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On the Motion of Free Interface in Ideal Incompressible MHD

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Abstract

For the free boundary problem of the plasma–vacuum interface to 3D ideal incompressible magnetohydrodynamics, the a priori estimates of smooth solutions are proved in Sobolev norms by adopting a geometrical point of view and some quantities such as the second fundamental form and the velocity of the free interface are estimated. In the vacuum region, the magnetic fields are described by the div–curl system of pre-Maxwell dynamics, while at the interface the total pressure is continuous and the magnetic fields are tangential to the interface, but we do not need any restrictions on the size of the magnetic fields on the free interface. We introduce the "fictitious particle" endowed with a fictitious velocity field in vacuum to reformulate the problem to a fixed boundary problem under the Lagrangian coordinates. The L^2-norms of any order covariant derivatives of the magnetic fields both in vacuum and on the boundaries are bounded in terms of initial data and the second fundamental forms of the free interface and the rigid wall. The estimates of the curl of the electric fields in vacuum are also obtained, which are also indispensable in elliptic estimates.

1. Introduction

In the present paper, we are concerned with the free boundary problem of ideal incompressible magnetohydrodynamics (MHD). It consists of finding a bounded variable domain $\Omega_t^+ \subset \mathbb{R}^3$ filled with inviscid incompressible electrically conducting homogeneous plasma (the density is a positive constant), together with the vector field of velocity $\mathbf{v}(t,x) = (v_1, v_2, v_3)$, the scalar pressure $p(t,x)$ and the magnetic field $\mathbf{H}(t,x) = (H_1, H_2, H_3)$ satisfying the equations of MHD. The

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boundary Γ_t of Ω_t^+ is the free surface of the plasma. It is assumed that the plasma is surrounded by a vacuum region Ω_t^- and that the whole domain $\Omega = \Omega_t^+ \cup \Gamma_t \cup \Omega_t^-$ is independent of time and bounded by a fixed perfectly conducting rigid wall W such that $W \cap \Gamma_t = \emptyset$. Both Ω_t^+ and Ω are simply connected. The magnetic fields should be found not only in Ω_t^+ but also in Ω_t^-.

In the plasma region Ω_t^+, the ideal MHD equations apply, i.e., for $t > 0$

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mu\big(\mathbf{H} \cdot \nabla \mathbf{H} - \frac{1}{2}\nabla |\mathbf{H}|^2\big), & \text{(1.1a)} \\ \mathbf{H}_t + \mathbf{v} \cdot \nabla \mathbf{H} = \mathbf{H} \cdot \nabla \mathbf{v}, & \text{(1.1b)} \\ \operatorname{div} \mathbf{v} = 0, \quad \operatorname{div} \mathbf{H} = 0. & \text{(1.1c)} \end{cases}$$

Let $\hat{\mathbf{H}}$ be the magnetic field in the vacuum Ω_t^-. Since the vacuum has no density, velocity, electric current (i.e., $\hat{\mathbf{E}}_t = 0$), except the magnetic field, we have the pre-Maxwell equations in vacuum

$$\nabla \times \hat{\mathbf{H}} = 0, \quad \operatorname{div} \hat{\mathbf{H}} = 0, \quad \hat{\mathbf{H}}_t = -\nabla \times \hat{\mathbf{E}}, \quad \operatorname{div} \hat{\mathbf{E}} = 0. \tag{1.2}$$

At the wall W, the tangential component of the electric field and the normal component of the magnetic field must vanish, i.e.,

$$\mathbf{n} \times \hat{\mathbf{E}} = 0, \quad \mathbf{n} \cdot \hat{\mathbf{H}} = 0, \quad \text{on } W, \tag{1.3}$$

where $\mathbf{n}$ is the inward drawn unit normal to the boundary W of Ω_t^-.

The plasma–vacuum interface is now free to move since the plasma is surrounded by vacuum. Hence, $\mathbf{v} \cdot \mathbf{n}|_{\Gamma_t}$ is unknown and arbitrary where $\mathbf{n}$ is the unit normal to Γ_t pointing from the plasma to the vacuum. Thus, we need some non-trivial jump conditions that must be satisfied to connect the fields across the interface. These arise from the divergence $\mathbf{H}$ equation, Faraday's law and the momentum equation. A convenient way to obtain the desired relations is to assume that the plasma surface Γ_t is moving with a normal velocity

$$V_{\mathbf{n}} \mathbf{n} = (\mathbf{v} \cdot \mathbf{n})\mathbf{n}, \tag{1.4}$$

where $V_{\mathbf{n}}$ is the velocity of evolution of Γ_t in the direction $\mathbf{n}$. The jump conditions are straightforward to derive in a reference frame moving with the fluid surface. Once these conditions are obtained, all that is then required are to convert back to the laboratory frame using the corresponding Galilean transformation (cf. [8]). From Maxwell's equations, we know that, at the interface Γ_t, the magnetic field and the electric field must satisfy the conditions

$$[\![\mathbf{n} \cdot \mathbf{H}]\!] = 0, \quad \text{and} \quad [\![\mathbf{n} \times \mathbf{E} - (\mathbf{n} \cdot \mathbf{v})\mathbf{H}]\!] = 0, \tag{1.5}$$

where $[\![f]\!] \equiv \hat{f} - f$ is the jump in a quantity across the interface. We assume that the plasma is a perfect conductor, i.e., $\mathbf{E} + \mathbf{v} \times \mathbf{H} = 0$. This implies that in the plasma, $[\mathbf{n} \cdot \mathbf{H}]_{\Gamma_t}$ and $[\mathbf{n} \times \mathbf{E} - (\mathbf{n} \cdot \mathbf{v})\mathbf{H}]_{\Gamma_t}$ are both automatically zero. Therefore, (1.5) reduces to

$$\mathbf{n} \cdot \mathbf{H} = \mathbf{n} \cdot \hat{\mathbf{H}} = 0, \quad \mathbf{n} \times \hat{\mathbf{E}} = (\mathbf{v} \cdot \mathbf{n})\hat{\mathbf{H}}, \quad \text{on } \Gamma_t. \tag{1.6}$$

The first one also means that the magnetic fields are not pointing into the vacuum on the interface.

We also have the following pressure balance condition (also cf. [8,19]) on the interface Γ_t:

$$\left[\!\left[p + \frac{\mu}{2}|\mathbf{H}|^2 \right]\!\right] = 0, \quad \text{on } \Gamma_t. \tag{1.7}$$

For convenience, we denote

$$P = q^+ - q^-, \quad q^+ = p + \frac{\mu}{2}|\mathbf{H}|^2, \quad \text{and} \quad q^- = \frac{\mu}{2}|\hat{\mathbf{H}}|^2.$$

The system can be written as

$$\begin{cases}
\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla q^+ = \mu\mathbf{H}\cdot\nabla\mathbf{H}, & \text{in } \Omega_t^+, \quad (1.8a)\\
\mathbf{H}_t + \mathbf{v}\cdot\nabla\mathbf{H} = \mathbf{H}\cdot\nabla\mathbf{v}, & \text{in } \Omega_t^+, \quad (1.8b)\\
\operatorname{div}\mathbf{v} = 0, \quad \operatorname{div}\mathbf{H} = 0, & \text{in } \Omega_t^+, \quad (1.8c)\\
\nabla\times\hat{\mathbf{H}} = 0,\ \operatorname{div}\hat{\mathbf{H}} = 0,\ \hat{\mathbf{H}}_t = -\nabla\times\hat{\mathbf{E}},\ \operatorname{div}\hat{\mathbf{E}} = 0, & \text{in } \Omega_t^-, \quad (1.8d)\\
P = 0, \quad \mathbf{H}\cdot\mathbf{n} = \hat{\mathbf{H}}\cdot\mathbf{n} = 0, & \text{on } \Gamma_t, \quad (1.8e)\\
\mathbf{n}\times\hat{\mathbf{E}} = (\mathbf{v}\cdot\mathbf{n})\hat{\mathbf{H}}, & \text{on } \Gamma_t, \quad (1.8f)\\
\mathbf{n}\times\hat{\mathbf{E}} = 0, \quad \hat{\mathbf{H}}\cdot\mathbf{n} = 0, & \text{on } W, \quad (1.8g)\\
\mathbf{v}(0,x) = \mathbf{v}_0(x), \quad \mathbf{H}(0,x) = \mathbf{H}_0(x), & \text{in } \Omega^+, \quad (1.8h)\\
\hat{\mathbf{H}}(0,x) = \hat{\mathbf{H}}_0(x), \quad \hat{\mathbf{E}}(0,x) = \hat{\mathbf{E}}_0(x), & \text{in } \Omega^-, \quad (1.8i)\\
\Omega_t^+|_{t=0} = \Omega^+, \quad \Omega_t^-|_{t=0} = \Omega^-, \quad \Gamma_t|_{t=0} = \Gamma, & (1.8j)
\end{cases}$$

where $\hat{\mathbf{E}}_0$ satisfies the boundary condition (1.8f) and (1.8g), i.e., $N \times \hat{\mathbf{E}}_0 = (\mathbf{v}_0 \cdot N)\hat{\mathbf{H}}_0$ on Γ and $\mathbf{n} \times \hat{\mathbf{E}}_0 = 0$ on W, where N denotes the unit normal to Γ pointing from the plasma to the vacuum.

We will prove a priori bounds for the interface problem (1.8) in Sobolev s-paces under the following generalized Rayleigh-Taylor sign condition for the total pressure P

$$\nabla_N P \leqslant -\varepsilon < 0 \quad \text{on } \Gamma, \tag{1.9}$$

where $\nabla_N = N^i\partial_i$ indicates the normal derivative. In fact, if this condition holds initially, then we can verify that it holds true within a period. For the free boundary problem of incompressible fluids in vacuum, without magnetic fields, the natural physical condition (cf. [2,4,6,7,14–16,22,26,27,29]) reads that $\nabla_N p \leqslant -\varepsilon < 0$ on Γ, which excludes the possibility of the Rayleigh–Taylor type instability (see [7]). In [25], Trakhinin showed that the simultaneous failure of the non-collinearity condition

$$|\mathbf{H}\times\hat{\mathbf{H}}| \geqslant \delta > 0 \quad \text{on } \Gamma \tag{1.10}$$

with a fixed constant δ (see also [20,21,24]) and the Rayleigh–Taylor sign condition (1.9) for those points on the interface where the unperturbed plasma and vacuum non-zero magnetic fields are collinear leads to Rayleigh–Taylor instability for the free boundary problem of linearized full compressible MHD system in half spaces. In some special cases, intuitively, (1.9) is a necessary condition for the plasmas on the interface to accelerate. For instance, assuming $\mathbf{H} = \hat{\mathbf{H}}$ and $\mathbf{H} \cdot N = 0$ (initially) on Γ, from (1.8a), and $\nabla \times \hat{\mathbf{H}} = 0$, we can get on Γ

$$
\begin{aligned}
N \cdot (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla_N q^+ = \mu(N^i H^j \partial_j H_i) &= \mu \left(N^i \hat{H}^j \partial_j \hat{H}_i \right) \\
&= \mu(N^i \hat{H}_j \partial_i \hat{H}^j) = \nabla_N q^-,
\end{aligned}
$$

which implies that

$$
-\nabla_N P = N \cdot (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) > 0,
$$

where $N \cdot (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v})$ is the normal component of the acceleration. This is analogous to the fact that $-\nabla_N p = N \cdot (\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) > 0$ for the free boundary problem of incompressible Euler equations. Of course, the rigorous proof of the ill-posedness of the nonlinear free boundary problem for the incompressible Euler equations in vacuum under the violation of the Rayleigh–Taylor sign condition $\nabla_N p \leqslant -\varepsilon < 0$ is a difficult mathematical problem (see [7,25]).

The condition (1.9) also provides some kind of possibility for solving the (local-in-time) well-posedness of free interface problem of the nonlinear ideal incompressible MHD system because under this initial assumption we can prove some a priori estimates for the solutions of the nonlinear system instead of linearized equations.

Up until now, there was no well-posedness result for the free boundary problem of the ideal incompressible MHD system except a few results about the linearized equations. This is due to the difficulties caused by the strong coupling between the velocity field and the magnetic field. Hao and Luo studied the a priori estimates for the free boundary problem of ideal incompressible MHD flows in [11] with a bounded initial domain homeomorphic to a ball, provided that the size of the magnetic field is invariant on the free boundary. For the special case where the magnetic field is zero on the free boundary and in vacuum, Lee proved the local existence and uniqueness of a plasma–vacuum free boundary problem of incompressible viscous-diffusive MHD flow in three-dimensional space with infinite depth setting in [12], and then got a local unique solution of free boundary MHD without kinetic viscosity and magnetic diffusivity via a zero kinetic viscosity-magnetic diffusivity limit in [13]. For the incompressible viscous MHD equations, a free boundary problem in a simply connected domain of $\mathbb{R}^3$ was studied by a linearization technique and the construction of a sequence of successive approximations in [18] with an irrotational condition for magnetic fields in a part of the domain. The well-posedness of the linearized plasma–vacuum interface problem in an ideal incompressible MHD was studied in [17] in an unbounded plasma domain. The linearized plasma–vacuum problem in an ideal full compressible MHD was investigated in [20,24], and the well-posedness of the nonlinear free boundary problem of this was proved in [21] using the Nash-Moser iteration method. A stationary problem was studied in [9]. In [5], the a priori estimates for smooth solutions of the free boundary problem for current-vortex sheets in an ideal incompressible two-fluid MHD was proved in the domain $\mathbb{T}^2 \times (-1, 1)$ under some linearized stability conditions on the jump function of the velocity field and the magnetic fields. The problem of the existence of current-vortex sheets in full compressible MHD was studied in [3,23].

Regarding the cases without magnetic fields, the free surface problem of the incompressible Euler equations of fluids has attracted much attention in recent decades and important progress has been made for flows with or without vorticity or surface tension. We refer readers to [1,4,6,7,10,14–16,22,26–29] and references therein.

In this paper, we prove a priori estimates for the free interface problem (1.8) and take into account the magnetic field not only in plasma but also in a vacuum. We do not need the restricted boundary condition $|\mathbf{H}| \equiv \text{const} \geqslant 0$ on the free interface assumed in [11]. What makes this problem difficult is that the regularity of the boundary enters to the highest order and energies interchange between plasma and vacuum. Roughly speaking, the energies of plasma and vacuum will exchange via the pressure balance relation $q^+ = q^-$ on the free interface. However, we have to investigate the estimates of the magnetic field and the electric field in a vacuum, although the electric field is only a secondary variable in order to obtain the energy estimates. We also introduce the "fictitious particle" endowed with a fictitious velocity field to reformulate the free boundary problem to a fixed boundary problem. We can show that the norms of the magnetic field in a vacuum depend only on the norms of initial data and the second fundamental forms of the interface and the wall.

Our result is an important and necessary step towards the proof of the (local-in-time) well-posedness of the free boundary problem of an ideal incompressible MHD system for which we have found a suitable initial condition [i.e., the generalized Rayleigh–Taylor sign condition (1.9)] for the initial data and provided some a priori properties of solutions of the present system under this condition.

Now, we derive the conserved energy. Letting $D_t := \partial_t + v^k \partial_k$ be the material derivative, it holds for any function F on $\overline{\Omega_t^+}$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t^+} F \mathrm{d}x = \int_{\Omega_t^+} D_t F \mathrm{d}x, \tag{1.11}$$

since $\operatorname{div} \mathbf{v} = 0$, and then for any function F on $\overline{\Omega_t^-}$ (we can extend it to Ω by a smooth cut-off function)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t^-} F \mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} F \mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t^+} F \mathrm{d}x \\ &= \int_{\Omega} \partial_t F \mathrm{d}x - \int_{\Omega_t^+} D_t F \mathrm{d}x \\ &= \int_{\Omega_t^-} \partial_t F \mathrm{d}x - \int_{\Omega_t^+} \mathbf{v} \cdot \nabla F \mathrm{d}x \\ &= \int_{\Omega_t^-} \partial_t F \mathrm{d}x - \int_{\Gamma_t} \mathbf{v_n} F \mathrm{d}S, \end{aligned} \tag{1.12}$$

where $\mathbf{n}$ is the outward unit normal to Γ_t corresponding to Ω_t^+.

Throughout the paper, we use the Einstein summation convention, that is, when an index variable appears twice in both the subscript and the superscript of a single term it indicates summation of that term over all the values of the index.

From (1.11) and (1.12), we have, by using the boundary conditions (1.8e), (1.8f) and (1.8g), that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t^+}\left(\frac{1}{2}|\mathbf{v}|^2+\frac{\mu}{2}|\mathbf{H}|^2\right)\mathrm{d}x+\frac{\mu}{2}\int_{\Omega_t^-}|\hat{\mathbf{H}}|^2\mathrm{d}x\right)\\
&=\int_{\Omega_t^+}\left(v^i D_t v_i+\mu H^i D_t H_i\right)\mathrm{d}x+\frac{\mu}{2}\int_{\Omega_t^-}\partial_t|\hat{\mathbf{H}}|^2\mathrm{d}x-\frac{\mu}{2}\int_{\Gamma_t}\mathbf{v_n}|\hat{\mathbf{H}}|^2\mathrm{d}S\\
&=\int_{\Omega_t^+}\left[v^i(-\partial_i p+\mu H^k\partial_k H_i-\frac{\mu}{2}\partial_i|\mathbf{H}|^2)+\mu H^i H^k\partial_k v_i\right]\mathrm{d}x\\
&\quad+\mu\int_{\Omega_t^-}\operatorname{div}(\hat{\mathbf{H}}\times\hat{\mathbf{E}})\mathrm{d}x-\frac{\mu}{2}\int_{\Gamma_t}\mathbf{v_n}|\hat{\mathbf{H}}|^2\mathrm{d}S\\
&=-\int_{\Gamma_t}\left(p+\frac{\mu}{2}|\mathbf{H}|^2\right)\mathbf{v_n}\mathrm{d}S+\mu\int_{\Gamma_t}(\mathbf{H}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{B})\mathrm{d}S\\
&\quad+\mu\int_{\Gamma_t}\hat{\mathbf{H}}\cdot(\mathbf{n}\times\hat{\mathbf{E}})\mathrm{d}S-\frac{\mu}{2}\int_{\Gamma_t}\mathbf{v_n}|\hat{\mathbf{H}}|^2\mathrm{d}S+\mu\int_W\hat{\mathbf{H}}\cdot(\hat{\mathbf{E}}\times\mathbf{n})\mathrm{d}S\\
&=-\int_{\Gamma_t}\left(p+\frac{\mu}{2}|\mathbf{H}|^2-\frac{\mu}{2}|\hat{\mathbf{H}}|^2\right)\mathbf{v_n}\mathrm{d}S\\
&=0,
\end{aligned}$$

due to

$$\begin{aligned}
\frac{1}{2}\partial_t|\hat{\mathbf{H}}|^2&=-\hat{\mathbf{H}}\cdot(\nabla\times\hat{\mathbf{E}})\\
&=\operatorname{div}(\hat{\mathbf{H}}\times\hat{\mathbf{E}})-\hat{\mathbf{E}}\cdot(\nabla\times\hat{\mathbf{H}})\\
&=\operatorname{div}(\hat{\mathbf{H}}\times\hat{\mathbf{E}}),
\end{aligned}$$

in view of $\nabla\times\hat{\mathbf{H}}=0$ in Ω_t^-. Thus, the conserved physical energy can be given by

$$\begin{aligned}
E_0(t):=&\int_{\Omega_t^+}\left(\frac{1}{2}|\mathbf{v}(t)|^2+\frac{\mu}{2}|\mathbf{H}(t)|^2\right)\mathrm{d}x\\
&+\int_{\Omega_t^-}\frac{\mu}{2}|\hat{\mathbf{H}}(t)|^2\mathrm{d}x\equiv E_0(0),
\end{aligned}$$

for any $t>0$.

The higher order energy has an interface boundary part and some interior parts in plasma since those of the magnetic field in vacuum can be bounded by $E_0(0)$ and the bounds of the second fundamental form of both the interface and the wall as shown in Propositions 3.1, 3.2 and 4.1. The boundary part controls the norms of the second fundamental form of the free interface, the interior part in plasma controls the norms of the velocity, magnetic fields and hence the pressure. We will prove that the time derivatives of the energies are controlled by themselves. A crucial point in the construction of the higher order energy norms is that the time derivatives of the interior parts will, after integrating by parts, contribute some boundary terms that cancel the leading-order terms in the corresponding time derivatives of the boundary integrals. To this end, we need to project the equations for the total pressure P to the tangent space of the boundary.

The orthogonal projection Π to the tangent space of the boundary of a $(0, r)$ tensor α is defined to be the projection of each component along the normal:

$$(\Pi\alpha)_{i_1\cdots i_r} = \Pi_{i_1}^{j_1}\cdots\Pi_{i_r}^{j_r}\alpha_{j_1\cdots j_r}, \quad \text{where } \Pi_i^j = \delta_i^j - \mathbf{n}_i\mathbf{n}^j,$$

with $\mathbf{n}^j = \delta^{ij}\mathbf{n}_i = \mathbf{n}_j$.

Let $\bar{\partial}_i = \Pi_i^j\partial_j$ be a tangential derivative. If $q = \text{const}$ on Γ_t, it follows that $\bar{\partial}_i q = 0$ there and

$$\left(\Pi\partial^2 q\right)_{ij} = \theta_{ij}\nabla_{\mathbf{n}}q,$$

where $\theta_{ij} = \bar{\partial}_i\mathbf{n}_j$ is the second fundamental form of Γ_t.

The higher order energies are defined as follows: for $r \geqslant 1$

$$\begin{aligned} E_r(t) = &\int_{\Omega_t^+}\delta^{ij}Q(\partial^r v_i, \partial^r v_j)\mathrm{d}x + \mu\int_{\Omega_t^+}\delta^{ij}Q(\partial^r H_i, \partial^r H_j)\mathrm{d}x \\ &+ \int_{\Omega_t^+}\left(|\partial^{r-1}\nabla\times v|^2 + \mu|\partial^{r-1}\nabla\times B|^2\right)\mathrm{d}x \\ &+ \operatorname{sgn}(r-1)\int_{\Gamma_t}Q(\partial^r P, \partial^r P)\vartheta\,\mathrm{d}S, \end{aligned}$$

where $\operatorname{sgn}(s)$ is the sign function of the real number s (so we do not need the boundary integral for $r = 1$) and $\vartheta = (-\nabla_{\mathbf{n}}P)^{-1}$ is a weight. Here Q is a positive definite quadratic form which, when restricted to the boundary, is the inner product of the tangential components, i.e., $Q(\alpha, \beta) = \langle\Pi\alpha, \Pi\beta\rangle$, and in the interior $Q(\alpha, \alpha)$ increases to the norm $|\alpha|^2$. To be more specific, let

$$Q(\alpha, \beta) = q^{i_1 j_1}\cdots q^{i_r j_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r},$$

where

$$q^{ij} = \delta^{ij} - \eta(d)^2\mathbf{n}^i\mathbf{n}^j, \quad d(x) = \text{dist}\,(x, \Gamma_t), \quad \mathbf{n}^i = -\delta^{ij}\partial_j d.$$

Here η is a smooth cutoff function satisfying $0 \leqslant \eta(d) \leqslant 1$, $\eta(d) = 1$ when $d < d_0/4$ and $\eta(d) = 0$ when $d > d_0/2$. d_0 is a fixed number that is smaller than the injectivity radius of the normal exponential map, defined to be the largest number ι_0 such that the map

$$\Gamma_t \times (-\iota_0, \iota_0) \to \left\{x \in \mathbb{R}^3 : \text{dist}\,(x,\ \Gamma_t) < \iota_0\right\}, \tag{1.13}$$

given by

$$(\bar{x}, \iota) \to x = \bar{x} + \iota\mathbf{n}\,(\bar{x}),$$

is an injection. We can also define ι_0 for W similarly, such that it is independent of t and denoted by ι_0'. In fact, both the second fundamental form of W and $1/\iota_0'$ are invariant with respect to time and known quantities.

Now, we can state the main result as follows.

Theorem 1.1. *Let*

$$\mathscr{K}(0) = \max\left(\|\theta(0,\cdot)\|_{L^\infty(\Gamma)}, 1/\iota_0(0)\right), \tag{1.14}$$

and

$$\mathscr{E}(0) = \|1/\left(\nabla_N P(0,\cdot)\right)\|_{L^\infty(\Gamma)} = 1/\varepsilon > 0. \tag{1.15}$$

Assume $E_s(0) < \infty$ *for* $s = 0, 1, \ldots, 4$ *and* $\nabla \times \hat{\mathbf{E}}_0 \in L^2(\Omega^-)$. *Then there exists a continuous function* $\mathscr{T} > 0$ *such that if*

$$T \leqslant \mathscr{T}(\mathscr{K}(0), \quad \mathscr{E}(0), \quad E_0(0), \ldots, E_4(0), \mathrm{Vol}\,\Omega), \tag{1.16}$$

then any smooth solution of the free boundary problem for MHD equations (1.8) *with* (1.9) *in* $[0, T]$ *satisfies the estimate*

$$\sum_{s=0}^{4} E_s(t) \leqslant 2\sum_{s=0}^{4} E_s(0), \quad 0 \leqslant t \leqslant T. \tag{1.17}$$

The rest of this paper is organized as follows. In Section 2, we use the Lagrangian coordinates to transform the free interface problem to a fixed boundary problem. Sections 3 and 4 are devoted to the estimates of the magnetic field and the electric field in vacuum, respectively. In Section 5, we prove the higher order energy estimates. In the derivation of the higher order energy estimates in Section 5, some a priori assumptions are made, which will be justified in Section 6. In order to make this paper more readable, we give an appendix for some estimates from [4] used in the previous sections.

2. Reformulation in Lagrangian Coordinates

We may think that the velocity field of the "fictitious particles" in a vacuum is $\mathbf{v}$ on the boundary. Then, we can extend the velocity from the boundary to the interior of vacuum by a cut-off function such that

$$\mathbf{v}(t,x) = \begin{cases} \mathbf{v}(t,\bar{x}), & \text{near } \Gamma_t, \\ \text{smooth}, & \text{otherwise}, \\ 0, & \text{near } W, \end{cases}$$

and $\operatorname{div}\mathbf{v} = 0$ for $x \in \Omega_t^-$ as long as $\Gamma_t \cap W = \emptyset$ in $[0, T]$, where $\bar{x} \in \Gamma_t$ satisfies $\operatorname{dist}(x, \Gamma_t) = |x - \bar{x}|$, and $|\mathbf{v}(t,x)| \leqslant \|\mathbf{v}(t)\|_{L^\infty(\Gamma_t)}$ for $x \in \Omega_t^-$ by construction.

Assume that we are given a velocity vector field $\mathbf{v}(t,x)$ defined in a set $\mathscr{D} \subset [0,T] \times \mathbb{R}^3$ such that the interface of $\Omega_t^+ = \{x : (t,x) \in \mathscr{D}\}$ and Ω_t^- moves with the velocity, i.e., $(1, \mathbf{v}) \in T(\partial\mathscr{D})$ which denotes the tangent space of $\partial\mathscr{D}$. We will now introduce Lagrangian or co-moving coordinates, that is, coordinates that are constant along the integral curves of the velocity vector field so that the free boundary becomes fixed in these coordinates (cf. [4]). Let $x = x(t, y) = f_t(y)$ be the trajectory of the particles given by

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \mathbf{v}(t, x(t, y)), & (t, y) \in [0, T] \times \Omega, \\ x(0, y) = f_0(y), & y \in \Omega, \end{cases} \tag{2.1}$$

where, when $t = 0$, we can start with either the Euclidean coordinates in Ω or some other coordinates $f_0 : \Omega \to \Omega$ where f_0 is a diffeomorphism in which the domain Ω becomes simple. For simplicity, we will assume $f_0(y) = y$ in this paper. Then, the Jacobian determinant $\det(\partial y/\partial x) \equiv 1$ due to the divergence-free property of $\mathbf{v}$. For each t, we will then have a change of coordinates $f_t : \Omega \to \Omega$, taking $y \to x(t, y)$. The Euclidean metric δ_{ij} in Ω then induces a metric

$$g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \tag{2.2}$$

in Ω for each fixed t.

We use the covariant differentiation in Ω with respect to the metric $g_{ab}(t, y)$, because it corresponds to differentiation in Ω under the change of coordinates $\Omega \ni y \to x(t, y) \in \Omega$, and we will work in both coordinates systems. This also avoids possible singularities in the change of coordinates. We denote covariant differentiation in the y_a-coordinate by ∇_a, $a = 1, 2, 3$, and differentiation in the x_i-coordinate by ∂_i, $i = 1, 2, 3$. The covariant differentiation of a $(0, r)$ tensor $k(t, y)$ is the $(0, r + 1)$ tensor given by

$$\nabla_a k_{a_1\cdots a_r} = \frac{\partial k_{a_1\cdots a_r}}{\partial y^a} - \Gamma^d_{aa_1} k_{d\cdots a_r} - \cdots - \Gamma^d_{aa_r} k_{a_1\cdots d},$$

where the Christoffel symbols Γ^c_{ab} are given by

$$\Gamma^c_{ab} = \frac{g^{cd}}{2} \left(\frac{\partial g_{bd}}{\partial y^a} + \frac{\partial g_{ad}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^d} \right) = \frac{\partial y^c}{\partial x^i} \frac{\partial^2 x^i}{\partial y^a \partial y^b},$$

where g^{cd} is the inverse of g_{ab}. If $w(t, x)$ is the $(0, r)$ tensor expressed in the x-coordinates, then the same tensor $k(t, y)$ expressed in the y-coordinates is given by

$$k_{a_1\cdots a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} w_{i_1\cdots i_r}(t, x), \quad x = x(t, y),$$

and by the transformation properties for tensors,

$$\nabla_a k_{a_1\cdots a_r} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial w_{i_1\cdots i_r}}{\partial x^i}. \tag{2.3}$$

Covariant differentiation is constructed so that the norms of tensors are invariant under changes of coordinates,

$$g^{a_1 b_1} \cdots g^{a_r b_r} k_{a_1\cdots a_r} k_{b_1\cdots b_r} = \delta^{i_1 j_1} \cdots \delta^{i_r j_r} w_{i_1\cdots i_r} w_{j_1\cdots j_r}. \tag{2.4}$$

Furthermore, we express in the y-coordinates,

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}. \tag{2.5}$$

Since the curvature vanishes in the x-coordinates, it must do so in the y-coordinates, and hence

$$[\nabla_a, \nabla_b] = 0.$$

Let us introduce the notation $k_{a\cdots}{}^{b}{}_{\cdots c} = g^{bd} k_{a\cdots d\cdots c}$, and recall that the covariant differentiation commutes with lowering and rising indices: $g^{ce}\nabla_a k_{b\cdot e\cdots d} = \nabla_a g^{ce} k_{b\cdot e\cdots d}$. We also introduce a notation for the material derivative:

$$D_t = \frac{\partial}{\partial t}\bigg|_{y=\text{const}} = \frac{\partial}{\partial t}\bigg|_{x=\text{const}} + v^k \frac{\partial}{\partial x^k}.$$

Then we have, from [4, Lemma 2.2], that

$$D_t k_{a_1\cdots a_r} = \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \left(D_t w_{i_1\cdots i_r} + \frac{\partial v^\ell}{\partial x^{i_1}} w_{\ell\cdots i_r} + \cdots + \frac{\partial v^\ell}{\partial x^{i_r}} w_{i_1\cdots \ell} \right). \tag{2.6}$$

We recall a result concerning time derivatives of the change of coordinates and commutators between time derivatives and space derivatives (cf. [4,11, Lemma 2.1]).

Lemma 2.1. *Let $x = f_t(y)$ be the change of variables given by (2.1), and let g_{ab} be the metric given by (2.2). Let $v_i = \delta_{ij} v^j = v^i$, and set*

$$u_a(t, y) = v_i(t, x)\frac{\partial x^i}{\partial y^a}, \qquad u^a = g^{ab} u_b, \tag{2.7}$$

$$h_{ab} = \frac{1}{2} D_t g_{ab}, \qquad h^{ab} = g^{ac} h_{cd} g^{db}. \tag{2.8}$$

Then

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a}\frac{\partial v^i}{\partial x^k}, \quad D_t \frac{\partial y^a}{\partial x^i} = -\frac{\partial y^a}{\partial x^k}\frac{\partial v^k}{\partial x^i}, \tag{2.9}$$

$$D_t g_{ab} = \nabla_a u_b + \nabla_b u_a, \quad D_t g^{ab} = -2h^{ab}, \quad D_t d\mu_g = \operatorname{tr} h\, d\mu_g, \tag{2.10}$$

where $d\mu_g$ is the Riemannian volume element on Ω in the metric g.

We now recall the estimates of commutators between the material derivative D_t and space derivatives ∂_i or covariant derivatives ∇_a.

Lemma 2.2. (cf. [4]) *Let ∂_i be given by (2.5). Then*

$$[D_t, \partial_i] = -(\partial_i v^k)\partial_k. \tag{2.11}$$

Furthermore,

$$[D_t, \partial^r] = -\sum_{s=0}^{r-1} \mathtt{C}_r^{s+1}(\partial^{1+s} v) \cdot \partial^{r-s}, \tag{2.12}$$

where $\mathtt{C}_r^s$ denotes the binomial coefficient defined by $\frac{r!}{(r-s)!s!}$ for $0 \leqslant s \leqslant r$, the symmetric dot product is defined to be in components

$$\left((\partial^{1+s} v) \cdot \partial^{r-s}\right)_{i_1 \cdots i_r} = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \left(\partial^{1+s}_{i_{\sigma_1} \cdots i_{\sigma_{1+s}}} v^k\right) \partial^{r-s}_{k i_{\sigma_{s+2}} \cdots i_{\sigma_r}},$$

and Σ_r denotes the collection of all permutations of $\{1, 2, \ldots, r\}$.

Lemma 2.3. (cf. [4,11]) *Let $T_{a_1 \cdots a_r}$ be a $(0, r)$ tensor. We have*

$$\begin{aligned}[D_t, \nabla_a] T_{a_1 \cdots a_r} = & - (\nabla_{a_1} \nabla_a u^d) T_{d a_2 \cdots a_r} \\ & - \cdots - (\nabla_{a_r} \nabla_a u^d) T_{a_1 \cdots a_{r-1} d}.\end{aligned} \tag{2.13}$$

If $\Delta = g^{cd} \nabla_c \nabla_d$ and q is a function, we have

$$[D_t, g^{ab} \nabla_a] T_b = -2h^{ab} \nabla_a T_b - (\Delta u^e) T_e, \tag{2.14}$$

$$[D_t, \nabla] q = 0, \tag{2.15}$$

$$[D_t, \Delta] q = -2h^{ab} \nabla_a \nabla_b q - (\Delta u^e) \nabla_e q. \tag{2.16}$$

Furthermore, for $r \geqslant 2$,

$$\left[D_t, \nabla^r\right] q = \sum_{s=1}^{r-1} -\mathtt{C}_r^{s+1} \left(\nabla^{s+1} u\right) \cdot \nabla^{r-s} q, \tag{2.17}$$

where the symmetric dot product is defined to be in components

$$\left((\nabla^{s+1} u) \cdot \nabla^{r-s} q\right)_{a_1 \cdots a_r} = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \left(\nabla^{s+1}_{a_{\sigma_1} \cdots a_{\sigma_{s+1}}} u^d\right) \nabla^{r-s}_{d a_{\sigma_{s+2}} \cdots a_{\sigma_r}} q.$$

Remark 2.1. It follows from (2.17) that for $r \geqslant 2$ and a function q,

$$D_t \nabla^r q + \nabla^r u \cdot \nabla q = \nabla^r D_t q - \operatorname{sgn}(r-2) \sum_{s=1}^{r-2} \mathtt{C}_r^{s+1} \left(\nabla^{s+1} u\right) \cdot \nabla^{r-s} q.$$

Denote

$$H_i = \delta_{ij} H^j = H^i, \quad \beta_a = H_j \frac{\partial x^j}{\partial y^a}, \quad \beta^a = g^{ab}\beta_b, \quad |\beta|^2 = \beta_a \beta^a,$$

$$\hat{H}_i = \delta_{ij} \hat{H}^j = \hat{H}^i, \quad \varpi_a = \hat{H}_j \frac{\partial x^j}{\partial y^a}, \quad \varpi^a = g^{ab}\varpi_b, \quad |\varpi|^2 = \varpi^a \varpi_a,$$

and

$$\hat{E}_i = \delta_{ij} \hat{E}^j = \hat{E}^i, \quad \Xi_a = \hat{E}_j \frac{\partial x^j}{\partial y^a}, \quad \Xi^a = g^{ab}\Xi_b, \quad |\Xi|^2 = \Xi^a \Xi_a.$$

It follows from (2.4) that

$$|\beta| = |\mathbf{H}|, \quad |\varpi| = |\hat{\mathbf{H}}|, \quad |\Xi| = |\hat{\mathbf{E}}|, \quad H_j = \frac{\partial y^a}{\partial x^j}\beta_a, \quad \hat{H}_j = \frac{\partial y^a}{\partial x^j}\varpi_a,$$
$$\hat{E}_j = \frac{\partial y^a}{\partial x^j}\Xi_a. \tag{2.18}$$

From (2.9) and (2.3), we have

$$\begin{aligned} D_t\varpi_a &= D_t\left(\hat{H}_j \frac{\partial x^j}{\partial y^a}\right) = \frac{\partial x^j}{\partial y^a} D_t \hat{H}_j + \hat{H}_j D_t \frac{\partial x^j}{\partial y^a} \\ &= \frac{\partial x^j}{\partial y^a}\left(-(\nabla \times \hat{\mathbf{E}})_j + v^k \partial_k \hat{H}_j\right) + \hat{H}_j \frac{\partial x^k}{\partial y^a}\frac{\partial v^j}{\partial x^k} \\ &= -\frac{\partial x^j}{\partial y^a}(\nabla \times \hat{\mathbf{E}})_j + \frac{\partial x^j}{\partial y^a}\frac{\partial x^k}{\partial y^b}\frac{\partial y^b}{\partial x^l} v^l \partial_k \hat{H}_j \\ &\quad + \hat{H}_j \frac{\partial x^k}{\partial y^a}\frac{\partial v^l}{\partial x^k}\frac{\partial x^j}{\partial y^b}\frac{\partial y^b}{\partial x^l} \\ &= -\frac{\partial x^j}{\partial y^a}(\nabla \times \hat{\mathbf{E}})_j + u^b \nabla_b \varpi_a + \varpi_b \nabla_a u^b. \end{aligned}$$

Due to $\det(\partial y/\partial x) \equiv 1$, we get

$$\begin{aligned} \frac{\partial x^j}{\partial y^a}\left(\nabla \times \hat{\mathbf{E}}\right)_j &= \frac{\partial x^1}{\partial y^a}\left(\frac{\partial \hat{E}_3}{\partial x^2} - \frac{\partial \hat{E}_2}{\partial x^3}\right) + \frac{\partial x^2}{\partial y^a}\left(\frac{\partial \hat{E}_1}{\partial x^3} - \frac{\partial \hat{E}_3}{\partial x^1}\right) \\ &\quad + \frac{\partial x^3}{\partial y^a}\left(\frac{\partial \hat{E}_2}{\partial x^1} - \frac{\partial \hat{E}_1}{\partial x^2}\right) \\ &= \frac{\partial x^1}{\partial y^a}\frac{\partial x^1}{\partial y^d}\frac{\partial y^d}{\partial x^1}\left(\frac{\partial x^2}{\partial y^b}\frac{\partial y^b}{\partial x^2}\frac{\partial x^3}{\partial y^c}\frac{\partial y^c}{\partial x^3}\frac{\partial \hat{E}_3}{\partial x^2}\right. \\ &\quad \left. - \frac{\partial x^3}{\partial y^b}\frac{\partial y^b}{\partial x^3}\frac{\partial x^2}{\partial y^c}\frac{\partial y^c}{\partial x^2}\frac{\partial \hat{E}_2}{\partial x^3}\right) \\ &\quad + \frac{\partial x^2}{\partial y^a}\frac{\partial x^2}{\partial y^d}\frac{\partial y^d}{\partial x^2}\left(\frac{\partial x^3}{\partial y^b}\frac{\partial y^b}{\partial x^3}\frac{\partial x^1}{\partial y^c}\frac{\partial y^c}{\partial x^1}\frac{\partial \hat{E}_1}{\partial x^3}\right. \end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial x^1}{\partial y^b}\frac{\partial y^b}{\partial x^1}\frac{\partial x^3}{\partial y^c}\frac{\partial y^c}{\partial x^3}\frac{\partial \hat{E}_3}{\partial x^1}\Bigg) \\
&+ \frac{\partial x^3}{\partial y^a}\frac{\partial x^3}{\partial y^d}\frac{\partial y^d}{\partial x^3}\Bigg(\frac{\partial x^1}{\partial y^b}\frac{\partial y^b}{\partial x^1}\frac{\partial x^2}{\partial y^c}\frac{\partial y^c}{\partial x^2}\frac{\partial \hat{E}_2}{\partial x^1} \\
&- \frac{\partial x^2}{\partial y^b}\frac{\partial y^b}{\partial x^2}\frac{\partial x^1}{\partial y^c}\frac{\partial y^c}{\partial x^1}\frac{\partial \hat{E}_1}{\partial x^2}\Bigg) \\
&= g_{ad}\nabla_b \varXi_c \Bigg[\frac{\partial y^d}{\partial x^1}\left(\frac{\partial y^b}{\partial x^2}\frac{\partial y^c}{\partial x^3} - \frac{\partial y^b}{\partial x^3}\frac{\partial y^c}{\partial x^2}\right) \\
&\quad + \frac{\partial y^d}{\partial x^2}\left(\frac{\partial y^b}{\partial x^3}\frac{\partial y^c}{\partial x^1} - \frac{\partial y^b}{\partial x^1}\frac{\partial y^c}{\partial x^3}\right) \\
&\quad + \frac{\partial y^d}{\partial x^3}\left(\frac{\partial y^b}{\partial x^1}\frac{\partial y^c}{\partial x^2} - \frac{\partial y^b}{\partial x^2}\frac{\partial y^c}{\partial x^1}\right)\Bigg] \\
&= g_{ad}\nabla_b \varXi_c \det\left(\frac{\partial(y^d, y^b, y^c)}{\partial(x^1, x^2, x^3)}\right) \\
&= g_{ad}\nabla_b \varXi_c \varepsilon^{dbc} \det\left(\frac{\partial y}{\partial x}\right) \\
&= (\nabla \times \varXi)_a,
\end{aligned}
$$

where ε_{ijk} denotes the Levi-Civita symbol defined as follows:

$$
\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (3, 1, 2) \text{ or } (2, 3, 1), \\ -1 & \text{if } (i, j, k) \text{ is } (1, 3, 2), (3, 2, 1) \text{ or } (2, 1, 3), \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i, \end{cases}
$$

which satisfies $\varepsilon^{ijk} \equiv \varepsilon_{ijk}$ in both Eulerian coordinates and Lagrangian coordinates, and the ith component of the curl of the vector F reads

$$
(\nabla \times F)^i = \varepsilon^{ijk}\partial_j F_k, \quad \text{and} \quad (\nabla \times F)^i = \varepsilon^{iab}\nabla_a F_b,
$$

in Eulerian coordinates and Lagrangian coordinates, respectively.

Thus, we have obtained

$$
D_t \varpi_a = -(\nabla \times \varXi)_a + u^b \nabla_b \varpi_a + \varpi_b \nabla_a u^b.
$$

Similarly, we get

$$
\nabla \times \varpi = 0.
$$

We also have those equations for $D_t u_a$ and $D_t \beta_a$, one can see [11] for details.

Thus, the system (1.8) can be written in the Lagrangian coordinates, for $t > 0$, as

$$\begin{cases} D_t u_a + \nabla_a q^+ = u^c \nabla_a u_c + \mu \beta^d \nabla_d \beta_a, & \text{in } \Omega^+, \quad (2.19a) \\ D_t \beta_a = \beta^d \nabla_d u_a + \beta^c \nabla_a u_c, & \text{in } \Omega^+, \quad (2.19b) \\ D_t \varpi_a = -(\nabla \times \Xi)_a + u^b \nabla_b \varpi_a + \varpi_b \nabla_a u^b, & \text{in } \Omega^-, \quad (2.19c) \\ \nabla_a u^a = 0 \text{ and } \nabla_a \beta^a = 0, & \text{in } \Omega^+, \quad (2.19d) \\ \nabla_a u^a = 0,\ \nabla \times \varpi = 0,\ \nabla_a \varpi^a = 0, \text{ and } \nabla_a \Xi^a = 0, & \text{in } \Omega^-, \quad (2.19e) \\ P = 0, \quad \beta_a N^a = \varpi_a N^a = 0, & \text{on } \Gamma, \quad (2.19f) \\ N \times \Xi = u_N \varpi, & \text{on } \Gamma, \quad (2.19g) \\ u = 0,\ \varpi_a N^a = 0,\ \Xi \times N = 0, & \text{on } W, \quad (2.19h) \end{cases}$$

where N is the unit normal vector pointing into the interior of Ω^-.

Obviously, the energy defined by

$$E_0(t) = \int_{\Omega^+} \left(\frac{1}{2} |u(t)|^2 + \frac{\mu}{2} |\beta(t)|^2 \right) \mathrm{d}\mu_g + \int_{\Omega^-} \frac{\mu}{2} |\varpi(t)|^2 \mathrm{d}\mu_g$$

is conserved. Of course, it is the equivalent one as in Eulerian coordinates. It can be easily verified by using the Gauss formula:

$$\begin{aligned} \int_{\Omega^+} \nabla_a F^a \mathrm{d}\mu_g &= \int_\Gamma N_a F^a \mathrm{d}\mu_\gamma, \\ \int_{\Omega^-} \nabla_a F^a \mathrm{d}\mu_g &= -\int_{\Gamma \cup W} N_a F^a \mathrm{d}\mu_\gamma, \end{aligned} \tag{2.20}$$

where F is a smooth vector-valued function, $N_a = g_{ab} N^b$ denotes the unit conormal, $g^{ab} N_a N_b = 1$, N^a denotes the outward (or inward) unit normal to Γ (and W) corresponding to Ω^+ (or Ω^-), $\mathrm{d}\mu_\gamma$ is the volume element on boundaries, and the induced metric γ on the tangent space to the boundary $T(\Gamma)$ (and $T(W)$) extended to be 0 on the orthogonal complement in $T(\Omega^+)$ (and $T(\Omega^-)$) is then given by

$$\gamma_{ab} = g_{ab} - N_a N_b, \quad \gamma^{ab} = g^{ab} - N^a N^b.$$

The orthogonal projection of a (r, s) tensor S to the boundaries is given by

$$(\Pi S)^{a_1 \cdots a_r}_{b_1 \cdots b_s} = \gamma^{a_1}_{c_1} \cdots \gamma^{a_r}_{c_r} \gamma^{d_1}_{b_1} \cdots \gamma^{d_s}_{b_s} S^{c_1 \cdots c_r}_{d_1 \cdots d_s},$$

where

$$\gamma^c_a = \delta^c_a - N_a N^c. \tag{2.21}$$

Covariant differentiation on the boundary $\overline{\nabla}$ is given by

$$\overline{\nabla} S = \Pi \nabla S,$$

and $\overline{\nabla}$ is invariantly defined since the projection and the covariant derivatives are. The second fundamental form of the boundary is given by

$$\theta_{ab} = (\overline{\nabla} N)_{ab} = \gamma^c_a \nabla_c N_b.$$

We need to extend the normal to a vector field defined and regular everywhere in the interior of Ω^+ and Ω^- such that when the geodesic distance to the boundaries $d(t, y) \leqslant \iota_0/4$, it is the normal to the set $\{y : \ d(t, y) = \text{const}\}$, and in the interior it drops off to 0. We also denote the extension of the normal by N^a which satisfies $|\nabla N| \leqslant 2\|\theta\|_{L^\infty(\Gamma\cup W)}$. Then we extend γ to the pseudo-Riemannian metric γ given by $\gamma_{ab} = g_{ab} - N_a N_b$ which satisfies $\nabla\gamma|_{L^\infty(\Omega)} \leqslant C(\|\theta\|_{L^\infty(\Gamma\cup W)} + 1/\iota_0)$. One can see [4] for more details about the derivation of the gradient estimates of the extensions N and γ.

3. The Estimates of the Magnetic Field in Vacuum

Since Ω^+ and Ω are simply connected, the equations $\nabla_a \varpi^a = 0$ and $\nabla \times \varpi = 0$ imply $\varpi(t, y) = \nabla\varphi(t, y)$, where φ is a solution of the Neumann problem

$$\begin{cases} \Delta\varphi = 0, & \text{in } \Omega^-, \quad (3.1)\\ \nabla_N\varphi = 0, & \text{on } \Gamma \cup W. \quad (3.2)\end{cases}$$

For the derivatives of ϖ, we have the following L^2-estimates.

Proposition 3.1. *Let $r \geqslant 0$ be an integer. If $|\theta| + 1/\iota_0 \leqslant K$ on Γ, $|\nabla N| \leqslant CK$ in Ω^-, then it holds*

$$\|\nabla^{r+1}\varpi\|^2_{L^2(\Omega^-)} \leqslant CK^{2(r+1)}E_0(0), \quad \|\nabla^r\varpi\|^2_{L^2(\Gamma\cup W)} \leqslant CK^{2r+1}E_0(0).$$

Proof. We use the induction argument to show, for any integer $s \geqslant 0$, that

$$\|\nabla^{s+1}\varpi\|^2_{L^2(\Omega^-)} \leqslant CK^{2(s+1)}E_0(0), \quad \|\nabla^s\varpi\|^2_{L^2(\Gamma\cup W)} \leqslant CK^{2s+2}E_0(0).$$

We first prove the case $s = 0$. By (3.1)–(3.2) and the Hölder inequality, we have

$$\begin{aligned}
\|\nabla^2\varphi\|^2_{L^2(\Omega^-)} &= \int_{\Omega^-} g^{ab}g^{cd}\nabla_c\nabla_a\varphi\nabla_d\nabla_b\varphi \mathrm{d}\mu_g \\
&= \int_{\Omega^-} g^{ab}g^{cd}\nabla_a(\nabla_c\varphi\nabla_d\nabla_b\varphi)\mathrm{d}\mu_g \\
&\quad - \int_{\Omega^-} g^{cd}\nabla_c\varphi\nabla_d\Delta\varphi \mathrm{d}\mu_g \\
&= -\int_{\Gamma\cup W} N^b g^{cd}\nabla_c\varphi\nabla_d\nabla_b\varphi \mathrm{d}\mu_\gamma \\
&= -\int_{\Gamma\cup W} g^{cd}\nabla_c\varphi\nabla_d(N^b\nabla_b\varphi)\mathrm{d}\mu_\gamma \\
&\quad + \int_{\Gamma\cup W} g^{cd}\nabla_c\varphi\nabla_d N^b\nabla_b\varphi \mathrm{d}\mu_\gamma \\
&= -\int_{\Gamma\cup W} \nabla_N\varphi\nabla^2_N\varphi \mathrm{d}\mu_\gamma - \int_{\Gamma\cup W} \gamma^{cd}\nabla_c\varphi\nabla_d(\nabla_N\varphi)\mathrm{d}\mu_\gamma \\
&\quad + \int_{\Gamma\cup W} g^{cd}\nabla_c\varphi\nabla_d N^b\nabla_b\varphi \mathrm{d}\mu_\gamma
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Gamma\cup W} \gamma^{cd}\nabla_c\varphi\nabla_d N^b\nabla_b\varphi \mathrm{d}\mu_\gamma \\
&\leqslant C\|\theta\|_{L^\infty(\Gamma\cup W)}\|\varpi\|^2_{L^2(\Gamma\cup W)}.
\end{aligned}
$$

We get, from Gauss' formula, Hölder's inequalities and Cauchy's inequality, that

$$
\begin{aligned}
\|\varpi\|^2_{L^2(\Gamma\cup W)} &= \int_{\Gamma\cup W} \varpi^c\varpi_c \mathrm{d}\mu_\gamma \\
&= \int_{\Gamma\cup W} N_a N^a \varpi^c\varpi_c \mathrm{d}\mu_\gamma \\
&= -\int_{\Omega^-} \nabla_a(N^a\varpi^c\varpi_c)\mathrm{d}\mu_g \\
&= -\int_{\Omega^-} \nabla_a N^a\varpi^c\varpi_c \mathrm{d}\mu_g \\
&\quad - 2\int_{\Omega^-} N^a\nabla_a\varpi^c\varpi_c \mathrm{d}\mu_g \\
&\leqslant C\|\mathrm{tr}\,(\nabla N)\|_{L^\infty(\Omega^-)}\|\varpi\|^2_{L^2(\Omega^-)} \\
&\quad + C\|\nabla\varpi\|_{L^2(\Omega^-)}\|\varpi\|_{L^2(\Omega^-)} \\
&\leqslant C\|\mathrm{tr}\,(\nabla N)\|_{L^\infty(\Omega^-)}\|\varpi\|^2_{L^2(\Omega^-)} \\
&\quad + \delta\|\nabla\varpi\|^2_{L^2(\Omega^-)} + (C/\delta)\|\varpi\|^2_{L^2(\Omega^-)},
\end{aligned} \tag{3.3}
$$

for any $\delta > 0$. Thus, it follows that

$$
\begin{aligned}
\|\nabla\varpi\|^2_{L^2(\Omega^-)} &\leqslant C\|\theta\|_{L^\infty(\Gamma\cup W)}\|\nabla N\|_{L^\infty(\Omega^-)}\|\varpi\|^2_{L^2(\Omega^-)} \\
&\quad + C\delta\|\theta\|_{L^\infty(\Gamma\cup W)}\|\nabla\varpi\|^2_{L^2(\Omega^-)} \\
&\quad + (C/\delta)\|\theta\|_{L^\infty(\Gamma\cup W)}\|\varpi\|^2_{L^2(\Omega^-)}.
\end{aligned}
$$

Taking δ so small that $C\delta K = 1/2$, we get

$$
\begin{aligned}
\|\nabla\varpi\|^2_{L^2(\Omega^-)} &\leqslant C\|\theta\|_{L^\infty(\Gamma\cup W)}\|\nabla N\|_{L^\infty(\Omega^-)}\|\varpi\|^2_{L^2(\Omega^-)} \\
&\quad + (C/\delta)\|\theta\|_{L^\infty(\Gamma\cup W)}\|\varpi\|^2_{L^2(\Omega^-)} \\
&\leqslant CK^2\|\varpi\|^2_{L^2(\Omega^-)} \leqslant\; CK^2 E_0(0).
\end{aligned}
$$

We also have, with the help of (3.3), that

$$
\|\varpi\|^2_{L^2(\Gamma\cup W)} \leqslant CK\|\varpi\|^2_{L^2(\Omega^-)} \leqslant\; CKE_0(0).
$$

Now, we assume that the claims are true for the cases $s = r - 1$, then we prove the case $s = r$.

From Gauss' formula, Hölder's inequality and Cauchy's inequality, it follows

$$
\begin{aligned}
\|\nabla^r\varpi\|^2_{L^2(\Gamma\cup W)} &= \int_{\Gamma\cup W} N_a N^a|\nabla^r\varpi|^2\mathrm{d}\mu_\gamma = -\int_{\Omega^-}\nabla_a(N^a|\nabla^r\varpi|^2)\mathrm{d}\mu_g \\
&\leqslant CK\|\nabla^r\varpi\|^2_{L^2(\Omega^-)} + C\|\nabla^r\varpi\|_{L^2(\Omega^-)}\|\nabla^{r+1}\varpi\|_{L^2(\Omega^-)}.
\end{aligned} \tag{3.4}
$$

Since $\operatorname{div}\varpi = 0$ and $\nabla \times \varpi = 0$, we have, from (A.7) and (A.8), that

$$|\nabla^{r+1}\varpi|^2 \leqslant C_1 g^{bc}\gamma^{af}\gamma^{AF}\nabla_A^r\nabla_a\varpi_b\nabla_F^r\nabla_f\varpi_c,$$
$$\int_{\Omega^-}|\nabla^{r+1}\varpi|^2 \mathrm{d}\mu_g \leqslant C_2\int_{\Omega^-}\Big(g^{bc}N^aN^f\gamma^{AF}\nabla_A^r\nabla_b\varpi_a\nabla_F^r\nabla_c\varpi_f + K^2|\nabla^r\varpi|^2\Big)\mathrm{d}\mu_g.$$

Noticing that $\nabla \times \varpi = 0$, it follows from the Gauss formula that

$$\begin{aligned}(C_1^{-1}+C_2^{-1})\|\nabla^{r+1}\varpi\|^2_{L^2(\Omega^-)} \leqslant & \int_{\Omega^-} g^{bc}\gamma^{af}\gamma^{AF}\nabla_A^r\nabla_a\varpi_b\nabla_F^r\nabla_c\varpi_f\mathrm{d}\mu_g \\ & + \int_{\Omega^-}\Big(g^{bc}N^aN^f\gamma^{AF}\nabla_A^r\nabla_a\varpi_b\nabla_F^r\nabla_c\varpi_f \\ & + K^2|\nabla^r\varpi|^2\Big)\mathrm{d}\mu_g \\ = & \int_{\Omega^-} g^{bc}\nabla_c\Big(g^{af}\gamma^{AF}\nabla_A^r\nabla_a\varpi_b\nabla_F^r\varpi_f\Big)\mathrm{d}\mu_g \qquad (3.5)\\ & - \int_{\Omega^-} g^{bc}g^{af}\nabla_c\Big(\gamma^{AF}\Big)\nabla_A^r\nabla_a\varpi_b\nabla_F^r\varpi_f\mathrm{d}\mu_g \qquad (3.6)\\ & + \int_{\Omega^-}K^2|\nabla^r\varpi|^2\mathrm{d}\mu_g. \qquad (3.7)\end{aligned}$$

By the Hölder inequalities and the Cauchy inequality, we get

$$|(3.6)| \leqslant CK\|\nabla^r\varpi\|_{L^2(\Omega^-)}\|\nabla^{r+1}\varpi\|_{L^2(\Omega^-)}.$$

We write

$$\begin{aligned}(3.5) = & -\int_{\Gamma\cup W} N^b g^{af}\gamma^{AF}\nabla_A^r\nabla_a\varpi_b\nabla_F^r\varpi_f\mathrm{d}\mu_\gamma \\ = & -\int_{\Gamma\cup W}\gamma^{af}\nabla_a(N^b\gamma^{AF}\nabla_A^r\varpi_b\nabla_F^r\varpi_f \\ & - N_fN^cN^b\gamma^{AF}\nabla_A^r\varpi_b\nabla_F^r\varpi_c)\mathrm{d}\mu_\gamma \qquad (3.8)\\ & + \int_{\Gamma\cup W}\gamma^{af}\nabla_a(N^b\gamma^{AF})\nabla_A^r\varpi_b\nabla_F^r\varpi_f\mathrm{d}\mu_\gamma \qquad (3.9)\\ & + \int_{\Gamma\cup W}\gamma^{af}N^b\gamma^{AF}\nabla_A^r\varpi_b\nabla_F^r\nabla_a\varpi_f\mathrm{d}\mu_\gamma \qquad (3.10)\\ & + \int_{\Gamma\cup W}N^aN^fN^b\gamma^{AF}\nabla_A^r\varpi_b\nabla_F^r\nabla_a\varpi_f\mathrm{d}\mu_\gamma \qquad (3.11)\\ & - \int_{\Gamma\cup W}\gamma^{af}\nabla_aN_fN^cN^b\gamma^{AF}\nabla_A^r\varpi_b\nabla_F^r\varpi_c\mathrm{d}\mu_\gamma. \qquad (3.12)\end{aligned}$$

In view of the Gauss formula, (3.8) vanishes. Due to div $\varpi = 0$, (3.10) + (3.11) = 0. From the Hölder inequality, we have

$$|(3.9) + (3.12)| \leqslant CK\|\nabla^r \varpi\|^2_{L^2(\Gamma\cup W)}.$$

Thus, from (3.4) and the Cauchy inequality, we get

$$\begin{aligned}\|\nabla^{r+1}\varpi\|^2_{L^2(\Omega^-)} &\leqslant CK\|\nabla^r \varpi\|^2_{L^2(\Gamma\cup W)} + CK\|\nabla^r \varpi\|_{L^2(\Omega^-)}\|\nabla^{r+1}\varpi\|_{L^2(\Omega^-)} \\ &\quad + CK^2\|\nabla^r \varpi\|^2_{L^2(\Omega^-)} \\ &\leqslant CK^2\|\nabla^r \varpi\|^2_{L^2(\Omega^-)} + CK\|\nabla^r \varpi\|_{L^2(\Omega^-)}\|\nabla^{r+1}\varpi\|_{L^2(\Omega^-)} \\ &\leqslant CK^2\|\nabla^r \varpi\|^2_{L^2(\Omega^-)} + \frac{1}{2}\|\nabla^{r+1}\varpi\|^2_{L^2(\Omega^-)},\end{aligned}$$

namely,

$$\|\nabla^{r+1}\varpi\|^2_{L^2(\Omega^-)} \leqslant CK^2\|\nabla^r \varpi\|^2_{L^2(\Omega^-)}, \tag{3.13}$$

and then

$$\|\nabla^r \varpi\|^2_{L^2(\Gamma\cup W)} \leqslant CK\|\nabla^r \varpi\|^2_{L^2(\Omega^-)}. \tag{3.14}$$

Therefore, by the induction argument, we have obtained the desired results. □

Proposition 3.2. *Suppose that for $\iota_1 \geqslant 1/K_1$,*

$$\begin{aligned}|N(\bar{x}_1) - N(\bar{x}_2)| \leqslant \varepsilon_1, \quad &\text{whenever } |\bar{x}_1 - \bar{x}_2| \\ &\leqslant \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \Gamma \cup W,\end{aligned}$$

and

$$C_0^{-1}\gamma^0_{ab}(y)Z^aZ^b \leqslant \gamma_{ab}(t,y)Z^aZ^b \leqslant C_0\gamma^0_{ab}(y)Z^aZ^b, \quad \text{if } Z \in T(\Omega^-),$$

where $\gamma^0_{ab}(y) = \gamma_{ab}(0,y)$. Then, it holds for any integer $r \geqslant 0$

$$\|\nabla^r \varpi\|_{L^\infty(\Gamma\cup W)} \leqslant C(r, K, K_1, \mathrm{Vol}\,\Omega^-)E_0^{1/2}(0),$$

and

$$\|\Pi\nabla^r|\varpi|^2\|_{L^2(\Gamma\cup W)} \leqslant C(r, K, K_1, \mathrm{Vol}\,\Omega^-)E_0(0).$$

Proof. From (A.14) and (A.17), it follows

$$\begin{aligned}\|\nabla^s \varpi\|_{L^\infty(\Gamma\cup W)} &\leqslant C\|\nabla^{s+1}\varpi\|_{L^4(\Gamma\cup W)} + C(K_1)\|\nabla^s \varpi\|_{L^4(\Gamma\cup W)} \\ &\leqslant C\left(K, K_1, \mathrm{Vol}\,\Omega^-\right)\sum_{\ell=0}^{2}\|\nabla^{s+\ell}\varpi\|_{L^2(\Omega^-)} \\ &\leqslant C\left(K, K_1, \mathrm{Vol}\,\Omega^-\right)\sum_{\ell=0}^{2}K^{s+\ell}E_0^{1/2}(0) \\ &\leqslant C\left(s, K, K_1, \mathrm{Vol}\,\Omega^-\right)E_0^{1/2}(0).\end{aligned}$$

By the Hölder inequality, Proposition 3.1, (A.14), and the Cauchy inequality, we have

$$\begin{aligned}\|\Pi\nabla^r|\varpi|^2\|_{L^2(\Gamma\cup W)} \leqslant & C\sum_{m=0}^{[r/2]}\|\nabla^m\varpi\|_{L^\infty(\Gamma\cup W)}\|\nabla^{r-m}\varpi\|_{L^2(\Gamma\cup W)}\\ \leqslant & C\sum_{m=0}^{[r/2]}\Big(\|\nabla^{m+1}\varpi\|_{L^4(\Gamma\cup W)}+C(K_1)\|\nabla^m\varpi\|_{L^4(\Gamma\cup W)}\Big)\\ & K^{r-m+1/2}E^{1/2}(0)\\ \leqslant & C(K_1,\mathrm{Vol}\,\Omega^-)E^{1/2}(0)\\ & \sum_{m=0}^{[r/2]}\sum_{\ell=0}^{2}\|\nabla^{m+\ell}\varpi\|_{L^2(\Omega^-)}K^{r-m+1/2}\\ \leqslant & C(K_1,\mathrm{Vol}\,\Omega^-)\sum_{m=0}^{[r/2]}\sum_{\ell=0}^{2}K^{m+\ell}K^{r-m+1/2}E_0(0)\\ \leqslant & C(r,K,K_1,\mathrm{Vol}\,\Omega^-)E_0(0).\end{aligned}$$

□

4. The Estimates of the Electric Field in Vacuum

Although the electric field can be regarded as a secondary variable due to

$$\nabla\times\hat{\mathbf{E}}=-\hat{\mathbf{H}}_t,\ \text{and}\quad \mathrm{div}\,\hat{\mathbf{E}}=0,$$

we have to use the estimates of the electric field in vacuum in order to get the energy estimates. In fact, we can prove the following estimates.

Proposition 4.1. *If $|\theta|+1/\iota_0\leqslant K$ on Γ and $\nabla\times\Xi_0\in L^2(\Omega^-)$, then it holds for any integer $r\geqslant 0$*

$$\begin{aligned}&\|\nabla^r(\nabla\times\Xi)\|^2_{L^2(\Omega^-)}+\|\nabla^r(\nabla\times\Xi)\|^2_{L^2(\Gamma\cup W)}\\ &\quad\leqslant C(r,K)\left[\|\nabla\times\Xi_0\|^2_{L^2(\Omega^-)}+E_0(0)\sup_{\tau\in[0,t]}\sum_{\ell=0}^{2}\|\nabla^\ell u(\tau)\|^2_{L^2(\Gamma)}\right]\\ &\quad\times\exp\left(C(K)\int_0^t\|u_N(\tau)\|_{L^\infty(\Gamma)}d\tau\right),\end{aligned}$$

where Ξ_0 is the initial datum of Ξ at $t=0$.

Proof. For convenience, we denote $B=\nabla\times\Xi$ in this section. Then, in Lagrangian coordinates, we have from (2.19c) that

$$B_a=-D_t\varpi_a+u^b\nabla_b\varpi_a+\varpi_b\nabla_a u^b.$$

From (A.5), we have on boundaries $\Gamma \cup W$

$$\begin{aligned} N^a B_a =& - D_t(N^a \varpi_a) + D_t N^a \varpi_a + N^a u^b \nabla_b \varpi_a + N^a \varpi_b \nabla_a u^b \\ =& - 2h^a_d N^d \varpi_a - h_{NN} N^a \varpi_a + N^a u^b \nabla_b \varpi_a + N^a \varpi_b \nabla_a u^b \\ =& - (\nabla_a u_d + \nabla_d u_a) N^d \varpi^a + N^a u^b \nabla_b \varpi_a + N^a \varpi_b \nabla_a u^b \\ =& - \nabla_a u_d N^d \varpi^a + N^d u^a \nabla_a \varpi_d. \end{aligned}$$

By (2.14) and the fact $\nabla_c \varpi^c = 0$, we also get in Ω^-

$$\begin{aligned} \nabla_a B^a =& g^{ac} \nabla_c B_a = -g^{ac} \nabla_c D_t \varpi_a + \nabla_c \left(u^b \nabla_b \varpi^c\right) + g^{ac} \nabla_c \left(\varpi_b \nabla_a u^b\right) \\ =& \left[D_t, g^{ac} \nabla_c\right] \varpi_a - D_t \left(\nabla_c \varpi^c\right) + \nabla_c u^b \nabla_b \varpi^c + g^{ac} \nabla_c \varpi_b \nabla_a u^b + \varpi_b \Delta u^b \\ =& - 2h^{ac} \nabla_c \varpi_a - \left(\Delta u^e\right) \varpi_e + 2h^{ac} \nabla_c \varpi_a + \varpi_b \Delta u^b \\ =& 0, \end{aligned}$$

and by (2.13) and the fact $\nabla \times \varpi = 0$,

$$\begin{aligned} (\nabla \times B)^c =& \varepsilon^{cea} \nabla_e B_a = -\varepsilon^{cea} \nabla_e D_t \varpi_a + \varepsilon^{cea} \nabla_e \left(u^b \nabla_b \varpi_a\right) + \varepsilon^{cea} \nabla_e \left(\varpi_b \nabla_a u^b\right) \\ =& \varepsilon^{cea} \left[D_t, \nabla_e\right] \varpi_a - \varepsilon^{cea} D_t \nabla_e \varpi_a + \varepsilon^{cea} \nabla_e u^b \nabla_b \varpi_a + \varepsilon^{cea} \nabla_e \varpi_b \nabla_a u^b \\ =& - \varepsilon^{cea} \nabla_a \nabla_e u^d \varpi_d + \varepsilon^{cea} \nabla_e u^b \nabla_b \varpi_a - \varepsilon^{cea} \nabla_a \varpi_b \nabla_e u^b \\ =& 0. \end{aligned}$$

Thus, we have $\nabla_a B^a = 0$ and $\nabla \times B = 0$, which yields $B(t, y) = \nabla \psi(t, y)$, and ψ is a solution of the following Neumann problem

$$\begin{cases} \Delta \psi = 0, & \text{in } \Omega^-, \quad (4.1) \\ \nabla_N \psi = f, & \text{on } \Gamma, \quad (4.2) \\ \nabla_N \psi = 0, & \text{on } W, \quad (4.3) \end{cases}$$

where $f = N^d u^a \nabla_a \varpi_d - \nabla_a u_d N^d \varpi^a$.

From (4.1)–(4.3) and Hölder's inequality, we get

$$\begin{aligned} \|\overline{\nabla} B\|^2_{L^2(\Omega^-)} =& \int_{\Omega^-} g^{ab} \gamma^{cd} \nabla_c \nabla_a \psi \nabla_d \nabla_b \psi \mathrm{d}\mu_g \\ =& \int_{\Omega^-} g^{ab} \nabla_a (\gamma^{cd} \nabla_c \psi \nabla_d \nabla_b \psi) \mathrm{d}\mu_g \\ & - \int_{\Omega^-} g^{ab} \nabla_a \gamma^{cd} \nabla_c \psi \nabla_d \nabla_b \psi \mathrm{d}\mu_g \\ =& - \int_{\Gamma \cup W} N^b \gamma^{cd} \nabla_c \psi \nabla_d \nabla_b \psi \mathrm{d}\mu_\gamma \\ & - \int_{\Omega^-} g^{ab} \nabla_a \gamma^{cd} \nabla_c \psi \nabla_d \nabla_b \psi \mathrm{d}\mu_g \\ =& - \int_{\Gamma \cup W} \gamma^{cd} \nabla_c \psi \nabla_d \left(N^b \nabla_b \psi\right) \mathrm{d}\mu_\gamma \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\Gamma \cup W} \gamma^{cd} \nabla_c \psi \nabla_d N^b \nabla_b \psi \, \mathrm{d}\mu_\gamma \\
&- \int_{\Omega^-} g^{ab} \nabla_a \gamma^{cd} \nabla_c \psi \nabla_d \nabla_b \psi \, \mathrm{d}\mu_g \\
\leqslant\ & C \|\Pi B\|_{L^2(\Gamma)} \|\overline{\nabla} f\|_{L^2(\Gamma)} \\
&+ C \|\theta\|_{L^\infty(\Gamma \cup W)} \|B\|^2_{L^2(\Gamma \cup W)} \\
&+ CK \|B\|_{L^2(\Omega^-)} \|\nabla B\|_{L^2(\Omega^-)}.
\end{aligned}
$$

Similar to (3.3), we have for any $\delta > 0$

$$
\|B\|^2_{L^2(\Gamma \cup W)} \leqslant C(K + 1/\delta) \|B\|^2_{L^2(\Omega^-)} + \delta \|\nabla B\|^2_{L^2(\Omega^-)}. \tag{4.4}
$$

Thus, in view of (A.7) and Cauchy's inequality, it follows that

$$
\begin{aligned}
\|\nabla B\|^2_{L^2(\Omega^-)} &\leqslant C \|\overline{\nabla} B\|^2_{L^2(\Omega^-)} \\
&\leqslant C \|\overline{\nabla} f\|^2_{L^2(\Gamma)} + C(K+1)(K + 1/\delta) \|B\|^2_{L^2(\Omega^-)} \\
&\quad + C(K+1)\delta \|\nabla B\|^2_{L^2(\Omega^-)} + CK^2 \|B\|^2_{L^2(\Omega^-)} \\
&\quad + \frac{1}{4} \|\nabla B\|^2_{L^2(\Omega^-)}.
\end{aligned}
$$

Taken δ so small that $C(K+1)\delta < 1/4$, it yields

$$
\|\nabla B\|^2_{L^2(\Omega^-)} \leqslant C \|\overline{\nabla} f\|^2_{L^2(\Gamma)} + C(K) \|B\|^2_{L^2(\Omega^-)}. \tag{4.5}
$$

Similar to the derivation of (2.19c), we can get, due to $\hat{\mathbf{E}}_t = 0$, that

$$
D_t B_a = u^b \nabla_b B_a + \nabla_a u^b B_b. \tag{4.6}
$$

It follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^-} |B|^2 \mathrm{d}\mu_g = - \int_\Gamma u_N |B|^2 \mathrm{d}\mu_\gamma,
$$

which yields

$$
\|B(t)\|^2_{L^2(\Omega^-)} \leqslant \|B_0\|^2_{L^2(\Omega^-)} + \int_0^t \|u_N(\tau)\|_{L^\infty(\Gamma)} \|B(\tau)\|^2_{L^2(\Gamma)} \mathrm{d}\tau, \tag{4.7}
$$

where $B_0 = B(t)|_{t=0}$. From (4.5), (4.7) and (4.4), we obtain

$$\begin{aligned}\|B(t)\|^2_{L^2(\Omega^-)} + \|\nabla B\|^2_{L^2(\Omega^-)} &\leqslant C\|\overline{\nabla} f\|^2_{L^2(\Gamma)} + C(K)\|B_0\|^2_{L^2(\Omega^-)} \\ &\quad + C(K)\int_0^t \|u_N(\tau)\|_{L^\infty(\Gamma)}\|B(\tau)\|^2_{L^2(\Gamma)}\mathrm{d}\tau \\ &\leqslant C\|\overline{\nabla} f\|^2_{L^2(\Gamma)} + C(K)\|B_0\|^2_{L^2(\Omega^-)} \\ &\quad + C(K)\int_0^t \|u_N(\tau)\|_{L^\infty(\Gamma)}\Big(\|B(\tau)\|^2_{L^2(\Omega^-)} \\ &\quad + \|\nabla B(\tau)\|^2_{L^2(\Omega^-)}\Big)\mathrm{d}\tau,\end{aligned}$$

which implies, by Grönwall's inequality, that

$$\begin{aligned}&\|B(t)\|^2_{L^2(\Omega^-)} + \|\nabla B\|^2_{L^2(\Omega^-)} \\ &\quad\leqslant \left[C\sup_{\tau\in[0,t]}\|\overline{\nabla} f(\tau)\|^2_{L^2(\Gamma)} + C(K)\|B_0\|^2_{L^2(\Omega^-)}\right] \\ &\qquad\times\exp\left(C(K)\int_0^t \|u_N(\tau)\|_{L^\infty(\Gamma)}\mathrm{d}\tau\right).\end{aligned}$$

By the definition of f and Proposition 3.1, we have

$$\begin{aligned}\|\overline{\nabla} f\|_{L^2(\Gamma)} &\leqslant \|\overline{\nabla}(N^d u^a \nabla_a \varpi_d - \nabla_a u_d N^d \varpi^a)\|_{L^2(\Gamma)} \\ &\leqslant \|\theta\|_{L^\infty(\Gamma)}\|u\|_{L^2(\Gamma)}\|\nabla\varpi\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)}\|\nabla\varpi\|_{L^2(\Gamma)} \\ &\quad + \|u\|_{L^2(\Gamma)}\|\nabla^2\varpi\|_{L^2(\Gamma)} + \|\nabla^2 u\|_{L^2(\Gamma)}\|\varpi\|_{L^2(\Gamma)} \\ &\quad + \|\theta\|_{L^\infty(\Gamma)}\|\nabla u\|_{L^2(\Gamma)}\|\varpi\|_{L^2(\Gamma)} \\ &\leqslant C(K)\left[\|u\|_{L^2(\Gamma)} + \|\nabla u\|_{L^2(\Gamma)} + \|\nabla^2 u\|_{L^2(\Gamma)}\right]E^{1/2}(0) \\ &\leqslant C(K)E^{1/2}(0)\sum_{\ell=0}^2 \|\nabla^\ell u\|_{L^2(\Gamma)}.\end{aligned}$$

Therefore, we obtain, combining with (4.4), that

$$\begin{aligned}&\|B\|^2_{L^2(\Omega^-)} + \|\nabla B\|^2_{L^2(\Omega^-)} + \|B\|^2_{L^2(\Gamma\cup W)} \\ &\quad\leqslant C(K)\left[\|B_0\|^2_{L^2(\Omega^-)} + E_0(0)\sup_{\tau\in[0,t]}\sum_{\ell=0}^2\|\nabla^\ell u(\tau)\|^2_{L^2(\Gamma)}\right] \\ &\qquad\times\exp\left(C(K)\int_0^t \|u_N(\tau)\|_{L^\infty(\Gamma)}\mathrm{d}\tau\right).\end{aligned}$$

Since $\nabla_a B^a = 0$ and $\nabla\times B = 0$, it is similar to ϖ. One can verify that the lines between (3.4) and (3.14) also hold if ϖ is replaced by B everywhere in those lines. Thus, we can obtain for any $r \geqslant 1$

$$\|\nabla^{r+1}B\|^2_{L^2(\Omega^-)} \leqslant CK^2\|\nabla^r B\|^2_{L^2(\Omega^-)}, \tag{4.8}$$

and

$$\|\nabla^r B\|^2_{L^2(\Gamma\cup W)} \leqslant CK\|\nabla^r B\|^2_{L^2(\Omega^-)}. \tag{4.9}$$

Hence, we get for any $r \geqslant 0$

$$\begin{aligned}&\|\nabla^r B\|^2_{L^2(\Omega^-)} + \|\nabla^r B\|^2_{L^2(\Gamma\cup W)}\\ &\quad\leqslant C(K, \mathrm{Vol}\,\Omega^+)\left[\|B_0\|^2_{L^2(\Omega^-)} + E_0(0)\sup_{\tau\in[0,t]}\sum_{\ell=0}^{3}\|\nabla^\ell u(\tau)\|^2_{L^2(\Omega^+)}\right]\\ &\qquad\times\exp\left(C(K)\int_0^t\|u_N(\tau)\|_{L^\infty(\Gamma)}\mathrm{d}\tau\right).\end{aligned}$$

Changing B back to $\nabla\times\Xi$, we obtain the desired results. □

5. The General r-th Order Energy Estimates

We can get that for $r \geqslant 1$ (cf. [11])

$$\begin{aligned}&D_t\nabla^r u_a + \nabla^r\nabla_a q^+\\ &\quad= (\nabla_a u_c - \mathrm{sgn}(r-1)\nabla_c u_a)\,\nabla^r u^c + \mu\beta^c\nabla_c\nabla^r\beta_a + r\mu\nabla\beta\cdot\nabla^r\beta_a\\ &\qquad+\mathrm{sgn}(r-1)\mu\nabla^r\beta^c\nabla_c\beta_a + \mathrm{sgn}\,((r-1)(r-2))\,\mathcal{P}_a(\beta),\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}D_t\nabla^r\beta_a =&\, (\nabla_a u_c + \mathrm{sgn}(r-1)\nabla_c u_a)\nabla^r\beta^c - \mathrm{sgn}(r-1)\nabla^r u^c\nabla_c\beta_a\\ &+\beta^c\nabla_c\nabla^r u_a + r\nabla\beta\cdot\nabla^r u_a + \mathrm{sgn}\,((r-1)(r-2))\,\mathcal{Q}_a,\end{aligned} \tag{5.2}$$

where

$$\mathcal{P}_a(\beta) := -\sum_{s=2}^{r-1}\mathtt{C}_r^s\nabla^s u^c\nabla^{r-s}\nabla_c u_a + \mu\sum_{s=2}^{r-1}\mathtt{C}_r^s\nabla^s\beta^c\nabla^{r-s}\nabla_c\beta_a,$$

and

$$\mathcal{Q}_a := -\sum_{s=2}^{r-1}\mathtt{C}_r^s\nabla^s u^c\nabla^{r-s}\nabla_c\beta_a + \sum_{s=2}^{r-1}\mathtt{C}_r^s\nabla^s\beta^c\nabla^{r-s}\nabla_c u_a.$$

Define the r-th order energy as

$$\begin{aligned}E_r(t) =& \int_{\Omega^+} g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\mathrm{d}\mu_g\\ &+\mu\int_{\Omega^+} g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a\beta_b\nabla_F^{r-1}\nabla_f\beta_d\mathrm{d}\mu_g\\ &+\int_{\Omega^+}|\nabla^{r-1}\nabla\times u|^2\mathrm{d}\mu_g + \mu\int_{\Omega^+}|\nabla^{r-1}\nabla\times\beta|^2\mathrm{d}\mu_g\\ &+\int_\Gamma\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\,\vartheta\mathrm{d}\mu_\gamma,\end{aligned}$$

where the weight $\vartheta = -1/\nabla_N P$ as before.

Theorem 5.1. *Let $1 \leqslant r \leqslant 4$ be an integer, then there exists a $T > 0$ such that the following holds: For any smooth solution of MHD (2.19) satisfying*

$$|\beta| \leqslant M_1 \quad \textit{for } r = 2, \qquad \text{in } [0,T]\times\Omega^+, \tag{5.3}$$

$$|\nabla P| + |\nabla u| + |\nabla\beta| \leqslant M, \qquad \text{in } [0,T]\times\Omega, \tag{5.4}$$

$$|\theta| + 1/\iota_0 \leqslant K, \qquad \text{on } [0,T]\times\Gamma, \tag{5.5}$$

$$-\nabla_N P \geqslant \varepsilon > 0, \qquad \text{on } [0,T]\times\Gamma, \tag{5.6}$$

$$|u| + |\nabla^2 P| + |\nabla_N D_t P| \leqslant L, \qquad \text{on } [0,T]\times\Gamma, \tag{5.7}$$

we have, for $t \in [0,T]$,

$$E_r(t) \leqslant e^{C_1 t} E_r(0) + C_2 e^{C_3 t}\left(e^{C_4 t} - 1\right), \tag{5.8}$$

where the constant $C_i > 0$ depends on K, K_1, M, M_1, L, $1/\varepsilon$, $\mathrm{Vol}\,\Omega$, $E_0(0)$, $E_1(0)$, …, and $E_{r-1}(0)$; both C_2 and C_3 also depend on $\|\nabla\times\Xi_0\|_{L^2(\Omega^-)}$ if $r \geqslant 3$; $C_3 = 0$ for $r = 1, 2$.

Proof. Since $\mathrm{tr}\, h = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_r(t) = \int_{\Omega^+} D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b \nabla_F^{r-1}\nabla_f u_d\right)\mathrm{d}\mu_g \tag{5.9}$$

$$+\mu\int_{\Omega^+} D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \beta_b \nabla_F^{r-1}\nabla_f \beta_d\right)\mathrm{d}\mu_g \tag{5.10}$$

$$+\int_{\Omega^+} D_t|\nabla^{r-1}\nabla\times u|^2 \mathrm{d}\mu_g + \mu\int_{\Omega^+} D_t|\nabla^{r-1}\nabla\times\beta|^2\mathrm{d}\mu_g \tag{5.11}$$

$$+\int_\Gamma D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\right)\vartheta\,\mathrm{d}\mu_\gamma \tag{5.12}$$

$$+\int_\Gamma \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\left(\frac{\vartheta_t}{\vartheta} - h_{NN}\right)\vartheta\,\mathrm{d}\mu_\gamma. \tag{5.13}$$

Since the boundary integrals disappear for the case $r = 1$, it is easy to obtain the desired estimate and we omit the details. So we assume $r \geqslant 2$ from now on in the proofs.

We first estimate (5.9)–(5.10) and (5.12). From Lemmas 2.1 and A.1, we have in Ω^+,

$$\begin{aligned}
&D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\right)\\
&= -2\nabla_c u_e\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^c\nabla_F^{r-1}\nabla_f u^e\\
&\quad - 4r\nabla_c u_e\gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^d\nabla_F^{r-1}\nabla_f u_d\\
&\quad - 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a\nabla_b q^+\\
&\quad + 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b(\nabla_b u_c - \mathrm{sgn}(r-1)\nabla_c u_b)\nabla_A^{r-1}\nabla_a u^c\\
&\quad + 2\mu\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d\left(\beta^c\nabla_c\nabla_{Aa}^r\beta^d\right)\\
&\quad + 2r\mu\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d(\nabla\beta^c\nabla^{r-1}\nabla_c\beta^d)_{Aa}
\end{aligned}$$

$$+\ 2\mathrm{sgn}(r-1)\mu\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d\nabla_{Aa}^r\beta^c\nabla_c\beta_b$$
$$+\ 2\mathrm{sgn}((r-1)(r-2))\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d(\mathcal{P}_b(\beta))_{Aa},$$

and

$$\begin{aligned}
&D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a\beta_b\nabla_F^{r-1}\nabla_f\beta_d\right)\\
&\quad=-2\nabla_c u_e\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a\beta^c\nabla_F^{r-1}\nabla_f\beta^e\\
&\quad\quad-2r\nabla_c u_e\gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a\beta^d\nabla_F^{r-1}\nabla_f\beta_d\\
&\quad\quad+2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^b(\nabla_b u_c+\mathrm{sgn}(r-1)\nabla_c u_b)\nabla_A^{r-1}\nabla_a\beta^c\\
&\quad\quad-2\mathrm{sgn}(r-1)\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^b\nabla_c\beta_b\nabla_A^{r-1}\nabla_a u^c\\
&\quad\quad+2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^d\beta^c\nabla_c\nabla_{Aa}^r u_d\\
&\quad\quad+2r\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^d(\nabla\beta\cdot\nabla^r u_b)_{Aa}\\
&\quad\quad+2\mathrm{sgn}((r-1)(r-2))\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f\beta^d(\mathcal{Q}_b)_{Aa},
\end{aligned}$$

and in the interface Γ,

$$\begin{aligned}
D_t\left(\gamma^{af}\gamma^{AF}\nabla_{Aa}^r P\nabla_{Ff}^r P\right)=&-2r\nabla_c u_e\gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_{Aa}^r P\nabla_{Ff}^r P\\
&+2\gamma^{af}\gamma^{AF}\nabla_{Aa}^r P D_t\nabla_{Ff}^r P.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
(5.9)+(5.10)+(5.12)\leqslant\ &C\left(\|\nabla u\|_{L^\infty(\Omega)}+\|\nabla\beta\|_{L^\infty(\Omega^+)}\right)E_r(t)\\
&+CE_r^{1/2}(t)\sum_{s=2}^{r-1}\left(\|\nabla^s u\|_{L^4(\Omega^+)}+\|\nabla^s\beta\|_{L^4(\Omega^+)}\right)\\
&\times\left(\|\nabla^{r-s+1}u\|_{L^4(\Omega^+)}+\|\nabla^{r-s+1}\beta\|_{L^4(\Omega^+)}\right) \qquad (5.14)\\
&+2\int_\Gamma\gamma^{af}\gamma^{AF}\nabla_{Aa}^r P\left(D_t\nabla_{Ff}^r P-\frac{1}{\vartheta}N_b\nabla_{Ff}^r u^b\right)\vartheta\,\mathrm{d}\mu_\gamma \qquad (5.15)\\
&-\mu\int_\Gamma\gamma^{af}\gamma^{AF}\nabla_{Aa}^r|\varpi|^2N_b\nabla_{Ff}^r u^b\mathrm{d}\mu_\gamma \qquad (5.16)\\
&+2\int_{\Omega^+}\nabla_b\left(\gamma^{af}\gamma^{AF}\right)\nabla_{Ff}^r u^b\nabla_{Aa}^r q^+\mathrm{d}\mu_g \qquad (5.17)\\
&+2\mu\int_\Gamma N_c\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d\beta^c\nabla_{Aa}^r\beta^d\mathrm{d}\mu_\gamma \qquad (5.18)\\
&-2\mu\int_{\Omega^+}\nabla_c\left(\gamma^{af}\gamma^{AF}\right)\nabla_F^{r-1}\nabla_f u_d\beta^c\nabla_{Aa}^r\beta^d\mathrm{d}\mu_g. \qquad (5.19)
\end{aligned}$$

From Lemma A.7, it follows that

$$|(5.14)|\leqslant C\,(K,K_1,M,\mathrm{Vol}\,\Omega,1/\varepsilon)\left(1+\sum_{s=0}^{r-1}E_s(t)\right)E_r(t).$$

By Gauss' formula, Proposition 3.2 and (A.17), we get

$$
\begin{aligned}
(5.16) = & -\mu \int_\Gamma \gamma^{af} \nabla_f \Big[\gamma^{AF} \nabla^r_{Aa} |\varpi|^2 N_b \nabla_F^{r-1} u^b \\
& - N_a N^c \gamma^{AF} \nabla^r_{Ac} |\varpi|^2 N_b \nabla_F^{r-1} u^b \Big] \mathrm{d}\mu_\gamma \\
& + \mu \int_\Gamma \gamma^{af} \nabla_f \Big[\gamma^{AF} \nabla^r_{Aa} |\varpi|^2 N_b\Big] \nabla_F^{r-1} u^b \mathrm{d}\mu_\gamma \\
& - \mu \int_\Gamma \gamma^{af} \nabla_f N_a N^c \gamma^{AF} \nabla^r_{Ac} |\varpi|^2 N_b \nabla_F^{r-1} u^b \mathrm{d}\mu_\gamma \\
\leqslant & \, \mu C\left(r, K, K_1, \mathrm{Vol}\, \Omega^-\right) E_0(0) \|\nabla^{r-1} u\|_{L^2(\Gamma)} \\
\leqslant & \, \mu C\left(r, K, K_1, \mathrm{Vol}\, \Omega^-, \mathrm{Vol}\, \Omega^+\right) E_0(0) (\|\nabla^r u\|_{L^2(\Omega^+)} + \|\nabla^{r-1} u\|_{L^2(\Omega^+)}) \\
\leqslant & \, \mu C(r, K, K_1, \mathrm{Vol}\, \Omega) E_0(0) \left(E_r^{1/2}(t) + E_{r-1}^{1/2}(t)\right).
\end{aligned}
$$

Due to $\beta \cdot N = 0$ on Γ, (5.18) vanishes. From (A.14) and (A.17), it follows

$$
\|u\|_{L^\infty(\Gamma)} \leqslant C(K_1) \sum_{s=0}^{2} \|\nabla^s u\|_{L^2(\Omega^+)} \leqslant C(K_1) \sum_{s=0}^{2} E_s^{1/2}(t).
$$

From Lemma A.8, it follows, for $\iota_1 \geqslant 1/K_1$, that

$$
\begin{aligned}
\|\beta\|_{L^\infty(\Omega^+)} & \leqslant C \sum_{0 \leqslant s \leqslant 2} K_1^{n/2-s} \|\nabla^s \beta\|_{L^2(\Omega^+)} \\
& \leqslant C(K_1) \sum_{s=0}^{2} E_s^{1/2}(t).
\end{aligned} \tag{5.20}
$$

Thus, with the help of the Hölder inequality, we have for any $r \geqslant 3$

$$
(5.19) \leqslant CK \|\beta\|_{L^\infty(\Omega^+)} E_r(t) \leqslant C(K, K_1) \left(\sum_{s=0}^{2} E_s^{1/2}(t)\right) E_r(t).
$$

For $r = 1$, it is easy to verify that there exists a $T > 0$ such that $E_1(t)$ can be controlled by the initial energy $E_1(0)$ for $t \in [0, T]$, e.g., $E_1(t) \leqslant 2E_1(0)$. For $r = 2$, we have to assume the a priori bound $|\beta| \leqslant M_1$ on $[0, T] \times \Omega^+$, i.e., (5.3), in order to get a bound that is linear in the highest-order derivative or energy. Then, we have from (5.3) for $r = 2$

$$
(5.19) \leqslant CK \|\beta\|_{L^\infty(\Omega^+)} E_r(t) \leqslant C(K, M_1) E_r(t).
$$

From the Hölder inequality, we get

$$
(5.17) \leqslant CK E_r^{1/2}(t) \|\nabla^r q^+\|_{L^2(\Omega^+)}. \tag{5.21}
$$

From (1.8), it follows that

$$
\partial_j (D_t v^j) + \Delta q^+ = \mu \partial_j (H^k \partial_k H^j),
$$

which yields from (2.11)

$$\Delta q^+ = -\partial_j v^k \partial_k v^j + \mu \partial_j H^k \partial_k H^j.$$

Since the Laplacian operator Δ is invariant, it yields

$$\Delta q^+ = -\nabla_a u^b \nabla_b u^a + \mu \nabla_a \beta^b \nabla_b \beta^a. \tag{5.22}$$

We have a simple estimate from the assumption (5.4) and Hölder's inequality, i.e.,

$$\begin{aligned}\|\Delta q^+\|_{L^2(\Omega^+)} \leqslant & C\|\nabla u\|_{L^2(\Omega^+)}\|\nabla u\|_{L^\infty(\Omega^+)} \\ & + C\|\nabla \beta\|_{L^2(\Omega^+)}\|\nabla \beta\|_{L^\infty(\Omega^+)} \\ \leqslant & CME_1^{1/2}(t),\end{aligned} \tag{5.23}$$

which is a lower energy term.

For $m \geqslant 0$, it follows that

$$\begin{aligned}\nabla^m \Delta q^+ = & -\sum_{s=0}^{m} \complement_m^s \nabla^s \nabla_a u^b \nabla^{m-s} \nabla_b u^a \\ & + \mu \sum_{s=0}^{m} \complement_m^s \nabla^s \nabla_a \beta^b \nabla^{m-s} \nabla_b \beta^a.\end{aligned}$$

From (5.20), we get for $s \geqslant 0$

$$\begin{aligned}\|\nabla^s \beta\|_{L^\infty(\Omega^+)} \leqslant & C \sum_{\ell=0}^{2} K_1^{n/2-\ell} \|\nabla^{\ell+s}\beta\|_{L^2(\Omega^+)} \\ \leqslant & C(K_1) \sum_{\ell=0}^{2} E_{s+\ell}^{1/2}(t),\end{aligned} \tag{5.24}$$

and, similarly,

$$\|\nabla^s u\|_{L^\infty(\Omega^+)} \leqslant C(K_1) \sum_{\ell=0}^{2} E_{s+\ell}^{1/2}(t). \tag{5.25}$$

From Hölder's inequality, (5.24), Lemma A.7 and (5.25), we get,

$$\begin{aligned}\|\nabla \Delta q^+\|_{L^2(\Omega^+)} \leqslant & C\|\nabla u\|_{L^\infty(\Omega^+)}\|\nabla^2 u\|_{L^2(\Omega^+)} \\ & + C\|\nabla \beta\|_{L^\infty(\Omega^+)}\|\nabla^2 \beta\|_{L^2(\Omega^+)} \\ \leqslant & CME_2^{1/2}(t), \ \|\nabla^2 \Delta q^+\|_{L^2(\Omega^+)} \\ \leqslant & C\|\nabla u\|_{L^\infty(\Omega^+)}\|\nabla^3 u\|_{L^2(\Omega^+)} + C\|\nabla^2 u\|_{L^4(\Omega^+)}^2 \\ & + C\|\nabla \beta\|_{L^\infty(\Omega^+)}\|\nabla^3 \beta\|_{L^2(\Omega^+)} + C\|\nabla^2 \beta\|_{L^4(\Omega^+)}^2 \\ \leqslant & CME_3^{1/2}(t) + C\|\nabla u\|_{L^\infty(\Omega^+)} \sum_{s=0}^{2} \|\nabla^{s+1} u\|_{L^2(\Omega^+)} K_1^{2-s}\end{aligned} \tag{5.26}$$

$$+ C\|\nabla\beta\|_{L^\infty(\Omega^+)} \sum_{s=0}^{2} \|\nabla^{s+1}\beta\|_{L^2(\Omega^+)} K_1^{2-s}$$
$$\leqslant CME_3^{1/2}(t) + C(K_1)\left(E_1^{1/2}(t) + E_2^{1/2}(t) + E_3^{1/2}(t)\right), \tag{5.27}$$

and

$$\begin{aligned}\|\nabla^3\Delta q^+\|_{L^2(\Omega^+)} \leqslant\, & C\|\nabla u\|_{L^\infty(\Omega^+)}\|\nabla^4 u\|_{L^2(\Omega^+)} \\ & + C\|\nabla^2 u\|_{L^3(\Omega^+)}\|\nabla^3 u\|_{L^6(\Omega^+)} \\ & + C\|\nabla\beta\|_{L^\infty(\Omega^+)}\|\nabla^4\beta\|_{L^2(\Omega^+)} \\ & + C\|\nabla^2\beta\|_{L^3(\Omega^+)}\|\nabla^3\beta\|_{L^6(\Omega^+)} \\ \leqslant\, & CME_4^{1/2}(t) + C(\text{Vol}\,\Omega^+)E_3^{1/2}(t)E_4^{1/2}(t). \end{aligned} \tag{5.28}$$

From the definition of the projection and the fact that the measure in the energy is $(-\nabla_N P)^{-1}\mathrm{d}\mu_\gamma$, we have

$$\|\Pi\nabla^r P\|_{L^2(\Gamma)} \leqslant \|\nabla P\|_{L^\infty(\Gamma)}^{1/2} E_r^{1/2}(t).$$

Thus, by (A.9), (5.26), (5.23) and Proposition 3.2, we obtain for any $2 \leqslant r \leqslant 4$

$$\begin{aligned} &\|\nabla^r q^+\|_{L^2(\Gamma)} + \|\nabla^r q^+\|_{L^2(\Omega^+)} \\ &\quad\leqslant C\|\Pi\nabla^r q^+\|_{L^2(\Gamma)} + C(\tilde{K}, \text{Vol}\,\Omega^+)\sum_{s\leqslant r-1}\|\nabla^s\Delta q^+\|_{L^2(\Omega^+)} \\ &\quad\leqslant C\|\nabla P\|_{L^\infty(\Gamma)}^{1/2}E_r^{1/2}(t) + C(r, K, K_1, \text{Vol}\,\Omega^-)E_0(0) \\ &\qquad + C(K, K_1, \text{Vol}\,\Omega^+)\left[M + (r-2)\sum_{s=1}^{r-1}E_s^{1/2}(t)\right]E_r^{1/2}(t). \end{aligned} \tag{5.29}$$

Therefore,

$$\begin{aligned} (5.17) \leqslant\, & C(r, K, K_1, \text{Vol}\,\Omega^-)E_0^2(0) \\ & + \left[C\|\nabla P\|_{L^\infty(\Gamma)}^{1/2} + C(K, K_1, \text{Vol}\,\Omega^+)\right. \\ & \left.\times\left[M + (r-2)\sum_{s=1}^{r-1}E_s^{1/2}(t)\right]\right]E_r(t). \end{aligned} \tag{5.30}$$

Now, we turn to the estimates of (5.15). Due to $P = 0$ on Γ implying $\gamma_b^a\nabla_a P = 0$ on Γ, we have from (2.21), by noticing $\vartheta = -1/\nabla_N P$, that

$$-\vartheta^{-1}N_b = \nabla_N P N_b = N^a\nabla_a P N_b = \delta_b^a\nabla_a P - \gamma_b^a\nabla_a P = \nabla_b P. \tag{5.31}$$

By Hölder's inequality and (5.31), we get

$$|(5.15)| \leqslant C\|\vartheta\|_{L^\infty(\Gamma)}^{1/2}E_r^{1/2}(t)\|\Pi\left(D_t\left(\nabla^r P\right) + \nabla^r u\cdot\nabla P\right)\|_{L^2(\Gamma)}.$$

It follows from (2.17) that

$$D_t \nabla^r P + \nabla^r u \cdot \nabla P = \operatorname{sgn}(2-r) \sum_{s=1}^{r-2} \mathrm{C}_r^{s+1} (\nabla^{s+1} u) \cdot \nabla^{r-s} P + \nabla^r D_t P. \quad (5.32)$$

We first consider the estimates of the last term in (5.32). By (A.11), (A.12) and (A.17), we have, for $2 \leqslant r \leqslant 4$

$$\begin{aligned} &\|\Pi \nabla^r D_t P\|_{L^2(\Gamma)} \\ &\quad \leqslant 2\|\overline{\nabla}^{r-2}\theta\|_{L^2(\Gamma)} \|\nabla_N D_t P\|_{L^\infty(\Gamma)} + C \sum_{k=1}^{r-1} K^k \|\nabla^{r-k} D_t P\|_{L^2(\Gamma)} \\ &\quad \leqslant C(1/\varepsilon) L \left(\|\Pi \nabla^r P\|_{L^2(\Gamma)} + \sum_{k=1}^{r-1} K^k \|\nabla^{r-k} P\|_{L^2(\Gamma)} \right) \\ &\qquad + C \sum_{k=1}^{r-1} K^k \|\nabla^{r-k} D_t P\|_{L^2(\Gamma)}. \end{aligned} \quad (5.33)$$

By (A.10), this yields

$$\begin{aligned} &\|\nabla^k D_t q^+\|_{L^2(\Gamma)} \\ &\quad \leqslant \delta \|\Pi \nabla^{k+1} D_t q^+\|_{L^2(\Gamma)} + C(1/\delta, K, \mathrm{Vol}\, \Omega^+) \\ &\qquad \times \sum_{s \leqslant k-1} \|\nabla^s \Delta D_t q^+\|_{L^2(\Omega^+)}. \end{aligned} \quad (5.34)$$

From (2.16), (5.22) and Lemma 2.1, it follows that

$$\begin{aligned} \Delta D_t q^+ =& 4 g^{ac} \nabla_c u^b \nabla_a \nabla_b q^+ + (\Delta u^e) \nabla_e q^+ + 2 \nabla_e u^b \nabla_b u^a \nabla_a u^e \\ &- 2\mu \nabla_b u^a \nabla_a \beta^c \nabla_c \beta^b - 2\mu \nabla_b u^a \beta^c \nabla_a \nabla_c \beta^b \\ &+ 2\mu \nabla_b \beta^a \beta^e \nabla_e \nabla_a u^b. \end{aligned}$$

By (5.24), (5.29) and Lemma A.8, we get, for $s \leqslant 2$,

$$\begin{aligned} \|\nabla^s \Delta D_t q^+\|_{L^2(\Omega^+)} \leqslant & C \|\nabla u\|_{L^\infty(\Omega^+)} \|\nabla^{s+2} q^+\|_{L^2(\Omega^+)} \\ &+ s(s-1) C \|\nabla^3 u\|_{L^2(\Omega^+)} \|\nabla^2 q^+\|_{L^\infty(\Omega^+)} \\ &+ s C \|\nabla^2 u\|_{L^4(\Omega^+)} \|\nabla^{s+1} q^+\|_{L^4(\Omega^+)} \\ &+ C \|\nabla^{s+2} u\|_{L^2(\Omega^+)} \|\nabla q^+\|_{L^\infty(\Omega^+)} \\ &+ C \left(\|\nabla u\|_{L^\infty(\Omega^+)} \|\nabla u\|_{L^\infty(\Omega^+)} \right. \\ &\left. + \|\nabla \beta\|_{L^\infty(\Omega^+)} \|\nabla \beta\|_{L^\infty(\Omega^+)} \right) \|\nabla^{s+1} u\|_{L^2(\Omega^+)} \\ &+ s(s-1) C \|\nabla u\|_{L^\infty(\Omega^+)} \|\nabla^2 u\|_{L^4(\Omega^+)} \|\nabla^2 u\|_{L^4(\Omega^+)} \\ &+ C \|\nabla u\|_{L^\infty(\Omega^+)} \|\nabla \beta\|_{L^\infty(\Omega^+)} \|\nabla^{s+1} \beta\|_{L^2(\Omega^+)} \\ &+ s C \|\nabla^2 u\|_{L^4(\Omega^+)} \|\nabla^2 \beta\|_{L^4(\Omega^+)} \end{aligned}$$

$$\times \left((s-1)\|\nabla\beta\|_{L^\infty(\Omega^+)} + \|\beta\|_{L^\infty(\Omega^+)}\right)$$
$$+ s(s-1)C\|\nabla u\|_{L^\infty(\Omega^+)}\|\nabla^2\beta\|_{L^4(\Omega^+)}\|\nabla^2\beta\|_{L^4(\Omega^+)}$$
$$+ C\|\nabla u\|_{L^\infty(\Omega^+)}\|\beta\|_{L^\infty(\Omega^+)}\|\nabla^{s+2}\beta\|_{L^2(\Omega^+)}$$
$$+ sC\|\nabla^3 u\|_{L^2(\Omega^+)}\|\beta\|_{L^\infty(\Omega^+)}$$
$$\times \left((s-1)\|\nabla^2\beta\|_{L^\infty(\Omega^+)} + \|\nabla\beta\|_{L^\infty(\Omega^+)}\right)$$
$$+ s(s-1)C\|\nabla^3\beta\|_{L^2(\Omega^+)}\|\beta\|_{L^\infty(\Omega^+)}\|\nabla^2 u\|_{L^\infty(\Omega^+)}$$
$$+ s(s-1)C\|\nabla\beta\|_{L^\infty(\Omega^+)}\|\nabla^2\beta\|_{L^4(\Omega^+)}\|\nabla^2 u\|_{L^4(\Omega^+)}$$
$$+ s(s-1)C\|\nabla\beta\|_{L^\infty(\Omega^+)}\|\beta\|_{L^\infty(\Omega^+)}\|\nabla^4 u\|_{L^2(\Omega^+)}$$
$$+ s(s-1)C\|\nabla^2\beta\|_{L^\infty(\Omega^+)}\|\beta\|_{L^\infty(\Omega^+)}\|\nabla^3 u\|_{L^2(\Omega^+)}.$$

From Lemma A.7 and (5.25), it follows for $s \leqslant 2$ that

$$\|\nabla^{s+1}u\|_{L^4(\Omega^+)} \leqslant C\|\nabla^s u\|_{L^\infty(\Omega^+)}^{1/2}\left(\sum_{\ell=0}^{2}\|\nabla^{s+\ell}u\|_{L^2(\Omega^+)}K_1^{2-\ell}\right)^{1/2}$$
$$\leqslant C(K_1)\sum_{\ell=0}^{2}E_{s+\ell}^{1/2}(t).$$

All the terms with $L^4(\Omega^+)$ norms can be estimated in the same way with the help of (5.24), (5.25), the similar estimates of q^+ and the assumptions. Then, we obtain the bound which is linear about the highest-order derivative or the highest-order energy $E_r^{1/2}(t)$, i.e.,

$$\|\nabla^s \Delta D_t q^+\|_{L^2(\Omega^+)} \leqslant C\left(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega^+, E_0(0)\right)$$
$$\times\left(1+\sum_{\ell=0}^{r-1}E_\ell(t)\right)\left(1+E_r^{1/2}(t)\right). \tag{5.35}$$

Because of

$$D_t q^- = \frac{\mu}{2}D_t|\varpi|^2 = -\mu\varpi^a(\nabla\times\Xi)_a + \mu u^c\varpi^a\nabla_c\varpi_a,$$

it follows from Propositions 3.1 and 4.1 that

$$\|\nabla^k D_t q^-\|_{L^2(\Gamma)} \leqslant \mu C(K)E_0(0)\sum_{s=0}^{k+1}E_s^{1/2}(t)$$
$$+ \mu C(r, K, \mathrm{Vol}\,\Omega^+)E_0(0)\|\nabla\times\Xi_0\|_{L^2(\Omega^-)}\mathrm{e}^{C(K,L)t}.$$

Then, from (5.29), (5.33), (5.34), (5.35) and taking some small δ's which are independent of $E_r(t)$, we obtain, by the induction argument for r, that

$$\|\Pi\nabla^r D_t q^+\|_{L^2(\Gamma)} \leqslant \mu C(r, K)E_0(0)\|\nabla\times\Xi_0\|_{L^2(\Omega^-)}\mathrm{e}^{C(K,L)t}$$
$$+ (1+\mu)C\left(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0)\right)$$

$$\times \left(1 + \sum_{\ell=0}^{r-1} E_\ell(t)\right)\left(1 + E_r^{1/2}(t)\right). \tag{5.36}$$

Then, we have the similar bound for $\|\varPi \nabla^r D_t P\|_{L^2(\varGamma)}$.

For (5.32), it only remains to estimate

$$\|\varPi \left((\nabla^{s+1}u)\cdot\nabla^{r-s}P\right)\|_{L^2(\varGamma)} \quad \text{for } 1 \leqslant s \leqslant r-2.$$

For the cases $r = 3, 4$ and $s = r - 2$, we get, from (5.7) and Lemma A.10, that

$$\begin{aligned}
\|\varPi\left(\nabla^{r-1}u\cdot\nabla^2 P\right)\|_{L^2(\varGamma)} &\leqslant \|\nabla^{r-1}u\|_{L^2(\varGamma)}\|\nabla^2 P\|_{L^\infty(\varGamma)} \\
&\leqslant C(K, \mathrm{Vol}\,\varOmega^+)L(\mathrm{Vol}\,\varOmega^+)^{1/6}\left(\|\nabla^r u\|_{L^2(\varOmega)} + \|\nabla^{r-1}u\|_{L^2(\varOmega)}\right) \\
&\leqslant C(K, L, \mathrm{Vol}\,\varOmega^+)\left(E_{r-1}^{1/2}(t) + E_r^{1/2}(t)\right).
\end{aligned}$$

For the case $r = 4$ and $s = 1$, by (A.1), Lemma A.10, (5.29) and Proposition 3.1, we have

$$\begin{aligned}
\|\varPi\left(\nabla^2 u\cdot\nabla^3 P\right)\|_{L^2(\varGamma)} =& \|\varPi\nabla^2 u\cdot\varPi\nabla^3 P \\
&+ \varPi(\nabla^2 u\cdot N)\tilde{\otimes}\varPi(N\cdot\nabla^3 P)\|_{L^2(\varGamma)} \\
\leqslant& C\|\varPi\nabla^2 u\|_{L^4(\varGamma)}\|\varPi\nabla^3 P\|_{L^4(\varGamma)} \\
&+ C\|\varPi(N^a\nabla^2 u_a)\|_{L^4(\varGamma)}\|\varPi(\nabla_N\nabla^2 P)\|_{L^4(\varGamma)} \\
\leqslant& C\|\nabla^2 u\|_{L^4(\varGamma)}\|\nabla^3 P\|_{L^4(\varGamma)} \\
\leqslant& C(K, \mathrm{Vol}\,\varOmega^+)\left(\|\nabla^3 u\|_{L^2(\varOmega^+)} + \|\nabla^2 u\|_{L^2(\varOmega^+)}\right) \\
&\times\left(\|\nabla^4 q^+\|_{L^2(\varOmega^+)} + \|\nabla^3 q^+\|_{L^2(\varOmega^+)} + \|\nabla^3 q^-\|_{L^4(\varGamma)}\right) \\
\leqslant& C(K, K_1, \mathrm{Vol}\,\varOmega^+)(E_3^{1/2}(t) + E_2^{1/2}(t)) \\
&\times\left(\sum_{s=0}^{3} E_s(t) + \left(\sum_{\ell=0}^{2} E_\ell^{1/2}(t)\right)E_4^{1/2}(t)\right. \\
&\left.+ C(K, \mathrm{Vol}\,\varOmega^-)E_0(0)\right) \\
\leqslant& C(K, K_1, \mathrm{Vol}\,\varOmega)\sum_{s=0}^{3} E_s(t)\sum_{\ell=0}^{4} E_\ell^{1/2}(t).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
|(5.15)| \leqslant& C(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\varOmega, E_0(0)) \\
&\times\left(1 + \sum_{s=0}^{r-1} E_s(t)\right)\left(1 + E_r(t)\right) \\
&+ \mu C(r, K, \mathrm{Vol}\,\varOmega^+)E_0(0)\|\nabla\times\varXi_0\|_{L^2(\varOmega^-)}\mathrm{e}^{C(K,L)t}E_r^{1/2}(t).
\end{aligned}$$

Therefore, we have obtained

$$|(5.9) + (5.10) + (5.12)|$$

$$\leqslant C(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0))\Big(1 + \sum_{s=0}^{r-1} E_s(t)\Big)\big(1 + E_r(t)\big)$$
$$+ \mu C(r, K) E_0(0) \|\nabla \times \Xi_0\|_{L^2(\Omega^-)} \mathrm{e}^{C(K,L)t} E_r^{1/2}(t).$$

By a similar argument in [11, (5.65)], we get

$$|(5.11)| \leqslant C(K, K_1, M, \mathrm{Vol}\,\Omega, 1/\varepsilon)\left(1 + \sum_{s=0}^{r-1} E_s(t)\right) E_r(t).$$

From (A.5) and (2.15), we have

$$D_t(\nabla_N P) = -2h_d^a N^d \nabla_a P + h_{NN} \nabla_N P + \nabla_N D_t P,$$

which yields

$$\frac{\vartheta_t}{\vartheta} = -\frac{D_t \nabla_N P}{\nabla_N P} = \frac{2h_d^a N^d \nabla_a P}{\nabla_N P} - h_{NN} - \frac{\nabla_N D_t P}{\nabla_N P} = h_{NN} - \frac{\nabla_N D_t P}{\nabla_N P}.$$

Thus, we can easily obtain that (5.13) can be controlled by $C(K, M, L, 1/\varepsilon) E_r(t)$.

Note that there always exists a constant $C > 0$ such that $\|\nabla \times \Xi_0\|_{L^2(\Omega^-)}^2 \leqslant C E_0(0)$ at initial time. Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_r(t) \leqslant C(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0)) \times \left(1 + \sum_{s=0}^{r-1} E_s(t)\right)\big(1 + E_r(t)\big) + \mathrm{sgn}((r-1)(r-2))\mu^2 C(r, K) E_0(0) \mathrm{e}^{C(K,L)t},$$

which implies, by Grönwall's inequality, that

$$E_r(t) \leqslant E_r(0) \exp\Big(C\,(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0)) \times \int_0^t \Big(1 + \sum_{s=0}^{r-1} E_s(\tau)\Big)\mathrm{d}\tau\Big)$$
$$+ \Big\{C(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0)) + \mathrm{sgn}((r-1)(r-2))\mu^2 \mathrm{e}^{C(K,L)t}\Big\}$$
$$\times \int_0^t \left(1 + \sum_{s=0}^{r-1} E_s(\tau)\right) \exp\Big(C(K, K_1, M, M_1, L, 1/\varepsilon, \mathrm{Vol}\,\Omega, E_0(0)) \times \int_\tau^t \Big(1 + \sum_{s=0}^{r-1} E_s(s)\Big)\mathrm{d}s\Big)\mathrm{d}\tau.$$

By using induction for $r = 1, 2, 3, 4$ in turn, we obtain the desired estimates. □

6. Justification of A Priori Assumptions

Let $\mathscr{K}(t)$ and $\varepsilon(t)$ be the maximum and minimum values, respectively, such that (5.5) and (5.6) hold at time t:

$$\mathscr{K}(t) = \max\left(\|\theta(t,\cdot)\|_{L^\infty(\Gamma)}, 1/\iota_0(t)\right), \tag{6.1}$$

and

$$\mathscr{E}(t) = \|1/(\nabla_N q(t,\cdot))\|_{L^\infty(\Gamma)} = 1/\varepsilon(t). \tag{6.2}$$

Lemma 6.1. *Let $K_1 \geqslant 1/\iota_1$ be as in Definition A.2, $\mathscr{E}(t)$ as in (6.2). Then there are continuous functions G_j, $j = 1, 2, 3, 4, 5$, such that*

$$\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \beta\|_{L^\infty(\Omega)} + \|\beta\|_{L^\infty(\Omega)} \leqslant G_1(K_1, E_0, \ldots, E_4), \tag{6.3}$$

$$\|\nabla P\|_{L^\infty(\Omega)} + \|\nabla^2 P\|_{L^\infty(\Gamma)} \leqslant G_2(K_1, \mathscr{E}, E_0, \ldots, E_4, \mathrm{Vol}\,\Omega), \tag{6.4}$$

$$\|\theta\|_{L^\infty(\Gamma)} \leqslant G_3(K_1, \mathscr{E}, E_0, \ldots, E_4, \mathrm{Vol}\,\Omega), \tag{6.5}$$

$$\|u\|_{L^\infty(\Gamma)} + \|\nabla D_t P\|_{L^\infty(\Gamma)} \leqslant G_4(K_1, \mathscr{E}, E_0, \ldots, E_4, \mathrm{Vol}\,\Omega), \tag{6.6}$$

and

$$\|u\|_{L^\infty(\Gamma)} + \sum_{\ell=0}^{2} \|\nabla u\|_{L^2(\Gamma)} \leqslant G_5(K_1, \mathscr{E}, E_0, \ldots, E_4, \mathrm{Vol}\,\Omega). \tag{6.7}$$

Proof. (6.3) follows from (5.25), (5.24) and (5.20). (6.4) follows from Lemmas A.8 and A.6, Lemmas A.9–A.10, and (5.23). Since, from (A.2),

$$|\nabla^2 P| \geqslant |\Pi \nabla^2 P| = |\nabla_N P||\theta| \geqslant \mathscr{E}^{-1}|\theta|, \tag{6.8}$$

(6.5) follows from (6.4). (6.6) follows from Lemma A.6, (5.34), (5.35) and (5.36). (6.7) follows from Lemmas A.6 and A.10. □

Lemma 6.2. *Let $K_1 \geqslant 1/\iota_1$ and ε_1 be as in Definition A.2. Then*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} E_r\right| \leqslant C_r(K_1, \mathscr{E}, E_0, \ldots, E_4, \mathrm{Vol}\,\Omega) \sum_{s=0}^{r} E_s, \tag{6.9}$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}\right| \leqslant C_r\left(K_1, \mathscr{E}, E_0, \ldots, E_4, \mathrm{Vol}\,\Omega\right). \tag{6.10}$$

Proof. (6.9) is a consequence of Lemma 6.1 and the estimates in the proof of Theorems 5.1. (6.10) follows from (6.6) and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{-\nabla_N P(t,\cdot)}\right\|_{L^\infty(\Gamma)}\right| \leqslant C\left\|\frac{1}{-\nabla_N P(t,\cdot)}\right\|^2_{L^\infty(\Gamma)} \|\nabla_N D_t P(t,\cdot)\|_{L^\infty(\Gamma)}.$$

□

As a consequence of Lemma 6.2, we have the following:

Lemma 6.3. *There exists a continuous function* $\mathscr{T} > 0$ *depending on* K_1, $\mathscr{E}(0)$, $E_0(0), \ldots, E_{n+1}(0)$, $\mathrm{Vol}\,\Omega$ *such that for*

$$0 \leqslant t \leqslant \mathscr{T}(K_1, \mathscr{E}(0), E_0(0), \ldots, E_4(0), \mathrm{Vol}\,\Omega),$$

the following statements hold: we have

$$E_s(t) \leqslant 2E_s(0), \quad 0 \leqslant s \leqslant 4; \quad \mathscr{E}(t) \leqslant 2\mathscr{E}(0). \tag{6.11}$$

Furthermore,

$$\frac{g_{ab}(0, y)Y^a Y^b}{2} \leqslant g_{ab}(t, y)Y^a Y^b \leqslant 2g_{ab}(0, y)Y^a Y^b, \tag{6.12}$$

and with ε_1 *as in Definition A.2,*

$$|\mathbf{n}(x(t, \bar{y})) - \mathbf{n}(x(0, \bar{y}))| \leqslant \frac{\varepsilon_1}{16}, \qquad \bar{y} \in \Gamma, \tag{6.13}$$

$$|x(t, y) - x(t, y)| \leqslant \frac{\iota_1}{16}, \qquad y \in \Omega, \tag{6.14}$$

$$\left|\frac{\partial x(t, \bar{y})}{\partial y} - \frac{\partial(0, \bar{y})}{\partial y}\right| \leqslant \frac{\varepsilon_1}{16}, \qquad \bar{y} \in \Gamma. \tag{6.15}$$

Proof. Since the proof is similar to [11, Lemma 6.3], we omit the details. □

Now we use (6.12)–(6.15) to pick a K_1, i.e., ι_1, which depends only on its value at $t = 0$,

$$\iota_1(t) \geqslant \iota_1(0)/2.$$

Lemma 6.4. *Let* $\mathscr{T}$ *be as in Lemma 6.2. Pick* $\iota_1 > 0$ *such that*

$$|\mathbf{n}(x(0, y_1)) - \mathbf{n}(x(0, y_2))| \leqslant \frac{\varepsilon_1}{2}, \quad \text{whenever } |x(0, y_1) - x(0, y_2)| \leqslant 2\iota_1. \tag{6.16}$$

Then if $t \leqslant \mathscr{T}$, *we have*

$$|\mathbf{n}(x(t, y_1)) - \mathbf{n}(x(t, y_2))| \leqslant \varepsilon_1, \quad \text{whenever } |x(t, y_1) - x(t, y_2)| \leqslant 2\iota_1. \tag{6.17}$$

Proof. (6.17) follows from (6.16), (6.13) and (6.14) in view of triangle inequalities. □

Finally, Lemma 6.4 allows us to pick a K_1 depending only on initial conditions, while Lemma 6.3 gives us $\mathscr{T} > 0$ that depends only on the initial conditions and K_1 such that, by Lemma 6.4, $1/\iota_1 \leqslant K_1$ for $t \leqslant \mathscr{T}$. Thus, we immediately obtain Theorem 1.1.

Appendix A. Preliminaries and Some Estimates

Let us now recall some properties of the projection. Since $g^{ab} = \gamma^{ab} + N^a N^b$, we have

$$\Pi(S \cdot R) = \Pi(S) \cdot \Pi(R) + \Pi(S \cdot N)\tilde{\otimes}\Pi(N \cdot R), \tag{A.1}$$

where $S\tilde{\otimes}R$ denotes some partial symmetrization of the tensor product $S \otimes R$, i.e., a sum over some subset of the permutations of the indices divided by the number of permutations in that subset. Similarly, we let $S\tilde{\cdot}R$ denote a partial symmetrization of the dot product $S \cdot R$. Now we recall some identities:

$$\Pi\nabla^2 q = \overline{\nabla}^2 q + \theta\nabla_N q, \tag{A.2}$$

$$\Pi\nabla^3 q = \overline{\nabla}^3 q - 2\theta\tilde{\otimes}(\theta\tilde{\cdot}\overline{\nabla}q) + (\overline{\nabla}\theta)\nabla_N q + 3\theta\tilde{\otimes}\overline{\nabla}\nabla_N q, \tag{A.3}$$

and

$$\begin{aligned}\Pi\nabla^4 q = &\overline{\nabla}^4 q - \theta\tilde{\otimes}\left(5(\overline{\nabla}\theta)\tilde{\cdot}\overline{\nabla}q + 8\theta\tilde{\cdot}\overline{\nabla}^2 q\right) - 2(\overline{\nabla}\theta)\tilde{\otimes}(\theta\tilde{\cdot}\overline{\nabla}q) + (\overline{\nabla}^2\theta)\nabla_N q \\ &+ 4(\overline{\nabla}\theta)\tilde{\otimes}\overline{\nabla}\nabla_N q + 6\theta\tilde{\otimes}\overline{\nabla}^2\nabla_N q - 3\theta\tilde{\otimes}(\theta\tilde{\cdot}\theta)\nabla_N q + 3\theta\tilde{\otimes}\theta\nabla_N^2 q.\end{aligned} \tag{A.4}$$

Definition A.1. Let $\mathbf{n}(\bar{x})$ be the outward unit normal to Γ_t at $\bar{x} \in \Gamma_t$. Let dist $(x_1, x_2) = |x_1 - x_2|$ denote the Euclidean distance in $\mathbb{R}^n$, and for $\bar{x}_1, \bar{x}_2 \in \Gamma_t$, let dist $_{\Gamma_t}(\bar{x}_1, \bar{x}_2)$ denote the geodesic distance on the boundary.

Definition A.2. Let $0 < \varepsilon_1 < 2$ be a fixed number, and let $\iota_1 = \iota_1(\varepsilon_1)$ the largest number such that

$$|\mathbf{n}(\bar{x}_1) - \mathbf{n}(\bar{x}_2)| \leqslant \varepsilon_1 \quad \text{whenever } |\bar{x}_1 - \bar{x}_2| \leqslant \iota_1,\ \bar{x}_1, \bar{x}_2 \in \Gamma_t.$$

Lemma A.1. (cf. [4, Lemma 3.9]) *Let N be the unit normal to Γ, and let $h_{ab} = \frac{1}{2}D_t g_{ab}$. On $[0, T] \times \Gamma$, we have*

$$D_t N_a = h_{NN} N_a,\ D_t N^c = -2h^c_d N^d + h_{NN} N^c,\ D_t\gamma^{ab} = -2\gamma^{ac} h_{cd}\gamma^{db}, \tag{A.5}$$

where $h_{NN} = h_{ab}N^a N^b$. The volume element on Γ satisfies

$$D_t d\mu_\gamma = (\operatorname{tr} h - h_{NN})d\mu_\gamma. \tag{A.6}$$

Lemma A.2. (cf. [4, Lemma 5.5]) *Let $w_a = w_{Aa} = \nabla^r_A f_a$, $\nabla^r_A = \nabla_{a_1}\cdots\nabla_{a_r}$, f be a $(0, 1)$ tensor, and $[\nabla_a, \nabla_b] = 0$. Let* $\operatorname{div} w = \nabla_a w^a = \nabla^r \operatorname{div} f$, *and let* $(\nabla \times w)_{ab} = \nabla_a w_b - \nabla_b w_a = \nabla^r(\nabla \times f)_{ab}$. *Then,*

$$|\nabla w|^2 \leqslant C(g^{ab}\gamma^{cd}\gamma^{AB}\nabla_c w_{Aa}\nabla_d w_{Bb} + |\operatorname{div} w|^2 + |\nabla \times w|^2), \tag{A.7}$$

and

$$\begin{aligned}\int_\Omega |\nabla w|^2 d\mu_g \leqslant C\int_\Omega (&N^a N^f g^{cd}\gamma^{AF}\nabla_c w_{Aa}\nabla_d w_{Ff} \\ &+ |\operatorname{div} w|^2 + |\nabla \times w|^2 + K^2|w|^2)\mathrm{d}\mu_g.\end{aligned} \tag{A.8}$$

Lemma A.3. (cf. [4, Proposition 5.8]) *Let ι_0 and ι_1 be as in (1.13) and Definition A.2, and suppose that $|\theta|+1/\iota_0 \leqslant K$ and $1/\iota_1 \leqslant K_1$. Then with $\tilde{K} = \min(K, K_1)$ we have, for any $r \geqslant 2$ and $\delta > 0$,*

$$\begin{aligned} &\|\nabla^r q\|_{L^2(\Gamma)} + \|\nabla^r q\|_{L^2(\Omega)} \\ &\quad \leqslant C\|\Pi\nabla^r q\|_{L^2(\Gamma)} + C(\tilde{K}, \mathrm{Vol}\,\Omega) \sum_{s\leqslant r-1} \|\nabla^s \Delta q\|_{L^2(\Omega)}, \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} &\|\nabla^{r-1} q\|_{L^2(\Gamma)} + \|\nabla^r q\|_{L^2(\Omega)} \\ &\quad \leqslant \delta\|\Pi\nabla^r q\|_{L^2(\Gamma)} + C(1/\delta, K, \mathrm{Vol}\,\Omega) \sum_{s\leqslant r-2} \|\nabla^s \Delta q\|_{L^2(\Omega)}. \end{aligned} \tag{A.10}$$

Lemma A.4. (cf. [4, Proposition 5.9]) *Assume that $2 \leqslant r \leqslant 4$. Suppose that $|\theta| \leqslant K$ and $\iota_1 \geqslant 1/K_1$, where ι_1 is as in Definition A.2. If $q = 0$ on Γ, then*

$$\|\Pi\nabla^r q\|_{L^2(\Gamma)} \leqslant 2\|\overline{\nabla}^{r-2}\theta\|_{L^2(\Gamma)}\|\nabla_N q\|_{L^\infty(\Gamma)} + C\sum_{k=1}^{r-1} K^k\|\nabla^{r-k} q\|_{L^2(\Gamma)}. \tag{A.11}$$

If, in addition, $|\nabla_N q| \geqslant \varepsilon > 0$ and $|\nabla_N q| \geqslant 2\varepsilon\|\nabla_N q\|_{L^\infty(\Gamma)}$, then

$$\|\overline{\nabla}^{r-2}\theta\|_{L^2(\Gamma)} \leqslant C(1/\varepsilon)\left(\|\Pi\nabla^r q\|_{L^2(\Gamma)} + \sum_{k=1}^{r-1} K^k\|\nabla^{r-k} q\|_{L^2(\Gamma)}\right). \tag{A.12}$$

Lemma A.5. ([4, Lemma A.1]) *If α is a $(0, r)$ tensor, then with $a = k/m$ and a constant C that only depends on m and n, such that*

$$\|\overline{\nabla}^k\alpha\|_{L^s(\Gamma)} \leqslant C\|\alpha\|_{L^q(\Gamma)}^{1-a}\|\overline{\nabla}^m\alpha\|_{L^p(\Gamma)}^a,$$

if

$$\frac{m}{s} = \frac{k}{p} + \frac{m-k}{q}, \quad 2 \leqslant p \leqslant s \leqslant q \leqslant \infty.$$

Lemma A.6. ([4, Lemma A.2]) Suppose that for $\iota_1 \geqslant 1/K_1$

$$|\mathbf{n}(\bar{x}_1) - \mathbf{n}(\bar{x}_2)| \leqslant \varepsilon_1, \quad \text{whenever } |\bar{x}_1 - \bar{x}_2| \leqslant \iota_1,\ \bar{x}_1, \bar{x}_2 \in \Gamma_t,$$

and

$$C_0^{-1}\gamma_{ab}^0(y)Z^aZ^b \leqslant \gamma_{ab}(t, y)Z^aZ^b \leqslant C_0\gamma_{ab}^0(y)Z^aZ^b, \quad \text{if } Z \in T(\Omega^+),$$

where $\gamma_{ab}^0(y) = \gamma_{ab}(0, y)$. Then if α is a $(0, r)$ tensor,

$$\|\alpha\|_{L^{(n-1)p/(n-1-kp)}(\Gamma)} \leqslant C(K_1)\sum_{\ell=0}^{k} \|\nabla^\ell\alpha\|_{L^p(\Gamma)}, \quad 1 \leqslant p < \frac{n-1}{k}, \tag{A.13}$$

and

$$\|\alpha\|_{L^\infty(\Gamma)} \leqslant \delta\|\nabla^k\alpha\|_{L^p(\Gamma)} + C_\delta(K_1)\sum_{\ell=0}^{k-1}\|\nabla^\ell\alpha\|_{L^p(\Gamma)}, \quad k > \frac{n-1}{p}, \quad \text{(A.14)}$$

for any $\delta > 0$.

Lemma A.7. ([4, Lemma A.3]) *With notation as in Lemmas A.5 and A.6, we have*

$$\sum_{j=0}^{k}\|\nabla^j\alpha\|_{L^s(\Omega)} \leqslant C\|\alpha\|_{L^q(\Omega)}^{1-a}\left(\sum_{i=0}^{m}\|\nabla^i\alpha\|_{L^p(\Omega)}K_1^{m-i}\right)^a.$$

Lemma A.8. ([4, Lemma A.4]) *Suppose that $\iota_1 \geqslant 1/K_1$ and α is a $(0, r)$ tensor. Then*

$$\|\alpha\|_{L^{np/(n-kp)}(\Omega)} \leqslant C\sum_{\ell=0}^{k}K_1^{k-\ell}\|\nabla^\ell\alpha\|_{L^p(\Omega)}, \quad 1 \leqslant p < \frac{n}{k}, \quad \text{(A.15)}$$

and

$$\|\alpha\|_{L^\infty(\Omega)} \leqslant C\sum_{\ell=0}^{k}K_1^{n/p-\ell}\|\nabla^\ell\alpha\|_{L^p(\Omega)}, \quad k > \frac{n}{p}. \quad \text{(A.16)}$$

Lemma A.9. ([4, Lemma A.5]) *Suppose that $q = 0$ on Γ. Then*

$$\|q\|_{L^2(\Omega)} \leqslant C(\mathrm{Vol}\,\Omega)^{1/n}\|\nabla q\|_{L^2(\Omega)}, \ \|\nabla q\|_{L^2(\Omega)} \leqslant C(\mathrm{Vol}\,\Omega)^{1/2n}\|\Delta q\|_{L^2(\Omega)}.$$

Lemma A.10. ([4, Lemma A.7]) *Let α be a $(0, r)$ tensor. Assume that*

$$\mathrm{Vol}\,\Omega \leqslant V \text{ and } \|\theta\|_{L^\infty(\Gamma)} + 1/\iota_0 \leqslant K,$$

then there is a $C = C(K, V, r, n)$ such that

$$\|\alpha\|_{L^{(n-1)p/(n-p)}(\Gamma)} \leqslant C\|\nabla\alpha\|_{L^p(\Omega)} + C\|\alpha\|_{L^p(\Omega)}, \quad 1 \leqslant p < n, \quad \text{(A.17)}$$

and

$$\|\nabla^2\alpha\|_{L^2(\Omega)} \leqslant C\left(\|\Pi\nabla^2\alpha\|_{L^{2(n-1)/n}(\Gamma)} + \|\Delta\alpha\|_{L^2(\Omega)} + \|\nabla\alpha\|_{L^2(\Omega)}\right). \quad \text{(A.18)}$$

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