Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 261 (2016) 712–737
Journal of Differential Equations
www.elsevier.com/locate/jde

A priori estimates for the free boundary problem of incompressible neo-Hookean elastodynamics

Chengchun Hao [a,b,*], Dehua Wang [c]

[a] *Institute of Mathematics, Academy of Mathematics and Systems Science, and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, China*
[b] *BCMIIS, Beijing 100048, China*
[c] *Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA*

Received 4 February 2015
Available online 25 March 2016

Abstract

A free boundary problem for the incompressible neo-Hookean elastodynamics is studied in two and three spatial dimensions. The *a priori* estimates in Sobolev norms of solutions with the physical vacuum condition are established through a geometrical point of view of Christodoulou and Lindblad (2000) [3]. Some estimates on the second fundamental form and velocity of the free surface are also obtained.

MSC: 35A05; 76A10; 76D03

Keywords: Incompressible elastodynamics; Free boundary; *A priori* estimates; The second fundamental form

1. Introduction

We are concerned with the motion of neo-Hookean elastic waves in an incompressible material for which the deformation or strain is proportional to the stress. Precisely, we consider the free boundary problem of the following incompressible elastodynamic equations of neo-Hookean elastic materials:

* Corresponding author at: Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.
E-mail addresses: hcc@amss.ac.cn (C. Hao), dwang@math.pitt.edu (D. Wang).

http://dx.doi.org/10.1016/j.jde.2016.03.025

$$v_t + v \cdot \partial v + \partial p = \operatorname{div}(FF^\top), \tag{1.1a}$$

$$F_t + v \cdot \partial F = \partial v F, \tag{1.1b}$$

$$\operatorname{div} v = 0, \quad \operatorname{div} F^\top = 0, \tag{1.1c}$$

in a set

$$\mathscr{D} = \bigcup_{0 \leqslant t \leqslant T} \{t\} \times \mathscr{D}_t,$$

where $\mathscr{D}_t \subset \mathbb{R}^n$, $n = 2$ or 3, is the domain that the material occupies at time $t \in [0, T]$ for some $T > 0$; where $\partial = (\partial_1, \cdots, \partial_n)$ and div are the usual gradient operator and spatial divergence in the Eulerian coordinates with $\partial_i = \partial/\partial x^i$, respectively; $v(t,x) = (v_1(t,x), \cdots, v_n(t,x))$ is the velocity vector field of the fluid, $p(t,x)$ is the pressure, $F(t,x) = (F_{ij}(t,x))$ is the deformation tensor, $F^\top = (F_{ji})$ denotes the transpose of the $n \times n$ matrix F, $FF^\top$ is the Cauchy–Green tensor in the case of neo-Hookean elastic materials (cf. [9,14]); and the notations $(\partial v)_{ij} = \partial_j v_i$, $(\partial v F)_{ij} = (\partial v F)^{ij} = (\partial v)_{ik} F^{kj} = \partial_k v^i F^{kj}$, $\operatorname{div} v = \partial_i v^i$, $(\operatorname{div} F^\top)^i = \partial_j F^{ji}$ follow the Einstein summation convention: $v^i = \delta^{ij} v_j = v_i$ and $F^{ij} = \delta^{ik}\delta^{jl} F_{kl} = F_{ij}$. The boundary conditions on the free boundary:

$$\partial\mathscr{D} = \bigcup_{0 \leqslant t \leqslant T} \{t\} \times \partial\mathscr{D}_t$$

are prescribed as the following:

$$p = 0 \ \text{ on } \partial\mathscr{D}, \tag{1.2a}$$

$$\mathcal{N} \cdot F^\top = 0 \ \text{ on } \partial\mathscr{D}, \tag{1.2b}$$

$$(\partial_t + v \cdot \partial)|_{\partial\mathscr{D}} \in T(\partial\mathscr{D}), \tag{1.2c}$$

where $\mathcal{N}(t,x)$ is the exterior unit normal to the free surface $\partial\mathscr{D}_t$ and $T(\partial\mathscr{D})$ is the tangential space to $\partial\mathscr{D}$. The boundary condition (1.2a) implies that the pressure p vanishes outside the domain, (1.2b) indicates that the normal component of $F^\top$ (i.e., $\mathcal{N}_k F^{kj}$) vanishes on the boundary, and (1.2c) means that the free boundary moves with the velocity v of the material particles, i.e., $v \cdot \mathcal{N} = \kappa$ on $\partial\mathscr{D}_t$ with κ the normal velocity of $\partial\mathscr{D}_t$.

For a simply connected bounded domain $\mathscr{D}_0 \subset \mathbb{R}^n$ that is homeomorphic to the unit ball, and the initial data $(v_0(x), F_0(x))$ satisfying the constraint (1.1c): $\operatorname{div} v_0 = 0$, $\operatorname{div} F_0^\top = 0$, we shall establish *a priori* estimates for the set $\mathscr{D} \subset [0, T] \times \mathbb{R}^n$ and the vector fields v and F solving (1.1)–(1.2) with the initial conditions:

$$\{x : (0, x) \in \mathscr{D}\} = \mathscr{D}_0, \quad (v, F)|_{t=0} = (v_0(x), F_0(x)) \text{ for } x \in \mathscr{D}_0. \tag{1.3}$$

We will study the free boundary problem (1.1)–(1.3) under the following natural condition (cf. [2–4,7,8,10–13,16–18]):

$$\nabla_{\mathcal{N}} p \leqslant -\varepsilon < 0 \text{ on } \partial\mathscr{D}_t, \tag{1.4}$$

where $\nabla_{\mathcal{N}} = \mathcal{N}^i \partial_i$ and $\varepsilon > 0$ is a constant. We assume that (1.4) holds initially, and will verify that it still holds within a time period. Roughly speaking, the elastic body will not break up in the interior since the pressure is positive, the boundary moves according to the velocity, and the boundary is the level set of the pressure that, together with the Cauchy–Green tensor, determines the acceleration, thus the regularity of the boundary in quite involved, which is a difficult issue for this problem.

There have been some results for the free surface problem of the incompressible Euler equations of fluids in the recent decades, see for examples [1,3,4,7,10–13,16–18] and the references therein. For elastodynamics, there have been some studies on the fixed boundary problems, see for examples Ebin [5,6] for the global existence of small solutions to the three-dimensional incompressible and isotropic elasticity equations and the special case of incompressible neo-Hookean materials, and Sideris–Thomases [14,15] for the global existence of the three-dimensional incompressible elasticity. In this paper, we shall prove the *a priori* estimates for the free boundary problem (1.1)–(1.3) in all physical spatial dimensions $n = 2, 3$ by adopting a geometrical point of view used in Christodoulou–Lindblad [3] and establishing estimates on quantities such as the second fundamental form and the velocity of the free surface.

Define the material derivative by $D_t = \partial_t + v^k \partial_k$. We rewrite the system (1.1) as

$$D_t v_i + \partial_i p = \partial_k F_{ij} F^{kj}, \qquad \text{in } \mathscr{D}, \tag{1.5a}$$

$$D_t F_{ij} = \delta^{kl} \partial_k v_i F_{lj}, \qquad \text{in } \mathscr{D}, \tag{1.5b}$$

$$\partial_i v^i = 0, \quad \partial_j F^{ji} = 0, \qquad \text{in } \mathscr{D}. \tag{1.5c}$$

From (1.5), one has

$$\frac{1}{2}\frac{d}{dt}\int_{\mathscr{D}_t} (|v|^2 + |F|^2) dx = -\int_{\partial \mathscr{D}_t} p v^i \mathcal{N}_i dS + \int_{\partial \mathscr{D}_t} F_{ij} F^{kj} v^i \mathcal{N}_k dS, \tag{1.6}$$

where dS is the surface measure. We see that (1.6) and the boundary conditions (1.2) yield the conserved physical energy:

$$E_0(t) = \int_{\mathscr{D}_t} \left(\frac{1}{2} |v(t,x)|^2 + \frac{1}{2} |F(t,x)|^2 \right) dx. \tag{1.7}$$

Note that the identities $\operatorname{div} F^\top = 0$ in $\mathscr{D}$ and $\mathcal{N} \cdot F^\top = 0$ on $\partial\mathscr{D}$ are preserved, that is, they hold if $\operatorname{div} F_0^\top = 0$ in $\mathscr{D}_0$ and $N \cdot F_0^\top = 0$ on $\partial \mathscr{D}_0$ for initial data, where N denotes the exterior unit normal to the initial interface $\partial \mathscr{D}_0$, which can be verified in the Lagrangian coordinates.

The higher order energy norm has a boundary part and an interior part. Consider the following positive definite quadratic form Q of the form (see [3]):

$$Q(\alpha, \beta) = q^{i_1 j_1} \cdots q^{i_r j_r} \alpha_{i_1 \cdots i_r} \beta_{j_1 \cdots j_r}, \tag{1.8}$$

where

$$q^{ij} = \delta^{ij} - \eta(d)^2 \mathcal{N}^i \mathcal{N}^j, \quad d(x) = \operatorname{dist}(x, \partial \mathscr{D}_t), \quad \text{and } \mathcal{N}^i = -\delta^{ij} \partial_j d, \tag{1.9}$$

and η is a smooth cutoff function satisfying

$$0 \leqslant \eta(d) \leqslant 1, \quad \eta(d) = 1 \text{ for } d < \frac{d_0}{4}, \quad \eta(d) = 0 \text{ for } d > \frac{d_0}{2},$$

with d_0 a fixed number less than the injectivity radius of the normal exponential map ι_0 which is the largest number ι_0 such that the map

$$\partial \mathscr{D}_t \times (-\iota_0, \iota_0) \to \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial \mathscr{D}_t) < \iota_0\}, \tag{1.10}$$

defined by

$$(\bar{x}, \iota) \to x = \bar{x} + \iota \mathcal{N}(\bar{x}),$$

is an injection.

Let $\operatorname{sgn}(s)$ be the sign function of the real number s. Denote

$$(\operatorname{curl} F^\top)_{ijk} := \partial_i F_{jk} - \partial_j F_{ik}$$

and $\vartheta = (-\nabla_{\mathcal{N}} p)^{-1}$. Then, we define the higher order energies for $r \geqslant 1$ as:

$$\begin{aligned} E_r(t) = & \int_{\mathscr{D}_t} \left(\delta^{ij} Q(\partial^r v_i, \partial^r v_j) + \delta^{ij}\delta^{km} Q(\partial^r F_{ik}, \partial^r F_{jm})\right) dx \\ & + \int_{\mathscr{D}_t} \left(|\partial^{r-1}\operatorname{curl} v|^2 + |\partial^{r-1}\operatorname{curl} F^\top|^2\right) dx \\ & + \operatorname{sgn}(r-1) \int_{\partial\mathscr{D}_t} Q(\partial^r p, \partial^r p)\vartheta \, dS. \end{aligned} \tag{1.11}$$

The higher order energy norm has a boundary part (for $r \geqslant 2$) which controls the norms of the second fundamental form of the free surface, and an interior part which controls the norms of the velocity and thus the pressure. We will prove that the time derivatives of the energy norms are controlled by themselves. One advantage of the above higher order energy norms is that the time derivatives of the interior parts yield some boundary terms which have some cancellation with the leading-order terms in the time derivatives of the boundary integrals.

Now, we can state the main result of this paper as follows.

Theorem 1.1. *Let*

$$\mathcal{K}(0) = \max\left(\|\theta(0,\cdot)\|_{L^\infty(\partial\mathscr{D}_0)}, \frac{1}{\iota_0(0)}\right)$$

and

$$\mathcal{E}(0) = \left\| \frac{1}{\nabla_N p(0,\cdot)} \right\|_{L^\infty(\partial\mathscr{D}_0)}.$$

Then, there exists a continuous function $\mathscr{T} > 0$ *such that if*

$$T \leqslant \mathscr{T}(\mathcal{K}(0), \mathcal{E}(0), E_0(0), \cdots, E_{n+1}(0), \mathrm{Vol}\,\mathscr{D}_0),$$

any smooth solution of the free boundary problem (1.1)–(1.4) *satisfies*

$$\sum_{s=0}^{n+1} E_s(t) \leqslant 2 \sum_{s=0}^{n+1} E_s(0), \quad 0 \leqslant t \leqslant T.$$

We remark that Theorem 1.1 extends the result of [3] for the Euler equations of incompressible flow to the elastodynamics (1.1). Our proof will be based on the geometric point of view following [3]. We need to develop new ingredients in the proof to handle the deformation F and the interaction with the velocity v, which requires some new thoughts. For the well-posedness of incompressible Euler equations we refer the readers to [10,11] and the references therein. The well-posedness of the elastodynamics (1.1) is much harder. In this paper we shall explore all the symmetries of the equations and then we will be able to establish the sharp *a priori* estimates. Although the well-posedness does not follow directly, these estimates are crucial for the local existence of smooth solution for the system (1.1) which could be possibly obtained by improving the estimates of this paper together with the Nash–Moser technique.

The rest of the paper is organized as follows. In Section 2, we reformulate the problem to a fixed initial–boundary value problem in the Lagrangian coordinates. Sections 3 and 4 are devoted to the first and higher order energy estimates, respectively. Finally, we justify the *a priori* assumptions in Section 5.

2. Reformulation in Lagrangian coordinates

In this section, we can reformulate the free boundary problem to fix the boundaries by following the same terminology and lines of [3] (we omit the details). We first generalize from [3] the estimates of commutators between the material derivative D_t and covariant derivatives ∇_a:

Lemma 2.1. *Let* $T_{a_1 \cdots a_r}$ *be a* $(0, r)$ *tensor. Then*

$$\begin{aligned}[D_t, g^{ac} g^{bd} \nabla_a] T_{cd} = & -\Delta u_e T^{eb} - g^{bd} \nabla_d \nabla_a u_e T^{ae} - g^{ae} \nabla_e u_f \nabla_a T^{fb} \\ & - g^{be} \nabla_e u_f \nabla_a T^{af} - \nabla_f u^a \nabla_a T^{fb} - \nabla_f u^b \nabla_a T^{af},\end{aligned} \tag{2.1}$$

where g_{ab} *is the induced metric and* g^{ab} *is the inverse of* g_{ab}.

Proof. From [3, (2.13) and (2.20)], it follows that

$$\begin{aligned}&[D_t, g^{ac} g^{bd} \nabla_a] T_{cd} \\ &= D_t g^{ac} g^{bd} \nabla_a T_{cd} + g^{ac} D_t g^{bd} \nabla_a T_{cd} + g^{ac} g^{bd} D_t \nabla_a T_{cd} - g^{ac} g^{bd} \nabla_a D_t T_{cd} \\ &= -2h^{ac} g^{bd} \nabla_a T_{cd} - 2g^{ac} h^{bd} \nabla_a T_{cd} - g^{ac} g^{bd} (\nabla_c \nabla_a u^e T_{ed} + \nabla_d \nabla_a u^e T_{ce}) \\ &= -\Delta u_e T^{eb} - g^{bd} \nabla_d \nabla_a u_e T^{ae} - g^{ae} \nabla_e u_f \nabla_a T^{fb} \\ &\quad - g^{be} \nabla_e u_f \nabla_a T^{af} - \nabla_f u^a \nabla_a T^{fb} - \nabla_f u^b \nabla_a T^{af}. \quad \square\end{aligned}$$

Let $u_a = v_i \frac{\partial x^i}{\partial y^a}$, $u^a = g^{ab} u_b$ and

$$\mathbb{F}_{ab} = F_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}, \quad \mathbb{F}^{ab} = g^{ac} g^{bd} \mathbb{F}_{cd}, \quad |\mathbb{F}|^2 = \mathbb{F}_{ab}\mathbb{F}^{ab}.$$

Then, it follows from [3, (2.7)] that

$$|\mathbb{F}|^2 = |F|^2 \quad \text{and} \qquad F_{ij} = \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \mathbb{F}_{ab}. \tag{2.2}$$

From [3, (2.11)], (2.2), [3, (2.12)] and [3, (2.6)], we can rewrite the system (1.1) in the Lagrangian coordinates as

$$D_t u_a + \nabla_a p = \nabla_c \mathbb{F}_{ab}\mathbb{F}^{cb} + u^c \nabla_a u_c \qquad \text{in } [0,T] \times \Omega, \tag{2.3a}$$

$$D_t \mathbb{F}_{ab} = g^{dc} \nabla_d u_a \mathbb{F}_{cb} + \mathbb{F}_{cb} \nabla_a u^c + \mathbb{F}_{ac} \nabla_b u^c \qquad \text{in } [0,T] \times \Omega, \tag{2.3b}$$

$$\nabla_a u^a = 0, \quad \nabla_a \mathbb{F}^{ab} = 0 \qquad \text{in } [0,T] \times \Omega, \tag{2.3c}$$

$$p = 0, \quad N^a \mathbb{F}_{ab} = 0 \qquad \text{on } [0,T] \times \partial\Omega. \tag{2.3d}$$

From (1.7), we also have the conserved energy

$$E_0(t) = \int_\Omega \left(\frac{1}{2} |u(t,y)|^2 + \frac{1}{2} |\mathbb{F}(t,y)|^2 \right) d\mu_g. \tag{2.4}$$

We note that if

$$|\nabla u(t,y)| \leqslant C \quad \text{in } [0,T] \times \overline{\Omega}, \tag{2.5}$$

and $\operatorname{div} v = 0$ in $[0,T] \times \Omega$, then the divergence free property of $\mathbb{F}^\top$, i.e., $\operatorname{div} \mathbb{F}^\top = 0$, is preserved for all times under the Lagrangian coordinates or in view of the material derivative, i.e., $D_t \operatorname{div} \mathbb{F}^\top = 0$. Moreover, $N \cdot \mathbb{F}^\top = 0$ is also preserved for all time t in the lifespan $[0,T]$, that is, we have $N^a \mathbb{F}_{ab} = 0$ on $[0,T] \times \partial\Omega$ if $N \cdot \mathbb{F}^\top = 0$ on the boundary $\partial\Omega$ at initial time.

3. The first order energy estimates

In this section, we prove the first order energy estimate. From [3, (2.20)], (2.3a) and (2.3b), we have

$$D_t(\nabla_b u_a) + \nabla_b \nabla_a p = \nabla_b u^c \nabla_a u_c + \nabla_b \nabla_c \mathbb{F}_{ad} \mathbb{F}^{cd} + \nabla_c \mathbb{F}_{ad} \nabla_b \mathbb{F}^{cd}, \tag{3.1}$$

and

$$D_t(\nabla_c \mathbb{F}_{ab}) = g^{de} \nabla_c \mathbb{F}_{eb} (\nabla_d u_a + \nabla_a u_d) + \nabla_c \mathbb{F}_{ad} \nabla_b u^d + g^{de} \mathbb{F}_{eb} \nabla_c \nabla_d u_a. \tag{3.2}$$

We now derive the material derivative of $g^{bd} \gamma^{ae} \nabla_a u_b \nabla_e u_d$. From [3, (2.13), (2.11), (3.21), (3.22)] and (3.2), it follows that

$$\begin{aligned} & D_t\left(g^{bd}\gamma^{ae}\nabla_a u_b\nabla_e u_d + g^{bd}g^{cf}\gamma^{ae}\nabla_a\mathbb{F}_{bc}\nabla_e\mathbb{F}_{df}\right) \\ = & -2\gamma^{ae}\gamma^{fc}\nabla_e u_f\nabla_a u^d\nabla_c u_d - 2\gamma^{ae}\gamma^{fc}\nabla_e u_f\nabla_a\mathbb{F}^{ds}\nabla_c\mathbb{F}_{ds} \\ & - 2\nabla_b(\gamma^{ae}\nabla_e u^b\nabla_a p - \gamma^{ae}\nabla_e u^c\nabla_a\mathbb{F}_{cf}\mathbb{F}^{bf}) \\ & + 2(\nabla_b\gamma^{ae})(\nabla_e u^b\nabla_a p - \nabla_e u^c\nabla_a\mathbb{F}_{cf}\mathbb{F}^{bf}) \\ & + 2\gamma^{ae}\nabla_e u^d\nabla_c\mathbb{F}_{df}\nabla_a\mathbb{F}^{cf} + 2\gamma^{ae}\nabla_c u^d\nabla_e\mathbb{F}_{df}\nabla_a\mathbb{F}^{cf}. \end{aligned} \tag{3.3}$$

Then, we calculate the material derivatives of $|\mathrm{curl}\, u|^2$ and $|\mathrm{curl}\, \mathbb{F}^\top|^2$ where $\mathrm{curl}\, \mathbb{F}^\top$ is defined as

$$(\mathrm{curl}\, \mathbb{F}^\top)_{abc} := \nabla_a\mathbb{F}_{bc} - \nabla_b\mathbb{F}_{ac}.$$

Indeed, one has

$$\begin{aligned} & D_t(|\mathrm{curl}\, u|^2 + |\mathrm{curl}\, \mathbb{F}^\top|^2) \\ = & -4g^{ae}g^{bd}\nabla_e u^c(\mathrm{curl}\, u)_{ab}(\mathrm{curl}\, u)_{cd} + 4g^{ac}g^{bd}(\mathrm{curl}\, u)_{cd}\nabla_a\mathbb{F}^{ef}\nabla_e\mathbb{F}_{bf} \\ & - 4g^{aq}g^{bd}g^{ef}\nabla_q u^c(\mathrm{curl}\, \mathbb{F}^\top)_{abe}(\mathrm{curl}\, \mathbb{F}^\top)_{cdf} \\ & - 2g^{ac}g^{bd}g^{eq}\nabla_q u^f(\mathrm{curl}\, \mathbb{F}^\top)_{abe}(\mathrm{curl}\, \mathbb{F}^\top)_{cdf} \\ & + 4g^{ac}(\mathrm{curl}\, \mathbb{F}^\top)_{cdf}\nabla_a\mathbb{F}^{sf}\nabla_s u^d + 4g^{ac}g^{bd}(\mathrm{curl}\, \mathbb{F}^\top)_{cdf}\nabla_a\mathbb{F}^{qf}\nabla_b u_q \\ & + 4g^{ac}g^{ef}(\mathrm{curl}\, \mathbb{F}^\top)_{cdf}\nabla_a\mathbb{F}^{ds}\nabla_e u_s + 4\nabla_e\left[g^{ac}g^{bd}(\mathrm{curl}\, u)_{cd}\nabla_a\mathbb{F}_{bf}\mathbb{F}^{ef}\right]. \end{aligned} \tag{3.4}$$

Define the first order energy as

$$\begin{aligned} E_1(t) = & \int_\Omega \left(g^{bd}\gamma^{ae}\nabla_a u_b\nabla_e u_d + g^{bd}g^{cf}\gamma^{ae}\nabla_a\mathbb{F}_{bc}\nabla_e\mathbb{F}_{df}\right)d\mu_g \\ & + \int_\Omega \left(|\mathrm{curl}\, u|^2 + |\mathrm{curl}\, \mathbb{F}^\top|^2\right)d\mu_g. \end{aligned} \tag{3.5}$$

Then, we can establish the following estimate on the first order energy:

Theorem 3.1. *For any smooth solution of system (2.3) for $0 \leqslant t \leqslant T$ satisfying*

$$|\nabla p| \leqslant M, \quad |\nabla u| \leqslant M, \qquad \text{in } [0,T]\times\Omega, \tag{3.6}$$

$$|\theta| + |\nabla u| + \frac{1}{\iota_0} \leqslant K, \qquad \text{on } [0,T]\times\partial\Omega, \tag{3.7}$$

one has, for any $t \in [0,T]$,

$$E_1(t) \leqslant 2e^{CMt}E_1(0) + CK^2\left(\mathrm{Vol}\,\Omega + E_0(0)\right)\left(e^{CMt} - 1\right) \tag{3.8}$$

with some constant $C > 0$ which depends only on the dimension n.

Proof. It follows, from (3.3), (3.4) and Gauss' formula, that

$$
\begin{aligned}
\frac{d}{dt}E_1(t) &= \int_\Omega D_t\left(g^{bd}\gamma^{ae}\nabla_a u_b \nabla_e u_d + g^{bd}g^{cf}\gamma^{ae}\nabla_a \mathbb{F}_{bc}\nabla_e \mathbb{F}_{df}\right)d\mu_g \\
&\quad + \int_\Omega \left(g^{bd}\gamma^{ae}\nabla_a u_b \nabla_e u_d + g^{bd}g^{cf}\gamma^{ae}\nabla_a \mathbb{F}_{bc}\nabla_e \mathbb{F}_{df}\right)\operatorname{tr} h\, d\mu_g \\
&\quad + \int_\Omega D_t\left(|\operatorname{curl} u|^2 + |\operatorname{curl}\mathbb{F}^\top|^2\right)d\mu_g \\
&\quad + \int_\Omega \left(|\operatorname{curl} u|^2 + |\operatorname{curl}\mathbb{F}^\top|^2\right)\operatorname{tr} h\, d\mu_g \\
&= -2\int_\Omega \gamma^{ae}\gamma^{fc}\nabla_e u_f \nabla_a u^d \nabla_c u_d\, d\mu_g - 2\int_\Omega \gamma^{ae}\gamma^{fc}\nabla_e u_f \nabla_a \mathbb{F}^{ds}\nabla_c \mathbb{F}_{ds}\, d\mu_g \\
&\quad - 2\int_{\partial\Omega} N_b(\gamma^{ae}\nabla_e u^b \nabla_a p - \gamma^{ae}\nabla_e u^c \nabla_a \mathbb{F}_{cf}\mathbb{F}^{bf})\,d\mu_\gamma \qquad (3.9) \\
&\quad + 2\int_\Omega (\nabla_b \gamma^{ae})(\nabla_e u^b \nabla_a p - \nabla_e u^c \nabla_a \mathbb{F}_{cf}\mathbb{F}^{bf})\,d\mu_g \qquad (3.10) \\
&\quad + 2\int_\Omega \gamma^{ae}\nabla_e u^d \nabla_c \mathbb{F}_{df}\nabla_a \mathbb{F}^{cf}\, d\mu_g + 2\int_\Omega \gamma^{ae}\nabla_c u^d \nabla_e \mathbb{F}_{df}\nabla_a \mathbb{F}^{cf}\, d\mu_g \\
&\quad - 4\int_\Omega g^{ae}g^{bd}\nabla_e u^c (\operatorname{curl} u)_{ab}(\operatorname{curl} u)_{cd}\, d\mu_g \\
&\quad + 4\int_\Omega g^{ac}g^{bd}(\operatorname{curl} u)_{cd}\nabla_a \mathbb{F}^{ef}\nabla_e \mathbb{F}_{bf}\, d\mu_g \\
&\quad - 4\int_\Omega g^{aq}g^{bd}g^{ef}\nabla_q u^c (\operatorname{curl}\mathbb{F}^\top)_{abe}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\, d\mu_g \\
&\quad - 2\int_\Omega g^{ac}g^{bd}g^{eq}\nabla_q u^f (\operatorname{curl}\mathbb{F}^\top)_{abe}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\, d\mu_g \\
&\quad + 4\int_\Omega g^{ac}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\nabla_a \mathbb{F}^{sf}\nabla_s u^d\, d\mu_g \\
&\quad + 4\int_\Omega g^{ac}g^{bd}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\nabla_a \mathbb{F}^{qf}\nabla_b u_q\, d\mu_g \\
&\quad + 4\int_\Omega g^{ac}g^{ef}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\nabla_a \mathbb{F}^{ds}\nabla_e u_s\, d\mu_g \\
&\quad + 4\int_{\partial\Omega} N_e \mathbb{F}^{ef}g^{ac}g^{bd}(\operatorname{curl} u)_{cd}\nabla_a \mathbb{F}_{bf}\, d\mu_\gamma, \qquad (3.11)
\end{aligned}
$$

due to the fact $\operatorname{tr} h = 0$. Since $p = 0$ on the boundary $\partial\Omega$, it follows that $\overline{\nabla} p = 0$, i.e., $\gamma_a^d \nabla_d p = 0$, and then $\gamma^{ae}\nabla_a p = g^{ce}\gamma_c^a \nabla_a p = 0$ on $\partial\Omega$. In addition, $N \cdot \mathbb{F}^\top = 0$ on $\partial\Omega$. Thus, the integrals in (3.9) and (3.11) vanish.

For the term (3.10), we first have from [3, (3.6) and (3.4)],

$$\theta_{ab} = (\delta_a^c - N_a N^c)\nabla_c N_b = \nabla_a N_b - N_a \nabla_N N_b = \nabla_a N_b$$

since $\nabla_N N = 0$ in geodesic coordinates, and then

$$\begin{aligned}\nabla_b \gamma^{ae} &= \nabla_b (g^{ae} - N^a N^e) = -\nabla_b (N^a N^e) \\ &= -(\nabla_b N^a) N^e - (\nabla_b N^e) N^a = -\theta_b^a N^e - \theta_b^e N^a.\end{aligned}$$

Thus, by the Hölder inequality, (3.7) and [3, Lemma 5.5], one has

$$\begin{aligned}|(3.10)| &\leqslant CK \left(\|\nabla u\|_{L^2(\Omega)} \|\nabla p\|_{L^\infty(\Omega)} (\operatorname{Vol}\Omega)^{1/2} + \|\nabla u\|_{L^\infty(\Omega)} \|\mathbb{F}\|_{L^2(\Omega)} \|\nabla \mathbb{F}\|_{L^2(\Omega)} \right) \\ &\leqslant CKM \left((\operatorname{Vol}\Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t).\end{aligned}$$

For other terms, we can use the Hölder inequality directly. Hence, we obtain

$$\begin{aligned}\frac{d}{dt} E_1(t) &\leqslant CKM \left((\operatorname{Vol}\Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t) \\ &\quad + C \|\nabla u\|_{L^\infty(\Omega)} \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla \mathbb{F}\|^2_{L^2(\Omega)} + \|\operatorname{curl} u\|^2_{L^2(\Omega)} + \left\| \operatorname{curl} \mathbb{F}^\top \right\|^2_{L^2(\Omega)} \right) \\ &\leqslant CKM \left((\operatorname{Vol}\Omega)^{1/2} + E_0^{1/2}(0) \right) E_1^{1/2}(t) + CME_1(t).\end{aligned}$$

By the Gronwall inequality, it yields the desired estimate. □

4. The general *r*-th order energy estimates

In this section, we establish the higher order energy estimates. In view of [3, (2.16), (2.18)] and (1.5a), one has

$$\begin{aligned}D_t \nabla^r u_a &= D_t \nabla_{a_1} \cdots \nabla_{a_r} u_a = D_t \nabla_{a_1} \cdots \nabla_{a_{r-1}} \left(\frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial v_i}{\partial x^{i_r}} \right) \\ &= D_t \Big(\frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial^r v_i}{\partial x^{i_1} \cdots \partial x^{i_r}} \Big) \\ &= \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \left(D_t \frac{\partial^r v_i}{\partial x^{i_1} \cdots \partial x^{i_r}} + \frac{\partial v^l}{\partial x^{i_1}} \frac{\partial^r v_i}{\partial x^l \cdots \partial x^{i_r}} + \cdots \right. \\ &\qquad \left. + \frac{\partial v^l}{\partial x^{i_r}} \frac{\partial^r v_i}{\partial x^{i_1} \cdots \partial x^l} + \frac{\partial v^l}{\partial x^i} \frac{\partial^r v_l}{\partial x^{i_1} \cdots \partial x^{x_r}} \right) \\ &= \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \Big([D_t, \partial^r] v_i + \partial^r D_t v_i \Big) + \nabla u \cdot \nabla^r u_a + \nabla_a u^c \nabla^r u_c\end{aligned}$$

$$= -\nabla^r \nabla_a p + \nabla_a u^c \nabla^r u_c - \sum_{s=1}^{r-1} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a + \sum_{s=0}^{r} \binom{r}{s} \nabla^s \mathbb{F}^{cb} \nabla^{r-s} \nabla_c \mathbb{F}_{ab},$$

where

$$\left(\nabla^s \mathbb{F}^{cb} \nabla^{r-s} \nabla_c \mathbb{F}_{ab}\right)_{a_1 \cdots a_r} = \sum_{\Sigma_r} \nabla^s_{a_{\sigma_1} \cdots a_{\sigma_s}} \mathbb{F}^{cb} \nabla^{r-s}_{a_{\sigma_{s+1}} \cdots a_{\sigma_r}} \nabla_c \mathbb{F}_{ab}.$$

Then, using $\operatorname{div} \mathbb{F}^\top = 0$, we obtain, for $r \geqslant 2$,

$$D_t \nabla^r u_a + \nabla^r \nabla_a p = (\operatorname{curl} u)_{ac} \nabla^r u^c + \operatorname{sgn}(2-r) \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a + \nabla_c \left(\mathbb{F}^{cb} \nabla^r \mathbb{F}_{ab}\right) + \sum_{s=1}^{r} \binom{r}{s} \nabla^s \mathbb{F}^{cb} \nabla^{r-s} \nabla_c \mathbb{F}_{ab}. \tag{4.1}$$

Similarly, by $\operatorname{div} \mathbb{F}^\top = 0$ again, we have, for $r \geqslant 2$,

$$\begin{aligned} D_t \nabla^r \mathbb{F}_{ab} &= D_t \Big(\frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial^r F_{ij}}{\partial x^{i_1} \cdots \partial x^{i_r}} \Big) \\ &= \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \Big(D_t \frac{\partial^r F_{ij}}{\partial x^{i_1} \cdots \partial x^{i_r}} + \frac{\partial v^l}{\partial x^{i_1}} \frac{\partial^r F_{ij}}{\partial x^l \cdots \partial x^{i_r}} + \cdots \\ &\qquad + \frac{\partial v^l}{\partial x^{i_r}} \frac{\partial^r F_{ij}}{\partial x^{x_1} \cdots \partial x^l} + \frac{\partial v^l}{\partial x^i} \frac{\partial^r F_{lj}}{\partial x^{x_1} \cdots \partial x^{x_r}} + \frac{\partial v^l}{\partial x^j} \frac{\partial^r F_{il}}{\partial x^{x_1} \cdots \partial x^{x_r}} \Big) \\ &= \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \Big([D_t, \partial^r] F_{ij} + \partial^r D_t F_{ij} \Big) \\ &\quad + \nabla u \cdot \nabla^r \mathbb{F}_{ab} + \nabla_a u^c \nabla^r \mathbb{F}_{cb} + \nabla_b u^c \nabla^r \mathbb{F}_{ac} \\ &= \nabla_a u^c \nabla^r \mathbb{F}_{cb} + \nabla_b u^c \nabla^r \mathbb{F}_{ac} - \nabla^r u^c \nabla_c \mathbb{F}_{ab} + \nabla_c \left(g^{cd} \mathbb{F}_{db} \nabla^r u_a \right) \\ &\quad - \operatorname{sgn}(2-r) \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} \mathbb{F}_{ab} \\ &\quad + \sum_{s=1}^{r} \binom{r}{s} g^{cd} \nabla^s \mathbb{F}_{db} \nabla^{r-s} \nabla_c u_a. \end{aligned} \tag{4.2}$$

From [3, Lemmas 2.1 and 3.9] and (4.1), it follows that

$$\begin{aligned} &D_t \left(g^{bd} \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \right) \\ &= (D_t g^{bd}) \gamma^{af} \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d + r g^{bd} (D_t \gamma^{af}) \gamma^{AF} \nabla_A^{r-1} \nabla_a u_b \nabla_F^{r-1} \nabla_f u_d \\ &\quad + 2 g^{bd} \gamma^{af} \gamma^{AF} D_t (\nabla_A^{r-1} \nabla_a u_b) \nabla_F^{r-1} \nabla_f u_d \end{aligned}$$

$$
\begin{aligned}
&= -2\nabla_c u_e \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^c \nabla_F^{r-1}\nabla_f u^e \\
&\quad - 2r\nabla_c u_e \gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^d \nabla_F^{r-1}\nabla_f u_d \\
&\quad - 2\nabla_b\left(\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b \nabla_A^{r-1}\nabla_a p\right) \qquad (4.3)\\
&\quad + 2\nabla_b\left(\gamma^{af}\gamma^{AF}\right)\nabla_F^{r-1}\nabla_f u^b \nabla_A^{r-1}\nabla_a p \\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b (\mathrm{curl}\, u)_{bc}\nabla_A^{r-1}\nabla_a u^c \\
&\quad + 2\mathrm{sgn}(2-r)\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d \sum_{s=1}^{r-2}\binom{r}{s+1}\left((\nabla^{s+1}u)\cdot\nabla^{r-s}u^d\right)_{Aa} \\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^d \nabla_c\left(\mathbb{F}^{cb}\nabla_A^{r-1}\nabla_a \mathbb{F}_{db}\right) \qquad (4.4)\\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^d \sum_{s=1}^{r}\binom{r}{s}\left(\nabla^s \mathbb{F}^{cb}\nabla^{r-s}\nabla_c \mathbb{F}_{db}\right)_{Aa}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& D_t\left(g^{bd}g^{ce}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}_{bc}\nabla_F^{r-1}\nabla_f \mathbb{F}_{de}\right) \\
&= D_t(g^{bd})g^{ce}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}_{bc}\nabla_F^{r-1}\nabla_f \mathbb{F}_{de} \\
&\quad + g^{bd}D_t(g^{ce})\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}_{bc}\nabla_F^{r-1}\nabla_f \mathbb{F}_{de} \\
&\quad + rD_t(\gamma^{af})\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}_{bc}\nabla_F^{r-1}\nabla_f \mathbb{F}^{bc} \\
&\quad + 2\gamma^{af}\gamma^{AF}D_t(\nabla_A^{r-1}\nabla_a \mathbb{F}_{bc})\nabla_F^{r-1}\nabla_f \mathbb{F}^{bc} \\
&= -2\nabla_k u^m \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}^{kc}\nabla_F^{r-1}\nabla_f \mathbb{F}_{mc} \\
&\quad - 2\nabla_k u^m \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}^{dk}\nabla_F^{r-1}\nabla_f \mathbb{F}_{dm} \\
&\quad - 2r\nabla_d u_e \gamma^{ad}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}^{bc}\nabla_F^{r-1}\nabla_f \mathbb{F}_{bc} \\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}_{ec}\nabla_b u^e \nabla_F^{r-1}\nabla_f \mathbb{F}^{bc} \\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}_{be}\nabla_c u^e \nabla_F^{r-1}\nabla_f \mathbb{F}^{bc} \\
&\quad - 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f \mathbb{F}^{bc}\nabla_A^{r-1}\nabla_a u^e \nabla_e \mathbb{F}_{bc} \\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f \mathbb{F}^{bc}g^{ed}\mathbb{F}_{dc}\nabla_e \nabla_A^{r-1}\nabla_a u_b \qquad (4.5)\\
&\quad + 2\mathrm{sgn}(2-r)\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f \mathbb{F}^{bc}\sum_{s=1}^{r-2}\binom{r}{s+1}\left((\nabla^{1+s}u)\cdot\nabla^{r-s}\mathbb{F}_{bc}\right)_{Aa} \\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f \mathbb{F}^{bc}\sum_{s=1}^{r}\binom{r}{s}\left(g^{ed}\nabla^s \mathbb{F}_{dc}\nabla^{r-s}\nabla_e u_b\right)_{Aa}.
\end{aligned}
$$

For (4.4) and (4.5), one has

$$
\begin{aligned}
(4.4)+(4.5) &= 2\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^b \nabla_e\left(\mathbb{F}^{ec}\nabla_F^{r-1}\nabla_f\mathbb{F}_{bc}\right)\\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_e\nabla_A^{r-1}\nabla_a u^b\mathbb{F}^{ec}\nabla_F^{r-1}\nabla_f\mathbb{F}_{bc}\\
&= 2\nabla_e\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^b\mathbb{F}^{ec}\nabla_F^{r-1}\nabla_f\mathbb{F}_{bc}\right) \qquad (4.6)\\
&\quad - 2\nabla_e(\gamma^{af}\gamma^{AF})\nabla_A^{r-1}\nabla_a u^b\mathbb{F}^{ec}\nabla_F^{r-1}\nabla_f\mathbb{F}_{bc}. \qquad (4.7)
\end{aligned}
$$

The boundary integral stemmed from the integration of (4.6) over Ω will vanish since it involves the term $N_e\mathbb{F}^{ec}$ which is zero on the boundary. Since (3.9), especially the integral involving p, vanishes, we do not need the boundary integral in the first order energy $E_1(t)$. However, the boundary integral derived from the integral of (4.3) over Ω will be out of control for higher order energies. Thus, we have to include a boundary integral to overcome this difficulty.

Define the r-th order energy for an integer $r \geqslant 2$ as

$$
\begin{aligned}
E_r(t) = &\int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\, d\mu_g + \int_\Omega |\nabla^{r-1}\mathrm{curl}\, u|^2 d\mu_g\\
&+ \int_\Omega g^{bd}g^{ce}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a\mathbb{F}_{bc}\nabla_F^{r-1}\nabla_f\mathbb{F}_{de}\, d\mu_g + \int_\Omega |\nabla^{r-1}\mathrm{curl}\,\mathbb{F}^\top|^2 d\mu_g\\
&+ \int_{\partial\Omega} \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a p\nabla_F^{r-1}\nabla_f p\,\vartheta\, d\mu_\gamma,
\end{aligned}
$$

where $\vartheta = 1/(-\nabla_N p)$ as before. Then, we have the following theorem.

Theorem 4.1. *For the integer $r \in \{2, \cdots, n+1\}$, there exists a constant $T > 0$ such that, for any smooth solution to system (2.3) for $0 \leqslant t \leqslant T$ satisfying*

$$|\mathbb{F}| \leqslant M_1 \quad \text{for } r=2, \qquad \text{in } [0,T]\times\Omega, \tag{4.8}$$

$$|\nabla p| \leqslant M, \quad |\nabla u| \leqslant M, \quad |\nabla\mathbb{F}| \leqslant M, \qquad \text{in } [0,T]\times\Omega, \tag{4.9}$$

$$|\theta| + 1/\iota_0 \leqslant K, \qquad \text{on } [0,T]\times\partial\Omega, \tag{4.10}$$

$$-\nabla_N p \geqslant \varepsilon > 0, \qquad \text{on } [0,T]\times\partial\Omega, \tag{4.11}$$

$$|\nabla^2 p| + |\nabla_N D_t p| \leqslant L, \qquad \text{on } [0,T]\times\partial\Omega, \tag{4.12}$$

the following estimate holds for any $t \in [0,T]$,

$$E_r(t) \leqslant e^{C_1 t}E_r(0) + C_2\left(e^{C_1 t} - 1\right), \tag{4.13}$$

where the constants C_1 and C_2 depend on K, K_1, M, M_1, L, $1/\varepsilon$, $\mathrm{Vol}\,\Omega$, $E_0(0)$, $E_1(0)$, $\cdots$, and $E_{r-1}(0)$.

Proof. The derivative of $E_r(t)$ with respect to t is

$$\frac{d}{dt}E_r(t) = \int_\Omega D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b \nabla_F^{r-1}\nabla_f u_d\right)d\mu_g \tag{4.14}$$

$$+\int_\Omega D_t\left(g^{bd}g^{ce}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}_{bc}\nabla_F^{r-1}\nabla_f \mathbb{F}_{de}\right)d\mu_g \tag{4.15}$$

$$+\int_\Omega D_t|\nabla^{r-1}\mathrm{curl}\, u|^2 d\mu_g + \int_\Omega D_t|\nabla^{r-1}\mathrm{curl}\,\mathbb{F}^\top|^2 d\mu_g \tag{4.16}$$

$$+\int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b \nabla_F^{r-1}\nabla_f u_d \mathrm{tr}\, h d\mu_g \tag{4.17}$$

$$+\int_\Omega |\nabla^{r-1}\mathrm{curl}\, u|^2 \mathrm{tr}\, h d\mu_g + \int_\Omega |\nabla^{r-1}\mathrm{curl}\,\mathbb{F}^\top|^2 \mathrm{tr}\, h d\mu_g \tag{4.18}$$

$$+\int_\Omega g^{bd}g^{ce}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \mathbb{F}_{bc}\nabla_F^{r-1}\nabla_f \mathbb{F}_{de}\mathrm{tr}\, h d\mu_g \tag{4.19}$$

$$+\int_{\partial\Omega} D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a p \nabla_F^{r-1}\nabla_f p\right)\vartheta d\mu_\gamma \tag{4.20}$$

$$+\int_{\partial\Omega} \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a p \nabla_F^{r-1}\nabla_f p\left(\frac{\vartheta_t}{\vartheta} + \mathrm{tr}\, h - h_{NN}\right)\vartheta d\mu_\gamma. \tag{4.21}$$

Step 1: Estimate the integrals (4.14), (4.15) and (4.20)

From the previous derivations for the integrands in (4.14) and (4.15), (4.6), (4.7) and

$$D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a p \nabla_F^{r-1}\nabla_f p\right)$$
$$= -2r\nabla_c u_e \gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a p \nabla_F^{r-1}\nabla_f p + 2\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a p D_t\left(\nabla_F^{r-1}\nabla_f p\right),$$

we have

$$(4.14) + (4.15) + (4.20)$$
$$\leqslant C\left(\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\Omega)}\right)E_r(t)$$
$$+ CE_r^{1/2}(t)\sum_{s=1}^{r-2}\left\|\nabla^{s+1}u\right\|_{L^4(\Omega)}\left(\left\|\nabla^{r-s}u\right\|_{L^4(\Omega)} + \left\|\nabla^{r-s}\mathbb{F}\right\|_{L^4(\Omega)}\right) \tag{4.22}$$
$$+ CE_r^{1/2}(t)\sum_{s=2}^{r-1}\left\|\nabla^{s}\mathbb{F}\right\|_{L^4(\Omega)}\left(\left\|\nabla^{r-s+1}u\right\|_{L^4(\Omega)} + \left\|\nabla^{r-s+1}\mathbb{F}\right\|_{L^4(\Omega)}\right) \tag{4.23}$$

$$+2\int_{\partial\Omega} \gamma^{af}\gamma^{AF}\nabla^r_{Aa}p\left(D_t\nabla^r_{Ff}p-\frac{1}{\vartheta}N_b\nabla^r_{Ff}u^b\right)\vartheta d\mu_\gamma \tag{4.24}$$

$$+2\int_{\Omega} \nabla_b\left(\gamma^{af}\gamma^{AF}\right)\nabla^{r-1}_F\nabla_f u^b\nabla^{r-1}_A\nabla_a p d\mu_g \tag{4.25}$$

$$+\int_{\partial\Omega} N_e\gamma^{af}\gamma^{AF}\nabla^{r-1}_A\nabla_a u^b\mathbb{F}^{ec}\nabla^{r-1}_F\nabla_f\mathbb{F}_{bc}d\mu_\gamma \tag{4.26}$$

$$-\int_{\Omega} \nabla_e(\gamma^{af}\gamma^{AF})\nabla^{r-1}_A\nabla_a u^b\mathbb{F}^{ec}\nabla^{r-1}_F\nabla_f\mathbb{F}_{bc}d\mu_g. \tag{4.27}$$

Since $N\cdot\mathbb{F}^\top=0$ on $\partial\Omega$, (4.26) vanishes. From [3, Lemma A.4], we see that, for $\iota_1\geqslant 1/K_1$,

$$\|\mathbb{F}\|_{L^\infty(\Omega)}\leqslant C\sum_{0\leqslant s\leqslant 2}K_1^{n/2-s}\left\|\nabla^s\mathbb{F}\right\|_{L^2(\Omega)}\leqslant C(K_1)\sum_{s=0}^{2}E_s^{1/2}(t). \tag{4.28}$$

Using the Hölder inequality and the assumption (4.9), we obtain for any integers $r\geqslant 3$,

$$|(4.27)|\leqslant CK\,\|\mathbb{F}\|_{L^\infty(\Omega)}E_r(t)\leqslant C(K,K_1)\left(\sum_{s=0}^{2}E_s^{1/2}(t)\right)E_r(t). \tag{4.29}$$

For $r=2$, by (4.8), one has

$$|(4.27)|\leqslant CK\,\|\mathbb{F}\|_{L^\infty(\Omega)}E_r(t)\leqslant C(K,M_1)E_r(t). \tag{4.30}$$

Step 1.1: Estimate (4.25)

From Hölder's inequality, we have

$$|(4.25)|\leqslant CKE_r^{1/2}(t)\left\|\nabla^r p\right\|_{L^2(\Omega)}. \tag{4.31}$$

It follows from (2.3a) and [3, (2.21)] that

$$\Delta p=-\nabla_a u^b\nabla_b u^a+g^{cb}\nabla_a\mathbb{F}_{cd}\nabla_b\mathbb{F}^{ad}. \tag{4.32}$$

Then, for $r\geqslant 2$,

$$\nabla^{r-2}\Delta p=-\sum_{s=0}^{r-2}\binom{r-2}{s}\nabla^s\nabla_a u^b\nabla^{r-2-s}\nabla_b u^a$$
$$+\sum_{s=0}^{r-2}\binom{r-2}{s}g^{cb}\nabla^s\nabla_a\mathbb{F}_{cd}\nabla^{r-2-s}\nabla_b\mathbb{F}^{ad}.$$

In view of (4.28), one has, for $s\geqslant 0$,

$$\left\|\nabla^s \mathbb{F}\right\|_{L^\infty(\Omega)} \leqslant C \sum_{\ell=0}^{2} K_1^{n/2-\ell} \left\|\nabla^{\ell+s}\mathbb{F}\right\|_{L^2(\Omega)} \leqslant C(K_1)\sum_{\ell=0}^{2} E_{s+\ell}^{1/2}(t), \tag{4.33}$$

and

$$\left\|\nabla^s u\right\|_{L^\infty(\Omega)} \leqslant C(K_1)\sum_{\ell=0}^{2} E_{s+\ell}^{1/2}(t). \tag{4.34}$$

From the Hölder inequality, (4.33) and (4.34), we have, for $r \in \{3, 4\}$,

$$\begin{aligned}
&\left\|\nabla^{r-2}\Delta p\right\|_{L^2(\Omega)} \\
&\leqslant C\sum_{s=0}^{r-2}\left\|\nabla^s\nabla_a u^b \nabla^{r-2-s}\nabla_b u^a\right\|_{L^2(\Omega)} \\
&\quad + C\sum_{s=0}^{r-2}\left\|g^{cb}\nabla^s\nabla_a\mathbb{F}_{cd}\nabla^{r-2-s}\nabla_b\mathbb{F}^{ad}\right\|_{L^2(\Omega)} \\
&\leqslant C\left\|\nabla u\right\|_{L^\infty(\Omega)}\left\|\nabla^{r-1}u\right\|_{L^2(\Omega)} + C\left\|\nabla\mathbb{F}\right\|_{L^\infty(\Omega)}\left\|\nabla^{r-1}\mathbb{F}\right\|_{L^2(\Omega)} \\
&\quad + (r-3)C\left(\left\|\nabla^2 u\right\|_{L^\infty(\Omega)}\left\|\nabla^2 u\right\|_{L^2(\Omega)} + \left\|\nabla^2\mathbb{F}\right\|_{L^\infty(\Omega)}\left\|\nabla^2\mathbb{F}\right\|_{L^2(\Omega)}\right) \\
&\leqslant C(K_1)\sum_{\ell=1}^{r-1}E_\ell(t) + C(K_1)E_2^{1/2}(t)E_r^{1/2}(t).
\end{aligned} \tag{4.35}$$

For the case $r = 2$, we have the following estimate from the assumption (4.9) and the Hölder inequality:

$$\|\Delta p\|_{L^2(\Omega)} \leqslant C\|\nabla u\|_{L^2(\Omega)}\|\nabla u\|_{L^\infty(\Omega)} + C\|\nabla\mathbb{F}\|_{L^2(\Omega)}\|\nabla\mathbb{F}\|_{L^\infty(\Omega)} \leqslant CME_1^{1/2}(t), \tag{4.36}$$

which is a lower order energy term. Hence, from [3, (5.29)], (4.35) and (4.36), we have for any real number $\delta_r > 0$,

$$\begin{aligned}
\left\|\nabla^r p\right\|_{L^2(\Omega)} &\leqslant \delta_r\left\|\Pi\nabla^r p\right\|_{L^2(\partial\Omega)} + C(1/\delta_r, K, \operatorname{Vol}\Omega)\sum_{s\leqslant r-2}\left\|\nabla^s\Delta p\right\|_{L^2(\Omega)} \\
&\leqslant \delta_r\left\|\Pi\nabla^r p\right\|_{L^2(\partial\Omega)} + C(1/\delta_r, K, K_1, M, \operatorname{Vol}\Omega)\sum_{\ell=1}^{r-1}E_\ell(t) \\
&\quad + (r-2)C(1/\delta_r, K, K_1, M, \operatorname{Vol}\Omega)E_2^{1/2}(t)E_r^{1/2}(t).
\end{aligned} \tag{4.37}$$

Next, we estimate the boundary terms. Because $p = 0$ on the boundary $\partial\Omega$, from [3, (5.30)], we obtain for $r \geqslant 1$,

$$\left\|\Pi\nabla^r p\right\|_{L^2(\partial\Omega)} \leqslant C(K,K_1)\Big(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2)\sum_{k\leqslant r-3}\left\|\overline{\nabla}^k\theta\right\|_{L^2(\partial\Omega)}\Big)$$
$$\times \sum_{k\leqslant r-1}\left\|\nabla^k p\right\|_{L^2(\partial\Omega)}. \tag{4.38}$$

Due to [3, (4.20)], we have $\Pi\nabla^2 p = \theta\nabla_N p$. From (4.11), (4.10), [3, (A.20)], (4.9) and (4.37), we obtain

$$\|\theta\|_{L^2(\partial\Omega)} = \left\|\frac{\Pi\nabla^2 p}{\nabla_N p}\right\|_{L^2(\partial\Omega)} \leqslant \frac{1}{\varepsilon}\left\|\Pi\nabla^2 p\right\|_{L^2(\partial\Omega)}, \tag{4.39}$$

and

$$\begin{aligned}
\left\|\Pi\nabla^2 p\right\|_{L^2(\partial\Omega)} &\leqslant \|\theta\|_{L^\infty(\partial\Omega)}\|\nabla p\|_{L^2(\partial\Omega)} \\
&\leqslant C(K,\mathrm{Vol}\,\Omega)\left(\left\|\nabla^2 p\right\|_{L^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)}\right) \\
&\leqslant C(K,\mathrm{Vol}\,\Omega)\delta_2\left\|\Pi\nabla^2 p\right\|_{L^2(\partial\Omega)} + C(K,\mathrm{Vol}\,\Omega)(\mathrm{Vol}\,\Omega)^{1/2}M \\
&\quad + C(1/\delta_2,K,K_1,M,\mathrm{Vol}\,\Omega)E_1(t),
\end{aligned} \tag{4.40}$$

where the first term on the right hand side of (4.40) can be absorbed by the left hand side if we take δ_2 small such that $C(K,\mathrm{Vol}\,\Omega)\delta_2 \leqslant 1/2$. Then,

$$\left\|\Pi\nabla^2 p\right\|_{L^2(\partial\Omega)} + \left\|\nabla^2 p\right\|_{L^2(\Omega)} \leqslant C(K,K_1,M,\mathrm{Vol}\,\Omega)(1+E_1(t)), \tag{4.41}$$

$$\|\theta\|_{L^2(\partial\Omega)} \leqslant C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon)(1+E_1(t)). \tag{4.42}$$

From Theorem 3.1, there exists a constant $T>0$ such that $E_1(t)$ can be controlled by the initial energy $E_1(0)$ for $t\in[0,T]$, e.g., $E_1(t)\leqslant 2E_1(0)$. Then, from (4.38), (4.42), (4.9) and (4.41), we get

$$\begin{aligned}
\left\|\Pi\nabla^3 p\right\|_{L^2(\partial\Omega)} &\leqslant C(K,K_1)\left(K+\|\theta\|_{L^2(\partial\Omega)}\right)\sum_{k\leqslant 2}\left\|\nabla^k p\right\|_{L^2(\partial\Omega)} \\
&\leqslant C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon,E_1(0))\left\|\nabla^3 p\right\|_{L^2(\Omega)} \\
&\quad + C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon,E_1(0)).
\end{aligned}$$

It follows from (4.37) that

$$\begin{aligned}
\left\|\nabla^3 p\right\|_{L^2(\Omega)} &\leqslant \delta_3 C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon,E_1(0))\left\|\nabla^3 p\right\|_{L^2(\Omega)} \\
&\quad + \delta_3 C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon,E_1(0)) \\
&\quad + C(1/\delta_3,K,K_1,M,\mathrm{Vol}\,\Omega)(E_1(t)+E_2(t)) \\
&\quad + C(1/\delta_3,K,K_1,M,\mathrm{Vol}\,\Omega)E_2^{1/2}(t)E_3^{1/2}(t),
\end{aligned}$$

which, if we take $\delta_3 > 0$ so small that $\delta_3 C(K, K_1, M, \mathrm{Vol}\,\Omega, 1/\varepsilon, E_1(0)) \leqslant 1/2$, implies

$$\left\|\nabla^3 p\right\|_{L^2(\Omega)} \leqslant C(K, K_1, M, \mathrm{Vol}\,\Omega, 1/\varepsilon, E_1(0)) + C(K, K_1, M, \mathrm{Vol}\,\Omega) \sum_{\ell=1}^{2} E_\ell(t) + C(K, K_1, M, \mathrm{Vol}\,\Omega) E_2^{1/2}(t) E_3^{1/2}(t), \tag{4.43}$$

and thus

$$\left\|\Pi\nabla^3 p\right\|_{L^2(\partial\Omega)} \leqslant C(K, K_1, M, \mathrm{Vol}\,\Omega, 1/\varepsilon, E_1(0)) \left(1 + \sum_{\ell=1}^{2} E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t)\right).$$

Because

$$\begin{aligned}\overline{\nabla}_b \nabla_N p &= \gamma_b^d \nabla_d (N^a \nabla_a p) = (\delta_b^d - N_b N^d)((\nabla_d N^a)\nabla_a p + N^a \nabla_d \nabla_a p) \\ &= \theta_b^a \nabla_a p + N^a \nabla_b \nabla_a p - N_b N^d (\theta_d^a \nabla_a p + N^a \nabla_d \nabla_a p),\end{aligned}$$

we have from [3, (A.20)] that

$$\begin{aligned}&\left\|\overline{\nabla}\nabla_N p\right\|_{L^2(\partial\Omega)} \\ &\leqslant C \|\theta\|_{L^\infty(\partial\Omega)} \|\nabla p\|_{L^2(\partial\Omega)} + C \left\|\nabla^2 p\right\|_{L^2(\partial\Omega)} \\ &\leqslant C(K, \mathrm{Vol}\,\Omega) \left(\left\|\nabla^3 p\right\|_{L^2(\Omega)} + \left\|\nabla^2 p\right\|_{L^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)}\right) \\ &\leqslant C(K, K_1, M, \mathrm{Vol}\,\Omega, 1/\varepsilon, E_1(0)) + C(K, K_1, M, \mathrm{Vol}\,\Omega) \sum_{\ell=1}^{2} E_\ell(t) \\ &\quad + C(K, K_1, M, \mathrm{Vol}\,\Omega) E_2^{1/2}(t) E_3^{1/2}(t).\end{aligned}$$

Then, by [3, (4.21)], we have

$$(\overline{\nabla}\theta)\nabla_N p = \Pi\nabla^3 p - 3\theta \tilde{\otimes} \overline{\nabla}\nabla_N p$$

and

$$\begin{aligned}\left\|\overline{\nabla}\theta\right\|_{L^2(\partial\Omega)} &\leqslant \frac{1}{\varepsilon} \left(\left\|\Pi\nabla^3 p\right\|_{L^2(\partial\Omega)} + C \|\theta\|_{L^\infty(\partial\Omega)} \left\|\overline{\nabla}\nabla_N p\right\|_{L^2(\partial\Omega)}\right) \\ &\leqslant C(K, K_1, M, \mathrm{Vol}\,\Omega, 1/\varepsilon, E_1(0)) \Big(1 + \sum_{\ell=1}^{2} E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t)\Big).\end{aligned}$$

From (4.38) and [3, (A.20)],

$$\begin{aligned}&\left\|\Pi\nabla^4 p\right\|_{L^2(\partial\Omega)}\\&\leqslant C(K,K_1)\left(K+\|\theta\|_{L^2(\partial\Omega)}+\left\|\overline{\nabla}\theta\right\|_{L^2(\partial\Omega)}\right)\sum_{k\leqslant 4}\left\|\nabla^k p\right\|_{L^2(\Omega)}.\end{aligned} \tag{4.44}$$

Thus, by (4.37) we can absorb the highest order term $\left\|\nabla^4 p\right\|_{L^2(\Omega)}$ by the left hand side for $\delta_4>0$ small enough which is independent of the highest order energy $E_4(t)$, and

$$\begin{aligned}&\left\|\nabla^4 p\right\|_{L^2(\Omega)}+\left\|\Pi\nabla^4 p\right\|_{L^2(\partial\Omega)}\\&\leqslant C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon,E_1(0))\Big(1+\sum_{\ell=1}^{3}E_\ell(t)+E_2^{1/2}(t)E_4^{1/2}(t)\Big).\end{aligned}$$

Hence, from (4.41), (4.43) and (4.44), we have for $r\geqslant 2$,

$$\begin{aligned}&\left\|\nabla^r p\right\|_{L^2(\Omega)}\\&\leqslant C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon,E_1(0))\Big(1+\sum_{\ell=1}^{r-1}E_\ell(t)+(r-2)E_2^{1/2}(t)E_r^{1/2}(t)\Big),\end{aligned}$$

which, from (4.31), yields

$$\begin{aligned}|(4.25)|\leqslant{}& C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon,E_1(0))E_r^{1/2}(t)\\&\cdot\Big(1+\sum_{\ell=1}^{r-1}E_\ell(t)+(r-2)E_2^{1/2}(t)E_r^{1/2}(t)\Big).\end{aligned}$$

Step 1.2: Estimate (4.24)

The boundary condition $p=0$ on $\partial\Omega$ implies $\gamma_b^a\nabla_a p=0$ on $\partial\Omega$. Then we have, from [3, (3.4)] and $\vartheta=-1/\nabla_N p$,

$$-\vartheta^{-1}N_b=\nabla_N pN_b=N^a\nabla_a pN_b=\delta_b^a\nabla_a p-\gamma_b^a\nabla_a p=\nabla_b p. \tag{4.45}$$

From the Hölder inequality and (4.45), we get

$$\begin{aligned}|(4.24)|&\leqslant C\left\|\vartheta\right\|_{L^\infty(\partial\Omega)}^{1/2}E_r^{1/2}(t)\left\|\Pi\left(D_t\left(\nabla^r p\right)-\vartheta^{-1}N_b\nabla^r u^b\right)\right\|_{L^2(\partial\Omega)}\\&=C\left\|\vartheta\right\|_{L^\infty(\partial\Omega)}^{1/2}E_r^{1/2}(t)\left\|\Pi\left(D_t\left(\nabla^r p\right)+\nabla^r u\cdot\nabla p\right)\right\|_{L^2(\partial\Omega)}.\end{aligned} \tag{4.46}$$

It follows from [3, (2.23)] that

$$\begin{aligned}D_t\nabla^r p+\nabla^r u\cdot\nabla p&=[D_t,\nabla^r]p+\nabla^r D_t p+\nabla^r u\cdot\nabla p\\&=\mathrm{sgn}(2-r)\sum_{s=1}^{r-2}\binom{r}{s+1}(\nabla^{s+1}u)\cdot\nabla^{r-s}p+\nabla^r D_t p.\end{aligned} \tag{4.47}$$

Now, we consider the last term in (4.47). From [3, (5.30) and (A.20)], we have, for $2 \leqslant r \leqslant 4$,

$$\begin{aligned} &\left\| \Pi \nabla^r D_t p \right\|_{L^2(\partial\Omega)} \\ &\leqslant C(K, K_1, \mathrm{Vol}\,\Omega) \left(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2) \sum_{k \leqslant r-3} \left\| \overline{\nabla}^k \theta \right\|_{L^2(\partial\Omega)} \right) \\ &\quad \cdot \sum_{k \leqslant r} \left\| \nabla^k D_t p \right\|_{L^2(\Omega)} . \end{aligned} \tag{4.48}$$

It follows from [3, (5.29)] that

$$\begin{aligned} \left\| \nabla^r D_t p \right\|_{L^2(\Omega)} \leqslant{}& \delta \left\| \Pi \nabla^r D_t p \right\|_{L^2(\partial\Omega)} \\ &+ C(1/\delta, K, \mathrm{Vol}\,\Omega) \sum_{s \leqslant r-2} \left\| \nabla^s \Delta D_t p \right\|_{L^2(\Omega)} . \end{aligned} \tag{4.49}$$

From [3, (2.22)], (4.32), [3, Lemma 2.1], (3.1), (3.2) and (2.3), it follows that

$$\begin{aligned} \Delta D_t p ={}& 2 g^{ac} \nabla_c u^b \nabla_a \nabla_b p + (\Delta u^e) \nabla_e p + 2 \nabla_e u^b \nabla_b u^a \nabla_a u^e \\ &- 2 g^{ce} \nabla_e u^b \nabla_a \mathbb{F}_{cd} \nabla_b \mathbb{F}^{ad} - 2 \nabla_d u_f \nabla_a \mathbb{F}^{bd} \nabla_b \mathbb{F}^{af} \\ &+ 2 g^{ce} \nabla_c \mathbb{F}^{ad} \nabla_a \mathbb{F}_{eb} \nabla_d u^b - 2 g^{bd} \nabla_b u^a \nabla_a \nabla_c \mathbb{F}_{de} \mathbb{F}^{ce} \\ &+ 2 g^{ce} \nabla_b \mathbb{F}^{ad} \mathbb{F}_{ed} \nabla_a \nabla_c u^b . \end{aligned}$$

From (4.33), (4.37) and [3, Lemma A.4], it implies that, for $s \leqslant 2$,

$$\begin{aligned} &\left\| \nabla^s \Delta D_t p \right\|_{L^2(\Omega)} \\ &\leqslant C \|\nabla u\|_{L^\infty(\Omega)} \left\| \nabla^{s+2} p \right\|_{L^2(\Omega)} + s(s-1) C \left\| \nabla^3 u \right\|_{L^2(\Omega)} \left\| \nabla^2 p \right\|_{L^\infty(\Omega)} \\ &\quad + s C \left\| \nabla^2 u \right\|_{L^4(\Omega)} \left\| \nabla^{s+1} p \right\|_{L^4(\Omega)} + C \left\| \nabla^{s+2} u \right\|_{L^2(\Omega)} \|\nabla p\|_{L^\infty(\Omega)} \\ &\quad + C \left(\|\nabla u\|_{L^\infty(\Omega)} \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \right) \left\| \nabla^{s+1} u \right\|_{L^2(\Omega)} \\ &\quad + s(s-1) C \|\nabla u\|_{L^\infty(\Omega)} \left\| \nabla^2 u \right\|_{L^4(\Omega)} \left\| \nabla^2 u \right\|_{L^4(\Omega)} \\ &\quad + C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \left\| \nabla^{s+1} \mathbb{F} \right\|_{L^2(\Omega)} \\ &\quad + s C \left\| \nabla^2 u \right\|_{L^4(\Omega)} \left\| \nabla^2 \mathbb{F} \right\|_{L^4(\Omega)} \left((s-1) \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} + \|\mathbb{F}\|_{L^\infty(\Omega)} \right) \\ &\quad + s(s-1) C \|\nabla u\|_{L^\infty(\Omega)} \left\| \nabla^2 \mathbb{F} \right\|_{L^4(\Omega)} \left\| \nabla^2 \mathbb{F} \right\|_{L^4(\Omega)} \\ &\quad + C \|\nabla u\|_{L^\infty(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \left\| \nabla^{s+2} \mathbb{F} \right\|_{L^2(\Omega)} \end{aligned}$$

$$+ sC \left\| \nabla^3 u \right\|_{L^2(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \left((s-1) \left\| \nabla^2 \mathbb{F} \right\|_{L^\infty(\Omega)} + \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \right)$$
$$+ s(s-1)C \left\| \nabla^3 \mathbb{F} \right\|_{L^2(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \left\| \nabla^2 u \right\|_{L^\infty(\Omega)}$$
$$+ s(s-1)C \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \left\| \nabla^2 \mathbb{F} \right\|_{L^4(\Omega)} \left\| \nabla^2 u \right\|_{L^4(\Omega)}$$
$$+ s(s-1)C \|\nabla \mathbb{F}\|_{L^\infty(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \left\| \nabla^4 u \right\|_{L^2(\Omega)}$$
$$+ s(s-1)C \left\| \nabla^2 \mathbb{F} \right\|_{L^\infty(\Omega)} \|\mathbb{F}\|_{L^\infty(\Omega)} \left\| \nabla^3 u \right\|_{L^2(\Omega)}.$$

In view of [3, Lemma A.3] and (4.34), the following holds

$$\left\| \nabla^{s+1} u \right\|_{L^4(\Omega)} \leqslant C \left\| \nabla^s u \right\|_{L^\infty(\Omega)}^{1/2} \left(\sum_{\ell=0}^{2} \left\| \nabla^{s+\ell} u \right\|_{L^2(\Omega)} K_1^{2-\ell} \right)^{1/2}$$
$$\leqslant C(K_1) \sum_{\ell=0}^{2} E_{s+\ell}^{1/2}(t).$$

We can estimate all the terms with $L^4(\Omega)$ norms in the same way in view of (4.33), (4.34), the similar estimate of p and the assumptions. Hence, we obtain the bound which is linear with respect to the highest-order derivative or the highest-order energy $E_r^{1/2}(t)$, i.e.,

$$\left\| \nabla^s \Delta D_t p \right\|_{L^2(\Omega)}$$
$$\leqslant C(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0)) \Big(1 + \sum_{\ell=0}^{r-1} E_\ell(t) \Big) \big(1 + E_r^{1/2}(t) \big). \tag{4.50}$$

Therefore, by (4.48), (4.49), (4.50) and for some small δ independent of $E_r(t)$, we obtain, by the induction argument for r,

$$\left\| \Pi \nabla^r D_t p \right\|_{L^2(\partial\Omega)} \leqslant C(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0))$$
$$\cdot \Big(1 + \sum_{\ell=0}^{r-1} E_\ell(t) \Big) \big(1 + E_r^{1/2}(t) \big). \tag{4.51}$$

For the estimate of (4.47), it only remains to estimate

$$\left\| \Pi \left((\nabla^{s+1} u) \cdot \nabla^{r-s} p \right) \right\|_{L^2(\partial\Omega)} \quad \text{for } 1 \leqslant s \leqslant r-2.$$

For the cases $r = 3, 4$ and $s = r - 2$, we have, from (4.12) and [3, Lemma A.7], that

$$
\begin{aligned}
&\left\|\Pi\left((\nabla^{r-1}u)\cdot\nabla^2 p\right)\right\|_{L^2(\partial\Omega)}\\
&\leqslant \left\|\nabla^{r-1}u\right\|_{L^2(\partial\Omega)}\left\|\nabla^2 p\right\|_{L^\infty(\partial\Omega)}\leqslant CL\left\|\nabla^2 u\right\|_{L^{2(n-1)/(n-2)}(\partial\Omega)}\\
&\leqslant C(K,\mathrm{Vol}\,\Omega)L\left(\left\|\nabla^r u\right\|_{L^2(\Omega)}+\left\|\nabla^{r-1}u\right\|_{L^2(\Omega)}\right)\\
&\leqslant C(K,L,\mathrm{Vol}\,\Omega)\left(E_{r-1}^{1/2}(t)+E_r^{1/2}(t)\right).
\end{aligned}
$$

For the cases $n=3$, $r=4$ and $s=1$, from [3, (4.48)], [3, Lemma A.7] and (4.37), we have

$$
\begin{aligned}
&\left\|\Pi\left((\nabla^2 u)\cdot\nabla^3 p\right)\right\|_{L^2(\partial\Omega)}\\
&=\left\|\Pi\nabla^2 u\cdot\Pi\nabla^3 p+\Pi(\nabla^2 u\cdot N)\tilde{\otimes}\Pi(N\cdot\nabla^3 p)\right\|_{L^2(\partial\Omega)}\\
&\leqslant C\left\|\Pi\nabla^2 u\right\|_{L^4(\partial\Omega)}\left\|\Pi\nabla^3 p\right\|_{L^4(\partial\Omega)}+C\left\|\Pi(N^a\nabla^2 u_a)\right\|_{L^4(\partial\Omega)}\left\|\Pi(\nabla_N\nabla^2 p)\right\|_{L^4(\partial\Omega)}\\
&\leqslant C\left\|\nabla^2 u\right\|_{L^4(\partial\Omega)}\left\|\nabla^3 p\right\|_{L^4(\partial\Omega)}\\
&\leqslant C(K,\mathrm{Vol}\,\Omega)\left(\left\|\nabla^3 u\right\|_{L^2(\Omega)}+\left\|\nabla^2 u\right\|_{L^2(\Omega)}\right)\left(\left\|\nabla^4 p\right\|_{L^2(\Omega)}+\left\|\nabla^3 p\right\|_{L^2(\Omega)}\right)\\
&\leqslant C(K,K_1,\mathrm{Vol}\,\Omega)(E_3^{1/2}(t)+E_2^{1/2}(t))\left(\sum_{s=0}^{3}E_s(t)+\left(\sum_{\ell=0}^{2}E_\ell^{1/2}(t)\right)E_4^{1/2}(t)\right)\\
&\leqslant C(K,K_1,\mathrm{Vol}\,\Omega)\sum_{s=0}^{3}E_s(t)\sum_{\ell=0}^{4}E_\ell^{1/2}(t).
\end{aligned}
$$

Thus, we get

$$
|(4.24)|\leqslant C(K,K_1,M,M_1,L,1/\varepsilon,\mathrm{Vol}\,\Omega,E_0(0))\Big(1+\sum_{s=0}^{r-1}E_s(t)\Big)\big(1+E_r(t)\big).
$$

From [3, Lemma A.3], it follows that

$$
|(4.22)+(4.23)|\leqslant C(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon)\Big(1+\sum_{s=0}^{r-1}E_s(t)\Big)E_r(t).
$$

Therefore, we have shown that

$$
\begin{aligned}
&|(4.14)+(4.15)+(4.20)|\\
&\leqslant C(K,K_1,M,M_1,L,1/\varepsilon,\mathrm{Vol}\,\Omega,E_0(0))\Big(1+\sum_{s=0}^{r-1}E_s(t)\Big)\big(1+E_r(t)\big).
\end{aligned}
$$

Step 2: Estimate (4.16)–(4.19) and (4.21)

From [3, Lemma 2.1], (4.1) and (4.2), we have

$$
\begin{aligned}
& D_t\left(|\nabla^{r-1}\operatorname{curl} u|^2+|\nabla^{r-1}\operatorname{curl}\mathbb{F}^\top|^2\right)\\
=& -2(r+1)g^{ae}\nabla_e u^c g^{bd} g^{AF}\nabla_A^{r-1}(\operatorname{curl} u)_{ab}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}\\
& -2(r+1)g^{ae}\nabla_e u^c g^{bd} g^{ef} g^{AF}\nabla_A^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{abe}\nabla_F^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\\
& +2g^{ac}g^{bd}g^{es}\nabla_s u^f g^{AF}\nabla_A^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{abe}\nabla_F^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\\
& +4g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}(\operatorname{curl} u)_{be}\nabla_{Aa}^r u^e\\
& +4\operatorname{sgn}(2-r)g^{ac}g^{AF}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}\sum_{s=1}^{r-2}\binom{r}{s+1}\left((\nabla^{1+s}u)\cdot\nabla^{r-s}u^d\right)_{Aa}\\
& +4\operatorname{sgn}(2-r)g^{ac}g^{AF}\nabla_F^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\sum_{s=1}^{r-2}\binom{r}{s+1}\left((\nabla^{1+s}u)\cdot\nabla^{r-s}\mathbb{F}^{df}\right)_{Aa}\\
& +4g^{ac}g^{bd}g^{ef}g^{AF}\nabla_F^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\nabla_{Aa}^r\mathbb{F}_{se}\nabla_b u^s\\
& -4g^{ac}g^{bd}g^{ef}g^{AF}\nabla_F^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\nabla_{Aa}^r u^s\nabla_s\mathbb{F}_{be}\\
& +4g^{ac}g^{bd}g^{ef}g^{AF}\nabla_F^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\nabla_e u^s\nabla_{Aa}^r\mathbb{F}_{bs}\\
& +4\nabla_f\left(g^{ac}g^{bd}g^{AF}\mathbb{F}^{fe}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}\nabla_{Aa}^r\mathbb{F}_{be}\right)\\
& +4g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\operatorname{curl} u)_{cd}\sum_{s=1}^{r}\binom{r}{s}\left(\nabla^s\mathbb{F}^{ef}\nabla^{r-s}\nabla_e\mathbb{F}^{bf}\right)_{Aa}\\
& +4g^{ac}g^{AF}\nabla_F^{r-1}(\operatorname{curl}\mathbb{F}^\top)_{cdf}\sum_{s=1}^{r}\binom{r}{s}\left(\nabla^s\mathbb{F}^{bf}\nabla^{r-s}\nabla_b u^d\right)_{Aa}.
\end{aligned}
$$

Since $N\cdot\mathbb{F}^\top=0$ on $\partial\Omega$, by the Hölder inequality and the Gauss formula, we have

$$
(4.16)\leqslant C(K,K_1,M,\operatorname{Vol}\Omega,1/\varepsilon)\left(1+\sum_{s=0}^{r-1}E_s(t)\right)E_r(t). \tag{4.52}
$$

From [3, (3.21)–(3.22)] and [3, (2.22)], we get

$$
D_t(\nabla_N p)=-2h_d^a N^d\nabla_a p+h_{NN}\nabla_N p+\nabla_N D_t p,
$$

which implies

$$
\frac{\vartheta_t}{\vartheta}=-\frac{D_t\nabla_N p}{\nabla_N p}=\frac{2h_d^a N^d\nabla_a p}{\nabla_N p}-h_{NN}+\frac{\nabla_N D_t p}{\nabla_N p}. \tag{4.53}
$$

Hence, (4.21) can be controlled by $C(K, M, L, 1/\varepsilon)E_r(t)$. The remaining integrals (4.17), (4.18) and (4.19) vanish due to the fact $\operatorname{tr} h = 0$.

Therefore, we have

$$\frac{d}{dt}E_r(t) \leqslant C(K, K_1, M, M_1, L, 1/\varepsilon, \operatorname{Vol}\Omega, E_0(0))\Big(1 + \sum_{s=0}^{r-1} E_s(t)\Big)\big(1 + E_r(t)\big), \tag{4.54}$$

which implies the desired result (4.13) by the Gronwall inequality and the induction argument for $r \in \{2, \cdots, n+1\}$. □

5. Justification of a priori assumptions

In the derivation of the higher order energy estimates in Section 4, some *a priori* assumptions are made. In this section we shall justify these *a priori* assumptions.

Denote

$$\mathcal{K}(t) = \max\left(\|\theta(t,\cdot)\|_{L^\infty(\partial\Omega)}, 1/\iota_0(t)\right),$$
$$\mathcal{E}(t) = \|1/(\nabla_N p(t,\cdot))\|_{L^\infty(\partial\Omega)}, \quad \varepsilon(t) = \frac{1}{\mathcal{E}(t)}. \tag{5.1}$$

As in [3, Definition 3.5], let $0 < \varepsilon_1 < 2$ be a fixed number, take $\iota_1 = \iota_1(\varepsilon_1)$ to be the largest number such that $|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leqslant \varepsilon_1$ whenever $|\bar{x}_1 - \bar{x}_2| \leqslant \iota_1$ for $\bar{x}_1, \bar{x}_2 \in \partial\mathscr{D}_t$.

Lemma 5.1. *Let $K_1 \geqslant 1/\iota_1$. Then there are continuous functions G_j, $j = 1, 2, 3, 4$, such that*

$$\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla\mathbb{F}\|_{L^\infty(\Omega)} + \|\mathbb{F}\|_{L^\infty(\Omega)} \leqslant G_1(K_1, E_0, \cdots, E_{n+1}), \tag{5.2}$$
$$\|\nabla p\|_{L^\infty(\Omega)} + \left\|\nabla^2 p\right\|_{L^\infty(\partial\Omega)} \leqslant G_2(K_1, \mathcal{E}, E_0, \cdots, E_{n+1}, \operatorname{Vol}\Omega), \tag{5.3}$$
$$\|\theta\|_{L^\infty(\partial\Omega)} \leqslant G_3(K_1, \mathcal{E}, E_0, \cdots, E_{n+1}, \operatorname{Vol}\Omega), \tag{5.4}$$
$$\|\nabla D_t p\|_{L^\infty(\partial\Omega)} \leqslant G_4(K_1, \mathcal{E}, E_0, \cdots, E_{n+1}, \operatorname{Vol}\Omega). \tag{5.5}$$

Proof. The estimate (5.2) follows from (4.34), (4.33) and (4.28). By [3, Lemmas A.4 and A.2], we obtain

$$\|\nabla p\|_{L^\infty(\Omega)} \leqslant C(K_1)\sum_{\ell=0}^{2}\left\|\nabla^{\ell+1} p\right\|_{L^2(\Omega)}, \tag{5.6}$$

and

$$\left\|\nabla^2 p\right\|_{L^\infty(\partial\Omega)} \leqslant C(K_1)\sum_{\ell=0}^{n+1}\left\|\nabla^{\ell} p\right\|_{L^2(\partial\Omega)}. \tag{5.7}$$

Thus, the estimate (5.3) follows from (5.6), (5.7), [3, Lemmas A.5–A.7], (4.36), (4.41) and (4.43). Since $|\nabla^2 p| \geqslant |\Pi\nabla^2 p| = |\nabla_N p||\theta| \geqslant \mathcal{E}^{-1}|\theta|$ in view of [3, (4.20)], the estimate (5.4) follows from (5.3). The estimate (5.5) follows from [3, Lemma A.2], (4.49), (4.50) and (4.51). □

Lemma 5.2. *Let $K_1 \geqslant 1/\iota_1$. Then we have*

$$\left|\frac{d}{dt}E_r\right| \leqslant C_r(K_1, \mathcal{E}, E_0, \cdots, E_{n+1}, \mathrm{Vol}\,\Omega)\sum_{s=0}^{r} E_s, \tag{5.8}$$

and

$$\left|\frac{d}{dt}\mathcal{E}\right| \leqslant C_r(K_1, \mathcal{E}, E_0, \cdots, E_{n+1}, \mathrm{Vol}\,\Omega). \tag{5.9}$$

Proof. The first estimate (5.8) follows immediately from Lemma 5.1 and the proof of Theorems 3.1 and 4.1. The second estimate follows from

$$\left|\frac{d}{dt}\left\|\frac{1}{-\nabla_N p(t,\cdot)}\right\|_{L^\infty(\partial\Omega)}\right| \leqslant C\left\|\frac{1}{-\nabla_N p(t,\cdot)}\right\|^2_{L^\infty(\partial\Omega)}\|\nabla_N D_t p(t,\cdot)\|_{L^\infty(\partial\Omega)}$$

and (5.5). □

As a consequence of Lemma 5.2, we have the following result:

Lemma 5.3. *There exists a continuous function $\mathscr{T} > 0$ depending on K_1, $\mathcal{E}(0)$, $E_0(0)$, $\cdots$, $E_{n+1}(0)$ and $\mathrm{Vol}\,\Omega$ such that for*

$$0 \leqslant t \leqslant \mathscr{T}(K_1, \mathcal{E}(0), E_0(0), \cdots, E_{n+1}(0), \mathrm{Vol}\,\Omega), \tag{5.10}$$

one has

$$E_s(t) \leqslant 2E_s(0), \quad 0 \leqslant s \leqslant n+1, \quad \mathcal{E}(t) \leqslant 2\mathcal{E}(0). \tag{5.11}$$

Furthermore,

$$\frac{1}{2}g_{ab}(0,y)Y^aY^b \leqslant g_{ab}(t,y)Y^aY^b \leqslant 2g_{ab}(0,y)Y^aY^b, \tag{5.12}$$

and

$$|\mathcal{N}(x(t,\bar{y})) - \mathcal{N}(x(0,\bar{y}))| \leqslant \frac{\varepsilon_1}{16}, \qquad \bar{y} \in \partial\Omega, \tag{5.13}$$

$$|x(t,y) - x(t,y)| \leqslant \frac{\iota_1}{16}, \qquad y \in \Omega, \tag{5.14}$$

$$\left|\frac{\partial x(t,\bar{y})}{\partial y} - \frac{\partial(0,\bar{y})}{\partial y}\right| \leqslant \frac{\varepsilon_1}{16}, \qquad \bar{y} \in \partial\Omega. \tag{5.15}$$

Proof. Since the proofs are standard, we omit the details. One can refer to [3, Lemma 7.8] and [8, Lemma 6.3]. □

As a consequence of (5.13), (5.14) and the triangle inequality, we have the following result:

Lemma 5.4. *Let $\mathscr{T}$ be as in Lemma 5.3. There exists some $\iota_1 > 0$ such that, if*

$$|\mathcal{N}(x(0, y_1)) - \mathcal{N}(x(0, y_2))| \leqslant \frac{\varepsilon_1}{2}$$

when $|x(0, y_1) - x(0, y_2)| \leqslant 2\iota_1$, then

$$|\mathcal{N}(x(t, y_1)) - \mathcal{N}(x(t, y_2))| \leqslant \varepsilon_1$$

when $|x(t, y_1) - x(t, y_2)| \leqslant 2\iota_1$.

Lemmas 5.3 and 5.4 yield immediately the main Theorem 1.1.

Acknowledgments

The authors would like to thank Professor Tao Luo for some helpful discussions. C. Hao's research was partially supported by the National Science Foundation of China under Grants 11171327, 11461161007 and 11501323, and by the Youth Innovation Promotion Association, Chinese Academy of Sciences. D. Wang's research was supported in part by the National Science Foundation under Grant DMS-1312800.

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