{i.e., } f = \rho_0 J^{-1}, \tag{1.12}$$

thus, the initial density function ρ_0 can be viewed as a parameter in compressible Euler equations.

Using the identity $A_i^j = J^{-1}a_i^j$, we write (1.10) as

$$\rho_0 v_t^i + a_i^j \big(\rho_0^2 J^{-2}\big)_{,j} = 0 \qquad \text{in } \Omega \times (0,T], \tag{1.13a}$$

$$\rho_0 = 0 \qquad \text{on } \Gamma_1, \tag{1.13b}$$

$$a_1^3 = a_2^3 = 0,\ v_3 = 0 \qquad \text{on } \Gamma_0 \times (0,T], \tag{1.13c}$$

$$(v, \eta) = (u_0, e) \qquad \text{in } \Omega \times \{t = 0\}, \tag{1.13d}$$

with $\rho_0(x) \geqslant C\mathrm{dist}\,(x, \Gamma_1)$ for $x \in \Omega$ near Γ_1.

To understand the behavior of vacuum states is an important problem in gas and fluid dynamics. In particular, the physical vacuum, in which the boundary moves with a nontrivial finite normal acceleration, naturally arises in the study of the motion of gaseous stars or shallow water [10]. Despite its importance, there are only few mathematical results available near vacuum. The main difficulty lies in the fact that the physical systems become degenerate along the vacuum boundary. The existence and uniqueness for the three-dimensional compressible Euler equations modeling a liquid rather than a gas was established in [13] where the density is positive on the vacuum boundary. Trakhinin provided an alternative proof for the existence of a compressible liquid, employing a solution strategy based on symmetric hyperbolic systems combined with the Nash-Moser iteration in [20].

The local existence for the physical vacuum singularity can be found in the recent papers by Jang and Masmoudi [9,10] and by Coutand and Shkoller [6,7] for the one-dimensional and three-dimensional compressible gases. Coutand, Lindblad and Shkoller [4] established a priori estimates based on time differentiated energy estimates and elliptic estimates for normal derivatives for $\gamma = 2$ with $\rho_0 \in H^4(\Omega)$ where the energy function was given by

$$\begin{aligned}\bar{E}(t) :=& \sum_{\ell=0}^{4} \|\partial_t^{2\ell}\eta(t)\|_{4-\ell}^2 + \sum_{\ell=0}^{4}\Big[\|\rho_0\overline{\partial}^{4-\ell}\partial_t^{2\ell}\nabla\eta(t)\|_0^2 + \|\sqrt{\rho_0}\overline{\partial}^{4-\ell}\partial_t^{2\ell}v(t)\|_0^2\Big]\\ &+ \sum_{\ell=0}^{3}\|\rho_0\partial_t^{2\ell}J^{-2}(t)\|_{4-\ell}^2 + \|\mathrm{curl}_\eta\, v(t)\|_3^2 + \|\rho_0\overline{\partial}^4\mathrm{curl}_\eta\, v(t)\|_0^2,\end{aligned}$$

where the claimed interpolation estimate (cf. [4, Eq.(6.12)])

$$\|v\|_{L^2(0,T;H^{3.5}(\Omega))}^2 \leqslant C(\|v\|_3\|\eta\|_4)\big|_T^0 + C\|v_t\|_{L^2(0,T;H^3(\Omega))}\|\eta\|_{L^2(0,T;H^4(\Omega))} \tag{1.14}$$

had played a crucial role.

We will not attempt to address exhaustive references in this paper. For more related references, we refer the interested reader to [7,10] and references therein for a nice history of the analysis of compressible Euler equations.

In the present paper, we will use a similar argument as in [4] to consider the cases of general initial densities. We will rigorously prove a new mixed space-time interpolation inequality under the framework of Lebesgue spaces which will play a vital role in our energy estimates, rather than the framework of Lebesgue spaces for time but Sobolev spaces for spatial variables in bounded domain, namely (1.14), used in [4], then we can obtain some extra estimates of space-time derivatives of $v(t)$ in $L^3(\Omega)$. In order to deal with some sub-higher order terms (e.g., (5.164)) in the argument for the time derivatives, we have to investigate the fifth order energy estimates which are closed with themselves.

We now derive the physical energy of the system (1.13). From (1.11), the Piola identity (A.13) given in Section A.3, we get

$$\begin{aligned}\frac{1}{2}\partial_t(\rho_0|v|^2) =& \rho_0 v_i v_t^i = -a_i^j v^i\big(\rho_0^2 J^{-2}\big)_{,j} = -\Big(a_i^j v^i\left(\rho_0^2 J^{-2}\right)\Big)_{,j} + J_t\rho_0^2 J^{-2}\\ =& -\rho_0^2\partial_t J^{-1} - \left(a_i^j v^i \rho_0^2 J^{-2}\right)_{,j}.\end{aligned}$$

Since $\rho_0 = 0$ on the boundary Γ_1 and $a_i^3 v^i = 0$ on Γ_0, integrating over Ω yields, with the help of Gauss' theorem, that

$$\begin{aligned}E_0(t) :=& \int_\Omega\left(\frac{1}{2}\rho_0(x)|v(x,t)|^2 + \rho_0^2(x)J^{-1}(x,t)\right)dx\\ =& \frac{1}{2}\|\sqrt{\rho_0}v\|_0^2 + \|\rho_0 J^{-1/2}\|_0^2 = E_0(0)\end{aligned}$$

conserves for all $t \geqslant 0$.

Although the physical energy is a conserved quantity, it is far too weak for the purposes of constructing solutions. Instead, we introduce the following r^{th} order energy function

$$E_r(t) := \sum_{\ell=0}^{r}\Big[\|\partial_t^{2\ell}\eta(t)\|_{r-\ell}^2 + \|\rho_0^{1/2}\partial_t^{2\ell}\overline{\partial}^{r-\ell}v(t)\|_0^2 + \|\rho_0\partial_t^{2\ell}\overline{\partial}^{r-\ell}\nabla\eta(t)\|_0^2\Big]$$

$$+\sum_{\ell=0}^{r-1}\Big[\|\partial_t^{2\ell}v(t)\|_{H_3^{r-1-\ell}(\Omega)}+\|\rho_0\partial_t^{2\ell}J^{-2}(t)\|_{r-\ell}^2\Big]$$
$$+\|\mathrm{curl}_\eta\, v(t)\|_{r-1}^2+\|\rho_0\overline{\partial}^r\mathrm{curl}_\eta\, v(t)\|_0^2.$$

Now, we state our main result as follows.

Theorem 1.1. *Let $\gamma=2$. Suppose that $(\eta(t),v(t))$ is a smooth solution of the system (1.8) and (1.13) on a time interval $[0,T']$ satisfying the initial bounds $E_5(0)<\infty$ and that the initial density $\rho_0>0$ in Ω and $\rho_0\in H^5(\Omega)$ satisfies the physical vacuum condition (1.7). Then there exists a $T=T(E_5(0))>0$ so small that the energy function $E_5(t)$ constructed from the solution $(\eta(t),v(t))$ satisfies the a priori estimate*

$$\sup_{[0,T]}E_5(t)\leqslant C(E_5(0)).$$

Remark 1. The same arguments and the results hold true if the bottom boundary Γ_0 is also a moving vacuum boundary, i.e., by changing the boundary condition (1.1d) into $p=0$ on $\Gamma_0(t)\times(0,T]$, which will not cause any additional difficulties except for the transformation of coordinates.

Remark 2. For the general cases $\gamma>1$ with general densities, we give some further remarks. We think that they are much more different from the special case $\gamma=2$. They need to reform the energy function in order to get a priori estimates. For the cases $\gamma>2$, it seems to be similar to the case $\gamma=2$ due to $\gamma-1>\gamma/2$ in view of the exponent of the weight $\rho_0^{\gamma/2}$ and $\rho_0^{\gamma-1}$ and weighted Sobolev embedding relations given in Section A.1, but it is not easy to deal with the weight $\rho_0^{\gamma/2}$ in energy estimates in view of the higher order Hardy inequality. For the cases $1<\gamma<2$, one have to use the weight $\rho_0^{\gamma-1}$ instead of $\rho_0^{\gamma/2}$ in constructing the energy function according to the physical vacuum condition, the higher order Hardy inequality and weighted Sobolev embedding relations, especially for the cases $3/2\leqslant\gamma<2$, however one must deal with many extra, important and difficult remainder integrals in the estimates of every horizontal, time or mixed derivatives. For the cases $1<\gamma<3/2$, it might be different from and difficult than the above cases.

Throughout the paper, we will use the following notation: two-dimensional gradient vector or horizontal derivative $\overline{\partial}=(\partial_1,\partial_2)$, the $H_p^s(\Omega)$ interior norm $\|\cdot\|_{H_p^s(\Omega)}$, the $H^s(\Omega)$ interior norm $\|\cdot\|_s$ when $p=2$, and the $H^s(\Gamma)$ boundary norm $|\cdot|_s$. The component of a matrix M at the i^{th} row and the j^{th} column will be denoted by M_j^i. Sometimes, we will use "$\lesssim$" to stand for "$\leqslant C$" with a generic constant C. For more notations, one can read the appendix.

The rest of this paper is organized as follows. We give a mixed space-time interpolation inequality in Section 2, and derive the zero-th order energy estimates in Section 3 and the curl estimates in Section 4. Since the standard energy method is very problematic due to the degeneracy of ρ_0, we first derive the estimates of the horizontal and time derivatives in Sections 5.1-5.3 and then obtain the estimates of normal or full derivatives through the elliptic-type estimates in Section 6. We will complete the proof of the a priori estimates in Section 7. Finally, we will give some preliminaries in Appendix A. Precisely, we introduce some notations and weighted Sobolev spaces in Section A.1; we recall the higher-order Hardy-type inequality and Hodge's decomposition elliptic estimates in Section A.2; we give the properties of

the determinant J, the inverse of the deformation tensor A and the transpose of the cofactor matrix a in Section A.3.

2. **A mixed space-time interpolation inequality.** In this section, we prove a useful mixed space-time interpolation inequality which will play a vital role in our energy estimates.

Proposition 1 (Mixed interpolation inequality)**.** *Let $F(t,x)$ be a scalar or vector-valued function for $t\in[0,T]$, $T>0$ and $x\in\Omega\subset\mathbb{R}^3$. Assume that $F_t(0,\cdot)\in L^3(\Omega)$, $F\in L^\infty([0,T];L^6(\Omega))$ and $F_{tt}\in L^\infty([0,T];L^2(\Omega))$, then we have*

$$\|F_t\|^2_{L^3([0,T]\times\Omega)}\leqslant CT^{2/3}\Big[\|F_t(0)\|^2_{L^3(\Omega)}+\sup_{[0,T]}\|F(t)\|_{L^6(\Omega)}\|F_{tt}(t)\|_{L^2(\Omega)}\Big],\tag{2.1}$$

where C is a constant independent of T, Ω and F.

Proof. Notice that

$$2|F_t|(\partial_t|F_t|)F_t=\partial_t(|F_t|^2)F_t=2(F_t\cdot F_{tt})F_t\leqslant 2|F_t|^2|F_{tt}|$$

implies that $|\partial_t(|F_t|)F_t|\leqslant|F_t||F_{tt}|$. Then, by the Fubini theorem, integration by parts with respect to time, the fundamental theorem of calculus, the Hölder inequality and the Minkowski inequality, we have

$$\begin{aligned}
&\|F_t\|^3_{L^3([0,T]\times\Omega)}=\int_0^T\!\!\int_\Omega|F_t|^3dxdt=\int_\Omega\int_0^T|F_t|F_t\cdot F_tdtdx\\
=&\int_\Omega|F_t(T)|F_t(T)\cdot F(T)dx-\int_\Omega|F_t(0)|F_t(0)\cdot F(0)dx-\int_\Omega\int_0^T\partial_t(|F_t|F_t)\cdot Fdtdx\\
=&\int_\Omega\left(\int_0^T\partial_t(|F_t|F_t)dt\right)\cdot F(T)dx+\int_\Omega|F_t(0)|F_t(0)\cdot\left(\int_0^TF_tdt\right)dx\\
&-\int_0^T\!\!\int_\Omega|F_t|F_{tt}\cdot Fdxdt-\int_0^T\!\!\int_\Omega\partial_t(|F_t|)F_t\cdot Fdxdt\\
\leqslant&2\int_\Omega\left(\int_0^T|F_{tt}||F_t|dt\right)|F(T)|dx+\int_\Omega|F_t(0)|^2\left(\int_0^T|F_t|dt\right)dx\\
&+2\int_0^T\!\!\int_\Omega|F_t||F_{tt}||F|dxdt\\
\leqslant&C\|F(T)\|_{L^6(\Omega)}\int_0^T\||F_{tt}||F_t|\|_{L^{6/5}(\Omega)}dt+C\|F_t(0)\|^2_{L^3(\Omega)}\int_0^T\|F_t\|_{L^3(\Omega)}dt\\
&+C\|F_t\|_{L^3([0,T]\times\Omega)}\|F_{tt}\|_{L^2([0,T]\times\Omega)}\|F\|_{L^6([0,T]\times\Omega)}\\
\leqslant&CT^{2/3}\|F_t\|_{L^3([0,T]\times\Omega)}\left[\|F_t(0)\|^2_{L^3(\Omega)}+\sup_{[0,T]}\|F\|_{L^6(\Omega)}\|F_{tt}\|_{L^2(\Omega)}\right],
\end{aligned}$$

which implies the desired inequality by eliminating $\|F_t\|_{L^3([0,T]\times\Omega)}$ from both sides of the inequality. □

3. **A priori assumption and the zero-th order energy estimates.** We assume that we have smooth solutions η on a time interval $[0,T]$, and that for all such solutions, the time $T>0$ is taken sufficiently small so that for $t\in[0,T]$,

$$\frac{1}{2}\leqslant J(t)\leqslant\frac{3}{2}.\tag{3.1}$$

Once we establish the a priori bounds, we can ensure that our solution verifies the assumption (3.1) by means of the fundamental theorem of calculus. Then, by (1.9) and Sobolev's embedding $H^2(\Omega) \subset L^\infty(\Omega)$, we have for $t \in [0,T]$,

$$\|\eta(t)\|_{L^\infty(\Omega)} \leqslant C\|\eta(t)\|_2, \tag{3.2}$$

$$\|a(t)\|_{L^\infty(\Omega)} \leqslant C\|\nabla\eta(t)\|^2_{L^\infty(\Omega)} \leqslant C\|\eta(t)\|^2_3. \tag{3.3}$$

It follows from $a = JA$ and (3.1) that

$$\|A(t)\|_{L^\infty(\Omega)} \leqslant \|J^{-1}(t)a(t)\|_{L^\infty(\Omega)} \leqslant C\|\eta(t)\|^2_3. \tag{3.4}$$

Now, we prove the following zero-th order energy estimates.

Proposition 2. *It holds for* $r \geqslant 4$

$$\sup_{[0,T]}\left[\|\rho_0^{1/2}v\|_0^2 + \|\rho_0 J^{-1/2}\|_0^2\right] \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).$$

Proof. Since the proof is standard, we omit the details. □

4. **The curl estimates.** Taking the Lagrangian curl of (1.13a) yields that

$$\varepsilon_{lji}A^s_j v^i_{t,s} = 0, \quad \text{or} \quad \operatorname{curl}_\eta v_t = 0. \tag{4.1}$$

We can obtain the following proposition.

Proposition 3. *For all* $t \in (0,T]$, *we have for* $r \geqslant 4$

$$\sum_{\ell=0}^{r-1}\|\operatorname{curl}\partial_t^{2\ell}\eta(t)\|^2_{r-1-\ell} + \sum_{\ell=0}^{r}\|\rho_0\overline{\partial}^{r-\ell}\operatorname{curl}\partial_t^{2\ell}\eta(t)\|_0^2 \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)). \tag{4.2}$$

Proof. Since the proof is similar to those in [4], we omit the details. □

5. **The estimates for the horizontal and time derivatives.**

5.1. **The estimates for the horizontal derivatives.** We have the following estimates.

Proposition 4. *Let* $r \in \{4,5\}$. *For small* $\delta > 0$ *and the constant* M_0 *depending on* $1/\delta$, *it holds*

$$\begin{aligned}&\sup_{[0,T]}\left[\|\rho_0^{1/2}\overline{\partial}^r v(t)\|_0^2 + \|\rho_0\overline{\partial}^r\nabla\eta(t)\|_0^2 + \|\rho_0\overline{\partial}^r\operatorname{div}\eta(t)\|_0^2 + |\eta^\alpha|^2_{r-1/2}\right]\\ \leqslant& M_0 + \delta\sup_{[0,T]}E_r(t) + CTP(\sup_{[0,T]}E_r(t)),\end{aligned}$$

where $\eta^\alpha = \eta\cdot T_\alpha$ *for* $\alpha = 1,2$.

Proof. Letting $\overline{\partial}^r$ act on (1.13a), then we have

$$\sum_{l=0}^{r} C_r^l\overline{\partial}^{r-l}\rho_0\overline{\partial}^l v^i_t + \sum_{l=0}^{r} C_r^l\overline{\partial}^{r-l}a^j_i\overline{\partial}^l\left(\rho_0^2 J^{-2}\right)_{,j} = 0,$$

where C_r^l is the binomial coefficient. Taking the $L^2(\Omega)$-inner product with $\overline{\partial}^r v^i$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\overline{\partial}^r v|^2dx + \sum_{m=0}^{3}\mathfrak{I}_m = \sum_{m=4}^{6}\mathfrak{I}_m. \tag{5.1}$$

Here,

$$\mathfrak{I}_1 := \int_\Omega \overline{\partial}^r a_i^j \left(\rho_0^2 J^{-2}\right)_{,j} \overline{\partial}^r v^i dx, \quad \mathfrak{I}_2 := \int_\Omega a_i^j \left(\overline{\partial}^r \rho_0^2 J^{-2}\right)_{,j} \overline{\partial}^r v^i dx,$$

$$\mathfrak{I}_3 := \int_\Omega a_i^j \left(\rho_0^2 \overline{\partial}^r J^{-2}\right)_{,j} \overline{\partial}^r v^i dx, \quad \mathfrak{I}_4 := -\sum_{l=0}^{r-1} C_r^l \int_\Omega \overline{\partial}^{r-l} \rho_0 \overline{\partial}^l v_t^i \overline{\partial}^r v^i dx,$$

$$\mathfrak{I}_5 := -\sum_{l=1}^{r-1} C_r^l \int_\Omega \overline{\partial}^{r-l} a_i^j \overline{\partial}^l \left(\rho_0^2 J^{-2}\right)_{,j} \overline{\partial}^r v^i dx,$$

$$\mathfrak{I}_6 := -\sum_{l=1}^{r-1} C_r^l \int_\Omega a_i^j \left(\overline{\partial}^{r-l} \rho_0^2 \overline{\partial}^l J^{-2}\right)_{,j} \overline{\partial}^r v^i dx.$$

Step 1. Analysis of the integral $\mathfrak{I}_1$. We use the identity (A.13) to integrate by parts with respect to x_j to find that

$$\begin{aligned}\mathfrak{I}_1 &= -\int_\Omega \overline{\partial}^r a_i^j \rho_0^2 J^{-2} \overline{\partial}^r v^i{}_{,j}\, dx + \int_{\Gamma_0} \overline{\partial}^r a_i^3 \overline{\partial}^r v^i \rho_0^2 J^{-2} dx_1 dx_2 \\ &= -\int_\Omega \overline{\partial}^r a_i^j \overline{\partial}^r v^i{}_{,j}\, \rho_0^2 J^{-2} dx,\end{aligned}$$

due to the boundary conditions (1.13b) and (1.13c).

It follows from (A.14) that

$$\begin{aligned}\mathfrak{I}_1 &= -\int_\Omega \overline{\partial}^{r-1} \left(\overline{\partial} \eta^l{}_{,k}\, J^{-1} (a_i^j a_l^k - a_i^k a_l^j)\right) \overline{\partial}^r v^i{}_{,j}\, \rho_0^2 J^{-2} dx \\ &= -\int_\Omega \overline{\partial}^r \eta^l{}_{,k}\, A_l^k \overline{\partial}^r v^i{}_{,j}\, A_i^j \rho_0^2 J^{-1} dx \qquad (5.2) \\ &\quad + \int_\Omega \overline{\partial}^r \eta^l{}_{,k}\, A_i^k \overline{\partial}^r v^i{}_{,j}\, A_l^j \rho_0^2 J^{-1} dx \qquad (5.3) \\ &\quad - \sum_{s=1}^{r-1} C_{r-1}^s \int_\Omega \overline{\partial}^{r-s} \eta^l{}_{,k}\, \overline{\partial}^s \left(J^{-1} (a_i^j a_l^k - a_i^k a_l^j)\right) \overline{\partial}^r v^i{}_{,j}\, \rho_0^2 J^{-2} dx. \qquad (5.4)\end{aligned}$$

Since $v = \eta_t$, we get

$$\begin{aligned}(5.2) &= -\frac{1}{2}\frac{d}{dt} \int_\Omega |\mathrm{div}_\eta\, \overline{\partial}^r \eta|^2 \rho_0^2 J^{-1} dx + \int_\Omega \overline{\partial}^r \eta^l{}_{,k}\, A_l^k \overline{\partial}^r \eta^i{}_{,j}\, \partial_t A_i^j \rho_0^2 J^{-1} dx \\ &\quad + \frac{1}{2} \int_\Omega |\mathrm{div}_\eta\, \overline{\partial}^r \eta|^2 \rho_0^2 \partial_t J^{-1} dx \\ &= -\frac{1}{2}\frac{d}{dt} \int_\Omega |\mathrm{div}_\eta\, \overline{\partial}^r \eta|^2 \rho_0^2 J^{-1} dx + \frac{1}{2} \int_\Omega \overline{\partial}^r \eta^l{}_{,k}\, \overline{\partial}^r \eta^i{}_{,j}\, \partial_t (A_l^k A_i^j) \rho_0^2 J^{-1} dx \\ &\quad + \frac{1}{2} \int_\Omega |\mathrm{div}_\eta\, \overline{\partial}^r \eta|^2 \rho_0^2 \partial_t J^{-1} dx.\end{aligned}$$

For the integral (5.3), since $v = \eta_t$, it holds

$$\begin{aligned}&\overline{\partial}^r \eta^l{}_{,k}\, A_i^k A_l^j \overline{\partial}^r v^i{}_{,j} \\ =&\partial_t (\overline{\partial}^r \eta^l{}_{,k}\, A_i^k \overline{\partial}^r \eta^i{}_{,j}\, A_l^j) - \overline{\partial}^r v^l{}_{,k}\, A_i^k \overline{\partial}^r \eta^i{}_{,j}\, A_l^j - \overline{\partial}^r \eta^l{}_{,k}\, \overline{\partial}^r \eta^i{}_{,j}\, \partial_t (A_i^k A_l^j).\end{aligned}$$

It follows from (A.2) that

$$\overline{\partial}^r \eta^l{}_{,k}\, A_i^k A_l^j \overline{\partial}^r v^i{}_{,j}$$

$$=\frac{1}{2}\partial_t\left[|\nabla_\eta\overline{\partial}^r\eta|^2-|\mathrm{curl}_\eta\,\overline{\partial}^r\eta|^2\right]-\frac{1}{2}\overline{\partial}^r\eta^l{}_{,k}\,\overline{\partial}^r\eta^i{}_{,j}\,\partial_t(A_i^kA_l^j). \tag{5.5}$$

Thus, we have

$$\begin{aligned}(5.3)=&\frac{1}{2}\frac{d}{dt}\int_\Omega\left[|\nabla_\eta\overline{\partial}^r\eta|^2-|\mathrm{curl}_\eta\,\overline{\partial}^r\eta|^2\right]\rho_0^2J^{-1}dx\\&+\frac{1}{2}\int_\Omega\left[|\nabla_\eta\overline{\partial}^r\eta|^2-|\mathrm{curl}_\eta\,\overline{\partial}^r\eta|^2\right]\rho_0^2J^{-1}\mathrm{div}_\eta\,v dx\\&-\frac{1}{2}\int_\Omega\overline{\partial}^r\eta^l{}_{,k}\,\overline{\partial}^r\eta^i{}_{,j}\,\partial_t(A_i^kA_l^j)\rho_0^2J^{-1}dx.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^T\mathfrak{I}_1dt=&\frac{1}{2}\int_\Omega\left[|\nabla_\eta\overline{\partial}^r\eta(T)|^2-|\mathrm{div}_\eta\,\overline{\partial}^r\eta(T)|^2-|\mathrm{curl}_\eta\,\overline{\partial}^r\eta(T)|^2\right]\rho_0^2J^{-1}(T)dx\\&+\frac{2-1}{2}\int_0^T\int_\Omega\left[|\nabla_\eta\overline{\partial}^r\eta|^2-|\mathrm{div}_\eta\,\overline{\partial}^r\eta|^2-|\mathrm{curl}_\eta\,\overline{\partial}^r\eta|^2\right]\rho_0^2J^{-1}\mathrm{div}_\eta\,v dxdt\\&-\frac{1}{2}\int_0^T\int_\Omega\overline{\partial}^r\eta^l{}_{,k}\,\overline{\partial}^r\eta^i{}_{,j}\,\partial_t(A_i^kA_l^j-A_i^jA_l^k)\rho_0^2J^{-1}dxdt+\int_0^T(5.4)dt.\end{aligned}$$

It is clear that

$$\begin{aligned}&\left|\int_0^T\int_\Omega\left[|\nabla_\eta\overline{\partial}^r\eta|^2-|\mathrm{div}_\eta\,\overline{\partial}^r\eta|^2-|\mathrm{curl}_\eta\,\overline{\partial}^r\eta|^2\right]\rho_0^2J^{-1}\mathrm{div}_\eta\,v dxdt\right|\\\leqslant&CT\sup_{[0,T]}\|\rho_0\overline{\partial}^r\nabla\eta\|_0^2\|v\|_3\|\eta\|_3^4\leqslant CTP(\sup_{[0,T]}E_r(t)),\end{aligned}$$

and

$$\begin{aligned}&\left|\int_0^T\int_\Omega\overline{\partial}^r\eta^l{}_{,k}\,\overline{\partial}^r\eta^i{}_{,j}\,\partial_t(A_i^kA_l^j-A_i^jA_l^k)\rho_0^2J^{-1}dxdt\right|\\\leqslant&CT\sup_{[0,T]}\|\rho_0\overline{\partial}^r\nabla\eta\|_0^2\|v\|_3\|\eta\|_3^6\\\leqslant&CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

Now, we analyze the integral $\int_0^T(5.4)dt$. We will use integration by parts in time for the cases $s=1$ and $s=2$, while we have to use integration by parts with respect to spatial variables for the case $s=r-1$.

Case 1. $s=1$. From integration by parts with respect to time, we get

$$-(r-1)\int_0^T\int_\Omega\overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}\left(a_i^jA_l^k-A_i^ka_l^j\right)\overline{\partial}^rv^i{}_{,j}\,\rho_0^2J^{-2}dxdt \tag{5.6}$$

$$=-(r-1)\int_\Omega\overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}\left(a_i^jA_l^k-A_i^ka_l^j\right)\overline{\partial}^r\eta^i{}_{,j}\,\rho_0^2J^{-2}dx\Big|_{t=T} \tag{5.7}$$

$$+(r-1)\int_0^T\int_\Omega\overline{\partial}^{r-1}v^l{}_{,k}\,\overline{\partial}\left(a_i^jA_l^k-A_i^ka_l^j\right)\overline{\partial}^r\eta^i{}_{,j}\,\rho_0^2J^{-2}dxdt \tag{5.8}$$

$$+(r-1)\int_0^T\int_\Omega\overline{\partial}^{r-1}\eta^l{}_{,k}\,\partial_t\overline{\partial}\left(a_i^jA_l^k-A_i^ka_l^j\right)\overline{\partial}^r\eta^i{}_{,j}\,\rho_0^2J^{-2}dxdt \tag{5.9}$$

$$+(r-1)\int_0^T\int_\Omega\overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}\left(a_i^jA_l^k-A_i^ka_l^j\right)\overline{\partial}^r\eta^i{}_{,j}\,\rho_0^2\partial_tJ^{-2}dxdt. \tag{5.10}$$

It is clear that, by Hölder's inequality and the fundamental theorem of calculus twice,

$$\begin{aligned}|(5.7)| \leqslant & C\left\|\rho_0\int_0^T \overline{\partial}^{r-1}\partial_t\nabla\eta dt'\right\|_0 \|\rho_0\overline{\partial}^r\nabla\eta(T)\|_0\|\eta(T)\|_4^7\\ \leqslant & CT\left(\sup_{[0,T]}\left\|\rho_0\int_0^t \overline{\partial}^{r-1}\partial_t^2\nabla\eta dt'\right\|_0 + \|\rho_0\overline{\partial}^{r-1}\partial_t\nabla\eta(0)\|_0\right)\\ & \cdot\|\rho_0\overline{\partial}^r\nabla\eta(T)\|_0\|\eta(T)\|_4^7\\ \leqslant & CT^2\sup_{[0,T]}\left(\|\rho_0\overline{\partial}^{r-1}\partial_t^2\nabla\eta\|_0 + \sum_{l=1}^2\|\rho_0\overline{\partial}^{r-1}\partial_t^l\nabla\eta(0)\|_0\right)\\ & \cdot\|\rho_0\overline{\partial}^r\nabla\eta(T)\|_0\|\eta(T)\|_4^7\\ \leqslant & M_0 + CT^2P(\sup_{[0,T]}E_r(t)),\end{aligned}$$

$$\begin{aligned}|(5.8)| \leqslant & CT\sup_{[0,T]}\left(\left\|\rho_0\int_0^T \overline{\partial}^{r-1}\partial_t^2\nabla\eta dt\right\|_0 + \|\rho_0\overline{\partial}^{r-1}\partial_t\nabla\eta(0)\|_0\right)\|\rho_0\overline{\partial}^r\nabla\eta\|_0\|\eta\|_4^7\\ \leqslant & CT^2\sup_{[0,T]}\left(\|\rho_0\overline{\partial}^{r-1}\partial_t^2\nabla\eta\|_0 + \sum_{l=1}^2\|\rho_0\overline{\partial}^{r-1}\partial_t^l\nabla\eta(0)\|_0\right)\|\rho_0\overline{\partial}^r\nabla\eta\|_0\|\eta\|_4^7\\ \leqslant & M_0 + CT^2P(\sup_{[0,T]}E_r(t)),\end{aligned}$$

and

$$\begin{aligned}|(5.10)| \leqslant & CT\|\rho_0\|_2\sup_{[0,T]}\|\overline{\partial}^{r-1}\nabla\eta\|_0P(\|\eta\|_4)\|\nabla v\|_2\|\rho_0\overline{\partial}^r\nabla\eta\|_0\\ \leqslant & M_0 + CT^2P(\sup_{[0,T]}E_r(t)).\end{aligned}$$

We can rewrite (5.9) as, for $\beta\in\{1,2\}$

$$(5.9) = 3\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,\beta}\,\partial_t\overline{\partial}\left(a_i^3A_l^\beta - A_i^\beta a_l^3\right)J^{-2}\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2dxdt \tag{5.11}$$

$$+3\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\partial_t\overline{\partial}\left(a_i^\beta A_l^k - A_i^k a_l^\beta\right)J^{-2}\overline{\partial}^r\eta^i{}_{,\beta}\,\rho_0^2dxdt. \tag{5.12}$$

Obviously, we have from the Hölder inequality and Sobolev's embedding theorem that

$$\begin{aligned}|(5.11)| \leqslant & CT\sup_{[0,T]}\|\rho_0\overline{\partial}^r\eta\|_1P(\|\eta\|_4,\|v\|_3,\|\overline{\partial}\nabla v\|_1)\|\rho_0\overline{\partial}^r\nabla\eta\|_0\\ \leqslant & CT\sup_{[0,T]}\left(\|\rho_0\|_3\|\eta\|_r + \|\rho_0\overline{\partial}^r\nabla\eta\|_0\right)P(\|\eta\|_4,\|v\|_3,\|\overline{\partial}\nabla v\|_1)\|\rho_0\overline{\partial}^r\nabla\eta\|_0\\ \leqslant & M_0 + CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

By integration by parts, it yields

$$(5.12) = -3\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k\beta}\,\partial_t\overline{\partial}\left(a_i^\beta A_l^k - A_i^k a_l^\beta\right)\overline{\partial}^r\eta^iJ^{-2}\rho_0^2dxdt \tag{5.13}$$

$$-3\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\partial_t\overline{\partial}\Big(a_i^\beta A_l^k - A_i^k a_l^\beta\Big)_{,\beta}\,\overline{\partial}^r\eta^i J^{-2}\rho_0^2 dxdt \tag{5.14}$$

$$-3\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\partial_t\overline{\partial}\left(a_i^\beta A_l^k - A_i^k a_l^\beta\right)\overline{\partial}^r\eta^i\left(J^{-2}\rho_0^2\right)_{,\beta} dxdt. \tag{5.15}$$

It is easy to see that

$$|(5.13)| \leqslant CT\sup_{[0,T]}\|\rho_0\overline{\partial}^r\nabla\eta\|_0\|\overline{\partial}\nabla v\|_1 P(\|\eta\|_4)\|\rho_0\overline{\partial}^r\eta\|_1 \leqslant M_0 + CTP(\sup_{[0,T]}E_r(t)),$$

and

$$\begin{aligned}(5.15) \leqslant& CT\sup_{[0,T]}\|\overline{\partial}^{r-1}\nabla\eta\|_0\|\overline{\partial}\nabla v\|_1 P(\|\eta\|_4)\|\rho_0\overline{\partial}^r\eta\|_1\left(\|\rho_0\|_{L^\infty(\Omega)} + \|\overline{\partial}\rho_0\|_{L^\infty(\Omega)}\right)\\ \leqslant& M_0 + CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

In order to estimate (5.14), we first consider the estimate of $\|D^{r-1}v\|_{L^3([0,T]\times\Omega)}$ where D^{r-1} denotes all the derivatives ∂^θ for the multi-index $\theta=(\theta_1,\theta_2,\theta_3)$ and $0\leqslant|\theta|\leqslant r-1$. By Proposition 1 with $F=D^{r-1}\eta$ and the Sobolev embedding theorem, we have

$$\begin{aligned}\|D^{r-1}v\|^2_{L^3([0,T]\times\Omega)} \leqslant& CT^{2/3}\Big[\|D^{r-1}v(0)\|_{L^3(\Omega)} + \sup_{[0,T]}\|D^{r-1}v_t\|_{L^2(\Omega)}\|D^{r-1}\eta\|_{L^6(\Omega)}\Big]\\ \leqslant& M_0 + CT^{2/3}\sup_{[0,T]}\|v_t\|_{r-1}\|\eta\|_r\\ \leqslant& M_0 + CT^{2/3}P(\sup_{[0,T]}E_r(t)).\end{aligned}$$

Thus, we obtain

$$\|D^{r-1}v\|^2_{L^3([0,T]\times\Omega)} \leqslant M_0 + CT^{2/3}P(\sup_{[0,T]}E_r(t)). \tag{5.16}$$

By the Hölder inequality, the Sobolev embedding theorem, the Cauchy inequality and (5.16), we easily get

$$\begin{aligned}|(5.14)| \leqslant& CT^{2/3}\|\rho_0\|_2\|\overline{\partial}^2\nabla v\|_{L^3([0,T]\times\Omega)}\sup_{[0,T]}\|\overline{\partial}^3\nabla\eta\|_0\|\rho_0\overline{\partial}^4\eta\|_1\\ \leqslant& CT^{1/3}\|\overline{\partial}^2\nabla v\|^2_{L^3([0,T]\times\Omega)} + CT\|\rho_0\|_2^2\sup_{[0,T]}\|\overline{\partial}^3\nabla\eta\|_0^2\|\rho_0\overline{\partial}^4\eta\|_1^2\\ \leqslant& CT^{1/3}\left(M_0 + CT^{2/3}P(\sup_{[0,T]}E_r(t))\right) + M_0 + CTP(\sup_{[0,T]}E_r(t))\\ \leqslant& M_0 + CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

Hence, we obtain

$$|(5.6)| \leqslant M_0 + CTP(\sup_{[0,T]}E_r(t)).$$

Case 2. $s=2$. By integration by parts with respect to time, it yields

$$-C_{r-1}^2\int_0^T\int_\Omega \overline{\partial}^{r-2}\eta^l{}_{,k}\,\overline{\partial}^2(A_i^j a_l^k - a_i^k A_l^j)\overline{\partial}^r v^i{}_{,j}\,\rho_0^2 J^{-2}dxdt \tag{5.17}$$

$$= -C_{r-1}^2\int_\Omega \overline{\partial}^{r-2}\eta^l{}_{,k}\,\overline{\partial}^2(A_i^j a_l^k - a_i^k A_l^j)\overline{\partial}^r\eta^i{}_{,j}\,\rho_0^2 J^{-2}dx\Big|_{t=T} \tag{5.18}$$

$$+C_{r-1}^2\int_0^T\int_\Omega \overline{\partial}^{r-2}v^l{}_{,k}\,\overline{\partial}^2(A_i^jA_l^k-a_i^kA_l^j)\overline{\partial}^r\eta^i{}_{,j}\,\rho_0^2J^{-2}dxdt \tag{5.19}$$

$$+C_{r-1}^2\int_0^T\int_\Omega \overline{\partial}^{r-2}\eta^l{}_{,k}\,\overline{\partial}^2\partial_t(A_i^jA_l^k-a_i^kA_l^j)\overline{\partial}^r\eta^i{}_{,j}\,\rho_0^2J^{-2}dxdt \tag{5.20}$$

$$+C_{r-1}^2\int_0^T\int_\Omega \overline{\partial}^{r-2}\eta^l{}_{,k}\,\overline{\partial}^2(A_i^jA_l^k-a_i^kA_l^j)\overline{\partial}^r\eta^i{}_{,j}\,\rho_0^2\partial_tJ^{-2}dxdt. \tag{5.21}$$

Applying the fundamental theorem of calculus yields for a small $\delta>0$

$$\begin{aligned}|(5.18)|\leqslant&C\|\rho_0\|_2\|\overline{\partial}^{r-2}\nabla\eta(T)\|_1P(\|\eta(T)\|_4)\|\nabla\eta(T)\|_1\|\rho_0\overline{\partial}^r\nabla\eta(T)\|_0\\ \leqslant&C\|\rho_0\|_2P(\|\eta(T)\|_4,\|\eta(T)\|_r)\left(\left\|\int_0^T\nabla v dt\right\|_1+\|\nabla\eta(0)\|_1\right)\|\rho_0\overline{\partial}^r\nabla\eta(T)\|_0\\ \leqslant&M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

Similar to (5.14), we can get

$$|(5.19)|+|(5.20)|\leqslant M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).$$

By an L^6-L^6-L^6-L^2 Hölder inequality and the Sobolev embedding theorem, we get

$$|(5.21)|\leqslant M_0+CTP(\sup_{[0,T]}E_r(t)).$$

Therefore, we have obtained

$$|(5.17)|\leqslant M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).$$

Case 3. $s=r-1$. We write the space-time integral as, for $\beta\in\{1,2\}$

$$-\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}(a_i^jA_l^k-A_i^ka_l^j)\overline{\partial}^r v^i{}_{,j}\,\rho_0^2J^{-2}dxdt \tag{5.22}$$

$$=-\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}(a_i^\beta A_l^k-A_i^ka_l^\beta)\overline{\partial}^r v^i{}_{,\beta}\,\rho_0^2J^{-2}dxdt \tag{5.23}$$

$$-\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^{r-1}(a_i^3A_l^\beta-A_i^\beta a_l^3)\overline{\partial}^r v^i{}_{,3}\,\rho_0^2J^{-2}dxdt. \tag{5.24}$$

By integration by parts with respect to x_β, we have

$$(5.23)=\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k\beta}\,\overline{\partial}^{r-1}(a_i^\beta A_l^k-A_i^ka_l^\beta)\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt \tag{5.25}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}(a_i^\beta{}_{,\beta}\,A_l^k-A_i^ka_l^\beta{}_{,\beta})\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt \tag{5.26}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}(a_i^\beta A_l^k{}_{,\beta}-A_i^k{}_{,\beta}\,a_l^\beta)\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt \tag{5.27}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}(a_i^\beta A_l^k-A_i^ka_l^\beta)\overline{\partial}^r v^i(\rho_0^2)_{,\beta}\,J^{-2}dxdt \tag{5.28}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}(a_i^\beta A_l^k-A_i^ka_l^\beta)\overline{\partial}^r v^i\rho_0^2J^{-2}{}_{,\beta}\,dxdt. \tag{5.29}$$

From (A.14), it follows that

$$(5.25) = \int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k\beta}\,\overline{\partial}^{r-2}\left[\overline{\partial}\eta^p{}_{,q}\,JA_i^\beta(A_p^qA_l^k - A_p^kA_l^q)\right]\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt \tag{5.30}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k\beta}\,\overline{\partial}^{r-2}\left[\overline{\partial}\eta^p{}_{,q}\,JA_l^\beta(A_p^kA_i^q - A_p^qA_i^k)\right]\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt \tag{5.31}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k\beta}\,\overline{\partial}^{r-2}\left[\overline{\partial}\eta^p{}_{,q}\,JA_p^\beta(A_i^kA_l^q - A_i^qA_l^k)\right]\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt. \tag{5.32}$$

We can write

$$(5.30) = \int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k\beta}\,\overline{\partial}^{r-2}\left[\overline{\partial}\eta^p{}_{,\alpha}\,JA_i^\beta(A_p^\alpha A_l^k - A_p^kA_l^\alpha)\right]\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\alpha\beta}\,\overline{\partial}^{r-2}\left[\overline{\partial}\eta^p{}_{,3}\,JA_i^\beta(A_p^3A_l^\alpha - A_p^\alpha A_l^3)\right]\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt$$

$$=\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k\beta}\,\overline{\partial}^{r-1}\eta^p{}_{,\alpha}\,A_i^\beta(A_p^\alpha A_l^k - A_p^kA_l^\alpha)\overline{\partial}^r v^i\rho_0^2J^{-1}dxdt \tag{5.33}$$

$$+\sum_{m=0}^{r-3}C_{r-2}^m\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k\beta}\,\overline{\partial}^{m+1}\eta^p{}_{,\alpha}\,\overline{\partial}^{r-2-m}\left[JA_i^\beta(A_p^\alpha A_l^k - A_p^kA_l^\alpha)\right]$$

$$\cdot\,\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt \tag{5.34}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\alpha\beta}\,\overline{\partial}^{r-1}\eta^p{}_{,3}\,A_i^\beta(A_p^3A_l^\alpha - A_p^\alpha A_l^3)\overline{\partial}^r v^i\rho_0^2J^{-1}dxdt \tag{5.35}$$

$$+\sum_{m=0}^{r-3}C_{r-2}^m\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\alpha\beta}\,\overline{\partial}^{m+1}\eta^p{}_{,3}\,\overline{\partial}^{r-2-m}\left[JA_i^\beta(A_p^3A_l^\alpha - A_p^\alpha A_l^3)\right]$$

$$\cdot\,\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt. \tag{5.36}$$

It is easy to see that

$$\begin{aligned}|(5.33)| \leqslant& CT\|\rho_0\|_2^{1/2}\sup_{[0,T]}\|\overline{\partial}^2\nabla\eta\|_1\|\rho_0\overline{\partial}^r\eta\|_1P(\|\eta\|_3)\|\rho_0^{1/2}\overline{\partial}^r v\|_0\\ \leqslant& M_0 + CTP(\sup_{[0,T]}E_r(t)),\end{aligned}$$

and by an L^6-L^6-L^2 Hölder inequality and the Sobolev embedding theorem,

$$\begin{aligned}|(5.34)| + |(5.36)| \leqslant& CT\|\rho_0\|_2^{3/2}\sup_{[0,T]}P(\|\eta\|_4,\|\eta\|_r)\|\rho_0^{1/2}\overline{\partial}^r v\|_0\\ \leqslant& M_0 + CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

By using integration by parts with respect to time, we have

$$(5.35) = \int_\Omega \overline{\partial}\eta^l{}_{,\alpha\beta}\,\overline{\partial}^{r-1}\eta^p{}_{,3}\,A_i^\beta(A_p^3A_l^\alpha - A_p^\alpha A_l^3)\overline{\partial}^r\eta^i\rho_0^2J^{-1}dx\Big|_{t=T} \tag{5.37}$$

$$-\int_0^T\int_\Omega \overline{\partial}v^l{}_{,\alpha\beta}\,\overline{\partial}^{r-1}\eta^p{}_{,3}\,A_i^\beta(A_p^3A_l^\alpha - A_p^\alpha A_l^3)\overline{\partial}^r\eta^i\rho_0^2J^{-1}dxdt \tag{5.38}$$

$$-\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\alpha\beta}\,\overline{\partial}^{r-1}v^p{}_{,3}\,A_i^\beta(A_p^3A_l^\alpha - A_p^\alpha A_l^3)\overline{\partial}^r\eta^i\rho_0^2J^{-1}dxdt \tag{5.39}$$

$$-\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\alpha\beta}\,\overline{\partial}^{r-1}\eta^p{}_{,3}\,\partial_t\left[A_i^\beta(A_p^3A_l^\alpha - A_p^\alpha A_l^3)J^{-1}\right]\overline{\partial}^r\eta^i\rho_0^2dxdt. \tag{5.40}$$

Obviously, it yields by using the fundamental theorem of calculus twice

$$\begin{aligned}(5.37)\leqslant & C\left(\|\rho_0\|_3\|\eta\|_3+\left\|\rho_0\int_0^T\overline{\partial}^3\partial_t\nabla\eta dt\right\|_0\right)P(\|\eta(T)\|_3,\|\eta(T)\|_r)\|\rho_0\overline{\partial}^r\eta(T)\|_1\\ \leqslant & C\|\rho_0\|_3\|\eta\|_3+CT\left(\|\rho_0\overline{\partial}^3\partial_t\nabla\eta(0)\|_0+\left\|\rho_0\int_0^T\overline{\partial}^3\partial_t^2\nabla\eta dt\right\|_0\right)\\ & \cdot P(\|\eta(T)\|_3,\|\eta(T)\|_r)\|\rho_0\overline{\partial}^r\eta(T)\|_1\\ \leqslant & M_0+CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

By the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus, we get

$$\begin{aligned}(5.38)\leqslant & CT\sup_{[0,T]}\|\rho_0\overline{\partial}^3\partial_t\eta\|_1\|\overline{\partial}^{r-1}\nabla\eta\|_0\|\rho_0\overline{\partial}^r\eta\|_1P(\|\eta\|_3)\\ \leqslant & CT\sup_{[0,T]}\left(\|\rho_0\|_3\|v\|_3+\|\rho_0\overline{\partial}^3\partial_t\nabla\eta(0)\|_0+\left\|\rho_0\int_0^T\overline{\partial}^3\partial_t^2\nabla\eta\right\|_0\right)\\ & \|\rho_0\overline{\partial}^r\eta\|_1P(\|\eta\|_3,\|\eta\|_r)\\ \leqslant & M_0+CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(5.39)\leqslant & CT\sup_{[0,T]}\|\overline{\partial}^3\eta\|_1\|\rho_0\overline{\partial}^{r-1}\partial_t\nabla\eta\|_0\|\rho_0\overline{\partial}^r\eta\|_1P(\|\eta\|_3)\\ \leqslant & M_0+CTP(\sup_{[0,T]}E_r(t)),\end{aligned}$$

and

$$\begin{aligned}(5.40)\leqslant & CT\|\rho_0\|_2\sup_{[0,T]}\|\overline{\partial}^3\eta\|_1\|\overline{\partial}^{r-1}\nabla\eta\|_0\|\nabla v\|_1\|\rho_0\overline{\partial}^r\eta\|_1P(\|\eta\|_3)\\ \leqslant & M_0+CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

We can deal with (5.31) and (5.32) as the same arguments as for (5.30). Thus, we obtain

$$|(5.25)|\leqslant M_0+CTP(\sup_{[0,T]}E_r(t)). \tag{5.41}$$

We write

$$(5.26)=\int_0^T\int_\Omega\overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}\eta^p{}_{,q\beta}\,A_p^q(A_i^\beta A_l^k-A_i^kA_l^\beta)\overline{\partial}^r v^i\rho_0^2J^{-1}dxdt \tag{5.42}$$

$$+\int_0^T\int_\Omega\overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}\eta^p{}_{,\alpha\beta}\,A_p^\beta(A_i^kA_l^\alpha-A_i^\alpha A_l^k)\overline{\partial}^r v^i\rho_0^2J^{-1}dxdt \tag{5.43}$$

$$+\int_0^T\int_\Omega\overline{\partial}\eta^l{}_{,\alpha}\,\overline{\partial}^{r-1}\eta^p{}_{,3\beta}\,A_p^\beta(A_i^\alpha A_l^3-A_i^3A_l^\alpha)\overline{\partial}^r v^i\rho_0^2J^{-1}dxdt \tag{5.44}$$

$$+\sum_{m=0}^{r-2}C_{r-1}^m\int_0^T\int_\Omega\overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^m\eta^p{}_{,q\beta}\,\overline{\partial}^{r-1-m}\left[JA_p^q(A_i^\beta A_l^k-A_i^kA_l^\beta)\right]$$

$$\cdot\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt \tag{5.45}$$

$$+\sum_{m=0}^{r-2} C_{r-1}^{m}\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^m\eta^p{}_{,\alpha\beta}\,\overline{\partial}^{r-1-m}\left[JA_p^\beta(A_i^kA_l^\alpha-A_i^\alpha A_l^k)\right]\cdot\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt \tag{5.46}$$

$$+\sum_{m=0}^{r-2} C_{r-1}^{m}\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\alpha}\,\overline{\partial}^m\eta^p{}_{,3\beta}\,\overline{\partial}^{r-1-m}\left[JA_p^\beta(A_i^\alpha A_l^3-A_i^3 A_l^\alpha)\right]\cdot\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt. \tag{5.47}$$

It is easy to see that

$$\begin{aligned}|(5.42)|+|(5.43)|+|(5.44)|\leqslant& CT\|\rho_0\|_2^{1/2}\sup_{[0,T]}\|\eta\|_4\|\rho_0\overline{\partial}^r\nabla\eta\|_0P(\|\eta\|_3)\|\rho_0^{1/2}\overline{\partial}^r v\|_0\\ \leqslant& M_0+CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

By the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned}|(5.45)|+|(5.46)|+|(5.47)|\leqslant& CT\|\rho_0\|_2^{3/2}\sup_{[0,T]}\|\rho_0^{1/2}\overline{\partial}^r v\|_0P(\|\eta\|_4,\|\eta\|_r)\\ \leqslant& M_0+CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

Hence,

$$|(5.26)|\leqslant M_0+CTP(\sup_{[0,T]}E_r(t)). \tag{5.48}$$

By (A.10), we have

$$\begin{aligned}(5.27)=&\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}\left(\eta^p{}_{,q\beta}\,JA_p^k(A_i^qA_l^\beta-A_i^\beta A_l^q)\right)\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt\\ =&\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^{r-1}\eta^p{}_{,q\beta}\,A_p^k(A_i^qA_l^\beta-A_i^\beta A_l^q)\overline{\partial}^r v^i\rho_0^2J^{-1}dxdt \qquad (5.49)\\ &+\sum_{m=0}^{r-2} C_{r-1}^{m}\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,k}\,\overline{\partial}^m\eta^p{}_{,q\beta}\,\overline{\partial}^{r-1-m}\left(JA_p^k(A_i^qA_l^\beta-A_i^\beta A_l^q)\right)\cdot\overline{\partial}^r v^i\rho_0^2J^{-2}dxdt. \qquad (5.50)\end{aligned}$$

By the Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned}|(5.49)|\leqslant& CT\sup_{[0,T]}\|\eta\|_4\|\rho_0\overline{\partial}^r\nabla\eta\|_0P(\|\eta\|_3)\|\rho_0^{1/2}\overline{\partial}^r v\|_0\|\rho_0\|_2^{1/2}\\ \leqslant& M_0+CTP(\sup_{[0,T]}E_r(t)),\end{aligned}$$

and similar to (5.45)

$$|(5.50)|\leqslant M_0+CTP(\sup_{[0,T]}E_r(t)).$$

Thus,

$$|(5.27)|\leqslant M_0+CTP(\sup_{[0,T]}E_r(t)). \tag{5.51}$$

For (5.28) and (5.29), it is easy to have

$$|(5.28)|+|(5.29)|\leqslant CT(\|\rho_0\|_3^{3/2}+\|\rho_0\|_2^{3/2})\sup_{[0,T]}\|\rho_0^{1/2}\overline{\partial}^r v\|_0P(\|\eta\|_4,\|\eta\|_r)$$

$$\leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).$$

Hence,

$$|(5.23)| \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)). \tag{5.52}$$

By integration by parts with respect to time, it holds

$$(5.24) = -\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^{r-1}(a_i^3 A_l^\beta - A_i^\beta a_l^3)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2 J^{-2}dx\Big|_{t=T} \tag{5.53}$$

$$+\int_0^T\int_\Omega \overline{\partial}v^l{}_{,\beta}\,\overline{\partial}^{r-1}(a_i^3 A_l^\beta - A_i^\beta a_l^3)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2 J^{-2}dxdt \tag{5.54}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^{r-1}\partial_t(a_i^3 A_l^\beta - A_i^\beta a_l^3)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2 J^{-2}dxdt \tag{5.55}$$

$$+\int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^{r-1}(a_i^3 A_l^\beta - A_i^\beta a_l^3)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2\partial_t J^{-2}dxdt. \tag{5.56}$$

It is easy to see that by applying the fundamental theorem of calculus three times

$$\begin{aligned}|(5.53)| \leqslant& C\|\eta(T)\|_4\|\rho_0\overline{\partial}^r\nabla\eta(T)\|_0 P(\|\eta\|_3)\\ &\cdot\Bigg(\left\|\rho_0\int_0^T\overline{\partial}^{r-1}\partial_t\nabla\eta dt\right\|_0 + \|\rho_0\|_2\|\overline{\partial}^{r-2}\nabla\eta\|_1\left\|\int_0^T\overline{\partial}\nabla v dt\right\|_1\\ &+\|\rho_0\|_2\|\overline{\partial}^{r-3}\nabla\eta\|_1\left\|\int_0^T\overline{\partial}^2\nabla v dt\right\|_{L^3(\Omega)}\\ &+(1-\mathrm{sgn}(5-r))\|\rho_0\|_2\|\overline{\partial}^3\nabla\eta\|_1\left\|\int_0^T\overline{\partial}^{r-4}\nabla v dt\right\|_1\Bigg)\\ \leqslant& M_0 + CTP(\sup_{[0,T]} E_r(t)).\end{aligned}$$

By (A.14) and (A.10), we have

$$(5.54) = \int_0^T\int_\Omega \overline{\partial}v^l{}_{,\beta}\,\overline{\partial}^{r-2}\left(\overline{\partial}\eta^p{}_{,\alpha}\,JA_l^\beta(A_i^3A_p^\alpha - A_i^\alpha a_p^3)\right)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt \tag{5.57}$$

$$+\int_0^T\int_\Omega \overline{\partial}v^l{}_{,\beta}\,\overline{\partial}^{r-2}\left(\overline{\partial}\eta^p{}_{,\alpha}\,JA_p^\beta(A_l^3A_i^\alpha - A_l^\alpha a_i^3)\right)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt \tag{5.58}$$

$$+\int_0^T\int_\Omega \overline{\partial}v^l{}_{,\beta}\,\overline{\partial}^{r-2}\left(\overline{\partial}\eta^p{}_{,\alpha}\,JA_i^\beta(A_p^3A_l^\alpha - A_p^\alpha a_l^3)\right)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt. \tag{5.59}$$

We split (5.57) into two integrals, i.e.,

$$(5.57) = \int_0^T\int_\Omega \overline{\partial}v^l{}_{,\beta}\,\overline{\partial}^{r-1}\eta^p{}_{,\alpha}\,JA_l^\beta(A_i^3A_p^\alpha - A_i^\alpha a_p^3)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-1}dxdt \tag{5.60}$$

$$+\sum_{m=0}^{r-3}C_{r-2}^m\int_0^T\int_\Omega \overline{\partial}v^l{}_{,\beta}\,\overline{\partial}^{m+1}\eta^p{}_{,\alpha}\,\overline{\partial}^{r-2-m}\left(JA_l^\beta(A_i^3A_p^\alpha - A_i^\alpha a_p^3)\right)$$
$$\cdot\,\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt. \tag{5.61}$$

Obviously, we see that

$$|(5.60)| \leqslant CT\sup_{[0,T]}\|\overline{\partial}^2 v\|_1\|\rho_0\overline{\partial}^r\eta\|_1 P(\|\eta\|_3)\|\rho_0\overline{\partial}^r\nabla\eta\|_0 \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)),$$

and

$$|(5.61)| \leqslant CT\|\rho_0\|_2 \sup_{[0,T]} \|\overline{\partial}^2 v\|_1 P(\|\eta\|_r)\|\rho_0\overline{\partial}^r\nabla\eta\|_0 \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).$$

Both (5.58) and (5.59) can be dealt with as the same argument as for (5.57). Thus, we obtain

$$|(5.54)| \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).$$

By (A.15) and (A.11), we have

$$(5.55) = \int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^{r-1}\left[v^p{}_{,\alpha}\,JA_l^\beta(A_i^3A_p^\alpha - A_i^\alpha a_p^3)\right]\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt \tag{5.62}$$

$$+ \int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^{r-1}\left[v^p{}_{,\alpha}\,JA_p^\beta(A_l^3A_i^\alpha - A_l^\alpha a_i^3)\right]\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt \tag{5.63}$$

$$+ \int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^{r-1}\left[v^p{}_{,\alpha}\,JA_i^\beta(A_p^3A_l^\alpha - A_p^\alpha a_l^3)\right]\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt. \tag{5.64}$$

We write

$$(5.62) = \int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^{r-1}v^p{}_{,\alpha}\,A_l^\beta(A_i^3A_p^\alpha - A_i^\alpha a_p^3)\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-1}dxdt \tag{5.65}$$

$$+ \sum_{m=0}^{r-2} C_{r-1}^m \int_0^T\int_\Omega \overline{\partial}\eta^l{}_{,\beta}\,\overline{\partial}^m v^p{}_{,\alpha}\,\overline{\partial}^{r-1-m}\left[JA_l^\beta(A_i^3A_p^\alpha - A_i^\alpha a_p^3)\right]$$

$$\cdot\,\overline{\partial}^r\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt. \tag{5.66}$$

Then, by the same arguments as for (5.44) and (5.47), we can estimate (5.65) and (5.66), and then

$$|(5.55)| \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).$$

It is easy to see that (5.56) has the same bounds. Thus, we obtain the estimates of (5.24) and then of (5.22), i.e.,

$$|(5.22)| \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).$$

Case 4. $s = r-2$ **and** $r = 5$**.** Integration by parts with respect to time gives

$$\int_0^T\int_\Omega \overline{\partial}^2\eta^l{}_{,k}\,\overline{\partial}^3\left(J^{-1}(a_i^ja_l^k - a_i^ka_l^j)\right)\overline{\partial}^5v^i{}_{,j}\,\rho_0^2J^{-2}dxdt$$

$$= \int_\Omega \overline{\partial}^2\eta^l{}_{,k}\,\overline{\partial}^3\left(J^{-1}(a_i^ja_l^k - a_i^ka_l^j)\right)\overline{\partial}^5\eta^i{}_{,j}\,\rho_0^2J^{-2}dx\Big|_0^T$$

$$- \int_0^T\int_\Omega \overline{\partial}^2v^l{}_{,k}\,\overline{\partial}^3\left(J^{-1}(a_i^ja_l^k - a_i^ka_l^j)\right)\overline{\partial}^5\eta^i{}_{,j}\,\rho_0^2J^{-2}dxdt$$

$$- \int_0^T\int_\Omega \overline{\partial}^2\eta^l{}_{,k}\,\partial_t\overline{\partial}^3\left(J^{-1}(a_i^ja_l^k - a_i^ka_l^j)\right)\overline{\partial}^5\eta^i{}_{,j}\,\rho_0^2J^{-2}dxdt$$

$$- \int_0^T\int_\Omega \overline{\partial}^2\eta^l{}_{,k}\,\overline{\partial}^3\left(J^{-1}(a_i^ja_l^k - a_i^ka_l^j)\right)\overline{\partial}^5\eta^i{}_{,j}\,\rho_0^2\partial_tJ^{-2}dxdt,$$

which can be controlled by $M_0 + CTP(\sup_{[0,T]} E_5(t))$ from the Hölder inequality and the fundamental theorem of calculus.

Therefore, we obtain

$$\left|\int_0^T (5.4)dt\right| \leqslant M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).$$

Step 2. Analysis of the integral $\mathfrak{I}_2$. Similar to those of $\mathfrak{I}_1$, by (1.13b), (1.13c) and $\eta_t = v$, we have

$$\begin{aligned}\mathfrak{I}_2 =& -\int_\Omega a_i^j \overline{\partial}^r \rho_0^2 J^{-2} \overline{\partial}^r v^i{}_{,j}\, dx + \int_{\Gamma_0} a_i^3 \overline{\partial}^r \rho_0^2 J^{-2} \overline{\partial}^r v^i dx_1 dx_2\\ =& -\int_\Omega \overline{\partial}^r \rho_0^2 A_i^j J^{-1} \overline{\partial}^r \eta_t^i{}_{,j}\, dx.\end{aligned}$$

Integration by parts shows that

$$\begin{aligned}\int_0^T \mathfrak{I}_2(t)dt =& \int_0^T\int_\Omega \overline{\partial}^r \rho_0^2 (A_i^j J^{-1})_t \overline{\partial}^r \eta^i{}_{,j}\, dxdt - \int_\Omega \overline{\partial}^r \rho_0^2 A_i^j J^{-1} \overline{\partial}^r \eta^i{}_{,j}\, dx\Big|_{t=T}\\ =& \int_0^T\int_\Omega \overline{\partial}^r \rho_0^2 (A_i^j J^{-1})_t \overline{\partial}^r \eta^i{}_{,j}\, dxdt\\ & - \int_\Omega \overline{\partial}^r \rho_0^2 \left(\delta_i^j + \int_0^T (A_i^j J^{-1})_t dt\right) \overline{\partial}^r \eta^i{}_{,j}\,(T) dx,\end{aligned}$$

which yields, by Hölder's inequality and Sobolev's embedding theorem, that

$$\begin{aligned}\left|\int_0^T \mathfrak{I}_2(t)dt\right| \leqslant& C \left\|\frac{\overline{\partial}^r \rho_0^2}{\rho_0}\right\|_0 \left(T \sup_{[0,T]} \|\rho_0 \overline{\partial}^r \nabla\eta\|_0 \|v\|_3 \|\eta\|_3^4 + \|\rho_0 \overline{\partial}^r \operatorname{div}\eta\|_0\right)\\ \leqslant& M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)),\end{aligned}$$

where we require $\rho_0 \in H^{\max(4,r)}(\Omega)$ because, for $r \leqslant 5$,

$$\begin{aligned}\left\|\frac{\overline{\partial}^r \rho_0^2}{\rho_0}\right\|_0 \leqslant& \sum_{m=0}^r C_r^m \left\|\frac{\overline{\partial}^m \rho_0 \overline{\partial}^{r-m} \rho_0}{\rho_0}\right\|_0 \leqslant 2 \sum_{m=0}^{\lceil (r-1)/2\rceil} C_r^m \left\|\frac{\overline{\partial}^m \rho_0 \overline{\partial}^{r-m} \rho_0}{\rho_0}\right\|_0\\ \leqslant& C\|\overline{\partial}^r \rho_0\|_0 + C\left\|\frac{\overline{\partial}\rho_0}{\rho_0}\right\|_2 \|\overline{\partial}^{r-1}\rho_0\|_0 + C\left\|\frac{\overline{\partial}^2\rho_0}{\rho_0}\right\|_0 \|\overline{\partial}^{r-2}\rho_0\|_2\\ \leqslant& C\|\rho_0\|_r + C\|\rho_0\|_4 \|\rho_0\|_{r-1} + C\|\rho_0\|_3 \|\rho_0\|_r\\ \leqslant& C\|\rho_0\|_{\max(4,r)},\end{aligned} \tag{5.67}$$

by the higher order Hardy inequality.

Step 3. Analysis of the integral $\mathfrak{I}_3$. Similar to those of $\mathfrak{I}_1$, by (1.13b), (1.13c), (A.12) and $\eta_t = v$, we have

$$\begin{aligned}\mathfrak{I}_3 =& -\int_\Omega \rho_0^2 \overline{\partial}^r J^{-2} a_i^j \overline{\partial}^r v^i{}_{,j}\, dx + \int_{\Gamma_0} \rho_0^2 \overline{\partial}^r J^{-2} a_i^3 \overline{\partial}^r v^i dx_1 dx_2\\ =& 2\int_\Omega \rho_0^2 \overline{\partial}^{r-1} (J^{-3}\overline{\partial} J) a_i^j \overline{\partial}^r v^i{}_{,j}\, dx\\ =& 2\int_\Omega \overline{\partial}^r J A_i^j \overline{\partial}^r v^i{}_{,j}\, \rho_0^2 J^{-2} dx \qquad (5.68)\\ & + 2\sum_{s=0}^{r-2} C_{r-1}^s \int_\Omega \overline{\partial}^{r-1-s} J^{-3} \overline{\partial}^{s+1} J a_i^j \overline{\partial}^r v^i{}_{,j}\, \rho_0^2 dx. \qquad (5.69)\end{aligned}$$

Due to $v=\eta_t$, we get

$$
\begin{aligned}
(5.68) =& 2\int_\Omega \overline{\partial}^r \eta^k{}_{,l}\, A_k^l \overline{\partial}^r v^i{}_{,j}\, A_i^j \rho_0^2 J^{-1} dx && (5.70)\\
&+ 2\sum_{s=0}^{r-2} C_{r-1}^s \int_\Omega \overline{\partial}^{r-1-s} a_k^l \overline{\partial}^{s+1} \eta^k{}_{,l}\, \overline{\partial}^r v^i{}_{,j}\, A_i^j \rho_0^2 J^{-2} dx && (5.71)\\
=& \frac{d}{dt}\int_\Omega \overline{\partial}^r \eta^k{}_{,l}\, A_k^l \overline{\partial}^r \eta^i{}_{,j}\, A_i^j \rho_0^2 J^{-1} dx + (5.71)\\
&+ \int_\Omega \overline{\partial}^r \eta^k{}_{,l}\, \overline{\partial}^r \eta^i{}_{,j}\, A_k^l A_i^j \mathrm{div}_\eta\, v \rho_0^2 J^{-1} dx && (5.72)\\
&- \int_\Omega \overline{\partial}^r \eta^k{}_{,l}\, \overline{\partial}^r \eta^i{}_{,j}\, (A_k^l A_i^j)_t \rho_0^2 J^{-1} dx. && (5.73)
\end{aligned}
$$

Noticing that $\overline{\partial}^r \mathrm{div}\,\eta(0)=0$, integrating over $[0,T]$ yields

$$
\int_0^T (5.68) dt' = \int_\Omega |\mathrm{div}_\eta\, \overline{\partial}^r \eta(T)|^2 \rho_0^2 J^{-1}(T) dx + \int_0^T [(5.72)+(5.73)+(5.71)] dt.
$$

By Hölder's inequality and Sobolev's embedding theorem, we obtain

$$
\left| \int_0^T (5.72)+(5.73) dt \right| \leqslant CT \sup_{[0,T]} \|\rho_0 \overline{\partial}^r \nabla\eta\|_0^2 \|v\|_3 \|\eta\|_4^6 \leqslant CTP(\sup_{[0,T]} E_r(t)).
$$

By integration by parts, we have

$$
\begin{aligned}
\int_0^T (5.71) =& 2\sum_{s=0}^{r-2} C_{r-1}^s \int_0^T\!\!\int_\Omega \overline{\partial}^{r-1-s} a_k^l \overline{\partial}^{s+1} \eta^k{}_{,l}\, \overline{\partial}^r v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dx dt\\
=& -2\sum_{s=0}^{r-2} C_{r-1}^s \int_0^T\!\!\int_\Omega \overline{\partial}^{r-s} a_k^l \overline{\partial}^{s+1} \eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dx dt && (5.74)\\
&- 2\sum_{s=0}^{r-2} C_{r-1}^s \int_0^T\!\!\int_\Omega \overline{\partial}^{r-1-s} a_k^l \overline{\partial}^{s+2} \eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dx dt && (5.75)\\
&- 2\sum_{s=0}^{r-2} C_{r-1}^s \int_0^T\!\!\int_\Omega \overline{\partial}^{r-1-s} a_k^l \overline{\partial}^{s+1} \eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \overline{\partial}(\rho_0^2 J^{-2} A_i^j) dx dt. && (5.76)
\end{aligned}
$$

We first consider (5.74) and split it into four cases.

Case 1. $s=0$. By an L^2-L^∞-L^2 Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus for the norm $\|\rho_0\overline{\partial}^{r-1}\nabla v\|_0$, we can easily get

$$
\left| 2\int_0^T\!\!\int_\Omega \overline{\partial}^r a_k^l \overline{\partial}\eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dx dt \right| \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).
$$

Case 2. $s=1$. Integration by parts yields

$$
\begin{aligned}
&- 2C_{r-1}^1 \int_0^T\!\!\int_\Omega \overline{\partial}^{r-1} a_k^l \overline{\partial}^2 \eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dx dt && (5.77)\\
=& 2C_{r-1}^1 \int_0^T\!\!\int_\Omega \overline{\partial}^r a_k^l \overline{\partial}^2 \eta^k{}_{,l}\, \overline{\partial}^{r-2} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dx dt && (5.78)
\end{aligned}
$$

$$+2C_{r-1}^{1}\int_0^T\int_\Omega \overline{\partial}^{r-1}a_k^l\overline{\partial}^3\eta^k{}_{,l}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\rho_0^2J^{-2}A_i^j dxdt \tag{5.79}$$

$$+2C_{r-1}^{1}\int_0^T\int_\Omega \overline{\partial}^{r-1}a_k^l\overline{\partial}^2\eta^k{}_{,l}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\overline{\partial}(\rho_0^2J^{-2}A_i^j)dxdt. \tag{5.80}$$

By an L^2-L^6-L^3 Hölder inequality, (5.16) and the Sobolev embedding theorem, we get

$$|(5.78)|+|(5.80)|\leqslant M_0+CTP(\sup_{[0,T]}E_r(t)).$$

By using (A.14), it holds

$$(5.79)=2C_{r-1}^{1}\int_0^T\int_\Omega \overline{\partial}^{r-2}\left[\overline{\partial}\eta^p{}_{,q}\,J(A_k^lA_p^q-A_k^qA_p^l)\right]\overline{\partial}^3\eta^k{}_{,l}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\rho_0^2J^{-2}A_i^j dxdt$$

$$=2C_{r-1}^{1}\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^p{}_{,\beta}\,(A_k^lA_p^\beta-A_k^\beta A_p^l)\overline{\partial}^3\eta^k{}_{,l}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\rho_0^2J^{-1}A_i^j dxdt \tag{5.81}$$

$$+2C_{r-1}^{1}\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^p{}_{,3}\,(A_k^\beta A_p^3-A_k^3A_p^\beta)\overline{\partial}^3\eta^k{}_{,\beta}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\rho_0^2J^{-1}A_i^j dxdt \tag{5.82}$$

$$+2C_{r-1}^{1}\sum_{m=0}^{r-3}C_{r-2}^{m}\int_0^T\int_\Omega \overline{\partial}^{m+1}\eta^p{}_{,\beta}\,\overline{\partial}^{r-2-m}\left[J(A_k^lA_p^\beta-A_k^\beta A_p^l)\right]$$
$$\cdot\,\overline{\partial}^3\eta^k{}_{,l}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\rho_0^2J^{-2}A_i^j dxdt \tag{5.83}$$

$$+2C_{r-1}^{1}\sum_{m=0}^{r-3}C_{r-2}^{m}\int_0^T\int_\Omega \overline{\partial}^{m+1}\eta^p{}_{,3}\,\overline{\partial}^{r-2-m}\left[J(A_k^\beta A_p^3-A_k^3A_p^\beta)\right]$$
$$\cdot\,\overline{\partial}^3\eta^k{}_{,\beta}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\rho_0^2J^{-2}A_i^j dxdt. \tag{5.84}$$

By using the L^6-L^2-L^3 Hölder inequality for (5.81) and L^2-L^6-L^3 Hölder inequality for (5.82) on higher order terms, together with (5.16) and the Sobolev embedding theorem, we get

$$|(5.81)|+|(5.82)|\leqslant M_0+CTP(\sup_{[0,T]}E_r(t)).$$

From the L^∞-L^6-L^2-L^3 Hölder inequality for the case $m=0$ and L^6-L^∞-L^2-L^3 Hölder inequality for the case $1\leqslant m\leqslant r-3$ on higher order terms, together with (5.16) and the Sobolev embedding theorem, we can get the same bounds for (5.83) and (5.84). Then, we obtain

$$|(5.77)|\leqslant M_0+CTP(\sup_{[0,T]}E_r(t)).$$

Case 3. $s=2$. By integration by parts, it yields

$$-2C_{r-1}^{2}\int_0^T\int_\Omega \overline{\partial}^{r-2}a_k^l\overline{\partial}^3\eta^k{}_{,l}\,\overline{\partial}^{r-1}v^i{}_{,j}\,\rho_0^2J^{-2}A_i^j dxdt \tag{5.85}$$

$$=2C_{r-1}^{2}\int_0^T\int_\Omega \overline{\partial}^{r-1}a_k^l\overline{\partial}^3\eta^k{}_{,l}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\rho_0^2J^{-2}A_i^j dxdt \tag{5.86}$$

$$+2C_{r-1}^{2}\int_0^T\int_\Omega \overline{\partial}^{r-2}a_k^l\overline{\partial}^4\eta^k{}_{,l}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\rho_0^2J^{-2}A_i^j dxdt \tag{5.87}$$

$$+2C_{r-1}^{2}\int_0^T\int_\Omega \overline{\partial}^{r-2}a_k^l\overline{\partial}^3\eta^k{}_{,l}\,\overline{\partial}^{r-2}v^i{}_{,j}\,\overline{\partial}(\rho_0^2J^{-2}A_i^j)dxdt. \tag{5.88}$$

It is clear that $C_{r-1}^1(5.86) = C_{r-1}^2(5.79)$ while the latter has been just estimated. By an L^6-L^2-L^3 Hölder inequality for higher order terms, (5.16) and the Sobolev embedding theorem, we get

$$|(5.87)| + |(5.88)| \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).$$

That is, (5.85) has the same bound $M_0 + CTP(\sup_{[0,T]} E_r(t))$.

Case 4. $s = r-2$ **with** $r = 5$. It is easy to see that

$$\int_0^T\int_\Omega \overline{\partial}^2 a_k^l \overline{\partial}^4 \eta^k{}_{,l}\, \overline{\partial}^4 v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dxdt \tag{5.89}$$

can be bounded by the desired bound in view of Hölder's inequality and the fundamental theorem of calculus.

Next, we consider (5.75). Since for the case $s = 0$

$$-2\int_0^T\int_\Omega \overline{\partial}^{r-1} a_k^l \overline{\partial}^2 \eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dxdt = \frac{1}{r-1}(5.77),$$

and for the case $s = 1$

$$-2C_{r-1}^1\int_0^T\int_\Omega \overline{\partial}^{r-2} a_k^l \overline{\partial}^3 \eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dxdt = \frac{C_{r-1}^1}{C_{r-1}^2}(5.85),$$

we have the desired bounds. For the case $s = 2$, we have, by Hölder's inequality and the Sobolev embedding theorem and the fundamental theorem of calculus, that

$$\begin{aligned}
&\left|-2C_{r-1}^2\int_0^T\int_\Omega \overline{\partial}^{r-3} a_k^l \overline{\partial}^4 \eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dxdt\right| \\
\leqslant& CT\sup_{[0,T]} \|\eta\|_r P(\|\eta\|_3)\|\rho_0\overline{\partial}^4\nabla\eta\|_0 \left(\left\|\rho_0\int_0^T \overline{\partial}^{r-1}\partial_t^2\nabla\eta dt\right\|_0 + \|\rho_0\overline{\partial}^{r-1}\nabla v(0)\|_0\right) \\
\leqslant& M_0 + CTP(\sup_{[0,T]} E_r(t)).
\end{aligned}$$

For the case $s = r-2$ and $r = 5$,

$$\begin{aligned}
&\left|-2C_{r-1}^{r-2}\int_0^T\int_\Omega \overline{\partial} a_k^l \overline{\partial}^r \eta^k{}_{,l}\, \overline{\partial}^{r-1} v^i{}_{,j}\, \rho_0^2 J^{-2} A_i^j dxdt\right| \\
\leqslant& CT\sup_{[0,T]} \|\eta\|_4 P(\|\eta\|_3)\|\rho_0\overline{\partial}^r\nabla\eta\|_0 \left(\left\|\rho_0\int_0^T \overline{\partial}^{r-1}\partial_t^2\nabla\eta dt\right\|_0 + \|\rho_0\overline{\partial}^{r-1}\nabla v(0)\|_0\right) \\
\leqslant& M_0 + CTP(\sup_{[0,T]} E_r(t)).
\end{aligned}$$

For (5.76), by the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus, we can easily obtain the desired bounds. Thus,

$$\left|\int_0^T (5.71)\right| \leqslant M_0 + CTP(\sup_{[0,T]} E_r(t)).$$

Now, we turn to the estimates of $\int_0^T (5.69)dt$.

Case 1. $s = 0$. By (A.12) and integration by parts, we see that

$$2\int_0^T\int_\Omega \overline{\partial}^{r-1} J^{-3}\overline{\partial} J a_i^j \overline{\partial}^r v^i{}_{,j}\, \rho_0^2 dxdt \tag{5.90}$$

$$
\begin{aligned}
&= -6\int_0^T\int_\Omega \overline{\partial}^{r-2}(J^{-3}\overline{\partial}\eta^l{}_{,k}\,A_l^k)\overline{\partial}\eta^p{}_{,q}\,A_p^qA_i^j\overline{\partial}^r v^i{}_{,j}\,\rho_0^2J^2dxdt\\
&= -6\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}\eta^p{}_{,q}\,A_l^kA_p^qA_i^j\overline{\partial}^r v^i{}_{,j}\,\rho_0^2J^{-1}dxdt\\
&\quad -6\sum_{m=0}^{r-3}C_{r-2}^m\int_0^T\int_\Omega \overline{\partial}^{m+1}\eta^l{}_{,k}\,\overline{\partial}^{r-2-m}(J^{-3}A_l^k)\overline{\partial}\eta^p{}_{,q}\,A_p^qA_i^j\overline{\partial}^r v^i{}_{,j}\,\rho_0^2J^2dxdt\\
&=6\int_0^T\int_\Omega \overline{\partial}^r\eta^l{}_{,k}\,\overline{\partial}\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,\rho_0^2J^{-1}A_l^kA_p^qA_i^jdxdt &(5.91)\\
&\quad +6\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}^2\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,\rho_0^2J^{-1}A_l^kA_p^qA_i^jdxdt &(5.92)\\
&\quad +6\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,\overline{\partial}\left(\rho_0^2J^{-1}A_l^kA_p^qA_i^j\right)dxdt &(5.93)\\
&\quad +6\sum_{m=0}^{1}C_2^m\int_0^T\int_\Omega \overline{\partial}^{m+2}\eta^l{}_{,k}\,\overline{\partial}^{r-2-m}(J^{-3}A_l^k)\overline{\partial}\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,\rho_0^2J^2A_p^qA_i^jdxdt &(5.94)\\
&\quad +6\sum_{m=0}^{r-3}C_{r-2}^m\int_0^T\int_\Omega \overline{\partial}^{m+1}\eta^l{}_{,k}\,\overline{\partial}^{r-1-m}(J^{-3}A_l^k)\overline{\partial}\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,\rho_0^2J^2A_p^qA_i^jdxdt &(5.95)\\
&\quad +6\sum_{m=0}^{r-3}C_{r-2}^m\int_0^T\int_\Omega \overline{\partial}^{m+1}\eta^l{}_{,k}\,\overline{\partial}^{r-2-m}(J^{-3}A_l^k)\overline{\partial}^2\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,\rho_0^2J^2A_p^qA_i^jdxdt &(5.96)\\
&\quad +6\sum_{m=0}^{r-3}C_{r-2}^m\int_0^T\int_\Omega \overline{\partial}^{m+1}\eta^l{}_{,k}\,\overline{\partial}^{r-2-m}(J^{-3}A_l^k)\overline{\partial}\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,\overline{\partial}\left(\rho_0^2J^2A_p^qA_i^j\right)dxdt. &(5.97)
\end{aligned}
$$

By the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus, we can easily obtain the desired bound

$$M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)),$$

for (5.91) and (5.93)-(5.97).

For (5.92), we use integration by parts with respect to time to get

$$
\begin{aligned}
(5.92)=&6\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}^2\eta^p{}_{,q}\,\overline{\partial}^{r-1}\eta^i{}_{,j}\,\rho_0^2J^{-1}A_l^kA_p^qA_i^jdx\Big|_{t=T} &(5.98)\\
&-6\int_0^T\int_\Omega \overline{\partial}^{r-1}v^l{}_{,k}\,\overline{\partial}^2\eta^p{}_{,q}\,\overline{\partial}^{r-1}\eta^i{}_{,j}\,\rho_0^2J^{-1}A_l^kA_p^qA_i^jdxdt &(5.99)\\
&-6\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}^2v^p{}_{,q}\,\overline{\partial}^{r-1}\eta^i{}_{,j}\,\rho_0^2J^{-1}A_l^kA_p^qA_i^jdxdt &(5.100)\\
&-6\int_0^T\int_\Omega \overline{\partial}^{r-1}\eta^l{}_{,k}\,\overline{\partial}^2\eta^p{}_{,q}\,\overline{\partial}^{r-1}\eta^i{}_{,j}\,\partial_t\left(\rho_0^2J^{-1}A_l^kA_p^qA_i^j\right)dxdt. &(5.101)
\end{aligned}
$$

It is clear that (5.99) $=-$(5.92). Thus,

$$(5.92)=\frac{1}{2}\left((5.98)+(5.100)+(5.101)\right). \tag{5.102}$$

Moreover, integration by parts yields

$$(5.100) = 6\int_0^T\int_\Omega \overline{\partial}^r \eta^l{}_{,k}\,\overline{\partial}^2 v^p{}_{,q}\,\overline{\partial}^{r-2}\eta^i{}_{,j}\,\rho_0^2 J^{-1}A_l^k A_p^q A_i^j dxdt \tag{5.103}$$

$$+ 6\int_0^T\int_\Omega \overline{\partial}^{r-1} \eta^l{}_{,k}\,\overline{\partial}^3 v^p{}_{,q}\,\overline{\partial}^{r-2}\eta^i{}_{,j}\,\rho_0^2 J^{-1}A_l^k A_p^q A_i^j dxdt \tag{5.104}$$

$$+ 6\int_0^T\int_\Omega \overline{\partial}^{r-1} \eta^l{}_{,k}\,\overline{\partial}^2 v^p{}_{,q}\,\overline{\partial}^{r-2}\eta^i{}_{,j}\,\overline{\partial}\left(\rho_0^2 J^{-1}A_l^k A_p^q A_i^j\right) dxdt. \tag{5.105}$$

Since (5.104) = (5.92), we have from (5.102)

$$(5.92) = (5.98) + (5.101) + (5.103) + (5.105). \tag{5.106}$$

By integration by parts, we get for $t = T$

$$\begin{aligned}(5.98) = &-12\int_\Omega \overline{\partial}^r \eta^l{}_{,k}\,\overline{\partial}\eta^p{}_{,q}\,\overline{\partial}^{r-1}\eta^i{}_{,j}\,\rho_0^2 J^{-1}A_l^k A_p^q A_i^j dx\\ &-6\int_\Omega \overline{\partial}^{r-1} \eta^l{}_{,k}\,\overline{\partial}\eta^p{}_{,q}\,\overline{\partial}^{r-1}\eta^i{}_{,j}\,\overline{\partial}\left(\rho_0^2 J^{-1}A_l^k A_p^q A_i^j\right) dx,\end{aligned}$$

which yields the desired bound for (5.98) by using the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus.

Integration by parts implies

$$(5.101) = 6\int_0^T\int_\Omega \overline{\partial}^{r-2} \eta^l{}_{,k}\,\overline{\partial}^3 \eta^p{}_{,q}\,\overline{\partial}^{r-1}\eta^i{}_{,j}\,\partial_t\left(\rho_0^2 J^{-1}A_l^k A_p^q A_i^j\right) dxdt \tag{5.107}$$

$$+ 6\int_0^T\int_\Omega \overline{\partial}^{r-2} \eta^l{}_{,k}\,\overline{\partial}^2 \eta^p{}_{,q}\,\overline{\partial}^{r}\eta^i{}_{,j}\,\partial_t\left(\rho_0^2 J^{-1}A_l^k A_p^q A_i^j\right) dxdt \tag{5.108}$$

$$+ 6\int_0^T\int_\Omega \overline{\partial}^{r-2} \eta^l{}_{,k}\,\overline{\partial}^2 \eta^p{}_{,q}\,\overline{\partial}^{r-1}\eta^i{}_{,j}\,\partial_t\overline{\partial}\left(\rho_0^2 J^{-1}A_l^k A_p^q A_i^j\right) dxdt. \tag{5.109}$$

Due to (5.107) = −(5.101), it follows that

$$(5.101) = \frac{1}{2}((5.108) + (5.109)),$$

which implies the desired bound for (5.101) by using the Hölder inequality and the Sobolev embedding theorem. It is clear that both (5.103) and (5.101) have the same bound in view of the Hölder inequality and the Sobolev embedding theorem. Thus, we have obtained

$$|(5.90)| \leqslant M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).$$

Case 2. $s = 1$. From (A.12), it follows that

$$2C_{r-1}^1\int_0^T\int_\Omega \overline{\partial}^{r-2} J^{-3}\overline{\partial}^2 J a_i^j \overline{\partial}^r v^i{}_{,j}\,\rho_0^2 dxdt \tag{5.110}$$

$$\begin{aligned}=&-6C_{r-1}^1\int_0^T\int_\Omega \overline{\partial}^{r-3}\left(J^{-3}A_l^k\overline{\partial}\eta^l{}_{,k}\right)\overline{\partial}\left(JA_p^q\overline{\partial}\eta^p{}_{,q}\right)a_i^j\overline{\partial}^r v^i{}_{,j}\,\rho_0^2 dxdt\\ =&-6C_{r-1}^1\sum_{m=0}^{r-3} C_{r-3}^m\int_0^T\int_\Omega \overline{\partial}^{r-3-m}\left(J^{-3}A_l^k\right)\overline{\partial}^{m+1}\eta^l{}_{,k}\,\overline{\partial}\left(JA_p^q\right)\\ &\cdot\overline{\partial}\eta^p{}_{,q}\,\overline{\partial}^r v^i{}_{,j}\,\rho_0^2 JA_i^j dxdt\end{aligned} \tag{5.111}$$

$$-6C_{r-1}^{1}\sum_{m=0}^{r-3}C_{r-3}^{m}\int_0^T\int_\Omega \overline{\partial}^{r-3-m}\left(J^{-3}A_l^k\right)\overline{\partial}^{m+1}\eta^l{}_{,k}\,\overline{\partial}^{2}\eta^p{}_{,q}\,\overline{\partial}^{r}v^i{}_{,j}\,\rho_0^2J^2A_i^jA_p^q dxdt. \tag{5.112}$$

By using integration by parts with respect to $\overline{\partial}$ for (5.111) and the cases $m=0, r-3$ in (5.112), and integration by parts with respect to time for the case $m=1$ in (5.112), together with the Hölder inequality, the Sobolev embedding theorem, (5.16) and the fundamental theorem of calculus, we obtain

$$|(5.110)|\leqslant M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).$$

Case 3. $s=2$. By (A.12) and integration by parts, it yields

$$2C_{r-1}^{2}\int_0^T\int_\Omega \overline{\partial}^{r-3}J^{-3}\overline{\partial}^{3}J\overline{\partial}^{r}v^i{}_{,j}\,\rho_0^2JA_i^j dxdt \tag{5.113}$$

$$\begin{aligned}=&-6C_{r-1}^{2}\sum_{n=0}^{r-4}C_{r-4}^{n}\int_0^T\int_\Omega \overline{\partial}^{n+1}\eta^l{}_{,k}\,\overline{\partial}^{r-4-n}J^{-3}\overline{\partial}^{3}\eta^p{}_{,q}\,\overline{\partial}^{r}v^i{}_{,j}\,A_p^qA_i^jA_l^k\rho_0^2J^2dxdt\\&-6C_{r-1}^{2}\sum_{n=0}^{r-4}C_{r-4}^{n}\sum_{m=0}^{1}C_{2}^{m}\int_0^T\int_\Omega \overline{\partial}^{n+1}\eta^l{}_{,k}\,\overline{\partial}^{r-4-n}J^{-3}\overline{\partial}^{m+1}\eta^p{}_{,q}\,\overline{\partial}^{2-m}\left(JA_p^q\right)\\&\qquad\cdot\overline{\partial}^{r}v^i{}_{,j}\,A_i^jA_l^k\rho_0^2Jdxdt\\=&6C_{r-1}^{2}\sum_{n=0}^{r-4}C_{r-4}^{n}\int_0^T\int_\Omega \overline{\partial}^{n+2}\eta^l{}_{,k}\,\overline{\partial}^{r-4-n}J^{-3}\overline{\partial}^{3}\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,A_p^qA_i^jA_l^k\rho_0^2J^2dxdt\end{aligned} \tag{5.114}$$

$$+6C_{r-1}^{2}\sum_{n=0}^{r-4}C_{r-4}^{n}\int_0^T\int_\Omega \overline{\partial}^{n+1}\eta^l{}_{,k}\,\overline{\partial}^{r-4-n}J^{-3}\overline{\partial}^{4}\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\,A_p^qA_i^jA_l^k\rho_0^2J^2dxdt \tag{5.115}$$

$$+6C_{r-1}^{2}\sum_{n=0}^{r-4}C_{r-4}^{n}\int_0^T\int_\Omega \overline{\partial}^{n+1}\eta^l{}_{,k}\,\overline{\partial}^{3}\eta^p{}_{,q}\,\overline{\partial}^{r-1}v^i{}_{,j}\cdot\overline{\partial}(\overline{\partial}^{r-4-n}J^{-3}A_p^qA_i^jA_l^k\rho_0^2J^2)dxdt \tag{5.116}$$

$$+6C_{r-1}^{2}\sum_{n=0}^{r-4}C_{r-4}^{n}\sum_{m=0}^{1}C_{2}^{m}\int_0^T\int_\Omega \overline{\partial}^{n+2}\eta^l{}_{,k}\,\overline{\partial}^{r-4-n}J^{-3}\overline{\partial}^{m+1}\eta^p{}_{,q}\cdot\overline{\partial}^{2-m}\left(JA_p^q\right)\overline{\partial}^{r-1}v^i{}_{,j}\,A_i^jA_l^k\rho_0^2Jdxdt \tag{5.117}$$

$$+6C_{r-1}^{2}\sum_{n=0}^{r-4}C_{r-4}^{n}\sum_{m=0}^{1}C_{2}^{m}\int_0^T\int_\Omega \overline{\partial}^{n+1}\eta^l{}_{,k}\,\overline{\partial}^{r-4-n}J^{-3}\overline{\partial}^{m+2}\eta^p{}_{,q}\cdot\overline{\partial}^{2-m}\left(JA_p^q\right)\overline{\partial}^{r-1}v^i{}_{,j}\,A_i^jA_l^k\rho_0^2Jdxdt \tag{5.118}$$

$$+6C_{r-1}^{2}\sum_{n=0}^{r-4}C_{r-4}^{n}\sum_{m=0}^{1}C_{2}^{m}\int_0^T\int_\Omega \overline{\partial}^{n+1}\eta^l{}_{,k}\,\overline{\partial}^{r-4-n}J^{-3}\overline{\partial}^{m+1}\eta^p{}_{,q}\cdot\overline{\partial}^{3-m}\left(JA_p^q\right)\overline{\partial}^{r-1}v^i{}_{,j}\,A_i^jA_l^k\rho_0^2Jdxdt \tag{5.119}$$

$$+6C_{r-1}^2\sum_{n=0}^{r-4}C_{r-4}^n\sum_{m=0}^{1}C_2^m\int_0^T\int_\Omega \overline{\partial}^{n+1}\eta^l{}_{,k}\,\overline{\partial}^{m+1}\eta^p{}_{,q}\,\overline{\partial}^{2-m}\left(JA_p^q\right)\overline{\partial}^{r-1}v^i{}_{,j}$$
$$\cdot\,\overline{\partial}(\overline{\partial}^{r-4-n}J^{-3}A_i^jA_l^k\rho_0^2J)dxdt. \tag{5.120}$$

Due to the case $n=0$ in (5.114) is equal to C_{r-1}^2(5.92), we have obtained the desired bound for this case of (5.114). For (5.115)-(5.120) and the case $n=r-4$ with $r=5$ of (5.114), we can easily use the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus to get the desired bounds. Then, so does (5.113).

Case 4. $s=r-2$ **with** $r=5$**.** Integration by parts with respect to time yields

$$\begin{aligned}&\int_0^T\int_\Omega \overline{\partial}J^{-3}\overline{\partial}^4Ja_i^j\overline{\partial}^5v^i{}_{,j}\,\rho_0^2dxdt\\ =&\int_\Omega \overline{\partial}J^{-3}\overline{\partial}^4Ja_i^j\overline{\partial}^5\eta^i{}_{,j}\,\rho_0^2dx\Big|_0^T-\int_0^T\int_\Omega \partial_t(\overline{\partial}J^{-3}a_i^j)\overline{\partial}^4J\overline{\partial}^5\eta^i{}_{,j}\,\rho_0^2dxdt\\ &-\int_0^T\int_\Omega \overline{\partial}J^{-3}\partial_t\overline{\partial}^4Ja_i^j\overline{\partial}^5\eta^i{}_{,j}\,\rho_0^2dxdt-\int_0^T\int_\Omega \overline{\partial}J^{-3}\overline{\partial}^4Ja_i^j\overline{\partial}^5\eta^i{}_{,j}\,\rho_0^2dxdt,\end{aligned}$$

which can be easily controlled by the desired bound in view of the Hölder inequality and the fundamental theorem of calculus.

Therefore, combining with four cases, we obtain

$$\left|\int_0^T(5.69)dt\right|\leqslant M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)). \tag{5.121}$$

Step 4. Analysis of the remainder $\mathfrak{I}_4$.

For $l=0$, we have from the integration by parts in time, the fundamental theorem of calculus, the Hölder inequality, the Sobolev embedding theorem and the Cauchy inequality that

$$\begin{aligned}\left|\int_0^T\int_\Omega \overline{\partial}^r\rho_0v_t^i\overline{\partial}^rv^idxdt\right|\leqslant&\left|\int_0^T\int_\Omega \overline{\partial}^r\rho_0v_{tt}^i\overline{\partial}^r\eta^idxdt\right|+\left|\int_\Omega \overline{\partial}^r\rho_0v_t^i\overline{\partial}^r\eta^idx\Big|_0^T\right|\\ \leqslant&CT\|\overline{\partial}^r\rho_0\|_0\sup_{[0,T]}\|\overline{\partial}^r\eta\|_0\|v_{tt}\|_2\\ \leqslant&M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)),\end{aligned}$$

where we need the condition $\rho_0\in H^r(\Omega)$.

For $l=1$, we have, at a similar way, that

$$\begin{aligned}\left|\int_0^T\int_\Omega \overline{\partial}^{r-1}\rho_0\overline{\partial}v_t^i\overline{\partial}^rv^idxdt\right|\leqslant&CT\|\rho_0\|_{L^\infty(\Omega)}^{1/2}\left\|\frac{\overline{\partial}^{r-1}\rho_0}{\rho_0}\right\|_0\sup_{[0,T]}\|\rho_0^{1/2}\overline{\partial}^rv\|_0\|v_t\|_3\\ \leqslant&\|\rho_0\|_2^2\|\rho_0\|_r^4+\delta\sup_{[0,T]}\|\rho_0^{1/2}\overline{\partial}^rv\|_0^2+CT^4\|v_t\|_3^4\\ \leqslant&M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).\end{aligned}$$

For $l=2$, we get, by the Hölder inequality and the Sobolev embedding theorem,

$$\left|\int_0^T\int_\Omega \overline{\partial}^{r-2}\rho_0\overline{\partial}^2v_t^i\overline{\partial}^rv^idxdt\right|\leqslant CT\|\rho_0\|_{L^\infty(\Omega)}^{1/2}\left\|\frac{\overline{\partial}^{r-2}\rho_0}{\rho_0}\right\|_1\sup_{[0,T]}\|\rho_0^{1/2}\overline{\partial}^rv\|_0\|\overline{\partial}^2v_t\|_1$$

$$\leqslant \|\rho_0\|_2^2\|\rho_0\|_r^4 + \delta \sup_{[0,T]} \|\rho_0^{1/2}\overline{\partial}^r v\|_0^2 + CT^4\|v_t\|_3^4$$
$$\leqslant M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).$$

For $l = 3$, we obtain, by the Hölder inequality and the Sobolev embedding theorem, that

$$\left|\int_0^T\int_\Omega \overline{\partial}^{r-3}\rho_0\overline{\partial}^3 v_t^i\overline{\partial}^r v^i dxdt\right| \leqslant CT\|\rho_0\|_{L^\infty(\Omega)}^{1/2}\left\|\frac{\overline{\partial}^{r-3}\rho_0}{\rho_0}\right\|_2 \sup_{[0,T]}\|\rho_0^{1/2}\overline{\partial}^r v\|_0\|\overline{\partial}^3 v_t\|_0$$
$$\leqslant \|\rho_0\|_2^2\|\rho_0\|_r^4 + \delta \sup_{[0,T]} \|\rho_0^{1/2}\overline{\partial}^r v\|_0^2 + CT^4\|v_t\|_3^4$$
$$\leqslant M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).$$

For $l = 4$ with $r = 5$, we have, by the Hölder inequality and the Sobolev embedding theorem, that

$$\left|\int_0^T\int_\Omega \overline{\partial}^{r-4}\rho_0\overline{\partial}^4 v_t^i\overline{\partial}^r v^i dxdt\right| \leqslant CT\|\rho_0\|_{L^\infty(\Omega)}^{1/2}\left\|\frac{\overline{\partial}^{r-4}\rho_0}{\rho_0}\right\|_2 \sup_{[0,T]}\|\rho_0^{1/2}\overline{\partial}^r v\|_0\|\overline{\partial}^4 v_t\|_0$$
$$\leqslant \|\rho_0\|_2^2\|\rho_0\|_{r-1}^4 + \delta \sup_{[0,T]} \|\rho_0^{1/2}\overline{\partial}^r v\|_0^2 + CT^4\|v_t\|_4^4$$
$$\leqslant M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).$$

Step 5. Analysis of the remainders $\mathfrak{I}_5$ and $\mathfrak{I}_6$. By integration by parts, (A.13), (1.13b) and (1.13c) for $l = 1, \cdots, r-1$, we have

$$\int_0^T \mathfrak{I}_5 dt = \sum_{l=1}^{r-1} C_r^l \int_0^T\int_\Omega \overline{\partial}^{r-l} a_i^j \overline{\partial}^l(\rho_0^2 J^{-2})\overline{\partial}^r v^i{}_{,j}\, dxdt,$$

which can be written as, by integration by parts with respect to time,

$$\sum_{l=1}^{r-1} C_r^l \int_\Omega \overline{\partial}^{r-l} a_i^j \overline{\partial}^l(\rho_0^2 J^{-2})\overline{\partial}^r \eta^i{}_{,j}\, dx\Big|_{t=T} \tag{5.122}$$
$$-\sum_{l=1}^{r-1} C_r^l \int_0^T\int_\Omega \overline{\partial}^{r-l} a_{ti}^j \overline{\partial}^l(\rho_0^2 J^{-2})\overline{\partial}^r \eta^i{}_{,j}\, dxdt \tag{5.123}$$
$$-\sum_{l=1}^{r-1} C_r^l \int_0^T\int_\Omega \overline{\partial}^{r-l} a_i^j \overline{\partial}^l(\rho_0^2 \partial_t J^{-2})\overline{\partial}^r \eta^i{}_{,j}\, dxdt. \tag{5.124}$$

Case 1. $l = 1$. By using the fundamental theorem of calculus twice and (A.10), we get for (5.122)

$$\int_\Omega \overline{\partial}^{r-1} a_i^j(T)\overline{\partial}(\rho_0^2 J^{-2}(T))\overline{\partial}^r \eta^i{}_{,j}(T)dx$$
$$= \int_\Omega \overline{\partial}^{r-1} a_i^j(T)\overline{\partial}\rho_0^2 J^{-2}(T)\overline{\partial}^r \eta^i{}_{,j}(T)dx$$
$$- 2\int_\Omega \overline{\partial}^{r-1} a_i^j(T)\rho_0^2 J^{-2}(T)A_q^p(T)\overline{\partial}\eta^q{}_{,p}(T)\overline{\partial}^r \eta^i{}_{,j}(T)dx$$
$$= \int_\Omega\int_0^T \overline{\partial}^{r-1} a_{ti}^j dt J^{-2}(T)\overline{\partial}\rho_0^2\overline{\partial}^r \eta^i{}_{,j}(T)dx$$

$$
\begin{aligned}
&-2\int_\Omega\int_0^T \overline{\partial}^{r-1}a^j_{ti}dtJ^{-2}A^p_q\overline{\partial}\eta^q{}_{,p}\,\rho_0^2\overline{\partial}^r\eta^i{}_{,j}\,(T)dx\\
\leqslant& CT\|\rho_0\|_3P(\sup_{[0,T]}E_r(t))\left(1+\|\rho_0\overline{\partial}^{r-1}\nabla v_t\|_0\right)\|\rho_0\overline{\partial}^r\nabla\eta\|_0\\
\leqslant& M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).
\end{aligned}
$$

Similarly, we have for (5.123)

$$
\left|\int_0^T\int_\Omega \overline{\partial}^{r-1}a^j_{ti}\overline{\partial}(\rho_0^2J^{-2})\overline{\partial}^r\eta^i{}_{,j}\,dxdt\right|\leqslant M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).
$$

For (5.124), it is harder to be controlled than (5.122) and (5.123). We write it as

$$
\begin{aligned}
&-\int_0^T\int_\Omega \overline{\partial}^{r-1}a^j_i\overline{\partial}(\rho_0^2\partial_tJ^{-2})\overline{\partial}^r\eta^i{}_{,j}\,dxdt\\
=&2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^j_i\overline{\partial}\rho_0^2J^{-2}A^p_qv^q{}_{,p}\,\overline{\partial}^r\eta^i{}_{,j}\,dxdt &(5.125)\\
&+2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^j_i\rho_0^2\overline{\partial}(J^{-2}A^p_q)v^q{}_{,p}\,\overline{\partial}^r\eta^i{}_{,j}\,dxdt &(5.126)\\
&+2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^\beta_i\rho_0^2J^{-2}A^p_q\overline{\partial}v^q{}_{,p}\,\overline{\partial}^r\eta^i{}_{,\beta}\,dxdt &(5.127)\\
&+2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^3_i\rho_0^2J^{-2}A^p_q\overline{\partial}v^q{}_{,p}\,\overline{\partial}^r\eta^i{}_{,3}\,dxdt. &(5.128)
\end{aligned}
$$

It is easy to see that (5.125) and (5.126) are bounded by

$$
M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).
$$

By integration by parts, Hölder's inequality, (5.16) and Sobolev's embedding theorem, it holds

$$
\begin{aligned}
(5.127)=&-2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^\beta_i{}_{,\beta}\,A^p_q\overline{\partial}v^q{}_{,p}\,\overline{\partial}^r\eta^i\rho_0^2J^{-2}dxdt\\
&-2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^\beta_iA^p_q\overline{\partial}v^q{}_{,p}\,\overline{\partial}^r\eta^i\rho_0^2J^{-2}{}_{,\beta}\,dxdt\\
&-2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^\beta_iA^p_{q\,,\beta}\,\overline{\partial}v^q{}_{,p}\,\overline{\partial}^r\eta^i\rho_0^2J^{-2}dxdt\\
&-2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^\beta_iA^p_q\overline{\partial}v^q{}_{,p\beta}\,\overline{\partial}^r\eta^i\rho_0^2J^{-2}dxdt\\
&-2\int_0^T\int_\Omega \overline{\partial}^{r-1}a^\beta_iA^p_q\overline{\partial}v^q{}_{,p}\,\overline{\partial}^r\eta^i(\rho_0^2)_{,\beta}\,J^{-2}dxdt\\
\lesssim&\int_0^T\|\rho_0\overline{\partial}^ra\|_0\|\eta\|_2^2\|\overline{\partial}\nabla v\|_1\|\rho_0\overline{\partial}^r\eta\|_1dt\\
&+\int_0^T\|\rho_0\|_2\|\overline{\partial}^{r-1}a\|_0\|\eta\|_3^4\|\eta\|_r\|\overline{\partial}\nabla v\|_1\|\rho_0\overline{\partial}^r\eta\|_1dt\\
&+\int_0^T\|\rho_0\|_2\|\overline{\partial}^{r-1}a\|_0\|\eta\|_3^2\|\overline{\partial}\nabla v\|_1\|\rho_0\overline{\partial}^r\eta\|_1dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T \|\rho_0\|_2\|\overline{\partial}^{r-1}a\|_0\|\eta\|_3^2\|\overline{\partial}^2\nabla v\|_{L^3(\Omega)}\|\rho_0\overline{\partial}^r\eta\|_1 dt\\
&+\int_0^T \|\rho_0\|_3\|\overline{\partial}^{r-1}a\|_0\|A\|_2\|\overline{\partial}\nabla v\|_1\|\rho_0\overline{\partial}^r\eta\|_1 dt\\
\leqslant& M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).
\end{aligned}
$$

For (5.128), we can easily get the desired bound by an L^6-L^3-L^2 Hölder's inequality and the Sobolev embedding theorem because each component of a_i^3 is quadratic in $\overline{\partial}\eta$ due to (1.9).

Case 2. $l=2$. By the fundamental theorem of calculus, Hölder's inequality and the Sobolev embedding theorem, we can see that

$$
\left|\int_\Omega \overline{\partial}^{r-2}a_i^j\overline{\partial}^2(\rho_0^2J^{-2})\overline{\partial}^r\eta^i{}_{,j}\,dx\Big|_{t=T}\right|\leqslant M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).
$$

From (A.15), the Hölder inequality, the Sobolev embedding theorem and (5.16), it yields

$$
\begin{aligned}
&-\int_0^T\int_\Omega \overline{\partial}^{r-2}a_{ti}^j\overline{\partial}^2(\rho_0^2J^{-2})\overline{\partial}^r\eta^i{}_{,j}\,dxdt\\
=&\int_0^T\int_\Omega \overline{\partial}^{r-2}\left(Jv^q{}_{,p}\left(A_q^jA_i^p-A_q^pA_i^j\right)\right)\overline{\partial}^2(\rho_0^2J^{-2})\overline{\partial}^4\eta^i{}_{,j}\,dxdt\\
=&-2\int_0^T\int_\Omega J\overline{\partial}^{r-2}v^q{}_{,p}\left(A_q^jA_i^p-A_q^pA_i^j\right)\rho_0^2J^{-2}A_l^k\overline{\partial}^2\eta^l{}_{,k}\,\overline{\partial}^4\eta^i{}_{,j}\,dxdt+\text{remainders}\\
\leqslant& C\|\rho_0\|_2\int_0^T\|\overline{\partial}^{r-2}\nabla v\|_{L^3(\Omega)}\|\eta\|_r^7\|\rho_0\overline{\partial}^r\nabla\eta\|_0 dt+\text{remainders}\\
\leqslant& M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)),
\end{aligned}
$$

since the remainders can be easily controlled by the desired bound.

Similarly, we can get the bound for the last integral

$$
\left|\int_0^T\int_\Omega \overline{\partial}^{r-2}a_i^j\overline{\partial}^2(\rho_0^2\partial_tJ^{-2})\overline{\partial}^r\eta^i{}_{,j}\,dxdt\right|\leqslant M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).
$$

Case 3. $l=3$. By using the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus, the spatial integral (5.122) can be bounded by $M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t))$. Similarly, we can get the same bound for the first space-time integral (5.123). Since the norm $\|\rho_0\partial_t^2J^{-2}\|_3$ is contained in the energy function $E_r(t)$, the last space-time integral (5.124) have the same bound by the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus.

Case 4. $l=r-1$ **and** $r=5$. They can be easily controlled by the desired bound, especially with the help of the fundamental theorem of calculus for (5.124).

We can deal with the integrals in $\mathfrak{I}_6$ by using a similar argument and we omit the details for simplicity.

Step 6. Summing identities. Integrating (5.1) over $[0,T]$, by Hölder's inequality and Cauchy's inequality, we have, for sufficiently small T such that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho_0|\overline{\partial}^r v|^2dx+\frac{1}{2}\int_\Omega\left[|\nabla_\eta\overline{\partial}^r\eta|^2+|\mathrm{div}_\eta\,\overline{\partial}^r\eta|^2-|\mathrm{curl}_\eta\,\overline{\partial}^r\eta|^2\right]\rho_0^2J^{-1}dx\\
=&\frac{1}{2}\int_\Omega\rho_0|\overline{\partial}^r u_0|^2dx\\
&-\frac{1}{2}\int_0^T\int_\Omega\left[|\nabla_\eta\overline{\partial}^r\eta|^2+|\mathrm{div}_\eta\,\overline{\partial}^r\eta|^2-|\mathrm{curl}_\eta\,\overline{\partial}^r\eta|^2\right]\rho_0^2J^{-1}\mathrm{div}_\eta\, v dxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega\overline{\partial}^r\eta^l{}_{,k}\,\overline{\partial}^r\eta^i{}_{,j}\,\partial_t(A_i^kA_l^j-A_i^jA_l^k)\rho_0^2J^{-1}dxdt\\
&-\int_0^T\int_\Omega\overline{\partial}^r\rho_0^2(A_i^jJ^{-1})_t\overline{\partial}^r\eta^i{}_{,j}\,dxdt\\
&+\int_\Omega\overline{\partial}^r\rho_0^2\mathrm{div}_\eta\,\overline{\partial}^r\eta(T)J^{-1}dx-\int_0^T[(5.4)+(5.73)+(5.71)+(5.69)]dt\\
&+\int_0^T[\mathfrak{I}_4+\mathfrak{I}_5+\mathfrak{I}_6]dt\\
\leqslant&M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t)).
\end{aligned}$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}
&\int_\Omega|\nabla_\eta\overline{\partial}^r\eta|^2\rho_0^2J^{-1}dx=\int_\Omega(\nabla_\eta\overline{\partial}^r\eta)_i^j(\nabla_\eta\overline{\partial}^r\eta)_i^j\rho_0^2J^{-1}dx\\
=&\int_\Omega\overline{\partial}^r\eta^j{}_{,s}\,A_i^s\overline{\partial}^r\eta^j{}_{,p}\,A_i^p\rho_0^2J^{-1}dx\\
=&\int_\Omega\rho_0^2\left(\int_0^t(A_i^sJ^{-1})_t(t')dt'+A_i^s(0)\right)\overline{\partial}^r\eta^j{}_{,s}\left(\int_0^tA_{ti}^p(t')dt'+A_i^p(0)\right)\overline{\partial}^r\eta^j{}_{,p}\,dx\\
=&\int_\Omega\rho_0^2|\overline{\partial}^r\nabla\eta|^2dx+\int_\Omega\rho_0^2\left(\int_0^t(A_i^sJ^{-1})_t(t')dt'\right)\overline{\partial}^r\eta^j{}_{,s}\left(\int_0^tA_{ti}^p(t')dt'\right)\overline{\partial}^r\eta^j{}_{,p}\,dx\\
&+\int_\Omega\rho_0^2(\nabla\overline{\partial}^r\eta)_i^j\left(\int_0^tA_{ti}^p(t')dt'\right)\overline{\partial}^r\eta^j{}_{,p}\,dx\\
&+\int_\Omega\rho_0^2(\nabla\overline{\partial}^r\eta)_i^j\left(\int_0^t(A_i^sJ^{-1})_t(t')dt'\right)\overline{\partial}^r\eta^j{}_{,s}\,dx,
\end{aligned}$$

which yields

$$\begin{aligned}
&\sup_{[0,T]}\left|\|\rho_0\nabla_\eta\overline{\partial}^r\eta\|_0^2-\|\rho_0\overline{\partial}^r\nabla\eta\|_0^2\right|\\
\leqslant&CT^2\sup_{[0,T]}\|\rho_0\overline{\partial}^r\nabla\eta\|_0^2\|v\|_{r-1}^2\|\eta\|_r^8+CT\sup_{[0,T]}\|\rho_0\overline{\partial}^r\nabla\eta\|_0^2\|v\|_{r-1}\|\eta\|_r^4.
\end{aligned}$$

Thus, taking T so small that $CT\|v\|_{r-1}\|\eta\|_r^4\leqslant 1/6$, we obtain

$$\frac{1}{2}\sup_{[0,T]}\|\rho_0\overline{\partial}^r\nabla\eta\|_0\leqslant\sup_{[0,T]}\|\rho_0\nabla_\eta\overline{\partial}^r\eta\|_0\leqslant\frac{3}{2}\sup_{[0,T]}\|\rho_0\overline{\partial}^r\nabla\eta\|_0. \tag{5.129}$$

Similarly, we have

$$\begin{aligned}
&\int_\Omega |\mathrm{div}_\eta\, \overline{\partial}^r \eta|^2 \rho_0^2 J^{-1} dx \\
&= \int_\Omega \rho_0^2 \left(\overline{\partial}^r \mathrm{div}\, \eta + \overline{\partial}^r \eta^k{}_{,l} \int_0^t (J^{-1} A_k^l)_t dt' \right) \left(\overline{\partial}^r \mathrm{div}\, \eta + \overline{\partial}^r \eta^i{}_{,j} \int_0^t A_{ti}^j dt' \right) dx \\
&= \int_\Omega \rho_0^2 |\overline{\partial}^r \mathrm{div}\, \eta|^2 dx + \int_\Omega \rho_0^2 \overline{\partial}^r \mathrm{div}\, \eta \overline{\partial}^r \eta^i{}_{,j} \left(\int_0^t A_{ti}^j dt' \right) dx \\
&\quad + \int_\Omega \rho_0^2 \overline{\partial}^r \mathrm{div}\, \eta \overline{\partial}^r \eta^k{}_{,l} \left(\int_0^t (J^{-1} A_k^l)_t dt' \right) dx \\
&\quad + \int_\Omega \rho_0^2 J^{-1} \overline{\partial}^r \eta^k{}_{,l} \left(\int_0^t A_{tk}^l dt' \right) \overline{\partial}^r \eta^i{}_{,j} \left(\int_0^t A_{ti}^j dt' \right) dx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |\mathrm{curl}_\eta\, \overline{\partial}^r \eta|^2 \rho_0^2 J^{-1} dx \\
&= \int_\Omega \rho_0^2 \left(\overline{\partial}^r \mathrm{curl}\, \eta + \varepsilon_{\cdot ij} \overline{\partial}^r \eta^j{}_{,s} \int_0^t (A_i^s J^{-1})_t dt' \right) \\
&\qquad\qquad \cdot \left(\overline{\partial}^r \mathrm{curl}\, \eta + \varepsilon_{\cdot kl} \overline{\partial}^r \eta^l{}_{,p} \int_0^t A_{tk}^p dt' \right) dx \\
&= \int_\Omega \rho_0^2 |\overline{\partial}^r \mathrm{curl}\, \eta|^2 dx + \varepsilon_{\cdot ij} \int_\Omega \rho_0^2 \overline{\partial}^r \mathrm{curl}\, \eta \overline{\partial}^r \eta^j{}_{,s} \left(\int_0^t (A_i^s J^{-1})_t dt' \right) dx \\
&\quad + \varepsilon_{\cdot kl} \int_\Omega \rho_0^2 \overline{\partial}^r \mathrm{curl}\, \eta \overline{\partial}^r \eta^l{}_{,p} \left(\int_0^t A_{tk}^p dt' \right) dx \\
&\quad + \varepsilon_{\cdot ij} \varepsilon_{\cdot kl} \int_\Omega \rho_0^2 \overline{\partial}^r \eta^j{}_{,s} \left(\int_0^t (A_i^s J^{-1})_t dt' \right) \overline{\partial}^r \eta^l{}_{,p} \left(\int_0^t A_{tk}^p dt' \right) dx,
\end{aligned}$$

which yields for a sufficiently small T that

$$\begin{aligned}
&\frac{1}{2} \sup_{[0,T]} \|\rho_0 \overline{\partial}^r \mathrm{div}\, \eta\|_0 \leqslant \sup_{[0,T]} \|\rho_0 \mathrm{div}_\eta\, \overline{\partial}^r \eta\|_0 \leqslant \frac{3}{2} \sup_{[0,T]} \|\rho_0 \overline{\partial}^r \mathrm{div}\, \eta\|_0, \\
&\frac{1}{2} \sup_{[0,T]} \|\rho_0 \overline{\partial}^r \mathrm{curl}\, \eta\|_0 \leqslant \sup_{[0,T]} \|\rho_0 \mathrm{curl}_\eta\, \overline{\partial}^r \eta\|_0 \leqslant \frac{3}{2} \sup_{[0,T]} \|\rho_0 \overline{\partial}^r \mathrm{curl}\, \eta\|_0.
\end{aligned}$$

Thus, we obtain the desired inner estimates with the help of curl estimates.

Step 7. Boundary estimates. By Lemma A.3, (A.5), the fundamental theorem of calculus and Hölder's inequality, we obtain

$$\begin{aligned}
|\overline{\partial}^r \eta^\alpha|_{-1/2}^2 &\lesssim \|\overline{\partial}^r \eta\|_0^2 + \|\mathrm{curl}\, \overline{\partial}^{r-1} \eta\|_0^2 \\
&\lesssim \|\rho_0 \overline{\partial}^r \eta\|_0^2 + \|\rho_0 \nabla \overline{\partial}^r \eta\|_0^2 + \|\mathrm{curl}\, \overline{\partial}^{r-1} \eta\|_0^2 \\
&\lesssim \left\| \rho_0 \int_0^t \overline{\partial}^r v dt \right\|_0^2 + \|\rho_0 \nabla \overline{\partial}^r \eta\|_0^2 + \|\mathrm{curl}\, \overline{\partial}^{r-1} \eta\|_0^2 \\
&\lesssim \|\rho_0\|_2 T^2 \sup_{[0,T]} \|\rho_0^{1/2} \overline{\partial}^r v\|_0^2 + \|\rho_0 \nabla \overline{\partial}^r \eta\|_0^2 + \|\mathrm{curl}\, \overline{\partial}^{r-1} \eta\|_0^2,
\end{aligned}$$

which implies the desired estimates from curl estimates. □

5.2. **The estimates for the time derivatives.** We have the following estimates.

Proposition 5. *Let $r \in \{4,5\}$. Then, for a small $\delta > 0$ and the constant M_0 depending on $1/\delta$, we have*

$$\begin{aligned}&\sup_{[0,T]}\left[\|\rho_0^{1/2}\partial_t^{2r}v\|_0^2+\|\rho_0\partial_t^{2r}\nabla\eta\|_0^2+\|\rho_0\partial_t^{2r}\operatorname{div}\eta\|_0^2\right]\\ \leqslant& M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).\end{aligned}$$

Proof. Acting ∂_t^{2r} on (1.13a), and taking the $L^2(\Omega)$-inner product with $\partial_t^{2r}v^i$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial_t^{2r}v|^2dx+\mathfrak{I}_1+\mathfrak{I}_2=\mathfrak{I}_3,$$

where

$$\begin{aligned}&\mathfrak{I}_1:=\int_\Omega \partial_t^{2r}a_i^j\left(\rho_0^2J^{-2}\right)_{,j}\partial_t^{2r}v^idx,\qquad \mathfrak{I}_2:=\int_\Omega a_i^j\left(\rho_0^2\partial_t^{2r}J^{-2}\right)_{,j}\partial_t^{2r}v^idx,\\ &\mathfrak{I}_3:=-\sum_{l=1}^{2r-1}C_{2r}^l\int_\Omega \partial_t^{2r-l}a_i^j\left(\rho_0^2\partial_t^lJ^{-2}\right)_{,j}\partial_t^{2r}v^idx.\end{aligned}$$

Step 1. Analysis of the integral $\mathfrak{I}_1$. Noticing that $\partial_t^{2r}a_i^3\partial_t^{2r}v^i=0$ on Γ_0, integration by parts gives that

$$\begin{aligned}\mathfrak{I}_1=&-\int_\Omega \partial_t^{2r}a_i^j\partial_t^{2r}v^i{}_{,j}\,\rho_0^2J^{-2}dx+\int_{\Gamma_0}\partial_t^{2r}a_i^3\partial_t^{2r}v^i\rho_0^2J^{-2}dx_1dx_2\\ =&-\int_\Omega \partial_t^{2r-1}(v^l{}_{,k}\,J^{-1}(a_i^ja_l^k-a_i^ka_l^j))\partial_t^{2r}v^i{}_{,j}\,\rho_0^2J^{-2}dx\\ =&-\int_\Omega \partial_t^{2r-1}v^l{}_{,k}\,\partial_t^{2r}v^i{}_{,j}\,A_l^kA_i^j\rho_0^2J^{-1}dx &(5.130)\\ &+\int_\Omega \partial_t^{2r-1}v^l{}_{,k}\,\partial_t^{2r}v^i{}_{,j}\,A_i^kA_l^j\rho_0^2J^{-1}dx &(5.131)\\ &-\sum_{s=1}^{2r-1}C_{2r-1}^s\int_\Omega \partial_t^{2r-1-s}v^l{}_{,k}\,\partial_t^s(J^{-1}(a_i^ja_l^k-a_i^ka_l^j))\partial_t^{2r}v^i{}_{,j}\,\rho_0^2J^{-2}dx. &(5.132)\end{aligned}$$

Then, we have

$$\begin{aligned}(5.130)=&-\frac{1}{2}\frac{d}{dt}\int_\Omega|\operatorname{div}_\eta\partial_t^{2r-1}v|^2\rho_0^2J^{-1}dx\\ &+\frac{1}{2}\int_\Omega\partial_t^{2r-1}v^l{}_{,k}\,\partial_t^{2r-1}v^i{}_{,j}\,\partial_t(A_l^kA_i^j)\rho_0^2J^{-1}dx\\ &+\frac{1}{2}\int_\Omega|\operatorname{div}_\eta\partial_t^{2r-1}v|^2\rho_0^2\partial_tJ^{-1}dx.\end{aligned}$$

It follows from (A.2) that

$$\begin{aligned}\partial_t^{2r-1}v^l{}_{,k}\,A_i^kA_l^j\partial_t^{2r}v^i{}_{,j}=&\frac{1}{2}\partial_t\left[|\nabla_\eta\partial_t^{2r-1}v|^2-|\operatorname{curl}_\eta\partial_t^{2r-1}v|^2\right]\\ &-\frac{1}{2}\partial_t^{2r-1}v^l{}_{,k}\,\partial_t^{2r-1}v^i{}_{,j}\,\partial_t(A_i^kA_l^j).\end{aligned}$$

Thus, we have

$$\begin{aligned}(5.131)=&\frac{1}{2}\frac{d}{dt}\int_\Omega \left[|\nabla_\eta\partial_t^{2r-1}v|^2-|\mathrm{curl}_\eta\,\partial_t^{2r-1}v|^2\right]\rho_0^2J^{-1}dx\\&-\frac{1}{2}\int_\Omega \left[|\nabla_\eta\partial_t^{2r-1}v|^2-|\mathrm{curl}_\eta\,\partial_t^{2r-1}v|^2\right]\rho_0^2\partial_tJ^{-1}dx\\&-\frac{1}{2}\int_\Omega \partial_t^{2r-1}v^l{}_{,k}\,\partial_t^{2r-1}v^i{}_{,j}\,\partial_t(A_i^kA_l^j)\rho_0^2J^{-1}dx.\end{aligned}$$

Step 2. Analysis of the integral $\mathfrak{I}_2$. Similar to those of $\mathfrak{I}_1$, by noticing that $a_i^3\partial_t^{2r}v^i=0$ on Γ_0, we get

$$\begin{aligned}\mathfrak{I}_2=&-\int_\Omega a_i^j\partial_t^{2r}v^i{}_{,j}\,\rho_0^2\partial_t^{2r}J^{-2}dx+\int_{\Gamma_0}a_i^3\partial_t^{2r}v^i\rho_0^2\partial_t^{2r}J^{-2}dx_1dx_2\\=&2\int_\Omega A_i^j\partial_t^{2r}v^i{}_{,j}\,A_l^k\partial_t^{2r-1}v^l{}_{,k}\,\rho_0^2J^{-1}dx\\&+2\sum_{s=0}^{2r-2}C_{2r-1}^s\int_\Omega A_i^j\partial_t^{2r}v^i{}_{,j}\,\rho_0^2J\partial_t^{2r-1-s}(J^{-2}A_l^k)\partial_t^sv^l{}_{,k}\,dx\\=&-2\cdot(5.130)+(5.133).\end{aligned}\tag{5.133}$$

Thus, we obtain by integrating over $[0,T]$

$$\begin{aligned}&\frac{1}{2}\int_\Omega\rho_0|\partial_t^{2r}v|^2dx+\frac{1}{2}\int_\Omega\left[|\nabla_\eta\partial_t^{2r-1}v|^2+|\mathrm{div}_\eta\,\partial_t^{2r-1}v|^2-|\mathrm{curl}_\eta\,\partial_t^{2r-1}v|^2\right]\rho_0^2J^{-1}dx\\=&\frac{1}{2}\int_\Omega\rho_0|\partial_t^{2r}v(0)|^2dx\\&+\frac{1}{2}\int_\Omega\left[|\nabla\partial_t^{2r-1}v(0)|^2+|\mathrm{div}\,\partial_t^{2r-1}v(0)|^2-|\mathrm{curl}\,\partial_t^{2r-1}v(0)|^2\right]\rho_0^2dx\\&+\frac{1}{2}\int_0^T\int_\Omega\left[|\nabla_\eta\partial_t^{2r-1}v|^2+|\mathrm{div}_\eta\,\partial_t^{2r-1}v|^2-|\mathrm{curl}_\eta\,\partial_t^{2r-1}v|^2\right]\rho_0^2\partial_tJ^{-1}dxdt\\&+\frac{1}{2}\int_0^T\int_\Omega\partial_t^{2r-1}v^l{}_{,k}\,\partial_t^{2r-1}v^i{}_{,j}\,\partial_t(A_i^kA_l^j)\rho_0^2J^{-1}dxdt\\&+\frac{1}{2}\int_0^T\int_\Omega\partial_t^{2r-1}v^l{}_{,k}\,\partial_t^{2r-1}v^i{}_{,j}\,\partial_t(A_l^kA_i^j)\rho_0^2J^{-1}dxdt\\&-\int_0^T(5.132)dt-\int_0^T(5.133)dt+\int_0^T\mathfrak{I}_3dt.\end{aligned}$$

The first three space-time double integrals can be absorbed by the left hand side as long as T is sufficiently small.

Step 3. Analysis of the remainder $\int_0^T(5.132)dt$. Integration by parts with respect to time gives

$$\begin{aligned}&-\int_0^T(5.132)dt\\=&\sum_{s=1}^{2r-1}C_{2r-1}^s\int_0^T\int_\Omega\partial_t^{2r-1-s}v^l{}_{,k}\,\partial_t^s(J(A_i^jA_l^k-A_i^kA_l^j))\partial_t^{2r}v^i{}_{,j}\,\rho_0^2J^{-2}dxdt\end{aligned}$$

$$= \sum_{s=1}^{2r-1} C_{2r-1}^{s} \int_{\Omega} \partial_t^{2r-s}\eta^l{}_{,k}\, \partial_t^s (J(A_i^j A_l^k - A_i^k A_l^j))\partial_t^{2r}\eta^i{}_{,j}\, \rho_0^2 J^{-2} dx\Big|_0^T \tag{5.134}$$

$$- \sum_{s=1}^{2r-1} C_{2r-1}^{s} \int_0^T\int_{\Omega} \partial_t^{2r+1-s}\eta^l{}_{,k}\, \partial_t^s (J(A_i^j A_l^k - A_i^k A_l^j))\partial_t^{2r}\eta^i{}_{,j}\, \rho_0^2 J^{-2} dxdt \tag{5.135}$$

$$- \sum_{s=1}^{2r-1} C_{2r-1}^{s} \int_0^T\int_{\Omega} \partial_t^{2r-s}\eta^l{}_{,k}\, \partial_t^{s+1} (J(A_i^j A_l^k - A_i^k A_l^j))\partial_t^{2r}\eta^i{}_{,j}\, \rho_0^2 J^{-2} dxdt \tag{5.136}$$

$$- \sum_{s=1}^{2r-1} C_{2r-1}^{s} \int_0^T\int_{\Omega} \partial_t^{2r-s}\eta^l{}_{,k}\, \partial_t^{s} (J(A_i^j A_l^k - A_i^k A_l^j))\partial_t^{2r}\eta^i{}_{,j}\, \rho_0^2 \partial_t J^{-2} dxdt. \tag{5.137}$$

We first consider (5.135). For the cases $s = 1, 2$, it is easy to get the desired bounds by the Hölder inequality and the Sobolev embedding theorem. For the case $s = 3$,

$$\int_0^T\int_{\Omega} \partial_t^{2r-2}\eta^l{}_{,k}\, \partial_t^{3} (J(A_i^j A_l^k - A_i^k A_l^j))\partial_t^{2r}\eta^i{}_{,j}\, \rho_0^2 J^{-2} dxdt \tag{5.138}$$

$$= \int_0^T\int_{\Omega} \partial_t^{2r-2}\eta^l{}_{,k}\, \partial_t^{3} (J(A_i^\beta A_l^k - A_i^k A_l^\beta))\partial_t^{2r}\eta^i{}_{,\beta}\, \rho_0^2 J^{-2} dxdt \tag{5.139}$$

$$+ \int_0^T\int_{\Omega} \partial_t^{2r-2}\eta^l{}_{,\beta}\, \partial_t^{3} (J(A_i^3 A_l^\beta - A_i^\beta A_l^3))\partial_t^{2r}\eta^i{}_{,3}\, \rho_0^2 J^{-2} dxdt. \tag{5.140}$$

It is clear that (5.140) is easy to deal with by an L^6-L^3-L^2 Hölder inequality. For (5.139), integration by parts yields

$$(5.139) = -\int_0^T\int_{\Omega} \partial_t^{2r-2}\eta^l{}_{,k\beta}\, \partial_t^{3} (J(A_i^\beta A_l^k - A_i^k A_l^\beta))\partial_t^{2r}\eta^i \rho_0^2 J^{-2} dxdt \tag{5.141}$$

$$- \int_0^T\int_{\Omega} \partial_t^{2r-2}\eta^l{}_{,k}\, \partial_t^{3} (J(A_i^\beta A_l^k - A_i^k A_l^\beta))_{,\beta}\, \partial_t^{2r}\eta^i \rho_0^2 J^{-2} dxdt \tag{5.142}$$

$$- \int_0^T\int_{\Omega} \partial_t^{2r-2}\eta^l{}_{,k}\, \partial_t^{3} (J(A_i^\beta A_l^k - A_i^k A_l^\beta))\partial_t^{2r}\eta^i (\rho_0^2 J^{-2})_{,\beta}\, dxdt, \tag{5.143}$$

which, then, can be controlled easily by the desired bound by an L^2-L^3-L^6 Hölder inequality, in addition for (5.142), with the help of

$$\begin{aligned}
&\|\partial_t^{2m} D^{r-1-m}\eta_t\|^2_{L^3([0,T]\times\Omega)}\\
\leqslant& CT^{2/3}\Big[\|\partial_t^{2m} D^{r-1-m} v(0)\|^2_{L^3(\Omega)} + \sup_{[0,T]} \|\partial_t^{2m} D^{r-1-m}\eta\|_1 \|\partial_t^{2m+2} D^{r-1-m}\eta\|_0\Big]\\
\leqslant& CT^{2/3}\Big[\|\partial_t^{2m} D^{r-1-m} v(0)\|^2_{L^3(\Omega)} + \sup_{[0,T]} \|\partial_t^{2m}\eta\|_{r-m} \|\partial_t^{2m+2}\eta\|_{r-1-m}\Big]\\
\leqslant& M_0 + CTP(\sup_{[0,T]} E_r(t)),
\end{aligned} \tag{5.144}$$

for the integer $0 \leqslant m \leqslant r-1$ due to (2.1).

For the case $s = 4$,

$$\int_0^T\int_{\Omega} \partial_t^{2r-3}\eta^l{}_{,k}\, \partial_t^{4} (J(A_i^j A_l^k - A_i^k A_l^j))\partial_t^{2r}\eta^i{}_{,j}\, \rho_0^2 J^{-2} dxdt \tag{5.145}$$

$$= \int_0^T\int_{\Omega} \partial_t^{2r-3}\eta^l{}_{,k}\, \partial_t^{4} (J(A_i^\beta A_l^k - A_i^k A_l^\beta))\partial_t^{2r}\eta^i{}_{,\beta}\, \rho_0^2 J^{-2} dxdt \tag{5.146}$$

$$+\int_0^T\int_\Omega \partial_t^{2r-3}\eta^l{}_{,\beta}\,\partial_t^4(J(A_i^3A_l^\beta-A_i^\beta A_l^3))\partial_t^{2r}\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt. \tag{5.147}$$

It is clear that (5.147) is easy to be dealt with by an L^6-L^3-L^2 Hölder inequality. For (5.146), integration by parts yields

$$(5.146)=-\int_0^T\int_\Omega \partial_t^{2r-3}\eta^l{}_{,k\beta}\,\partial_t^4(J(A_i^\beta A_l^k-A_i^kA_l^\beta))\partial_t^{2r}\eta^i\rho_0^2J^{-2}dxdt \tag{5.148}$$

$$-\int_0^T\int_\Omega \partial_t^{2r-3}\eta^l{}_{,k}\,\partial_t^4(J(A_i^\beta A_l^k-A_i^kA_l^\beta))_{,\beta}\,\partial_t^{2r}\eta^i\rho_0^2J^{-2}dxdt \tag{5.149}$$

$$-\int_0^T\int_\Omega \partial_t^{2r-3}\eta^l{}_{,k}\,\partial_t^4(J(A_i^\beta A_l^k-A_i^kA_l^\beta))\partial_t^{2r}\eta^i(\rho_0^2J^{-2})_{,\beta}\,dxdt. \tag{5.150}$$

By an L^2-L^3-L^6 Hölder inequality, (5.148) and (5.150) can be easily estimated. We can also use an L^3-L^2-L^6 Hölder inequality to control (5.149) since $\partial_t^{2r-4}\nabla\eta_t\in L^3([0,T]\times\Omega)$ due to (5.144).

For the case $s=5$,

$$\int_0^T\int_\Omega \partial_t^{2r-4}\eta^l{}_{,k}\,\partial_t^5(J(A_i^jA_l^k-A_i^kA_l^j))\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2J^{-2}dxdt \tag{5.151}$$

$$=\int_0^T\int_\Omega \partial_t^{2r-4}\eta^l{}_{,k}\,\partial_t^5(J(A_i^\beta A_l^k-A_i^kA_l^\beta))\partial_t^{2r}\eta^i{}_{,\beta}\,\rho_0^2J^{-2}dxdt \tag{5.152}$$

$$+\int_0^T\int_\Omega \partial_t^{2r-4}\eta^l{}_{,\beta}\,\partial_t^5(J(A_i^3A_l^\beta-A_i^\beta A_l^3))\partial_t^{2r}\eta^i{}_{,3}\,\rho_0^2J^{-2}dxdt. \tag{5.153}$$

It is easy to see that (5.153) is well estimated by an L^6-L^3-L^2 Hölder inequality. For (5.152), integration by parts implies

$$(5.152)=-\int_0^T\int_\Omega \partial_t^{2r-4}\eta^l{}_{,k\beta}\,\partial_t^5(J(A_i^\beta A_l^k-A_i^kA_l^\beta))\partial_t^{2r}\eta^i\rho_0^2J^{-2}dxdt \tag{5.154}$$

$$-\int_0^T\int_\Omega \partial_t^{2r-4}\eta^l{}_{,k}\,\partial_t^5(J(A_i^\beta A_l^k-A_i^kA_l^\beta))_{,\beta}\,\partial_t^{2r}\eta^i\rho_0^2J^{-2}dxdt \tag{5.155}$$

$$-\int_0^T\int_\Omega \partial_t^{2r-4}\eta^l{}_{,k}\,\partial_t^5(J(A_i^\beta A_l^k-A_i^kA_l^\beta))\partial_t^{2r}\eta^i(\rho_0^2J^{-2})_{,\beta}\,dxdt. \tag{5.156}$$

By an L^2-L^3-L^6 Hölder inequality, (5.154) and (5.156) can be easily estimated. We can use an L^3-L^2-L^6 Hölder inequality to control (5.149) because of $\rho_0\partial_t^5\overline{\partial}\nabla\eta\in L^2(\Omega)$ in view of the fundamental theorem of calculus.

For the case $s=6$, integration by parts gives

$$\int_0^T\int_\Omega \partial_t^{2r-5}\eta^l{}_{,k}\,\partial_t^6(a_i^jA_l^k-A_i^ka_l^j)\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2J^{-2}dxdt \tag{5.157}$$

$$=-\int_0^T\int_\Omega \partial_t^{2r-5}\eta^l{}_{,kj}\,\partial_t^6(a_i^jA_l^k-A_i^ka_l^j)\partial_t^{2r}\eta^i\rho_0^2J^{-2}dxdt \tag{5.158}$$

$$-\int_0^T\int_\Omega \partial_t^{2r-5}\eta^l{}_{,k}\,\partial_t^6(a_i^jA_l^k-A_i^ka_l^j)\partial_t^{2r}\eta^i(\rho_0^2J^{-2})_{,j}\,dxdt. \tag{5.159}$$

For (5.158), it can be controlled by the bound

$$M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t))$$

by an L^3-L^2-L^6 Hölder inequality with the help of (5.144). For (5.159), it is easily to be controlled by the desired bound.

For the cases $s = 7$, and $s = 8, 9$ with $r = 5$, they are easy to be controlled by the desired bound via the Hölder inequality.

Next, we consider (5.136). For the case $s = 2r - 1$, it is easy to get the desired bounds by the Hölder inequality and the Sobolev embedding theorem. For other cases of s, it is the same as the cases of $s - 1$ in (5.135). Thus, we get the desired bounds for (5.136).

The spatial integral (5.134) can be treated similarly as for (5.136) with the help of the fundamental theorem of calculus for one lower order term in order to get the factor T. For the last integral (5.137), it is much easier to get the bound than (5.136), thus we omit the details. Therefore, we have obtained

$$\left|\int_0^T (5.132)dt\right| \leqslant M_0 + \delta \sup_{[0,T]} E_5(t) + CTP(\sup_{[0,T]} E_5(t)).$$

Step 4. Analysis of the remainder $\int_0^T (5.133)dt$. By integration by parts with respect to time, we get

$$\begin{aligned}
&-\int_0^T (5.133)dt \\
&= -2\sum_{s=0}^{2r-2} C_{2r-1}^s \int_0^T\int_\Omega \partial_t^{2r}v^i{}_{,j}\,\partial_t^{2r-1-s}(J^{-2}A_l^k)\partial_t^s v^l{}_{,k}\,\rho_0^2 JA_i^j dxdt \\
&= -2\sum_{s=0}^{2r-2} C_{2r-1}^s \int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-1-s}(J^{-2}A_l^k)\partial_t^s v^l{}_{,k}\,\rho_0^2 JA_i^j dx\Big|_0^T && (5.160)\\
&\quad + 2\sum_{s=0}^{2r-2} C_{2r-1}^s \int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-s}(J^{-2}A_l^k)\partial_t^s v^l{}_{,k}\,\rho_0^2 JA_i^j dxdt && (5.161)\\
&\quad + 2\sum_{s=0}^{2r-2} C_{2r-1}^s \int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-1-s}(J^{-2}A_l^k)\partial_t^{s+1} v^l{}_{,k}\,\rho_0^2 JA_i^j dxdt && (5.162)\\
&\quad + 2\sum_{s=0}^{2r-2} C_{2r-1}^s \int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-1-s}(J^{-2}A_l^k)\partial_t^s v^l{}_{,k}\,\rho_0^2 \partial_t(JA_i^j) dxdt. && (5.163)
\end{aligned}$$

We first consider the double integral (5.161). For the case $s = 0$, we write it as

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r}(J^{-2}A_l^k)v^l{}_{,k}\,\rho_0^2 JA_i^j dxdt \\
&= \sum_{m=0}^{2r} C_{2r}^m \int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^m J^{-2}\partial_t^{2r-m}A_l^k v^l{}_{,k}\,\rho_0^2 JA_i^j dxdt.
\end{aligned}$$

For $m = 0, 3, 2r - 1, 2r$, we can use the Hölder inequality, (5.144) and the Sobolev embedding theorem to get the desired bound. In particular, we have to use an L^2-L^3-L^6 Hölder inequality and (5.144) to deal with the integral involving the terms of the form $\partial_t^{2r}\nabla\eta\partial_t^{2r-3}\nabla\eta\partial_t^3\nabla\eta$ in order to get the bound. For $m = 1, 2, 4, \cdots, 2r-2$, we can only apply the Hölder inequality and the Sobolev embedding theorem to get the desired bound by noticing that $\rho_0\partial_t^{2\ell}J^{-2} \in H^{r-\ell}(\Omega)$ for $0 \leqslant \ell \leqslant r - 1$.

For the case $s=1$, since $v_t\in H^{r-1}(\Omega)$, or $\nabla v_t\in L^\infty(\Omega)$, it is similar to and easier than those of the case $s=0$. We omit the details.

For the case $s=2$, we have

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-2}(J^{-3}a_l^k)\partial_t^2 v^l{}_{,k}\,\rho_0^2 a_i^j\,dxdt\\&=\sum_{m=0}^{2r-2}C_{2r-2}^m\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^m J^{-3}\partial_t^{2r-2-m}a_l^k\partial_t^2 v^l{}_{,k}\,\rho_0^2 a_i^j\,dxdt.\end{aligned}$$

For $m=0$, i.e.,

$$\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-2}a_l^k\partial_t^2 v^l{}_{,k}\,\rho_0^2 a_i^j J^{-3}\,dxdt,\tag{5.164}$$

we must use $E_5(t)$ to control $\|\partial_t^2\nabla v\|_{L^\infty(\Omega)}$ when $r=4$; while it is easy to get the desired bound for $r=5$. For $m=1$, we can use an L^2-L^3-L^6 Hölder inequality and (5.144) to obtain the desired bound, i.e., $M_0+\delta\sup_{[0,T]}E_r(t)+CTP(\sup_{[0,T]}E_r(t))$. For $m=2,\cdots,2r-2$, they are controlled by the desired bounds by using the Hölder inequality and the Sobolev embedding theorem.

For the case $s=3$, we get

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-3}(J^{-2}A_l^k)\partial_t^3 v^l{}_{,k}\,\rho_0^2 JA_i^j\,dxdt\\&=\sum_{m=0}^{2r-3}C_{2r-3}^m\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^m J^{-2}\partial_t^{2r-3-m}A_l^k\partial_t^3 v^l{}_{,k}\,\rho_0^2 JA_i^j\,dxdt.\end{aligned}$$

For $m=0$, we can use an L^2-L^3-L^6 Hölder inequality and (5.144) to obtain the desired bound. For $m=1,\cdots,2r-3$, they are controlled by the desired bounds by using the Hölder inequality and the Sobolev embedding theorem.

For the case $s=4$, we can use an L^2-L^6-L^3 Hölder inequality, and (5.144) for $r=4$ additionally, to obtain the desired bound. For the cases $s=5,\cdots,2r-2$, we can use the Hölder inequality and the Sobolev embedding theorem to get the desired bounds.

Next, we consider the integral (5.162). For the case $s=0$, we have

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-1}(J^{-2}A_l^k)\partial_t v^l{}_{,k}\,\rho_0^2 JA_i^j\,dxdt\\&=\sum_{m=0}^{2r-1}C_{2r-1}^m\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^m J^{-2}\partial_t^{2r-1-m}A_l^k\partial_t v^l{}_{,k}\,\rho_0^2 JA_i^j\,dxdt,\end{aligned}$$

which can be controlled by the desired bound by using the Hölder inequality and the Sobolev embedding theorem.

For the case $s=1$, it follows that

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-2}(J^{-2}A_l^k)\partial_t^2 v^l{}_{,k}\,\rho_0^2 JA_i^j\,dxdt\\&=\sum_{m=0}^{2r-2}C_{2r-2}^m\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^m J^{-2}\partial_t^{2r-2-m}A_l^k\partial_t^2 v^l{}_{,k}\,\rho_0^2 JA_i^j\,dxdt.\end{aligned}$$

For $m=0$, we can have to use the fact $v\in H^4(\Omega)$, which is contained in $E_5(t)$, for all cases $r=4,5$ to obtain

$$\begin{aligned}&\left|\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,J^{-2}\partial_t^{2r-2}A_l^k\partial_t^2 v^l{}_{,k}\,\rho_0^2 JA_i^j dxdt\right|\\ \leqslant& M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).\end{aligned}$$

For $m=1$, we can use an L^2-L^3-L^6 Hölder inequality, (5.144) and the Sobolev embedding theorem to get the desired bound. For other cases of m, we can use the Hölder inequality and the Sobolev embedding theorem to get the desired bound by noticing that $\rho_0\partial_t^{2\ell}J^{-2}\in H^{r-\ell}(\Omega)$ for $0\leqslant\ell\leqslant r-1$ with the help of the fundamental theorem of calculus if necessary.

For the case $s=2$, we get

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^{2r-3}(J^{-2}A_l^k)\partial_t^3 v^l{}_{,k}\,\rho_0^2 JA_i^j dxdt\\ =&\sum_{m=0}^{2r-3}C_{2r-3}^m\int_0^T\int_\Omega \partial_t^{2r}\eta^i{}_{,j}\,\partial_t^m J^{-2}\partial_t^{2r-3-m}A_l^k\partial_t^3 v^l{}_{,k}\,\rho_0^2 JA_i^j dxdt,\end{aligned}$$

which can be controlled by the desired bound by using the Hölder inequality and the Sobolev embedding theorem, in addition, with the help of (5.144) for $m=0$.

For other cases of s, we can use similar argument to get the desired bounds and omit the details.

For the spatial integral (5.160), we can use the same argument as for (5.162) to get the desired bound with the help of the fundamental theorem of calculus. For the double integral (5.163), it is easier to get the bound than either (5.161) or (5.162) and thus we omit the details. Therefore, we obtain the estimates for $\int_0^T$ (5.133)dt, i.e.,

$$\left|\int_0^T (5.133)dt\right|\leqslant M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).$$

Step 5. Analysis of the remainder $\int_0^T\mathfrak{I}_3dt$. By integration by parts with respect to the spatial variables and the time variable, respectively, we obtain

$$\begin{aligned}\int_0^T\mathfrak{I}_3dt=&\sum_{l=1}^{2r-1}C_{2r}^l\int_0^T\int_\Omega \partial_t^{2r-l}a_i^j\partial_t^l J^{-2}\partial_t^{2r}v^i{}_{,j}\,\rho_0^2dxdt\\ =&\sum_{l=1}^{2r-1}C_{2r}^l\int_\Omega \partial_t^{2r-l}a_i^j\partial_t^l J^{-2}\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2dx\Big|_0^T &(5.165)\\ &-\sum_{l=1}^{2r-1}C_{2r}^l\int_0^T\int_\Omega \partial_t^{2r+1-l}a_i^j\partial_t^l J^{-2}\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2dxdt &(5.166)\\ &-\sum_{l=1}^{2r-1}C_{2r}^l\int_0^T\int_\Omega \partial_t^{2r-l}a_i^j\partial_t^{l+1} J^{-2}\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2dxdt, &(5.167)\end{aligned}$$

due to $\partial_t^{2r-l}a_i^3\partial_t^{2r}v^i=0$ on Γ_0.

We first consider (5.166). For the case $l=1$,

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t^{2r} a_i^j \partial_t J^{-2} \partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2 dxdt\\
&=\int_0^T\int_\Omega \partial_t^{2r-1}(\partial_t\eta^p{}_{,q}\,J^{-1}(a_i^j a_p^q - a_i^q a_p^j))\partial_t J^{-2}\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2 dxdt\\
&=\sum_{s=1}^{2r-1} C_{2r-1}^s \int_0^T\int_\Omega \partial_t^{2r-s}\eta^p{}_{,q}\,\partial_t^s(J^{-1}(a_i^j a_p^q - a_i^q a_p^j))\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2\partial_t J^{-2} dxdt \qquad (5.168)\\
&\quad+\int_0^T\int_\Omega \partial_t^{2r}\eta^p{}_{,q}\,\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2\partial_t J^{-2} J(A_i^j A_p^q - A_i^q A_p^j) dxdt. \qquad (5.169)
\end{aligned}$$

Since $\partial_t J^{-2}\in L^\infty(\Omega)$, we can use a similar argument as in (5.132) to get the estimates of (5.168). For (5.169), we easily have

$$|(5.169)|\leqslant CT\sup_{[0,T]}\|\rho_0\partial_t^{2r}\nabla\eta\|_0^2\|\nabla v\|_2 P(E_4(t))\leqslant CTP(\sup_{[0,T]}E_r(t)).$$

For the case $l=2$, similar to the case $l=1$, we can get the bound easily since $\partial_t^2 J^{-2}\in L^\infty(\Omega)$ and we omit the details. For the case $l=3$, we get

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t^{2r-2} a_i^j \partial_t^3 J^{-2} \partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2 dxdt\\
&=\int_0^T\int_\Omega \partial_t^{2r-3}(\partial_t\eta^p{}_{,q}\,J^{-1}(a_i^j a_p^q - a_i^q a_p^j))\partial_t^3 J^{-2}\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2 dxdt\\
&=\sum_{s=1}^{2r-3} C_{2r-3}^s \int_0^T\int_\Omega \partial_t^{2r-2-s}\eta^p{}_{,q}\,\partial_t^s(J^{-1}(a_i^j a_p^q - a_i^q a_p^j))\partial_t^3 J^{-2}\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2 dxdt \qquad (5.170)\\
&\quad+\int_0^T\int_\Omega \partial_t^{2r-2}\eta^p{}_{,q}\,\partial_t^3 J^{-2}\partial_t^{2r}\eta^i{}_{,j}\,\rho_0^2 J^{-1}(a_i^j a_p^q - a_i^q a_p^j) dxdt. \qquad (5.171)
\end{aligned}$$

In (5.170), we use an L^3-L^6-L^2 Hölder inequality and (5.144) for the higher order terms of the cases $s=1$ and $s=2r-3$ and an L^6-L^6-L^6-L^2 Hölder inequality for the other cases to get the desired bounds. For (5.171), since $\rho_0\partial_t^4 J^{-2}\in H^{r-2}(\Omega)\subset L^\infty(\Omega)$, we can get the desired bound easily in view of the Hölder inequality, the Sobolev embedding theorem and the fundamental theorem of calculus. For the case $l=4$, we have the desired bound as a similar argument as for the case $l=3$. For the case $l=2r-3$, we can use an L^6-L^3-L^2 Hölder inequality and (5.144) to get the desired bound. For the case $l=2r-2$, we use an L^3-L^6-L^2 Hölder inequality and the Sobolev embedding theorem to get the bound due to $\rho_0\partial_t^{2(r-1)}J^{-2}\in H^1(\Omega)\subset L^6(\Omega)$. For the case $l=2r-1$, it is similar to the case $s=1$ in (5.161) and we omit the details. For the other cases, we can easily get the desired bounds by using the Hölder inequality and the Sobolev embedding theorem.

Next, we consider (5.167). Since the cases $1\leqslant l\leqslant 2r-2$ are identical to the cases $2\leqslant l\leqslant 2r-1$ of (5.166) estimated just discussed up to some constant multipliers, we only need to consider the remainder case $l=2r-1$. We can apply (1.11) to split the integral of the case $l=2r-1$ into two integrals. One of them can be used an L^2-L^2 Hölder inequality to get the estimates, the other one can be dealt with as the same arguments as for the case $l=2r-1$ of (5.166) or the case $s=1$ in (5.161). Thus, we omit the details.

For the spatial integral (5.165), it can be estimated as the same arguments as for (5.166) or (5.167) with the help of the fundamental theorem of calculus whose details are omitted.

Step 6. Summing inequalities. As the same argument as in the estimates of the horizontal derivatives, we can obtain the desired result by combining the previous inequalities. □

5.3. **The estimates for the mixed time-horizontal derivatives.** We have the following estimates.

Proposition 6. *Let $r \in \{4,5\}$ and $1 \leqslant m \leqslant r-1$. For $\delta > 0$ and the constant M_0 depend on $1/\delta$, we have*

$$\begin{aligned}\sup_{[0,T]} \Big[&\|\rho_0^{1/2}\partial_t^{2m}\overline{\partial}^{r-m} v\|_0^2 + \|\rho_0 \nabla \partial_t^{2m}\overline{\partial}^{r-m}\eta\|_0^2 \\ &+ \|\rho_0 \mathrm{div}\, \partial_t^{2m}\overline{\partial}^{r-m}\eta\|_0^2 + |\partial_t^{2m}\eta^\alpha|^2_{r-m-1/2}\Big] \\ \leqslant & M_0 + \delta \sup_{[0,T]} E_r(t) + CTP(\sup_{[0,T]} E_r(t)).\end{aligned}$$

Proof. Since the ideas is similiar to those in Propositions 4 and 5, we omit the lenthy details. □

6. **The elliptic-type estimates for the normal derivatives.** Our energy estimates provide a priori control of horizontal and time derivatives of η; it remains to gain a priori control of the normal derivatives of η. This is accomplished via a bootstrapping procedure relying on the fact $\partial_t^9 v(t)$ is bounded in $L^2(\Omega)$.

Proposition 7. *For $t \in [0,T]$, it holds that*

$$\sup_{[0,T]} \Big[\|\partial_t^7 v(t)\|_1^2 + \|\rho_0 \partial_t^8 J^{-2}(t)\|_1^2\Big] \leqslant M_0 + \delta \sup_{[0,T]} E_5(t) + CTP(\sup_{[0,T]} E_5(t)).$$

Proof. From (1.13a), we have for $\beta = 1, 2$

$$\rho_0 a_i^3 J^{-2}{}_{,3} + 2a_i^3 \rho_{0,3} J^{-2} = -v_t^i - \rho_0 a_i^\beta J^{-2}{}_{,\beta} - 2a_i^\beta \rho_{0,\beta} J^{-2}. \tag{6.1}$$

Acting ∂_t^8 on (6.1), we get

$$\begin{aligned}&\rho_0 a_i^3 \partial_t^8 J^{-2}{}_{,3} + 2a_i^3 \rho_{0,3} \partial_t^8 J^{-2} \\ =& -\partial_t^9 v^i - \rho_0 \partial_t^8 (a_i^\beta J^{-2}{}_{,\beta}) - 2\rho_{0,\beta}\partial_t^8 (a_i^\beta J^{-2}) \\ &- \partial_t^8 a_i^3 [\rho_0 J^{-2}{}_{,3} + 2\rho_{0,3} J^{-2}] - \sum_{l=1}^{7} C_8^l \partial_t^{8-l} a_i^3 \partial_t^l [\rho_0 J^{-2}{}_{,3} + 2\rho_{0,3} J^{-2}].\end{aligned}$$

By (A.5), the fundamental theorem of calculus and Hölder's inequality, we have

$$\begin{aligned}\|\partial_t^9 v(t)\|_0^2 \leqslant & C\int_\Omega \rho_0^2 \left(|\partial_t^9 v|^2 + |\nabla \partial_t^9 v|^2\right) dx \\ \leqslant & C\int_\Omega \rho_0^2 \left|\int_0^t \partial_t^{10} v dt' + \partial_t^9 v(0)\right|^2 dx + C\|\rho_0 \nabla \partial_t^9 v\|_0^2 \\ \leqslant & Ct \int_\Omega \rho_0^2 \int_0^t |\partial_t^{10} v|^2 dt' + C\int_\Omega \rho_0^2 |\partial_t^9 v(0)|^2 dx + C\|\rho_0 \partial_t^9 \nabla v\|_0^2 \\ \leqslant & Ct^2 \|\rho_0\|_{L^\infty(\Omega)} \sup_{[0,t]} \|\rho_0^{1/2}\partial_t^{10} v\|_0^2 + \|\rho_0\|_{L^\infty(\Omega)}^2 \|\partial_t^9 v(0)\|_0^2 + C\|\rho_0 \partial_t^9 \nabla v\|_0^2.\end{aligned}$$

By the fundamental inequality of algebra, the fundamental theorem of calculus, the Hölder inequality and the Sobolev embedding theorems, we see that

$$\begin{aligned}
&\|\rho_0\partial_t^8(a_i^\beta J^{-2}{}_{,\beta})\|_0^2\\
\leqslant& C\sum_{l=0}^{4}\|\rho_0\partial_t^{8-2l}a_i^\beta\partial_t^{2l}J^{-2}{}_{,\beta}\|_0^2\\
&+C\sum_{l=1}^{4}\left\|\rho_0\partial_t^{9-2l}a_i^\beta\left(\int_0^t\partial_t^{2l}J^{-2}{}_{,\beta}\,dt'+\partial_t^{2l}J^{-2}{}_{,\beta}(0)\right)\right\|_0^2\\
\leqslant& M_0+Ct^2\sup_{[0,t]}\sum_{l=0}^{4}\|\rho_0\partial_t^{2l}J^{-2}\|_{5-l}^2P(\|\partial_t^{8-2l}\eta\|_{l+1},\cdots,\|\eta_{tt}\|_4,\|\eta_t\|_4,\|\eta\|_5).
\end{aligned}$$

Similarly, we also have, that

$$\|\partial_t^8a_i^3\rho_0J^{-2}{}_{,3}\|_0^2\leqslant M_0+Ct^2\sup_{[0,t]}\|\rho_0J^{-2}\|_5^2P(\|\partial_t^8\eta\|_1,\cdots,\|\eta_{tt}\|_4,\|\eta_t\|_4,\|\eta\|_5),$$

and

$$\begin{aligned}
\|\rho_{0,\beta}\,\partial_t^8(a_i^\beta J^{-2})\|_0^2\leqslant& M_0+Ct^2\left\|\frac{\rho_{0,\beta}}{\rho_0}\right\|_{L^\infty(\Omega)}^2\sup_{[0,t]}\sum_{l=0}^{4}\|\rho_0\partial_t^{2l}J^{-2}\|_{5-l}^2\\
&\cdot P(\|\partial_t^{8-2l}\eta\|_{l+1},\cdots,\|\eta_{tt}\|_4,\|\eta_t\|_4,\|\eta\|_5).
\end{aligned}$$

By the fundamental theorem of calculus, we get

$$\begin{aligned}
\|\partial_t^8a_i^3\rho_{0,3}\,J^{-2}\|_0^2\leqslant& M_0+Ct^2\|\rho_{0,3}\|_{L^\infty(\Omega)}^2\\
&\sup_{[0,t]}P(\|\partial_t^8\eta\|_1,\cdots,\|\partial_t^{8-2l}\eta\|_{l+1},\cdots,\|\eta_{tt}\|_4,\|\eta_t\|_4,\|\eta\|_5).
\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}
&\sum_{l=1}^{7}\left\|\partial_t^{8-l}a_i^3\partial_t^l\left[\rho_0J^{-2}{}_{,3}+2\rho_{0,3}\,J^{-2}\right]\right\|_0^2\\
\leqslant& M_0+Ct^2\sup_{[0,t]}\sum_{l=0}^{4}\|\rho_0\partial_t^{2l}J^{-2}\|_{5-l}^2P(\|\partial_t^{8-2l}\eta\|_{l+1}^2,\cdots,\|\eta_{tt}\|_4,\|\eta_t\|_4,\|\eta\|_5)\\
&+Ct^2\|\rho_{0,3}\|_{L^\infty(\Omega)}^2\sup_{[0,t]}P(\|\partial_t^8\eta\|_1,\cdots,\|\partial_t^{8-2l}\eta\|_{l+1},\cdots,\|\eta_{tt}\|_4,\|\eta_t\|_4,\|\eta\|_5).
\end{aligned}$$

Thus, we have obtained, for all $t\in[0,T]$, that

$$\left\|\rho_0a_i^3\partial_t^8J^{-2}{}_{,3}+2a_i^3\rho_{0,3}\,\partial_t^8J^{-2}\right\|_0^2\leqslant M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).$$

It follows that

$$\begin{aligned}
&\|\rho_0a_\cdot^3\partial_t^8J^{-2}{}_{,3}\|_0^2+4\|a_\cdot^3\rho_{0,3}\,\partial_t^8J^{-2}\|_0^2\\
\leqslant& M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t))-4\int_\Omega\rho_0\rho_{0,3}\,|a_\cdot^3|^2\partial_t^8J^{-2}{}_{,3}\,\partial_t^8J^{-2}dx.
\end{aligned}$$

By Hölder's inequality and the fundamental theorem of calculus, we get

$$\begin{aligned}
&\left|-4\int_\Omega\rho_0\rho_{0,3}\,|a_\cdot^3|^2\partial_t^8J^{-2}{}_{,3}\,\partial_t^8J^{-2}dx\right|\\
\leqslant& M_0+\delta\|\rho_0\partial_t^8J^{-2}{}_{,3}\|_0^2+\delta\|\rho_{0,3}\,\partial_t^8J^{-2}\|_0^2+CTP(\sup_{[0,T]}E_5(t)),
\end{aligned}$$

and then, by the fundamental theorem of calculus once again,

$$\|\rho_0\partial_t^8 J^{-2}{}_{,3}\|_0^2+\|\rho_{0,3}\,\partial_t^8 J^{-2}\|_0^2\leqslant M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).$$

By integration by parts with respect to x_3, the Hölder inequality and noticing that $\rho_0=0$ on Γ_1, it holds

$$\begin{aligned}\|\rho_0\partial_t^8 J^{-2}(t)\|_0^2&=\int_\Omega\partial_3x_3(\rho_0\partial_t^8J^{-2})^2dx\\&=-2\int_\Omega x_3\rho_0\partial_t^8J^{-2}(\rho_{0,3}\,\partial_t^8J^{-2}+\rho_0\partial_t^8J^{-2}{}_{,3})dx\\&\leqslant 2\|\rho_0\partial_t^8J^{-2}(t)\|_0\big[\|\rho_{0,3}\,\partial_t^8J^{-2}\|_0+\|\rho_0\partial_t^8J^{-2}{}_{,3}\|_0\big],\end{aligned}$$

which implies that

$$\begin{aligned}\|\rho_0\partial_t^8 J^{-2}(t)\|_0^2&\leqslant 8\Big[\|\rho_{0,3}\,\partial_t^8J^{-2}\|_0^2+\|\rho_0\partial_t^8J^{-2}{}_{,3}\|_0^2\Big]\\&\leqslant M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).\end{aligned}\tag{6.2}$$

Since we can get, by Proposition 6, that

$$\|\rho_0\overline{\partial}\partial_t^8J^{-2}\|_0^2\leqslant M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)),$$

we have

$$\|\rho_0\partial_t^8J^{-2}\|_1^2\leqslant M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).$$

It follows from (A.5) and (6.2) that

$$\begin{aligned}\|\partial_t^8J^{-2}\|_0^2&\leqslant C\|\rho_0\partial_t^8J^{-2}\|_0^2+C\|\rho_0\nabla\partial_t^8J^{-2}\|_0^2\\&\leqslant M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).\end{aligned}$$

Due to (1.11), we see that

$$\begin{aligned}\partial_t^8J^{-2}&=-2\partial_t^7(J^{-2}A_i^jv^i{}_{,j})\\&=-2J^{-2}A_i^j\partial_t^7v^i{}_{,j}-2\partial_t^7(J^{-2}A_i^j)v^i{}_{,j}-2\sum_{l=1}^6C_7^l\partial_t^l(J^{-2}A_i^j)\partial_t^{7-l}v^i{}_{,j},\end{aligned}$$

namely, in view of the fundamental theorem of calculus,

$$\begin{aligned}\operatorname{div}\partial_t^7v=&-\frac{1}{2}\partial_t^8J^{-2}-\partial_t^7v^i{}_{,j}\int_0^t(J^{-2}A_i^j)_tdt'-\partial_t^7(J^{-2}A_i^j)v^i{}_{,j}\\&-\sum_{l=1}^6C_7^l\partial_t^l(J^{-2}A_i^j)\partial_t^{7-l}v^i{}_{,j}.\end{aligned}$$

We can easily estimate last three terms by using the fundamental theorem of calculus and the Hölder inequality. Thus, we obtain

$$\|\operatorname{div}\partial_t^7v\|_0^2\leqslant M_0+\delta\sup_{[0,T]}E_5(t)+CTP(\sup_{[0,T]}E_5(t)).$$

According to Proposition 3, we have

$$\|\operatorname{curl}\partial_t^7v\|_0^2\leqslant M_0+CTP(\sup_{[0,T]}E_5(t)).$$

With the boundary estimates on $\partial_t^7 v^\alpha$ or $\partial_t^8 \eta^\alpha$ given by Proposition 6, we obtain, from Lemma A.4, that

$$\|\partial_t^7 v\|_1^2 \leqslant M_0 + \delta \sup_{[0,T]} E_5(t) + CTP(\sup_{[0,T]} E_5(t)).$$

Thus, we complete the proof. □

Having a good bound for $\partial_t^7 v(t)$ in $H^1(\Omega)$, we proceed with the bootstrapping. We have the following estimates and omit the lengthy proofs since their ideas are similar to Proposition 7.

Proposition 8. *For $t \in [0,T]$, it holds that*

$$\sup_{[0,T]} \left[\|\partial_t^5 v(t)\|_2^2 + \|\rho_0 \partial_t^6 J^{-2}(t)\|_2^2 \right] \leqslant M_0 + \delta \sup_{[0,T]} E_5(t) + CTP(\sup_{[0,T]} E_5(t)).$$

Proposition 9. *For $t \in [0,T]$, it holds that*

$$\sup_{[0,T]} \left[\|v_{ttt}(t)\|_3^2 + \|\rho_0 \partial_t^4 J^{-2}(t)\|_3^2 \right] \leqslant M_0 + \delta \sup_{[0,T]} E_5(t) + CTP(\sup_{[0,T]} E_5(t)).$$

Proposition 10. *For $t \in [0,T]$, it holds that*

$$\sup_{[0,T]} \left[\|v_t(t)\|_4^2 + \|\rho_0 \partial_t^2 J^{-2}(t)\|_4^2 \right] \leqslant M_0 + \delta \sup_{[0,T]} E_5(t) + CTP(\sup_{[0,T]} E_5(t)).$$

Proposition 11. *For $t \in [0,T]$, it holds that*

$$\sup_{[0,T]} \left[\|\eta(t)\|_5^2 + \|\rho_0 J^{-2}(t)\|_5^2 \right] \leqslant M_0 + \delta \sup_{[0,T]} E_5(t) + CTP(\sup_{[0,T]} E_5(t)).$$

7. **The a priori bound.** For the estimates of $\|\mathrm{curl}_\eta\, v(t)\|_4^2$ and $\|\rho_0 \overline{\partial}^5 \mathrm{curl}_\eta\, v(t)\|_0^2$, it is similar to those in [4] so that we omit the repeated proofs. Combining the inequalities provided by energy estimates, the additional elliptic estimates and the curl estimates shows, with the help of Proposition 1, that

$$\sup_{[0,T]} E_5(t) \leqslant C(E_5(0)) + CTP(\sup_{[0,T]} E_5(t)).$$

According to the polynomial-type inequality (A.3), by taking $T > 0$ sufficiently small, we obtain the a priori bound

$$\sup_{[0,T]} E_5(t) \leqslant 2C(E_5(0)).$$

Therefore, we complete the proof of the main theorem.

Appendix A. Preliminaries.

A.1. **Notations and weighted Sobolev spaces.** We make use of the Levi-Civita permutation symbol

$$\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) = \begin{cases} 1, & \text{even permutation of } \{1,2,3\}, \\ -1, & \text{odd permutation of } \{1,2,3\}, \\ 0, & \text{otherwise,} \end{cases}$$

and the basic identity regarding the i^{th} component of the curl of a vector field u:

$$(\mathrm{curl}\, u)_i = \varepsilon_{ijk} u^k{}_{,j}\,,$$

where it means that we have taken the sum with respect to the repeated scripts j and k. Since $v^i{}_{,j} = \partial u^i/\partial \eta^k \cdot \partial \eta^k/\partial x^j$, it follows that $\partial u^i/\partial \eta^k = \partial x^j/\partial \eta^k \cdot v^i{}_{,j}$, i.e., $u^i{}_{,k} = A^j_k v^i{}_{,j}$. The chain rule shows that

$$(\operatorname{curl} u(\eta))_i = (\operatorname{curl}_\eta v)_i := \varepsilon_{ijk} v^k{}_{,s} A^s_j,$$

where the right-hand side defines the Lagrangian curl operator curl_η. Similarly, we have

$$\operatorname{div} u(\eta) = \operatorname{div}_\eta v := v^j{}_{,s} A^s_j,$$

and the right-hand side defines the Lagrangian divergence operator div_η. We also use the notation for any vector field F

$$(\nabla_\eta F)^i_j = F^i{}_{,s} A^s_j. \tag{A.1}$$

For any vector field F, we have

$$\operatorname{curl}_\eta F = \operatorname{curl}(FA) = \begin{pmatrix} F^3{}_{,j} A^j_2 - F^2{}_{,j} A^j_3 \\ F^1{}_{,j} A^j_3 - F^3{}_{,j} A^j_1 \\ F^2{}_{,j} A^j_1 - F^1{}_{,j} A^j_2 \end{pmatrix},$$

and then

$$|\nabla_\eta F|^2 = |\operatorname{curl}_\eta F|^2 + (\nabla_\eta F) \cdot (\nabla_\eta F)^\top, \tag{A.2}$$

where the superscript $\top$ denotes the transpose of the matrix.

As a generalization of the standard Gronwall inequality, we introduce a polynomial type inequality. For a constant $M_0 \geqslant 0$, suppose that $f(t) \geqslant 0$, $t \mapsto f(t)$ is continuous, and

$$f(t) \leqslant M_0 + C\, t\, P(f(t)), \tag{A.3}$$

where P denotes a polynomial function, and C is a generic constant. Then for t taken sufficiently small, we have the bound (cf. [4, 5])

$$f(t) \leqslant 2M_0.$$

For integers $k \geqslant 0$ and a smooth, open domain Ω of $\mathbb{R}^3$, we define the Sobolev space $H^k(\Omega)$ ($H^k(\Omega; \mathbb{R}^3)$) to be the completion of $C^\infty(\Omega)$ ($C^\infty(\Omega; \mathbb{R}^3)$) in the norm

$$\|u\|_k := \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^2 \, dx \right)^{1/2},$$

for a multi-index $\alpha \in \mathbb{Z}^3_+$, with the standard convention $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. For real numbers $s \geqslant 0$, the Sobolev spaces $H^s(\Omega)$ and the norms $\|\cdot\|_s$ are defined by interpolation. We will write $H^s(\Omega)$ instead of $H^s(\Omega; \mathbb{R}^3)$ for vector-valued functions. In the case that $s \geqslant 3$, the above definition also holds for domains Ω of class H^s.

Our analysis will often make use of the following subspace of $H^1(\Omega)$:

$$\dot{H}^1_0 = \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } \Gamma, (x_1, x_2) \mapsto u(x_1, x_2) \text{ is periodic}\},$$

where, as usual, the vanishing of u on Γ is understood in the sense of trace.

For functions $u \in H^k(\Gamma)$, $k \geqslant 0$, we set

$$|u|_k := \left(\sum_{|\alpha| \leqslant k} \int_\Gamma |\overline{\partial}^\alpha u(x)|^2 dx \right)^{1/2},$$

for a multi-index $\alpha \in \mathbb{Z}^2_+$. For real $s \geqslant 0$, the Hilbert space $H^s(\Gamma)$ and the boundary norm $|\cdot|_s$ is defined by interpolation. The negative-order Sobolev spaces $H^{-s}(\Gamma)$ are defined via duality: for real $s \geqslant 0$,

$$H^{-s}(\Gamma) := [H^s(\Gamma)]'.$$

The derivative loss inherent to this degenerate problem is a consequence of the weighted embedding we now describe.

Using d to denote the distance function to the boundary Γ_1, i.e.,

$$d(x) = \operatorname{dist}(x, \Gamma_1),$$

and letting $p = 1$ or 2, the weighted Sobolev space $H^1_{d^p}(\Omega)$, with norm given by

$$\int_\Omega d(x)^p (|F(x)|^2 + |\nabla F(x)|^2) dx$$

for any $F \in H^1_{d^p}(\Omega)$, satisfies the following embedding:

$$H^1_{d^p}(\Omega) \hookrightarrow H^{1-\frac{p}{2}}(\Omega);$$

that is, there is a constant $C > 0$ depending only on Ω and p, such that

$$\|F\|^2_{1-p/2} \leqslant C \int_\Omega d(x)^p \big(|F(x)|^2 + |\nabla F(x)|^2\big) dx. \tag{A.4}$$

See, for example, Section 8.8 in Kufner [12]. From this embedding relation and (1.7), we obtain

$$\|F\|^2_0 \leqslant C \int_\Omega \rho_0^2 \big(|F(x)|^2 + |\nabla F(x)|^2\big) dx. \tag{A.5}$$

A.2. **Higher-order Hardy-type inequality and Hodge-type elliptic estimates.** We will make fundamental use of the following generalization of the well-known Hardy inequality to higher-order derivatives, see [7, Lemma 3.1]:

Lemma A.1 (Higher-order Hardy-type inequality)**.** *Let $s \geqslant 1$ be a given integer, Ω and $d(x)$ be defined as above, and suppose that*

$$u \in H^s(\Omega) \cap \dot{H}^1_0(\Omega),$$

then $\frac{u}{d} \in H^{s-1}(\Omega)$ and

$$\left\| \frac{u}{d} \right\|_{s-1} \leqslant C \|u\|_s. \tag{A.6}$$

The normal trace theorem provides the existence of the normal trace $w \cdot N$ of a velocity field $w \in L^2(\Omega)$ with $\operatorname{div} w \in L^2(\Omega)$ (see, e.g., [19]). For our purposes, the following form is most useful (see, e.g., [2]):

Lemma A.2 (Normal trace theorem)**.** *Let w be a vector field defined on Ω such that $\overline{\partial} w \in L^2(\Omega)$ and $\operatorname{div} w \in L^2(\Omega)$, and let N denote the outward unit normal vector to Γ. Then the normal trace $\overline{\partial} w \cdot N$ exists in $H^{-1/2}(\Gamma)$ with the estimate*

$$|\overline{\partial} w \cdot N|^2_{-1/2} \leq C \Big[\|\overline{\partial} w\|^2_{L^2(\Omega)} + \|\operatorname{div} w\|^2_{L^2(\Omega)} \Big], \tag{A.7}$$

for some constant C independent of w.

Lemma A.3 (Tangential trace theorem). *Let $\overline{\partial} w \in L^2(\Omega)$ so that* $\operatorname{curl} w \in L^2(\Omega)$, *and let T_1, T_2 denote the unit tangent vectors on Γ, so that any vector field u on Γ can be uniquely written as $u^\alpha T_\alpha$. Then*

$$|\overline{\partial} w \cdot T_\alpha|^2_{-1/2} \leq C\left[\|\overline{\partial} w\|^2_{L^2(\Omega)} + \|\operatorname{curl} w\|^2_{L^2(\Omega)}\right], \qquad \alpha = 1, 2, \tag{A.8}$$

for some constant C independent of w.

Combining (A.7) and (A.8), we have

$$|\overline{\partial} w|_{-1/2} \leq C\left[\|\overline{\partial} w\|_{L^2(\Omega)} + \|\operatorname{div} w\|_{L^2(\Omega)} + \|\operatorname{curl} w\|_{L^2(\Omega)}\right] \tag{A.9}$$

for some constant C independent of w.

The construction of our higher-order energy function is based on the following Hodge-type elliptic estimate (see, e.g., [4]):

Lemma A.4. *For an H^r domain Ω, $r \geqslant 3$, if $F \in L^2(\Omega; \mathbb{R}^3)$ with* $\operatorname{curl} F \in H^{s-1}(\Omega)$, $\operatorname{div} F \in H^{s-1}(\Omega)$, *and $F \cdot N|_\Gamma \in H^{s-1/2}(\Gamma)$ for $1 \leqslant s \leqslant r$, then there exists a constant $\bar{C} > 0$ depending only on Ω such that*

$$\|F\|_s \leqslant \bar{C}\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + |\overline{\partial} F \cdot N|_{s-3/2}\right),$$

$$\|F\|_s \leqslant \bar{C}\Big(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + \sum_{\alpha=1}^{2} |\overline{\partial} F \cdot T_\alpha|_{s-3/2}\Big),$$

where N denotes the outward unit normal to Γ and T_α are tangent vectors for $\alpha = 1, 2$.

A.3. **Properties of J, A and a.** From the derivative formula of matrices and determinants, we have

$$A^k_{i\,,s} = -A^k_l \eta^l_{\,,js} A^j_i, \tag{A.10}$$

$$A^k_{ti} = \partial_t A^k_i = -A^k_l v^l_{\,,j} A^j_i, \tag{A.11}$$

$$J_{,s} = J A^j_l \eta^l_{\,,js} = a^j_l \eta^l_{\,,js}. \tag{A.12}$$

It follows from $a = JA$, (A.10) and (A.12) that the columns of every adjoint matrix are divergence-free, i.e., the Piola identity,

$$a^k_{i\,,k} = 0, \tag{A.13}$$

which will play a vital role in our energy estimates. We also have

$$a^k_{i\,,s} = J\eta^l_{\,,js}\,(A^k_i A^j_l - A^j_i A^k_l) = J^{-1}\eta^l_{\,,js}\,(a^k_i a^j_l - a^j_i a^k_l), \tag{A.14}$$

$$a^k_{ti} = J v^l_{\,,j}\,(A^k_i A^j_l - A^j_i A^k_l) = J^{-1} v^l_{\,,j}\,(a^k_i a^j_l - a^j_i a^k_l). \tag{A.15}$$

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