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Well-posedness for the linearized free boundary problem of incompressible ideal magnetohydrodynamics equations

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Abstract

The well-posedness theory is studied for the linearized free boundary problem of incompressible ideal magnetohydrodynamics equations in a bounded domain. We express the magnetic field in terms of the velocity field and the deformation tensors in Lagrangian coordinates, and substitute it into the momentum equation to get an equation of the velocity in which the initial magnetic field serves only as a parameter. Then, the velocity equation is linearized with respect to the position vector field whose time derivative is the velocity. In this formulation, a key idea is to use the Lie derivative of the magnetic field taking the advantage that the magnetic field is tangential to the free boundary and divergence free. This paper contributes to the program of developing geometric approaches to study the well-posedness of free boundary problems of ideal magnetohydrodynamics equations under the condition of Taylor sign type for general free boundaries not restricted to graphs.

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Contents

1. Introduction

This paper is concerned with the well-posedness of the linearized motion of the following incompressible ideal magnetohydrodynamics (MHD) equations with free boundary

$$\begin{cases} v_t + v\cdot\nabla v + \nabla p = \mu H\cdot\nabla H, & \text{in } \mathcal{D}, \quad (1.1)\\ H_t + v\cdot\nabla H = H\cdot\nabla v, & \text{in } \mathcal{D}, \quad (1.2)\\ \operatorname{div} v = 0, \quad \operatorname{div} H = 0, & \text{in } \mathcal{D}, \quad (1.3)\end{cases}$$

where v is the velocity field, H is the magnetic field, p is the total pressure including the fluid pressure and the magnetic pressure, and $\mu > 0$ is the vacuum permeability, $\mathcal{D} = \cup_{0\leqslant t\leqslant T}\{t\}\times\Omega_t$, $\Omega_t \subset \mathbb{R}^n$ is the domain occupied by the fluid at time t.

We also require boundary conditions on the free boundary $\partial\mathcal{D}$:

$$H \cdot \mathcal{N} = 0, \quad p = 0, \quad \text{on } \partial\mathcal{D}, \tag{1.4}$$

$$(\partial_t + v^k \partial_k)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \tag{1.5}$$

where $\mathcal{N}$ is the exterior unit normal to $\Gamma_t := \partial\Omega_t$. The condition $p = 0$ indicates that the total pressure vanishes outside the domain. Here the fluid considered is an incompressible ideal case. Roughly speaking, the velocity determines the motion of the boundary, and the boundary is the level set of the total pressure that determines the acceleration together with the magnetic tension. The condition $H \cdot \mathcal{N} = 0$ comes from the assumption that the boundary Γ_t is a perfect conductor, and should be understood as the constraint on the initial data since it will hold true for all $t \in [0, T]$ if it holds initially as showed in [16]. The condition (1.5) means that the boundary moves with the velocity v of the fluid particles at the boundary.

Given a domain $\Omega \subset \mathbb{R}^n$ that is homeomorphic to the unit ball, and initial data (v_0, H_0) satisfying the constrain (1.3), we expect to find a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ and vector fields (v, H) solving (1.1)-(1.5) with initial conditions

$$\{x : (0, x) \in \mathcal{D}\} = \Omega; \quad v = v_0, \; H = H_0, \text{ on } \{0\} \times \Omega. \tag{1.6}$$

Then, let $\Omega_t = \{x : (t, x) \in \mathcal{D}\}$. Motivated by the Taylor sign condition on the fluid pressure for the Euler equations, we raised an analogous condition based on the total pressure for ideal MHD in [16]:

$$\nabla_{\mathcal{N}} p \leqslant -c_0 < 0 \text{ on } \partial\mathcal{D}, \tag{1.7}$$

where $\nabla_{\mathcal{N}} = \mathcal{N}^i \partial_{x^i}$. Here we have used the summation convection over repeated upper and lower indices. In [16], we have proved a priori estimates in standard Sobolev spaces for the free boundary problem of incompressible ideal MHD system (1.1)-(1.6) under the condition (1.7). We also showed in [17] that the above free boundary problem (1.1)-(1.6) under consideration would be ill-posed at least for the case $n = 2$ if the condition (1.7) was violated. Thus, it will be much reasonable and necessary to require this condition (1.7) in the studies of well-posedness of the considering free boundary problem of incompressible ideal MHD equations.

However, all the symmetries of the nonlinear equations were used for the a priori estimates in [16], and so only hold for perturbations of the equations that preserved all the symmetries. Thus, those a priori estimates for solutions for the nonlinear problem cannot be used for the linearized equations which do not preserve the symmetries. Of course, the results in [16] are important to raising the meaningful and reasonable condition (1.7) for the well-posedness.

Magnetic fields are essential in many important physical situations ([11,31,44]), for example, solar flares in astrophysics due to the coupling between magnetic and thermomechanical degrees of freedom for which magnetic reconnection is thought to be the mechanism responsible for the conversion of magnetic energy into heat and fluid motion ([11]). Moreover, interface problems in MHD are crucial to the theoretical and practical study of producing energy by fusion. In the study of the ideal MHD free boundary problems, a priori estimates were derived in [16] with a bounded initial domain homeomorphic to a ball, provided that the size of the magnetic field to be invariant on the free boundary. A priori estimates for the low regularity solution of this problem were given in [25] for the bounded domain with small volume. Ill-posedness was showed in [17] for the two-dimensional problem if the condition (1.7) was violated. A local existence result was established in [14] for which the detailed proof is given for an initial flat domain of the form $\mathbb{T}^2 \times (0, 1)$, where $\mathbb{T}^2$ is a two-dimensional period box in x_1 and x_2 so that the free boundary is a graph for a short time within which the local existence was proved. For a general free surface not restricted to the case of a graph, it might be feasible to use several coordinate charts to deal with the general free boundaries not restricted to the case of graphs. However, this is quite technically involved since one has to solve several free boundary problems simultaneously. In particular, it might be extremely difficult in the study of long time problem by this approach of using several coordinates chats for general free boundaries. The aim of the present paper contributes to the program of the study of the ideal MHD free surface problem with a free surface being a closed curved surface with large curvature by the geometric approach motivated by [8], [21] and [23] for the Euler equations, and developed in [16,25].

Besides the motivation of serving as a step of the iteration scheme for the nonlinear problem for the case of the general free boundary without reducing the problem to the case that the free boundary is a graph, the study of the linearized problem has its own interest since the analysis and estimates for it may help for the design of the effective numerical computation schemes. Indeed, the well-posedness of the nonlinear problem does not imply that for the linearized problem, for the reason mentioned earlier that the linearized problem does not preserve the symmetries of the nonlinear problem provided by the physical laws. Indeed, the well-posedness of the linearized problem is even unknown for the case when the free boundary is a graph, since the approach used in [14] for the proof of the nonlinear problem is to use the parabolic approximation motivated by [9] without using linearization.

In this paper, we prove the existence of solutions in Sobolev spaces for linearized equations using a new type of estimate, motivated by the work of the free boundary problem of incompressible Euler equations of Lindblad in [21]. It is crucially important to prove the existence and obtain estimates for the linearized equations or some modification, in order to prove the existence theory for the nonlinear problem by using iteration schemes. In the most usual way, it is to linearize the equations with respect to both the velocity field and the magnetic field. However, this strategy does not work for the problem considered in this paper to prove the well-posedness for the linearized system of this type, where many operators can not be controlled and the relations between the velocity field and the magnetic field are also destroyed. In order to preserve these important relations, we seek a new way to linearize the equations. Since the magnetic field can be expressed in terms of the velocity field and the deformation tensors in the Lagrangian coordinates, we first solve the equation (1.2) and substitute the magnetic field into the equation (1.1) to obtain an equation of the velocity in which the initial magnetic field serves only as a parameter. Then, we linearize this equation with respect to the position vector field whose time derivative is the velocity in the Lagrangian coordinates. We believe this strategy is more suitable for the proof of the well-posedness of the nonlinear problem by putting it in an iterative scheme motivated by the linearization investigated in this paper. This is the first new idea of this paper. As in [21], we project the linearized equation onto an equation in the interior using the orthogonal projection onto divergence-free vector fields in the L^2 inner product, which removes a difficult term, the differential of linearization of the pressure, and reduces a higher-order term, the linearization of the free boundary, to an unbounded symmetric operator on divergence-free vector fields. Doing so, the linearized equation turns to an evolution equation in the interior for this so-called normal operator that is positive due to the condition (1.7) and leads to energy bounds. The operator is time dependent and nonelliptic, the existence of regular solutions cannot be obtained via standard energy methods or semigroup approaches. It is effective to use Lie derivatives with respect to divergence-free vector fields tangential at the boundary, motivated by [21]. The estimates of all derivatives can be obtained from those of tangential derivatives, the divergence and the curl. We replace the normal operator by a sequence of bounded operators converging to it that are still symmetric and positive and have uniformly estimates in order to get the existence of solutions.

When adopting the strategy mentioned above, certain analytic obstacles appear due to the presence of magnetic fields. A large part of the paper is to deal with the coupling of the magnetic field with the perturbation of the velocity field. In the linearized equation, there appears a term involving coupling of the perturbation of the velocity field and the initial magnetic field, which is due to the magnetic tension force. Dealing with this term directly causes essential analytic difficulties. A novel idea is to regard it in terms of the Lie derivatives of the magnetic field. This provides advantages when we commute the vector field of the magnetic with other vector fields used in [21] for the problem of incompressible equations by making full use of the fact that the magnetic field belongs to the tangential spaces of the boundary when restricted there, and it is divergence free. Moreover, the perturbation of the pressure in the linearized evolution equation is obtained as a functional of the perturbation of the velocity field and magnetic field by solving a Dirichlet problem of Poisson equations. The presence of magnetic field creates considerable difficulties in estimating it coupled with other quantities.

Fluids free boundary problems arising from physical, engineering and medical models are both important in applications and challenging in partial differential equations theory. Examples include water waves, evolution of boundaries of stars, vortex sheets, multi-phase flow, reacting flow, shock waves, biomedical modeling such as turmor growth, cell deformation and etc. The most fundamental and simplest setting is for incompressible fluids for which the local well-posedness in Sobolev spaces for inviscid irrotational flow was obtained first in [40,41] for 2D and 3D, respectively. Substantial progresses for the cases without the irrotational assumption, finite depth water waves, lower regularity, uniform estimates with respect to surface tension and etc have been made in [1–4,7–9,12,18,24,29,32,33,43] and etc. For more references, one may refer to the excellent survey [18]. For compressible inviscid flow, the local-in-time well-posedness of smooth solutions was established for liquids in [22,35] (see also [10] for the zero surface tension limits), and the study of the effects of heat-conductivity to fluid free surface can be found in [26].

We conclude this introduction by reviewing related results of MHD free boundary problems. For the case where the magnetic field is zero on the free boundary and in vacuum, the local existence and uniqueness of the free boundary problem of incompressible viscous-diffusive MHD flow in three-dimensional space with infinite and finite depth setting was proved in [19] and [20] where also a local unique solution was obtained for the free boundary MHD without kinetic viscosity and magnetic diffusivity via zero kinetic viscosity-magnetic diffusivity limit. The convergence rates of inviscid limits for the free boundary problems of the three-dimensional incompressible MHD with or without surface tension was studied in [6], where the magnetic field is constant on the surface and outside of the unbounded domain. For the incompressible viscous MHD equations, a free boundary problem in a simply connected domain of $\mathbb{R}^3$ was studied by a linearization technique and the construction of a sequence of successive approximations in [28] with an irrotational condition for magnetic fields in a part of the domain. The plasma-vacuum system was investigated in [15] where the a priori estimates were derived in a bounded domain. The well-posedness of the linearized plasma-vacuum interface problem in incompressible ideal MHD was studied in [27] in an unbounded plasma domain. For other related results of inviscid MHD equations related to this paper with free boundaries or interfaces, one may refer to [5,13,30,34,36–39,42].

The rest of the paper is organized as follows. In Section 2, we reformulate the free boundary problem to a fixed boundary problem by using the Lagrangian coordinates, and then linearize the equation. The linearized equation is projected onto the divergence-free vector fields in Section 3. We derive the lowest-order energy estimates in Section 4. In Section 5, the linearized problem is changed into the case of homogeneous initial data and an inhomogeneous term which vanishes to any order as time tends to zero. Next, the a priori bounds of the linearized equation with homogeneous initial data will be derived in Section 6 including those of tangential derivatives and the curl. Then, a smoothed-out equation will be studied according to the normal operator and the existence of weak solution of it will be proved in Section 7. The existence of smooth solutions for the linearized equation will be proved in Section 8. In Section 9, we turn to the energy estimates of the original linearized equation with inhomogeneous initial data and an inhomogeneous term and give the main result and its proof in Section 10. The appendix is on preliminaries about the Lie derivatives.

2. Lagrangian coordinates and the linearization of equations

2.1. Lagrangian reformulation

In this section, we use the Lagrangian coordinates to reformulate the free boundary problem to a fixed boundary problem. Lagrangian coordinates $x = x(t, y) = f_t(y)$ are given by

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega. \tag{2.1}$$

Since $\operatorname{div} v = 0$, $f_t : \Omega \to \Omega_t$ is a volume-preserving diffeomorphism. The free boundary becomes fixed in the new y-coordinates. We take f_0 as the identity operator for simplicity, that is, $x(0, y) = y$ and Ω is just the unit ball. For convenience, the letters a, b, c, d, e, and f will refer to quantities in the Lagrangian frame, whereas the letters i, j, k, l, m, and n will refer to ones in the Eulerian frame, e.g., $\partial_a = \partial/\partial y^a$ and $\partial_i = \partial/\partial x^i$.

Denote

$$D_t = \partial_t + v^k \partial_k, \quad \partial_k = \frac{\partial}{\partial x^k} = \frac{\partial y^a}{\partial x^k}\frac{\partial}{\partial y^a}. \tag{2.2}$$

Then, we get

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a}\frac{\partial v^i}{\partial x^k} \text{ and } D_t \frac{\partial y^a}{\partial x^i} = -\frac{\partial y^a}{\partial x^j}\frac{\partial v^j}{\partial x^i}. \tag{2.3}$$

From (1.2) and (2.3), we have

$$D_t\left(H^i \frac{\partial y^a}{\partial x^i}\right) = D_t H^i \frac{\partial y^a}{\partial x^i} + H^i D_t \frac{\partial y^a}{\partial x^i} = H^j \partial_j v^i \frac{\partial y^a}{\partial x^i} - H^i \partial_i v^k \frac{\partial y^a}{\partial x^k} = 0,$$

which yields

$$H^i(t, x(t, y))\frac{\partial y^a}{\partial x^i} = H^i(0, x(0, y)) \left.\frac{\partial y^a}{\partial x^i}\right|_{t=0} = \bar{H}_0^i(y)\delta_i^a = \bar{H}_0^a(y),$$

and

$$H^j(t, x(t, y)) = \bar{H}_0^a(y)\frac{\partial x^j(t, y)}{\partial y^a}, \tag{2.4}$$

where $\bar{H}_0^a(y) = H_0^a(x(0, y))$. Then,

$$H^k\partial_k H^i = \bar{H}_0^a \frac{\partial x^k}{\partial y^a}\frac{\partial y^c}{\partial x^k}\partial_c(\bar{H}_0^b \frac{\partial x^i}{\partial y^b}) = \bar{H}_0^a \partial_a(\bar{H}_0^b \partial_b x^i).$$

For convenience, denote the differential operator

$$B := B^a(y)\frac{\partial}{\partial y^a}, \text{ with } B^a(y) := \sqrt{\mu}\bar{H}_0^a(y),$$

then (1.1)-(1.5) can be written as

$$\begin{cases} D_t^2 x^i + \partial_i P = B^2 x^i, & \text{in } [0, T] \times \Omega, \\ \kappa := \det\left(\dfrac{\partial x}{\partial y}\right) = 1, & \text{in } [0, T] \times \Omega, \\ P = 0, & \text{on } \Gamma, \end{cases} \tag{2.5}$$

where $P = P(t, y) = p(t, x(t, y))$, ∂_i is thought of as the differential operator in y given in (2.2) and D_t is the time derivative. The initial conditions read

$$x|_{t=0} = y, \quad D_t x|_{t=0} = v_0, \tag{2.6}$$

satisfying the constraint $\operatorname{div} v_0 = 0$. Taking the divergence of (2.5) gives the Laplacian of P:

$$\Delta P = -(\partial_i D_t x^k)(\partial_k D_t x^i) + \partial_i(B^2 x^i). \tag{2.7}$$

The condition (1.7) turns to be

$$\nabla_N P \leqslant -c_0 < 0, \text{ on } \Gamma, \tag{2.8}$$

where N is the exterior unit normal to Γ_t parameterized by $x(t, y)$.

2.2. Linearization

Now, we derive the linearized equations for (2.5). We assume that $(x(t, y), P(t, y))$ is a given smooth solution of (2.5) satisfying (2.7) for $t \in [0, T]$.

Let δ be a variation with respect to some parameter r in the Lagrangian coordinates:

$$\delta = \frac{\partial}{\partial r}\bigg|_{(t,y)=\text{const}}.$$

We think of $x(t, y)$ and $P(t, y)$ as depending on r and differentiating with respect to r, say, $\bar{x}(t, y, r)$ and $\bar{P}(t, y, r)$ respectively. Namely, $(\bar{x}, \bar{P})|_{r=0} = (x, P)$. Differentiating (2.2) and using the formula for the derivative of the inverse of a matrix, $\delta M^{-1} = -M^{-1}(\delta M)M^{-1}$, we have the following commutator formula

$$[\delta, \partial_i] = -(\partial_i \delta x^k)\partial_k. \tag{2.9}$$

Let

$$(\delta x, \delta P) = \left(\frac{\partial \bar{x}}{\partial r}, \frac{\partial \bar{P}}{\partial r}\right)\Big|_{r=0}, \tag{2.10}$$

which satisfies $\operatorname{div} \delta x = 0$ and $\delta P|_\Gamma = 0$.

From (2.5) and (2.9), noting that $[D_t, \delta] = 0$ and $[\delta, B] = 0$, we obtain

$$D_t^2 \delta x^i = -\delta \partial_i P + B^2 \delta x^i = (\partial_i \delta x^k)\partial_k P - \partial_i \delta P + B^2 \delta x^i. \tag{2.11}$$

From (2.5) again, we have

$$\partial_i P = -D_t^2 x^i + B^2 x^i = -D_t v^i + B^2 x^i, \tag{2.12}$$

and then

$$(\partial_i \delta x^k)\partial_k P = \partial_i(\delta x^k \partial_k P) + \delta x^k(\partial_k D_t v^i - \partial_k(B^2 x^i)). \tag{2.13}$$

It follows from (2.11) and (2.13) that

$$D_t^2 \delta x^i + \partial_i \delta P - \partial_i(\delta x^k \partial_k P) - \delta x^k(\partial_k D_t v^i - \partial_k(B^2 x^i)) - B^2 \delta x^i = 0. \tag{2.14}$$

Now, we introduce new variables.

Let $g_{ab} = \delta_{ij} \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}$ be the metric δ_{ij} expressed in the Lagrangian coordinates, g^{ab} be the inverse of g_{ab},

$$\dot{g}_{ab} = D_t g_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}(\partial_k v_i + \partial_i v_k), \text{ and } \omega_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}(\partial_i v_k - \partial_k v_i) \tag{2.15}$$

be the time derivatives of the metric and the vorticity in the Lagrangian coordinates, respectively. It follows that

$$\frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}\partial_k v_i = \frac{1}{2}(\dot{g}_{ab} - \omega_{ab}). \tag{2.16}$$

Denote

$$W^a = \delta x^i \frac{\partial y^a}{\partial x^i}, \quad \delta x^i = W^b \frac{\partial x^i}{\partial y^b}, \quad q = \delta P. \tag{2.17}$$

Multiplying (2.14) by $\frac{\partial x^i}{\partial y^a}$ and summing over i, we obtain

$$\delta_{il}\frac{\partial x^l}{\partial y^a}D_t^2\delta x^i-\partial_a(W^c\partial_c P)-W^b\delta_{il}\frac{\partial x^l}{\partial y^a}\partial_b D_t v^i+\partial_a q \\ +\delta_{il}\partial_a x^l W^d\partial_d(B^2x^i)-\delta_{il}\partial_a x^l B^2(W^c\partial_c x^i)=0. \tag{2.18}$$

The first term in (2.18) can be written as

$$\delta_{il}\frac{\partial x^l}{\partial y^a}D_t^2\delta x^i=g_{ab}D_t^2W^b+(\dot{g}_{ab}-\omega_{ab})D_tW^b+\frac{\partial x^i}{\partial y^a}W^b\partial_b D_t v_i.$$

It follows from (2.18) that

$$g_{ab}D_t^2W^b+(\dot{g}_{ab}-\omega_{ab})D_tW^b-\partial_a(W^c\partial_c P)+\partial_a q \\ +\delta_{il}\partial_a x^l W^d\partial_d(B^2x^i)-\delta_{il}\partial_a x^l B^2(W^c\partial_c x^i)=0, \tag{2.19}$$

which yields by acting g^{da}

$$D_t^2W^d+g^{da}(\dot{g}_{ab}-\omega_{ab})D_tW^b-g^{da}\partial_a(W^c\partial_c P)+g^{da}\partial_a q \\ +g^{da}\delta_{il}\partial_a x^l[W^c\partial_c(B^2x^i)-B^2(W^c\partial_c x^i)]=0. \tag{2.20}$$

From (2.18), we see that the energies will include $\|BW\|^2$, which is very complicated due to $\operatorname{div}(BW)\neq 0$. Indeed, we can regard B as a tangential derivative since $B=B^a\partial_a$ is independent of time and $\partial_a B^a=0$. Thus, we can use the Lie derivative corresponding to B given by

$$\mathcal{L}_B W^a=BW^a-\partial_b B^a W^b, \tag{2.21}$$

which is divergence-free due to $\operatorname{div}\mathcal{L}_B W=\partial_a(B^b\partial_b W^a-\partial_b B^a W^b)=0$ if $\operatorname{div}W=0$. We also have

$$\mathcal{L}_B\partial_c x^i=B\partial_c x^i+\partial_c B^d\partial_d x^i. \tag{2.22}$$

For more details and properties of Lie derivatives, one can see Appendix A.

It follows from (2.21) that

$$\mathcal{L}_B^2W^a=B^2W^a-2(\partial_c B^a)\mathcal{L}_B W^c-W^d\partial_d(BB^a),$$

and then

$$B^2(\partial_c x^iW^c)=\partial_c x^i\mathcal{L}_B^2W^c+2((\partial_c B^b)\partial_b x^i+B\partial_c x^i)\mathcal{L}_B W^c+W^c\partial_c(B^2x^i).$$

Hence, in view of (2.22), one has

$$g^{da}\delta_{il}\partial_a x^l[W^c\partial_c(B^2x^i)-B^2(W^c\partial_c x^i)]=-\mathcal{L}_B^2W^d-2g^{da}\delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i\mathcal{L}_B W^c.$$

We introduce some new notations. Denote

$$\dot{W}^a(t,y) := D_t W^a(t,y), \quad \ddot{W}^a := D_t^2 W^a. \tag{2.23}$$

Since $q = \delta P$, one has $q|_\Gamma = 0$. Thus, we have the following system, by virtue of (2.20) and (1.3),

$$\begin{cases} \ddot{W}^d - \mathcal{L}_B^2 W^d + g^{da}\partial_a q - g^{da}\partial_a(W^c\partial_c P) + g^{da}(\dot{g}_{ab} - \omega_{ab})\dot{W}^b \\ \qquad - 2g^{da}\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c = 0, \\ \operatorname{div} W = \kappa^{-1}\partial_a(\kappa W^a) = 0, \\ q|_\Gamma = 0, \\ W|_{t=0} = W_0, \ \dot{W}|_{t=0} = W_1, \end{cases} \tag{2.24}$$

where $\operatorname{div} W_0 = \operatorname{div} W_1 = 0$.

We can express (2.24) in one equation since $q = \delta P$ is determined as a functional of W and $\dot{W}$. Thus, we derive an elliptic equation for q.

2.3. *The equation of Δq*

In order to derive the equation of Δq, we calculate $\operatorname{div} \ddot{W}$ first. Denote

$$u^a := \frac{\partial y^a}{\partial x^i} v^i, \ \text{and } u_a = g_{ab} u^b.$$

It follows from $\operatorname{div} W = 0$, that $\operatorname{div} \ddot{W} = 0$. Taking the divergence of (2.24) yields

$$\begin{cases} \Delta q = \partial_d\big(g^{da}\partial_a(W^c\partial_c P) - g^{da}(\dot{g}_{ab} - \omega_{ab})\dot{W}^b + 2g^{da}\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c\big), \\ q|_\Gamma = 0, \end{cases}$$

since $\operatorname{div} \mathcal{L}_B^2 W = 0$.

We separate q into four parts:

$$q = \sum_{i=1}^4 q_i,$$

where q_i's satisfy the following Dirichlet problems of Poisson equations:

$$\begin{cases} \Delta q_1 = \Delta(W^c\partial_c P), & q_1|_\Gamma = 0, \\ \Delta q_2 = -\partial_d(g^{da}\dot{g}_{ab}\dot{W}^b), & q_2|_\Gamma = 0, \\ \Delta q_3 = \partial_d(g^{da}\omega_{ab}\dot{W}^b), & q_3|_\Gamma = 0, \\ \Delta q_4 = 2\partial_d(g^{da}\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c), & q_4|_\Gamma = 0. \end{cases}$$

Then, we can write (2.24) as

$$L_1 W := \ddot{W} - \mathcal{L}_B^2 W + \mathcal{A} W + \dot{\mathcal{G}} \dot{W} - \mathcal{C} \dot{W} + \mathcal{X} \mathcal{L}_B W = 0, \tag{2.25}$$

where

$$\begin{cases} \mathcal{A} W^d := -g^{da} \partial_a (\partial_c P W^c - q_1), & (2.26) \\ \dot{\mathcal{G}} \dot{W}^d := g^{da} (\dot{g}_{ab} \dot{W}^b + \partial_a q_2), & (2.27) \\ \mathcal{C} \dot{W}^d := g^{da} (\omega_{ab} \dot{W}^b - \partial_a q_3), & (2.28) \\ \mathcal{X} \mathcal{L}_B W^d := -2 g^{da} \delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c + g^{da} \partial_a q_4. & (2.29) \end{cases}$$

3. The projection onto divergence-free vector field

In this section, we recall some definitions on the projection onto divergence-free vector field, one can see [21] for details.

Let $\mathbb{P}$ be the orthogonal projection onto divergence-free vector fields in the inner product

$$\langle W, U \rangle = \int_\Omega g_{ab} W^a U^b dy.$$

Then,

$$\begin{cases} \mathbb{P} U^a = U^a - g^{ab} \partial_b q, \\ \Delta q = \partial_a (g^{ab} \partial_b q) = \operatorname{div} U = \partial_a U^a, \quad q|_\Gamma = 0, \end{cases}$$

because of $g_{ab} g^{bc} = \delta_a^c$ and

$$\langle W, (\mathbb{I} - \mathbb{P}) U \rangle = \int_\Gamma W^a N_a q dS - \int_\Omega q \operatorname{div} W dy = 0, \text{ if } \operatorname{div} W = 0,$$

where N_a is the exterior unit conormal and dS is the surface measure. Moreover, the projection of its gradient vanishes for a function vanishing on the boundary:

$$\mathbb{P}(g^{ab} \partial_b f) = 0, \text{ if } f|_\Gamma = 0.$$

Denote $\|W\| := \langle W, W \rangle^{1/2}$. Clearly, one has that

$$\|\mathbb{P} U\| \leqslant \|U\|, \quad \|(\mathbb{I} - \mathbb{P}) U\| \leqslant \|U\|.$$

The projection is continuous on the Sobolev spaces $H^r(\Omega)$ if the metric is sufficiently regular:

$$\|\mathbb{P} U\|_{H^r(\Omega)} \leqslant C_r \|U\|_{H^r(\Omega)}.$$

Furthermore, if the metric also depends smoothly on time t, then

$$\sum_{j=0}^{k} \|D_t^j \mathbb{P} U\|_{H^r(\Omega)} \leqslant C_{r,k} \sum_{j=0}^{k} \|D_t^j U\|_{H^r(\Omega)}. \tag{3.1}$$

For functions f vanishing on the boundary, we define operators on divergence-free vector fields ($\partial_a W^a = 0$)

$$\mathcal{A}_f W^a = \mathbb{P}\left(-g^{ab}\partial_b(W^c \partial_c f)\right). \tag{3.2}$$

$\mathcal{A}_f$ is symmetric, i.e., $\langle U, \mathcal{A}_f W\rangle = \langle \mathcal{A}_f U, W\rangle$.

Since P is the total pressure, the normal operator $\mathcal{A}$ in (2.26) is

$$\mathcal{A} = \mathcal{A}_P \geqslant 0, \quad \langle W, \mathcal{A}W\rangle \geqslant 0 \text{ if } \nabla_N P|_\Gamma \leqslant 0,$$

which is true in view of the condition (1.7). In fact,

$$\langle W, \mathcal{A}W\rangle = -\int_\Gamma |N \cdot W|^2 \nabla_N P dS \geqslant 0, \tag{3.3}$$

due to $P = 0$ and $q_1 = 0$ on the boundary Γ. It follows from the definition in (3.2),

$$\begin{cases} \mathcal{A}_{fP} W^a = -g^{ab}\partial_b(W^c \partial_c(fP)) + g^{ab}\partial_b q, \\ \Delta q = \Delta(W^c \partial_c(fP)), \quad q|_\Gamma = 0. \end{cases}$$

Then, for the divergence-free vector field U, we have

$$\begin{aligned} \langle U, \mathcal{A}_{fP} W\rangle &= -\int_\Omega (U^d \partial_d(W^c \partial_c(fP)) + U^d \partial_d q) dy \\ &= -\int_\Gamma U_N W_N \nabla_N(fP) dS \\ &= -\int_\Gamma U_N W_N f \nabla_N P dS, \end{aligned} \tag{3.4}$$

since $\nabla_N(fP) = f\nabla_N P$ and $fP = 0$ on the boundary, where $U_N = N_a U^a = N \cdot U$. It follows from the Cauchy-Schwarz inequality and the identity (3.3) that,

$$|\langle U, \mathcal{A}_{fP} W\rangle| \leqslant \|f\|_{L^\infty(\Gamma)} \langle U, \mathcal{A}U\rangle^{1/2} \langle W, \mathcal{A}W\rangle^{1/2}. \tag{3.5}$$

Moreover, since P vanishes on the boundary, so does $\dot{P} = D_t P$. Thus, we can define

$$\dot{\mathcal{A}} = \mathcal{A}_{\dot{P}}, \quad \dot{\mathcal{A}} W^a = -g^{ab}\partial_b(W^c \partial_c \dot{P} - q), \quad \Delta q = \Delta(W^c \partial_c \dot{P}),\ q|_\Gamma = 0,$$

which satisfies by (3.4)

$$|\langle W, \dot{\mathcal{A}} W\rangle| = \left| -\int_\Gamma |W_N|^2 \nabla_N \dot{P} dS \right| \leqslant \left\| \frac{\nabla_N \dot{P}}{\nabla_N P} \right\|_{L^\infty(\Gamma)} \langle W, \mathcal{A}W\rangle. \tag{3.6}$$

Indeed, $\dot{\mathcal{A}}$ is the time derivatives of the operator $\mathcal{A}$, considered as an operator with values in the 1-forms.

We define bounded projected multiplication operators for 2-forms α, as in [21], given by

$$\mathcal{M}_\alpha W^a = \mathbb{P}(g^{ab}\alpha_{bc}W^c), \quad \|\mathcal{M}_\alpha W\| \leqslant \|\alpha\|_{L^\infty(\Omega)}\|W\|. \tag{3.7}$$

In particular, the operators in (2.27) and (2.28) are bounded, projected multiplication operators:

$$\mathcal{G} = \mathcal{M}_g, \quad \mathcal{C} = \mathcal{M}_\omega, \quad \dot{\mathcal{G}} = \mathcal{M}_{\dot{g}}, \tag{3.8}$$

for the metric g, the vorticity ω, and the time derivative of the metric $\dot{g}$.

4. The lowest-order energy estimates

Now, we derive the energy estimates for the linearized equations

$$L_1 W = \ddot{W} - \mathcal{L}_B^2 W + \mathcal{A}W + \dot{\mathcal{G}}\dot{W} - \mathcal{C}\dot{W} + \mathcal{X}\mathcal{L}_B W = F, \tag{4.1}$$

where F is divergence-free.

We first compute the inner product of (4.1) with $\dot{W}$ and W. Since

$$D_t(g_{ab}\dot{W}^a\dot{W}^b) = \dot{g}_{ab}\dot{W}^a\dot{W}^b + 2g_{ab}\dot{W}^a\ddot{W}^b,$$

we get

$$\langle \ddot{W}, \dot{W}\rangle = \frac{1}{2}\frac{d}{dt}\langle \dot{W}, \dot{W}\rangle - \frac{1}{2}\langle \dot{W}, \dot{\mathcal{G}}\dot{W}\rangle, \tag{4.2}$$

where $\dot{G}$ is given by (3.8). From the symmetry of $\mathcal{A}$, it follows

$$\langle \mathcal{A}W, \dot{W}\rangle = \frac{1}{2}\frac{d}{dt}\langle \mathcal{A}W, W\rangle - \frac{1}{2}\langle \dot{\mathcal{A}}W, W\rangle,$$

where $\dot{\mathcal{A}}W^a = \mathcal{A}_{\dot{P}}W^a$ is defined by (3.2) with $f = \dot{P} = D_t P$. In addition,

$$\frac{1}{2}\frac{d}{dt}\langle W, W\rangle = \langle W, \dot{W}\rangle.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\langle (\mathcal{A}+I)W, W\rangle = \langle (\mathcal{A}+I)W, \dot{W}\rangle + \frac{1}{2}\langle \dot{\mathcal{A}}W, W\rangle. \tag{4.3}$$

We also have

$$
\begin{aligned}
-\langle \mathcal{L}_B^2 W, \dot{W}\rangle =& \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{L}_B W|^2 dy - \frac{1}{2}\int_\Omega \dot{g}_{ab}\mathcal{L}_B W^a \mathcal{L}_B W^b dy \\
&+ \int_\Omega \delta_{il}(\partial_b x^l \mathcal{L}_B \partial_a x^i + \mathcal{L}_B \partial_b x^l \partial_a x^i)\mathcal{L}_B W^a \dot{W}^b dy. \qquad (4.4)
\end{aligned}
$$

Since $q_4 = 0$ on Γ, it yields

$$
\langle \mathcal{X}\mathcal{L}_B W, \dot{W}\rangle = -2\int_\Omega \delta_{il}\partial_b x^l \mathcal{L}_B \partial_a x^i \mathcal{L}_B W^a \dot{W}^b dy.
$$

Hence, we can define the energy as

$$
E_0^2(t) = E(t) = \langle \dot{W}, \dot{W}\rangle + \langle (\mathcal{A}+I)W, W\rangle + \langle \mathcal{L}_B W, \mathcal{L}_B W\rangle. \qquad (4.5)
$$

Then, we have the following energy estimates.

Proposition 4.1. *Let*

$$
n_0(t) = \frac{1}{2}\left(1 + \left\|\frac{\nabla_N \dot{P}}{\nabla_N P}\right\|_{L^\infty(\Gamma)} + 2\|\dot{g}\|_{L^\infty(\Omega)} + 2\|\partial x\|_{L^\infty(\Omega)}\|\mathcal{L}_B \partial x\|_{L^\infty(\Omega)}\right).
$$

It holds

$$
E_0(t) \leqslant e^{\int_0^t n_0(\tau)d\tau}\left(E_0(0) + \int_0^t \|F(s)\| e^{-\int_0^s n_0(\tau)d\tau} ds\right). \qquad (4.6)
$$

Proof. Due to the antisymmetry of ω, we have $\langle \mathcal{C}\dot{W}, \dot{W}\rangle = 0$. Then, we get

$$
\begin{aligned}
\frac{1}{2}\dot{E}(t) =& \langle -\frac{1}{2}\dot{\mathcal{G}}\dot{W} + F, \dot{W}\rangle + \langle W, \dot{W}\rangle + \frac{1}{2}\langle \dot{\mathcal{A}}W, W\rangle + \frac{1}{2}\langle \dot{\mathcal{G}}\mathcal{L}_B W, \mathcal{L}_B W\rangle \\
&+ \int_\Omega \delta_{il}(\mathcal{L}_B \partial_b x^l \partial_a x^i - \partial_b x^l \mathcal{L}_B \partial_a x^i)\mathcal{L}_B W^a \dot{W}^b dy.
\end{aligned}
$$

Thus, we obtain

$$
|\dot{E}_0| \leqslant n_0(t)E_0 + \|F\|, \qquad (4.7)
$$

which yields the desired estimates. □

5. Turning initial data into an inhomogeneous divergence-free term

In this section, we want to change the initial value problem (4.1) and (2.24), i.e.,

$$\begin{cases} L_1 W = \ddot{W} - \mathcal{L}_B^2 W + \mathcal{A}W + \dot{\mathcal{G}}\dot{W} - \mathcal{C}\dot{W} + \mathcal{X}\mathcal{L}_B W = F, & (5.1a) \\ W|_{t=0} = W_0, \ \dot{W}|_{t=0} = W_1, & (5.1b) \end{cases}$$

into the case of homogeneous initial data and an inhomogeneous term F that vanishes to any order as $t \to 0$. As in [21], we can achieve it by subtracting off a power series solution in t to (5.1):

$$W_{0r}^a(t, y) = \sum_{s=0}^{r+2} \frac{t^s}{s!} W_s^a(y). \tag{5.2}$$

It is clear that W_{0r} is divergence-free if so does W_s. Here W_0 and W_1 are the initial data given in (5.1b), W_2 is obtained from (5.1) at $t = 0$:

$$W_2 = \ddot{W}(0) = F(0) + \mathcal{L}_B^2 W_0 - \mathcal{A}(0)W_0 - \dot{\mathcal{G}}(0)W_1 + \mathcal{C}(0)W_1 - \mathcal{X}(0)\mathcal{L}_B W_0.$$

The higher-order terms can be obtained by differentiating the equation with respect to time first and then taking the value at $t = 0$. Indeed, we can obtain an expression by doing so,

$$D_t^{k+2} W = M_k(W, D_t W, \cdots, D_t^{k+1} W) + D_t^k F,$$

from which we define inductively

$$W_{k+2} = M_k(W_0, W_1, \cdots, W_{k+1})|_{t=0} + D_t^k F|_{t=0},$$

where M_k is some linear operator of order at most 1 and that is all we need to derive. Next, we calculate the explicit form of M_k as a simple model case, since we will do similar derivations later on for other operators.

It is convenient to differentiate the corresponding operator with values in 1-forms, so we denote

$$\begin{aligned} \underline{L}_1 W_a :=& g_{ab} L_1 W^b \\ =& g_{ab}\ddot{W}^b - g_{ab}\mathcal{L}_B^2 W^b + \partial_a q - \partial_a(\partial_c P W^c) + (\dot{g}_{ab} - \omega_{ab})\dot{W}^b \\ & - 2\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c = g_{ab} F^b, \end{aligned} \tag{5.3}$$

where q is chosen such that the last terms are divergence-free, and afterwards project the result to the divergence-free vector fields. Denote

$$q^s = D_t^s q, \ P^s = D_t^s P, \ g_{ab}^s = D_t^s g_{ab}, \ \omega_{ab}^s = D_t^s \omega_{ab}, \ F_s = D_t^s F.$$

Applying the differential operator D_t^r to (5.3) and restricting t to 0, we obtain

$$\sum_{s=0}^{r}\mathtt{C}_r^s\left(g_{ab}^{r-s}W_{s+2}^b-g_{ab}^{r-s}\mathcal{L}_B^2W_s^b-\partial_a(\partial_cP^{r-s}W_s^c)\right)+\partial_aq^r$$
$$+\sum_{s=0}^{r}\mathtt{C}_r^s\left(g_{ab}^{r-s+1}-\omega_{ab}^{r-s}\right)W_{s+1}^b$$
$$-2\sum_{s=0}^{r}\mathtt{C}_r^s\sum_{s_1=0}^{r-s}\mathtt{C}_{r-s}^{s_1}\delta_{il}\partial_a(D_t^{r-s-s_1}x^l)\mathcal{L}_B\partial_c(D_t^{s_1}x^i)\mathcal{L}_BW_s^c=\sum_{s=0}^{r}\mathtt{C}_r^sg_{ab}^{r-s}F_s^b.$$

Then, we need to project all terms onto divergence-free vector fields. Let

$$\begin{cases}\mathcal{A}_sW^d:=\mathbb{P}(-g^{da}\partial_a(\partial_cP^sW^c)),\\ \mathcal{G}_sW^d:=\mathbb{P}(g^{da}g_{ab}^sW^b),\\ \mathcal{C}_sW^d:=\mathbb{P}(g^{da}\omega_{ab}^sW^b),\\ \mathcal{X}_s\mathcal{L}_BW^d:=-2\mathbb{P}\left(g^{da}\sum_{s_1=0}^{s}\mathtt{C}_s^{s_1}\delta_{il}\partial_a(D_t^{s-s_1}x^l)\mathcal{L}_B\partial_c(D_t^{s_1}x^i)\mathcal{L}_BW^c\right).\end{cases}$$

We have

$$W_{r+2}=-\sum_{s=0}^{r-1}\mathtt{C}_r^s\mathcal{G}_{r-s}W_{s+2}+\sum_{s=0}^{r}\mathtt{C}_r^s\left(\mathcal{G}_{r-s}\mathcal{L}_B^2W_s-\mathcal{A}_{r-s}W_s-\mathcal{G}_{r-s+1}W_{s+1}\right)$$
$$+\sum_{s=0}^{r}\mathtt{C}_r^s\left(\mathcal{C}_{r-s}W_{s+1}-\mathcal{X}_{r-s}\mathcal{L}_BW_s+\mathcal{G}_{r-s}F_s\right),$$

which defines inductively W_{r+2} from $W_0, W_1, \cdots, W_{r+1}$.

It is obvious that, by the definition of W_{0r} in (5.2),

$$D_t^s(L_1W_{0r}-F)|_{t=0}=0\ \text{ for } s\leqslant r,\quad W_{0r}|_{t=0}=W_0,\ \dot{W}_{0r}|_{t=0}=W_1.$$

Thus, we reduces (5.1) to the desired case of vanishing initial data and an inhomogeneous term that vanishes to any order r as $t\to 0$ by replacing W by $W-W_{0r}$ and F by $F-L_1W_{0r}$.

If the initial data are smooth, as similar as in [21], we can also construct a smooth approximate solution $\tilde{W}$ that satisfies the equation to all orders as $t\to 0$. We can realize it by multiplying the k-th term in (5.2) by a smooth cutoff function $\chi(t/\varepsilon_k)$ and summing up the infinite series where $\chi(s)=1$ for $|s|\leqslant\frac{1}{2}$ and $\chi(s)=0$ for $|s|\geqslant 1$. If we take $(\|\tilde{W}_k\|_k+1)\varepsilon_k\leqslant\frac{1}{2}$, then the sequence $\varepsilon_k>0$ can be chosen so small that the series converges in $C^m([0,T],H^m)$ for any m.

6. A priori estimates of the linearized equation with homogeneous initial data

6.1. The estimates of the one more order derivatives for the linearized equation

We take the time derivative to (5.3) to get

$$g_{ab}\dddot{W}^b-g_{ab}\mathcal{L}_B^2\dot{W}^b-\partial_a(\dot{W}^c\partial_cP)-\omega_{ab}\ddot{W}^b+\partial_a\dot{q}$$

$$
\begin{aligned}
=& -2\dot{g}_{ab}\ddot{W}^b+\dot{g}_{ab}\mathcal{L}_B^2 W^b+\partial_a(W^c\partial_c\dot{P})-(\ddot{g}_{ab}-\dot{\omega}_{ab})\dot{W}^b\\
&+2D_t(\delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i)\mathcal{L}_B W^c+2\delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i\mathcal{L}_B\dot{W}^c+\dot{g}_{ab}F^b+g_{ab}\dot{F}^b.
\end{aligned}
$$

Similar to (4.2) and (4.3), it holds

$$
\langle\dddot{W},\ddot{W}\rangle=\frac{1}{2}\frac{d}{dt}\langle\ddot{W},\ddot{W}\rangle-\frac{1}{2}\langle\ddot{W},\dot{\mathcal{G}}\ddot{W}\rangle,
$$

and

$$
\frac{1}{2}\frac{d}{dt}\langle\mathcal{A}\dot{W},\dot{W}\rangle=\langle\mathcal{A}\dot{W},\ddot{W}\rangle+\frac{1}{2}\langle\dot{\mathcal{A}}\dot{W},\dot{W}\rangle.
$$

We have, in view of (4.4),

$$
\begin{aligned}
-\langle\mathcal{L}_B^2\dot{W},\ddot{W}\rangle=&\frac{1}{2}\frac{d}{dt}\int_\Omega|\mathcal{L}_B\dot{W}|^2dy-\frac{1}{2}\int_\Omega\dot{g}_{ab}\mathcal{L}_B\dot{W}^a\mathcal{L}_B\dot{W}^b dy\\
&+\int_\Omega(\mathcal{L}_B g_{ab})\mathcal{L}_B\dot{W}^a\ddot{W}^b dy.
\end{aligned}
$$

Let

$$
E_{D_t}=E(D_tW)=\langle\ddot{W},\ddot{W}\rangle+\langle\dot{W},\mathcal{A}\dot{W}\rangle+\langle\mathcal{L}_B\dot{W},\mathcal{L}_B\dot{W}\rangle. \tag{6.1}
$$

Then, similarly, from the antisymmetry of $\dot{\omega}$, it holds that

$$
\begin{aligned}
\dot{E}_{D_t}=&2\langle\dot{F}+\dot{\mathcal{G}}F,\ddot{W}\rangle-2\langle\ddot{\mathcal{G}}\dot{W},\ddot{W}\rangle+2\langle\dot{\mathcal{C}}\dot{W},\ddot{W}\rangle-3\langle\dot{\mathcal{G}}\ddot{W},\ddot{W}\rangle+\langle\dot{W},\dot{\mathcal{A}}\dot{W}\rangle && \text{(6.2a)}\\
&+\langle\dot{\mathcal{G}}\mathcal{L}_B\dot{W},\mathcal{L}_B\dot{W}\rangle+4\langle D_t(\delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i)\mathcal{L}_B W^c,\ddot{W}^a\rangle && \text{(6.2b)}\\
&+2\langle\delta_{il}(\partial_a x^l\mathcal{L}_B\partial_c x^i-\mathcal{L}_B\partial_a x^l\partial_c x^i)\mathcal{L}_B\dot{W}^c,\ddot{W}^a\rangle && \text{(6.2c)}\\
&-2\langle\dot{\mathcal{A}}W,\ddot{W}\rangle. && \text{(6.2d)}
\end{aligned}
$$

Thus, we obtain

$$
|\,(6.2c)|\leqslant 2\|\partial x\|_{L^\infty(\Omega)}\|\mathcal{L}_B\partial x\|_{L^\infty(\Omega)}E_{D_t}.
$$

By Cauchy-Schwartz' inequalities, one has

$$
\begin{aligned}
|\,(6.2a)|\leqslant& 2(\|\dot{F}\|+\|\dot{g}\|_{L^\infty(\Omega)}\|F\|)E_{D_t}^{1/2}+2(\|\ddot{g}\|_{L^\infty(\Omega)}+\|\dot{\omega}\|_{L^\infty(\Omega)})E_{D_t}^{1/2}E_0\\
&+\left(3\|\dot{g}\|_{L^\infty(\Omega)}+\left\|\frac{\nabla_N\dot{P}}{\nabla_N P}\right\|_{L^\infty(\Omega)}\right)E_{D_t},
\end{aligned}
$$

and

$$|\,(6.2\text{b})| \leqslant \|\dot{g}\|_{L^\infty(\Omega)} E_{D_t} + 4\|D_t(\delta_{il}\partial x^l \mathcal{L}_B \partial x^i)\|_{L^\infty(\Omega)} E_0 E_{D_t}^{1/2}.$$

Now, it remainders to deal with the term $\langle \dot{\mathcal{A}} W, \ddot{W}\rangle$ in (6.2d). From (3.5), it follows that

$$|\langle \dot{\mathcal{A}} W, \ddot{W}\rangle| \leqslant \left\| \frac{\nabla_N \dot{P}}{\nabla_N P} \right\|_{L^\infty(\Gamma)} \langle W, \mathcal{A} W\rangle^{1/2} \langle \ddot{W}, \mathcal{A}\ddot{W}\rangle^{1/2}. \tag{6.3}$$

But this does not imply that the norm of $\dot{\mathcal{A}}$ is bounded by the norm of $\mathcal{A}$ because $\langle \ddot{W}, \mathcal{A}\ddot{W}\rangle$ is one more order derivative than the considering energies. However, we have

$$\langle \dot{\mathcal{A}} W, \ddot{W}\rangle = \frac{d}{dt}\langle \dot{\mathcal{A}} W, \dot{W}\rangle - \langle \ddot{\mathcal{A}} W, \dot{W}\rangle - \langle \dot{\mathcal{A}} \dot{W}, \dot{W}\rangle,$$

in which the last two terms can be bounded by E_{D_t} and E_0. Thus, we have to deal with this term in an indirect way, by including them in the energies and using (6.3). Let

$$D_{D_t} = 2\langle \dot{\mathcal{A}} W, \dot{W}\rangle,$$

then we get

$$\dot{D}_{D_t} = 2\langle \ddot{\mathcal{A}} W, \dot{W}\rangle + 2\langle \dot{\mathcal{A}} \dot{W}, \dot{W}\rangle + 2\langle \dot{\mathcal{A}} W, \ddot{W}\rangle,$$

and

$$|\dot{D}_{D_t} - 2\langle \dot{\mathcal{A}} W, \ddot{W}\rangle| \leqslant 2 \left\| \frac{\nabla_N \ddot{P}}{\nabla_N P} \right\|_{L^\infty(\Omega)} E_0 E_{D_t}^{1/2} + 2 \left\| \frac{\nabla_N \dot{P}}{\nabla_N P} \right\|_{L^\infty(\Omega)} E_{D_t},$$

$$|D_{D_t}| \leqslant 2 \left\| \frac{\nabla_N \dot{P}}{\nabla_N P} \right\|_{L^\infty(\Omega)} E_0 E_{D_t}^{1/2}.$$

Denote

$$\bar{n}_1(t) = \left\| \frac{\nabla_N \dot{P}}{\nabla_N P} \right\|_{L^\infty(\Omega)},$$

$$n_1(t) = \frac{5}{2}\|\dot{g}\|_{L^\infty(\Omega)} + \frac{5}{2}\left\| \frac{\nabla_N \dot{P}}{\nabla_N P} \right\|_{L^\infty(\Omega)} + \|\partial x\|_{L^\infty(\Omega)} \|\mathcal{L}_B \partial x\|_{L^\infty(\Omega)},$$

$$\tilde{n}_1(t) = \|\ddot{g}\|_{L^\infty(\Omega)} + \|\dot{\omega}\|_{L^\infty(\Omega)} + 2\|D_t(\delta_{il}\partial x^l \mathcal{L}_B \partial x^i)\|_{L^\infty(\Omega)} + \left\| \frac{\nabla_N \ddot{P}}{\nabla_N P} \right\|_{L^\infty(\Omega)},$$

and

$$f_1(t) = \|\dot{F}\| + \|\dot{g}\|_{L^\infty(\Omega)} \|F\|.$$

We then have the following energy estimates.

Proposition 6.1. *Let* $E_1^2(t) = E_{D_t}$*, and*

$$M_1(t) = 4\bar{n}_1^2(t)E_0^2(t) + 4e^{2\int_0^t n_1(\tau)d\tau}\int_0^t (\tilde{n}_1(\tau)E_0(\tau) + f_1(\tau))^2\,d\tau,$$

it holds for (5.3) *with zero initial data*

$$E_1^2(t) \leqslant M_1(t) + \int_0^t M_1(s)e^{t-s}ds.$$

Proof. From the above argument, we have obtained

$$\begin{aligned}\left|\frac{d}{dt}|E_{D_t} + D_{D_t}|\right| &= \left|\frac{d}{dt}(E_{D_t} + D_{D_t})\right| \\ &\leqslant 2n_1(t)|E_{D_t} + D_{D_t}| + 2(\tilde{n}_1(t)E_0(t) + f_1(t))E_1(t),\end{aligned}$$

which yields

$$|E_{D_t} + D_{D_t}| \leqslant 2e^{2\int_0^t n_1(\tau)d\tau}\int_0^t (\tilde{n}_1(s)E_0(s) + f_1(s))\,E_1(s)ds.$$

Therefore,

$$\begin{aligned}E_1^2(t) &\leqslant 2e^{2\int_0^t n_1(\tau)d\tau}\int_0^t (\tilde{n}_1(s)E_0(s) + f_1(s))\,E_1(s)ds + 2\bar{n}_1(t)E_0E_1 \\ &\leqslant e^{2\int_0^t n_1(\tau)d\tau}\Big(\frac{1}{2}e^{-2\int_0^t n_1(\tau)d\tau}\int_0^t E_1^2(s)ds \\ &\quad + 2e^{2\int_0^t n_1(\tau)d\tau}\int_0^t (\tilde{n}_1(s)E_0(s) + f_1(s))^2\,ds\Big) + 2\bar{n}_1^2(t)E_0^2 + \frac{1}{2}E_1^2,\end{aligned}$$

and then

$$E_1^2(t) \leqslant \int_0^t E_1^2(s)ds + M_1(t),$$

which implies the desired result by the Gronwall inequality. □

6.2. The more one order energy estimates with respect to $\mathcal{L}_B$

We now analyze the higher order energy functional. Let

$$\mathcal{A}_B = \mathcal{A}_{BP}, \quad \mathcal{G}_B = \mathcal{M}_{g^B}, \quad g^B_{ab} = \mathcal{L}_B g_{ab},$$
$$\dot{\mathcal{G}}_B = \mathcal{M}_{\dot{g}^B}, \quad \mathcal{C}_B = \mathcal{M}_{\omega^B}, \quad \omega^B_{ab} = \mathcal{L}_B \omega_{ab}.$$

From (6.14), it follows that

$$\begin{aligned} L_1 \mathcal{L}_B W^d =& \mathcal{L}_B \ddot{W}^d - \mathcal{L}_B^3 W^d + \mathcal{A}\mathcal{L}_B W^d + \dot{\mathcal{G}}\mathcal{L}_B \dot{W}^d - \mathcal{C}\mathcal{L}_B \dot{W}^d + \mathcal{X}\mathcal{L}_B^2 W^d \\ =& \mathcal{L}_B F^d - (\mathcal{A}_B W^d + \dot{\mathcal{G}}_B \dot{W}^d - \mathcal{C}_B \dot{W}^d + \mathcal{G}_B \ddot{W}^d - \mathcal{G}_B F^d) \\ &+ 2g^{ad}(\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i)_B \mathcal{L}_B W^c. \end{aligned}$$

We define, similar to the lowest-order energies,

$$E_B = E(\mathcal{L}_B W) = \langle \mathcal{L}_B \dot{W}, \mathcal{L}_B \dot{W}\rangle + \langle \mathcal{L}_B W, (\mathcal{A}+I)\mathcal{L}_B W\rangle + \langle \mathcal{L}_B^2 W, \mathcal{L}_B^2 W\rangle.$$

From (4.4), (A.8) and $B\cdot N|_\Gamma = 0$, we get

$$\begin{aligned} &-\langle \mathcal{L}_B^3 W, \mathcal{L}_B \dot{W}\rangle = -\int_\Omega g_{ad}\mathcal{L}_B^3 W^d \mathcal{L}_B \dot{W}^a dy \\ =& \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{L}_B^2 W|^2 dy - \frac{1}{2}\int_\Omega \dot{g}_{ab}\mathcal{L}_B^2 W^a \mathcal{L}_B^2 W^b dy + \int_\Omega (\mathcal{L}_B g_{ab})\mathcal{L}_B^2 W^a \mathcal{L}_B \dot{W}^b dy. \end{aligned}$$

One has

$$\langle \mathcal{X}\mathcal{L}_B^2 W, \dot{W}_T\rangle = -2\int_\Omega \delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B^2 W^c \dot{W}_T^a dy.$$

Thus, following from (A.3), we have

$$\begin{aligned} &-\langle \mathcal{L}_B^3 W, \mathcal{L}_B \dot{W}\rangle + \langle \mathcal{X}\mathcal{L}_B^2 W, \mathcal{L}_B \dot{W}\rangle \\ =& \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{L}_B^2 W|^2 dy - \frac{1}{2}\int_\Omega \dot{g}_{ab}\mathcal{L}_B^2 W^a \mathcal{L}_B^2 W^b dy \\ &+ \int_\Omega \delta_{il}(\mathcal{L}_B \partial_a x^l \partial_c x^i - \partial_a x^l \mathcal{L}_B \partial_c x^i)\mathcal{L}_B^2 W^a \mathcal{L}_B \dot{W}^b dy. \end{aligned}$$

Due to the antisymmetry of $\dot{\omega}$, one has

$$\dot{E}_B = 2\langle \mathcal{L}_B F + \mathcal{G}_B F, \mathcal{L}_B \dot{W}\rangle - 2\langle \dot{\mathcal{G}}_B \dot{W}, \mathcal{L}_B \dot{W}\rangle + 2\langle \mathcal{C}_B \dot{W}, \mathcal{L}_B \dot{W}\rangle - 2\langle \mathcal{G}_B \ddot{W}, \mathcal{L}_B \dot{W}\rangle \tag{6.4a}$$

$$-4\langle \mathcal{L}_B(\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i)\mathcal{L}_B W^c, \mathcal{L}_B \dot{W}^a\rangle \tag{6.4b}$$

$$-\langle \dot{\mathcal{G}}\mathcal{L}_B \dot{W}, \mathcal{L}_B \dot{W}\rangle + \langle \dot{\mathcal{G}}\mathcal{L}_B W, \mathcal{L}_B W\rangle + \langle \mathcal{L}_B W, \dot{\mathcal{A}}\mathcal{L}_B W\rangle + \langle \dot{\mathcal{G}}\mathcal{L}_B W, \mathcal{A}\mathcal{L}_B W\rangle \tag{6.4c}$$

$$+\int_\Omega \dot{g}_{ab}\mathcal{L}_B^2 W^a \mathcal{L}_B^2 W^b dy + 2\langle \mathcal{L}_B W, \mathcal{L}_B \dot{W}\rangle \tag{6.4d}$$

$$-2\langle \mathcal{A}_B W, \mathcal{L}_B \dot{W}\rangle. \tag{6.4e}$$

Now, we control the term $\langle \mathcal{A}_T W, \dot{W}_T\rangle$. As the same argument as in the estimates of $E_1(t)$, we have to deal with it in an indirect way, by including it in the energies. Let

$$D_B = 2\langle \mathcal{A}_B W, \mathcal{L}_B W\rangle,$$

then

$$\dot{D}_B = 2\langle \dot{\mathcal{A}}_B W, \mathcal{L}_B W\rangle + 2\langle \mathcal{A}_B \dot{W}, \mathcal{L}_B W\rangle + 2\langle \mathcal{A}_B W, \mathcal{L}_B \dot{W}\rangle.$$

Therefore, we obtain

$$\begin{aligned}\dot{E}_B + \dot{D}_B =& (6.26a) + (6.26b) + (6.26f) + (6.26g)\\ &+ 2\langle \dot{\mathcal{A}}_B W, \mathcal{L}_B W\rangle + 2\langle \mathcal{A}_B \dot{W}, \mathcal{L}_B W\rangle.\end{aligned} \tag{6.5}$$

From (3.7), (3.8) and (3.6), it yields

$$\begin{aligned}|(6.4a)| \leqslant& 2\big(\|\mathcal{L}_B F\| + \|\mathcal{L}_B g\|_{L^\infty(\Omega)}\|F\| + \|\mathcal{L}_B \dot{g}\|_{L^\infty(\Omega)} E_0\\ &+ \|\mathcal{L}_B \omega\|_{L^\infty(\Omega)} E_0 + \|\mathcal{L}_B g\|_{L^\infty(\Omega)} E_1\big) E_B^{1/2},\\ |(6.4b)| \leqslant& 4\|\mathcal{L}_B(\delta_{il}\partial x^l \mathcal{L}_B \partial x^i)\|_{L^\infty(\Omega)} E_0 E_B^{1/2},\\ |(6.4c) + (6.4d)| \leqslant& \left(1 + \|\dot{g}\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_N \dot{P}}{\nabla_N P}\right\|\right) E_B,\end{aligned}$$

and

$$|(6.5)| \leqslant 2\left(\left\|\frac{\nabla_N(B\dot{P})}{\nabla_N P}\right\| E_0 + \left\|\frac{\nabla_N(BP)}{\nabla_N P}\right\| E_1\right) E_B^{1/2}.$$

Let $E_1^B := E_B^{1/2}$ and

$$\begin{aligned}\bar{n}_1^B(t) =& \left\|\frac{\nabla_N(BP)}{\nabla_N P}\right\|_{L^\infty(\Omega)} E_0, \qquad n_1^B(t) = \frac{1}{2}\left(1 + \|\dot{g}\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_N \dot{P}}{\nabla_N P}\right\|\right),\\ \tilde{n}_1^B(t) =& \|\mathcal{L}_B \dot{g}\|_{L^\infty(\Omega)} E_0 + \|\mathcal{L}_B \omega\|_{L^\infty(\Omega)} E_0 + \|\mathcal{L}_B g\|_{L^\infty(\Omega)} E_1\\ &+ 2\|\mathcal{L}_B(\delta_{il}\partial x^l \mathcal{L}_B \partial x^i)\|_{L^\infty(\Omega)} E_0 + \left\|\frac{\nabla_N(B\dot{P})}{\nabla_N P}\right\| E_0 + \left\|\frac{\nabla_N(BP)}{\nabla_N P}\right\| E_1,\end{aligned}$$

$$f_1^B(t) = \|\mathcal{L}_B F\| + \|\mathcal{L}_B g\|_{L^\infty(\Omega)}\|F\|.$$

Then, we have the following estimates.

Proposition 6.2. *Let*

$$M_1^B(t) = 2\bar{n}_1^B(t) + 2\int_0^t (\tilde{n}_1^B(\tau) + f_1^B(\tau))d\tau,$$

it holds

$$E_1^B(t) \leqslant M_1^B(t) + 2\int_0^t n_1^B(s) M_1^B(s) \exp\left(2\int_s^t n_1^B(\tau)d\tau\right) ds.$$

Proof. From the above argument, we have obtained

$$\dot{E}_B + \dot{D}_B \leqslant 2E_1^B(f_1^B + \tilde{n}_1^B + n_1^B E_1^B).$$

Since $E_B(0) = D_B(0) = 0$, the integration over $[0, t]$ in time gives

$$E_B \leqslant 2E_1^B \bar{n}_1^B + 2\int_0^t E_1^B(n_1^B E_1^B + \tilde{n}_1^B + f_1^B)d\tau.$$

Taking the supremum on $[0, t]$ in time and dividing by $\sup_{[0,t]} E_1^B$, we get

$$E_1^B(t) \leqslant M_1^B(t) + 2\int_0^t n_1^B(\tau) E_1^B(\tau)d\tau.$$

By the Gronwall inequality, we can obtain the desired estimates. □

6.3. Construction of tangential vector fields and the div-curl decomposition

In the Euclidean coordinates, a basic estimate is that one can use derivatives of the curl, the divergence and the tangential derivatives to estimate derivatives of vector fields, as proved in [21, Lemma 11.1]. However, this estimate is not invariant under changes of coordinates, so it is desired to replace it by an inequality which also holds in the Lagrangian coordinates. Its higher-order versions will be derived as well afterwards. Both the curl and the divergence are invariant, but the other terms are not. There are two ways to make these terms to be invariant. One is to replace the differentiation by covariant differentiation as used in [8,16], and the other is to replace it by Lie derivatives with respect to tangential vector fields introduced below, as the same as used in [21]. A lower-order term involving only the norm of the 1-form itself multiplied by a constant relative to the coordinates appears in both ways.

Definition 6.3. Let c_1 be a constant satisfying

$$\sum_{a,b}(|g_{ab}|+|g^{ab}|)\leqslant c_1^2,\quad \left|\frac{\partial x}{\partial y}\right|^2+\left|\frac{\partial y}{\partial x}\right|^2\leqslant c_1^2,$$

and let K_1 denote a continuous function of c_1.

Indeed, the bound for the Jacobian of the coordinate and its inverse follows from the bound for the metric and its inverse, and the bound for the former implies an equivalent bound for the latter with c_1^2 multiplied by n.

Following [21], we now construct the tangential divergence-free vector fields which are independent of time and expressed of the form $T^a(y)\frac{\partial}{\partial y^a}$ in the Lagrangian coordinates. Due to $\det(\frac{\partial x}{\partial y})=1$, the divergence-free condition reduces to

$$\partial_a T^a=0.$$

The vector fields can be explicitly expressed since Ω is the unit ball in $\mathbb{R}^n$. The rotation vector fields $y^a\partial_b-y^b\partial_a$ span the tangent space of the boundary and are divergence-free in the interior. It is clear that $B=B^a\partial_a$ belongs to this space. Moreover, they also span the tangent space of the level sets of the distance function from the boundary in the Lagrangian coordinates $d(y)=\mathrm{dist}\,(y,\Gamma)=1-|y|$ for $y\neq 0$ away from the origin. We denote this set of vector fields by $\mathcal{S}_0$. Thus, $B\in\mathcal{S}_0$.

As in [21,23], a finite set of vector fields spanning the tangential space when $d\geqslant d_0$ and compactly supported in the set where $d\geqslant d_0/2$ can be constructed. This set of vector fields is denoted by $\mathcal{S}_1$. We use $\mathcal{S}=\mathcal{S}_0\cup\mathcal{S}_1$ to denote the family of space tangential vector fields, and let $\mathcal{T}=\mathcal{S}\cup\{D_t\}$ denote the family of space-time tangential vector fields.

Let the radial vector field be $R=y^a\partial_a$. Then, $\partial_a R^a=n$ is constant. Let $\mathcal{R}=\mathcal{S}\cup\{R\}$, which spans the full tangent space of the space everywhere. We use $\mathcal{U}=\mathcal{S}\cup\{R\}\cup\{D_t\}$ to denote the family of all vector fields. Note that the radial vector field commutes with the rotations, i.e.,

$$[R,S]=0,\quad S\in\mathcal{S}_0.$$

Moreover, the commutators of two vector fields in $\mathcal{S}_0$ are another vector field in $\mathcal{S}_0$. Set $\mathcal{R}_i=\mathcal{S}_i\cup\{R\}$, $\mathcal{T}_i=\mathcal{S}_i\cup\{D_t\}$ and $\mathcal{U}_i=\mathcal{T}_i\cup\{R\}$ for $i=0,1$.

Now, we recall some estimates as follows.

Lemma 6.4 *([21, Lemma 11.3]). In the Lagrangian frame, with $\underline{W}_a=g_{ab}W^b$, we have*

$$|\mathcal{L}_U W|\leqslant K_1\left(|\mathrm{curl}\,\underline{W}|+|\mathrm{div}\,W|+\sum_{S\in\mathcal{S}}|\mathcal{L}_S W|+[g]_1|W|\right),\quad U\in\mathcal{R},\tag{6.6}$$

$$|\mathcal{L}_U W|\leqslant K_1\left(|\mathrm{curl}\,\underline{W}|+|\mathrm{div}\,W|+\sum_{T\in\mathcal{T}}|\mathcal{L}_T W|+[g]_1|W|\right),\quad U\in\mathcal{U},\tag{6.7}$$

where $[g]_1=1+|\partial g|$. Furthermore,

$$|\partial W| \leqslant K_1 \left(|\mathcal{L}_R W| + \sum_{S \in \mathcal{S}} |\mathcal{L}_S W| + |W| \right). \tag{6.8}$$

When $d(y) \leqslant d_0$, we may replace the sums over $\mathcal{S}$ by the sums over $\mathcal{S}_0$ and the sum over $\mathcal{T}$ by the sum over $\mathcal{T}_0$.

Next, we recall the higher-order versions of the inequality in last lemma. The lemma will be applied to W replaced by $\mathcal{L}_U^J W$, and the divergence term will vanish in applications. We will be able to control The curl of $(\mathcal{L}_U^J \underline{W})_a = \mathcal{L}_U^J (g_{ab} W^b)$ will thus be controlled, which is different from the curl of $(\underline{\mathcal{L}_U^J W})_a = g_{ab} \mathcal{L}_U^J W^b$. However, the difference is lower order and therefore controlled. We first introduce some notation.

Definition 6.5. Let β be a function, a 1- or 2-form, or vector field, and let $\mathcal{V}$ be any of our families of vector fields. Set

$$|\beta|_s^{\mathcal{V}} = \sum_{|J| \leqslant s, J \in \mathcal{V}} \left| \mathcal{L}_S^J \beta \right|,$$

$$[\beta]_\mu^{\mathcal{V}} = \sum_{s_1 + \cdots + s_k \leqslant \mu, s_i \geqslant 1} |\beta|_{s_1}^{\mathcal{V}} \cdots |\beta|_{s_k}^{\mathcal{V}}, \quad [\beta]_0^{\mathcal{V}} = 1.$$

In particular, $|\beta|_r^{\mathcal{R}}$ and $|\beta|_r^{\mathcal{U}}$ are equivalent to $\sum_{|\alpha| \leqslant r} \left| \partial_y^\alpha \beta \right|$ and $\sum_{|\alpha|+k \leqslant r} \left| D_t^k \partial_y^\alpha \beta \right|$, respectively.

Lemma 6.6 *([21, Lemma 11.5]). With the convention that $|\mathrm{curl}\, \underline{W}|_{-1}^{\mathcal{V}} = |\mathrm{div}\, W|_{-1}^{\mathcal{V}} = 0$, we have*

$$|W|_r^{\mathcal{R}} \leqslant K_1 \left(|\mathrm{curl}\, \underline{W}|_{r-1}^{\mathcal{R}} + |\mathrm{div}\, W|_{r-1}^{\mathcal{R}} + |W|_r^{\mathcal{S}} + \sum_{s=1}^{r} |g|_s^{\mathcal{R}} |W|_{r-s}^{\mathcal{R}} \right),$$

$$|W|_r^{\mathcal{R}} \leqslant K_1 \sum_{s=1}^{r} [g]_s^{\mathcal{R}} \left(|\mathrm{curl}\, \underline{W}|_{r-1-s}^{\mathcal{R}} + |\mathrm{div}\, W|_{r-1-s}^{\mathcal{R}} + |W|_{r-s}^{\mathcal{S}} \right).$$

The same inequalities also hold with $\mathcal{R}$ replaced by $\mathcal{U}$ everywhere and $\mathcal{S}$ replaced by $\mathcal{T}$:

$$|W|_r^{\mathcal{U}} \leqslant K_1 \left(|\mathrm{curl}\, \underline{W}|_{r-1}^{\mathcal{U}} + |\mathrm{div}\, W|_{r-1}^{\mathcal{U}} + |W|_r^{\mathcal{T}} + \sum_{s=1}^{r} |g|_s^{\mathcal{U}} |W|_{r-s}^{\mathcal{U}} \right),$$

$$|W|_r^{\mathcal{U}} \leqslant K_1 \sum_{s=1}^{r} [g]_s^{\mathcal{U}} \left(|\mathrm{curl}\, \underline{W}|_{r-1-s}^{\mathcal{U}} + |\mathrm{div}\, W|_{r-1-s}^{\mathcal{U}} + |W|_{r-s}^{\mathcal{T}} \right).$$

6.4. Commutators between the linearized equation and Lie derivatives with respect to B

First, we commute tangential vector fields through the linearized equation, in order to get the higher-order energy estimates of tangential derivatives.

Let $T \in \mathcal{T}$ be a tangential vector field, and recall that $[\mathcal{L}_T, D_t] = 0$ and that if W is divergence-free, then so does $\mathcal{L}_T W$. Now, we apply Lie derivatives $\mathcal{L}_T^I = \mathcal{L}_{T_{i_1}} \cdots \mathcal{L}_{T_{i_r}}$ with the multi-index $I = (i_1, \cdots, i_r)$ to (5.3).

From (A.5), we have for $r = |I|$,

$$\mathcal{L}_T^I(g_{ab}\ddot{W}^b) = \sum_{I_1+I_2=I} \mathtt{C}_r^{|I_1|}\mathcal{L}_T^{I_1} g_{ab}\mathcal{L}_T^{I_2}\ddot{W}^b =: c^I_{I_1I_2}\mathcal{L}_T^{I_1} g_{ab}\mathcal{L}_T^{I_2}\ddot{W}^b,$$

where we sum over all $I_1 + I_2 = I$ and $c^I_{I_1I_2} = 1$ (only for the simplicity of summing over the repeated indices) in last expression.

From (A.2) and the identity

$$\begin{aligned} T(\partial_c P W^c) =&\partial_c(T^d\partial_d P)W^c - (\partial_c T^d)\partial_d P W^c + \partial_c P T^d\partial_d W^c \\ =&\partial_c(TP)W^c + \partial_c P\mathcal{L}_T W^c, \end{aligned}$$

one has

$$\mathcal{L}_T(\partial_a(\partial_c P W^c)) =\partial_a T(\partial_c P W^c) = \partial_a(\partial_c(TP)W^c + \partial_c P\mathcal{L}_T W^c).$$

Then we have inductively

$$\mathcal{L}_T^I(\partial_a(\partial_c P W^c)) =\partial_a T^I(\partial_c P W^c) = c^I_{I_1I_2}\partial_a(\partial_c(T^{I_1}P)\mathcal{L}_T^{I_2}W^c). \tag{6.9}$$

Hence, we obtain

$$\begin{aligned} &c^I_{I_1I_2}(\mathcal{L}_T^{I_1}g_{ab})\mathcal{L}_T^{I_2}\ddot{W}^b - c^I_{I_1I_2}(\mathcal{L}_T^{I_1}g_{ab})\mathcal{L}_T^{I_2}\mathcal{L}_B^2 W^b - c^I_{I_1I_2}\partial_a(\partial_c(T^{I_1}P)\mathcal{L}_T^{I_2}W^c) \\ =&-\partial_a T^I q - c^I_{I_1I_2}(\mathcal{L}_T^{I_1}(\dot{g}_{ab} - \omega_{ab}))\mathcal{L}_T^{I_2}\dot{W}^b + c^I_{I_1I_2}(\mathcal{L}_T^{I_1}g_{ab})\mathcal{L}_T^{I_2}F^b \\ &+ 2c^I_{I_1I_2}(\mathcal{L}_T^{I_1}(\delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i))\mathcal{L}_T^{I_2}\mathcal{L}_B W^c. \end{aligned} \tag{6.10}$$

Denote

$$\begin{aligned} W_I =&\mathcal{L}_T^I W,\ F_I = \mathcal{L}_T^I F,\ P_I = T^I P,\ q_I = T^I q, \\ (\cdot)_I =&\mathcal{L}_T^I(\cdot),\ g^I_{ab} = \mathcal{L}_T^I g_{ab},\ \omega^I_{ab} = \mathcal{L}_T^I\omega_{ab}, \end{aligned}$$

and $\dot{g}^I_{ab} = D_t\mathcal{L}_T^I g_{ab} = \mathcal{L}_T^I\dot{g}_{ab}$, $\dot{W}_I = D_t W_I = \mathcal{L}_T^I\dot{W}$, etc. Then, (6.10) can be written as

$$\begin{aligned} &c^I_{I_1I_2}g^{I_1}_{ab}\ddot{W}^b_{I_2} - c^I_{I_1I_2}g^{I_1}_{ab}(\mathcal{L}_B^2 W)^b_{I_2} - c^I_{I_1I_2}\partial_a(\partial_c P_{I_1}W^c_{I_2}) \\ =&-\partial_a q_I - c^I_{I_1I_2}(\dot{g}^{I_1}_{ab} - \omega^{I_1}_{ab})\dot{W}^b_{I_2} \\ &+ 2c^I_{I_1I_2}(\delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i)_{I_1}(\mathcal{L}_B W)^c I_2 + c^I_{I_1I_2}g^{I_1}_{ab}F^b_{I_2}. \end{aligned} \tag{6.11}$$

Next, we project each term onto the divergence-free vector fields and introduce some new notation for the operators

$$\begin{aligned}\mathcal{A}_I W^a = \mathcal{A}_{P_I} W^a, \quad \mathcal{G}_I W^a = \mathbb{P}(g^{ac} g^I_{cb} W^b),\\ \dot{\mathcal{G}}_I W^a = \mathbb{P}(g^{ac} \dot{g}^I_{cb} W^b), \quad \mathcal{C}_I W^a = \mathbb{P}(g^{ac} \omega^I_{cb} W^b),\end{aligned} \tag{6.12}$$

and $\tilde{c}_I^{I_1 I_2} = c^I_{I_1 I_2}$ if $I_2 \neq I$ while $\tilde{c}_I^{I_1 I_2} = 0$ if $I_2 = I$. Then, we can write (6.9) as

$$\mathbb{P}(g^{ba} \mathcal{L}_T^I (g_{ac} \mathcal{A} W^c)) = \mathcal{A} W^b_I + \tilde{c}_I^{I_1 I_2} \mathcal{A}_{I_1} W^b_{I_2}. \tag{6.13}$$

Thus, we are able to rewrite (6.11) as

$$\begin{aligned} L_1 W^d_I =& \ddot{W}^d_I - (\mathcal{L}_B^2 W)^d_I + \mathcal{A} W^d_I + \dot{\mathcal{G}} \dot{W}^d_I - \mathcal{C} \dot{W}^d_I + \mathcal{X}(\mathcal{L}_B W)^d_I \\ =& F^d_I - \tilde{c}_I^{I_1 I_2} (\mathcal{A}_{I_1} W^d_{I_2} + \dot{\mathcal{G}}_{I_1} \dot{W}^d_{I_2} - \mathcal{C}_{I_1} \dot{W}^d_{I_2} + \mathcal{G}_{I_1} \ddot{W}^d_{I_2} - \mathcal{G}_{I_1} F^d_{I_2}) \\ &+ 2 \tilde{c}_I^{I_1 I_2} g^{ad} (\delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i)_{I_1} (\mathcal{L}_B W)^c_{I_2}. \end{aligned} \tag{6.14}$$

Now, we define higher-order energies. For $I \in \mathcal{V}$ with $|I| = r \geqslant 2$, let

$$E_I = E(W_I) = \langle \dot{W}_I, \dot{W}_I \rangle + \langle W_I, (\mathcal{A} + I) W_I \rangle + \langle \mathcal{L}_B W_I, \mathcal{L}_B W_I \rangle.$$

For $\mathcal{V} \in \{\{D_t\}, \{B\}, \mathcal{B}, \mathcal{S}, \mathcal{T}, \mathcal{R}, \mathcal{U}\}$ where $\mathcal{B} = \{B, D_t\}$, let

$$|W|^{\mathcal{V}}_s = \sum_{|I| \leqslant s,\, T \in \mathcal{V}} \left| \mathcal{L}_T^I W \right|, \; |W|^{\mathcal{V}}_{s,B} = \sum_{|I| \leqslant s,\, T \in \mathcal{V}} \left| \mathcal{L}_B \mathcal{L}_T^I W \right|, \tag{6.15}$$

$$\|\partial q \mathring{\|}^{\mathcal{V}}_{s,\infty,P^{-1}} = \sum_{|I|=s, T \in \mathcal{V}} \left\| \frac{\nabla_N \mathcal{L}_T^I q}{\nabla_N P} \right\|_{L^\infty(\Omega)}, \; \|\partial q\|^{\mathcal{V}}_{s,\infty,P^{-1}} = \sum_{0 \leqslant l \leqslant s} \|\partial q \mathring{\|}^{\mathcal{V}}_{l,\infty,P^{-1}}, \tag{6.16}$$

$$\|f\|^{\mathcal{V}}_{s,\infty} = \sum_{|I| \leqslant s,\, T \in \mathcal{V}} \|\mathcal{L}_T^I f\|_{L^\infty(\Omega)}, \; F^{\mathcal{V}}_s = \sum_{|I| \leqslant s,\, T \in \mathcal{V}} \|F_I\|, \tag{6.17}$$

$$\mathring{E}^{\mathcal{V}}_s = \sum_{|I|=s,\, T \in \mathcal{V}} \sqrt{E_I}, \quad E^{\mathcal{V}}_s = \sum_{0 \leqslant l \leqslant s} \mathring{E}^{\mathcal{V}}_l. \tag{6.18}$$

6.5. *The higher-order energy estimates for time and $\mathcal{L}_B$ derivatives*

From (6.14), we have for $I \in \mathcal{B}$,

$$\begin{aligned} \dot{E}_I =& 2\langle \dot{W}_I, F_I \rangle - \langle \dot{\mathcal{G}} \dot{W}_I, \dot{W}_I \rangle + 2\langle \dot{W}_I, W_I \rangle + \langle \dot{\mathcal{G}} W_I, (\mathcal{A} + I) W_I \rangle \\ &+ \langle W_I, \dot{\mathcal{A}} W_I \rangle + \langle \dot{\mathcal{G}} \mathcal{L}_B W_I, \mathcal{L}_B W_I \rangle \qquad (6.19) \\ &- 2 \tilde{c}_I^{I_1 I_2} (\langle \dot{W}_I, \mathcal{A}_{I_1} W_{I_2} \rangle + \langle \dot{W}_I, \dot{\mathcal{G}}_{I_1} \dot{W}_{I_2} \rangle - \langle \dot{W}_I, \mathcal{C}_{I_1} \dot{W}_{I_2} \rangle \\ &+ \langle \dot{W}_I, \mathcal{G}_{I_1} \ddot{W}_{I_2} \rangle - \langle \dot{W}_I, \mathcal{G}_{I_1} F_{I_2} \rangle) \qquad (6.20) \\ &+ 4 \tilde{c}_I^{I_1 I_2} \langle \dot{W}^a_I, (\delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i)_{I_1} (\mathcal{L}_B W)^c_{I_2} \rangle \qquad (6.21) \\ &+ 2\langle \mathcal{L}_B \dot{W}_I, \mathcal{L}_B W_I \rangle + 2\langle \dot{W}_I, (\mathcal{L}_B^2 W)_I \rangle - 2\langle \dot{W}_I, \mathcal{X}(\mathcal{L}_B W)_I \rangle. \qquad (6.22) \end{aligned}$$

It is clear that

$$|\,(6.19)| \leqslant 2E_I^{1/2}\|F_I\| + \left(1 + \|\dot g\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_N \dot P}{\nabla_N P}\right\|_{L^\infty(\Omega)}\right) E_I,$$

and

$$|\,(6.21)| \leqslant 4\tilde c_I^{I_1 I_2}\|(\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i)_{I_1}\|_{L^\infty(\Omega)} E_{I_2}^{1/2} E_I^{1/2}.$$

To deal with the term $\langle \dot W_I, \mathcal{A}_{I_1} W_{I_2}\rangle$, we introduce

$$D_I = 2\tilde c_I^{I_1 I_2}\langle W_I, \mathcal{A}_{I_1} W_{I_2}\rangle,$$

then

$$\dot D_I = 2\tilde c_I^{I_1 I_2}(\langle \dot W_I, \mathcal{A}_{I_1} W_{I_2}\rangle + \langle W_I, \mathcal{A}_{I_1} \dot W_{I_2}\rangle + \langle W_I, \dot{\mathcal{A}}_{I_1} W_{I_2}\rangle).$$

Thus,

$$\begin{aligned}
&\left|\dot D_I + \ (6.20)\right| \\
&\leqslant 2\tilde c_I^{I_1 I_2}\left(\left\|\frac{\nabla_N P_{I_1}}{\nabla_N P}\right\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_N \dot P_{I_1}}{\nabla_N P}\right\|_{L^\infty(\Omega)} + \|\dot g^{I_1}\|_{L^\infty(\Omega)} + \|\omega^{I_1}\|_{L^\infty(\Omega)}\right) E_{I_2}^{1/2} E_I^{1/2} \\
&\quad + 2\tilde c_I^{I_1 I_2}\|g^{I_1}\|_{L^\infty(\Omega)}(\|\ddot W_{I_2}\| + \|F_{I_2}\|)E_I^{1/2},
\end{aligned}$$

where the term $\|\ddot W_{I_2}\|$ can be controlled by the energy norm taking one $T = D_t$.

Since $B \cdot N = 0$ on Γ, we get by (A.3),

$$\begin{aligned}
(6.22) =& -2\langle (\mathcal{L}_B g_{ab})\dot W_I^a, \mathcal{L}_B W_I^b\rangle + 4\langle \delta_{il}\partial_a x^l \mathcal{L}_B \partial_b x^i \dot W_I^a, \mathcal{L}_B W_I^b\rangle \\
=& 2\langle \delta_{il}(\partial_a x^l \mathcal{L}_B \partial_b x^i - \mathcal{L}_B \partial_a x^l \partial_b x^i)\dot W_I^a, \mathcal{L}_B W_I^b\rangle.
\end{aligned}$$

Then,

$$|\,(6.22)| \leqslant 4\|\partial x\|_{L^\infty(\Omega)}\|\mathcal{L}_B \partial x\|_{L^\infty(\Omega)} E_I.$$

Therefore,

$$\begin{aligned}
&\dot E_I + \dot D_I \\
&\leqslant 2E_I^{1/2}(\|F_I\| + \tilde c_I^{I_1 I_2}\|g^{I_1}\|_{L^\infty(\Omega)}(\|\ddot W_{I_2}\| + \|F_{I_2}\|)) \\
&\quad + \left(1 + \|\dot g\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_N \dot P}{\nabla_N P}\right\|_{L^\infty(\Omega)} + 4\|\delta_{il}\partial x^l \mathcal{L}_B \partial x^i\|_{L^\infty(\Omega)}\right) E_I \\
&\quad + 2\tilde c_I^{I_1 I_2}\left(\left\|\frac{\nabla_N P_{I_1}}{\nabla_N P}\right\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_N \dot P_{I_1}}{\nabla_N P}\right\|_{L^\infty(\Omega)} + \|\dot g^{I_1}\|_{L^\infty(\Omega)} + \|\omega^{I_1}\|_{L^\infty(\Omega)}\right)
\end{aligned}$$

$$\cdot E_{I_2}^{1/2} E_I^{1/2} + 4\tilde{c}_I^{I_1 I_2} \|(\delta_{il}\partial x^l \mathcal{L}_B \partial x^i)_{I_1}\|_{L^\infty(\Omega)} E_{I_2}^{1/2} E_I^{1/2}. \tag{6.23}$$

Noticing that $E_I(0) = D_I(0) = 0$, the integration over $[0, t]$ in time implies

$$\begin{aligned} E_I &\leqslant |D_I| + \int_0^t (6.23) d\tau \\ &\leqslant 2E_I \sum_{s=0}^{r-1} \mathbb{C}_r^s \|\partial \mathring{P}\|_{r-s,\infty,P^{-1}} \mathring{E}_s^{\mathcal{B}} + \int_0^t (6.23) d\tau. \end{aligned} \tag{6.24}$$

Let

$$\begin{aligned} n_r^{\mathcal{B}} =& 1 + (2r-1)\|\dot{g}\|_{L^\infty(\Omega)} + 2(r-1)\|\mathcal{L}_B g\|_{L^\infty(\Omega)} + \left\| \frac{\nabla_N \dot{P}}{\nabla_N P} \right\|_{L^\infty(\Omega)} \\ &+ 4\|\delta_{il}\partial_a x^l \mathcal{L}_B \partial_b x^i\|_{L^\infty(\Omega)}, \\ \tilde{n}_r^{\mathcal{B}} =& \int_0^t (\|\partial P\|_{r,\infty,P^{-1}} + \|\partial \dot{P}\|_{r,\infty,P^{-1}} + \|\dot{g}\|_{r,\infty}^{\mathcal{B}} + \|\omega\|_{r,\infty}^{\mathcal{B}} + \|g\|_{r,\infty}^{\mathcal{B}} \\ &\quad + \|\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i\|_{r,\infty}^{\mathcal{B}}) d\tau, \\ f_r^{\mathcal{B}} =& \int_0^t (1 + \|g\|_{r,\infty}^{\mathcal{B}}) F_r^{\mathcal{B}} d\tau. \end{aligned}$$

Taking the supremum on $[0, t]$ in time of (6.24), then summing up the order from 0 to r, and dividing by $\bar{E}_r^{\mathcal{B}} = \sup_{[0,t]} E_r^{\mathcal{B}}$, we get

$$\bar{E}_r^{\mathcal{B}} \leqslant C(\sup_{[0,t]} \|\partial P\|_{r-1,\infty,P^{-1}} + \tilde{n}_r^{\mathcal{B}}) \bar{E}_{r-1}^{\mathcal{B}} + C f_r^{\mathcal{B}} + \int_0^t n_r^{\mathcal{B}} \bar{E}_r^{\mathcal{B}} d\tau.$$

By the Gronwall inequality, we can obtain the following estimates.

Proposition 6.7. *Let*

$$M_r^{\mathcal{B}} = C\Big[(\sup_{[0,t]} \|\partial P\|_{r-1,\infty,P^{-1}} + \tilde{n}_r^{\mathcal{B}}) \bar{E}_{r-1}^{\mathcal{B}} + f_r^{\mathcal{B}}\Big],$$

it holds

$$\bar{E}_r^{\mathcal{B}}(t) \leqslant M_r^{\mathcal{B}}(t) + \int_0^t M_r^{\mathcal{B}}(s) n_r^{\mathcal{B}}(s) \exp\left(\int_s^t n_r^{\mathcal{B}}(\tau) d\tau\right) ds.$$

This is a recursion formula between $\bar{E}_r^{\mathcal{B}}$ and $\bar{E}_{r-1}^{\mathcal{B}}$, thus we can obtain inductively the estimates of $\bar{E}_r^{\mathcal{B}}$ and $E_r^{\mathcal{B}}$ since we have proved the estimates of $\bar{E}_1^{\mathcal{B}} = \sup_{[0,t]}(E_0 + E_1 + E_1^B)$ in Propositions 4.1, 6.1 and 6.9. Indeed, we have the following:

Proposition 6.8. *Assume that $x, P \in C^{r+2}([0,T]\times\Omega)$, $B \in C^{r+2}(\Omega)$, $P|_\Gamma = 0$, $\nabla_N P|_\Gamma \leqslant -c_0 < 0$, $B^a N_a|_\Gamma = 0$ and $\operatorname{div} V = 0$, where $V = D_t x$. Suppose that W is a solution of (5.3) where F is divergence-free and vanishing to order r as $t \to 0$. Then, there is a constant $C = C(x, P, B)$ depending only on the norm of (x, P, B), a lower bound for c_0, and an upper bound for T such that if $E_s^{\mathcal{B}}(0) = 0$ for $s \leqslant r$, then*

$$E_r^{\mathcal{B}}(t) \leqslant C \int_0^t \|F\|_r^{\mathcal{B}} d\tau, \quad \text{for } t \in [0,T]. \tag{6.25}$$

6.6. Estimates for the tangential derivatives

The obtained higher-order time and $\mathcal{L}_B$ derivatives are some kinds of tangential derivatives due to $\operatorname{div} B = 0$ and $B \cdot N|_\Gamma = 0$, but they do not give the estimates for all tangential derivatives. Thus, we need to derive the estimates for tangential derivatives $\sum_{T\in\mathcal{T}} |\mathcal{L}_T W|$ of W.

Let $T \in \mathcal{T}$, $W_T = \mathcal{L}_T W$, $F_T = \mathcal{L}_T F$, and similar notation as in (6.12):

$$\mathcal{A}_T = \mathcal{A}_{TP}, \quad \mathcal{G}_T = \mathcal{M}_{g^T}, \quad g_{ab}^T = \mathcal{L}_T g_{ab},$$
$$\dot{\mathcal{G}}_T = \mathcal{M}_{\dot{g}^T}, \quad \mathcal{C}_T = \mathcal{M}_{\omega^T}, \quad \omega_{ab}^T = \mathcal{L}_T \omega_{ab}.$$

Then, from (6.14), it follows that

$$\begin{aligned} L_1 W_T^d =& F_T^d - (\mathcal{A}_T W^d + \dot{\mathcal{G}}_T \dot{W}^d - \mathcal{C}_T \dot{W}^d + \mathcal{G}_T \ddot{W}^d - \mathcal{G}_T F^d) \\ &+ 2g^{ad}(\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i)_T \mathcal{L}_B W^c. \end{aligned}$$

We define, as for the lowest-order energies,

$$E_T = E(W_T) = \langle \dot{W}_T, \dot{W}_T\rangle + \langle W_T, (\mathcal{A}+I)W_T\rangle + \langle \mathcal{L}_B W_T, \mathcal{L}_B W_T\rangle.$$

From (4.4), (A.8) and $B\cdot N|_\Gamma = 0$, we obtain

$$\begin{aligned} &-\langle \mathcal{L}_T \mathcal{L}_B^2 W, \dot{W}_T\rangle \\ =& -\int_\Omega g_{ad}\mathcal{L}_{[T,B]}\mathcal{L}_B W^d \dot{W}_T^a dy - \int_\Omega g_{ad}\mathcal{L}_B\mathcal{L}_{[T,B]} W^d \dot{W}_T^a dy \\ &+ \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{L}_B W_T|^2 dy - \frac{1}{2}\int_\Omega \dot{g}_{ab}\mathcal{L}_B W_T^a \mathcal{L}_B W_T^b dy + \int_\Omega (\mathcal{L}_B g_{ab})\mathcal{L}_B W_T^a \dot{W}_T^b dy. \end{aligned}$$

One has

$$\langle \mathcal{X}\mathcal{L}_T\mathcal{L}_B W, \dot{W}_T\rangle = -2\int_\Omega \delta_{il}\partial_a x^l \mathcal{L}_B\partial_c x^i \mathcal{L}_{[T,B]}W^c\dot{W}_T^a dy$$
$$-2\int_\Omega \delta_{il}\partial_a x^l \mathcal{L}_B\partial_c x^i \mathcal{L}_B W_T^c\dot{W}_T^a dy.$$

Thus, by (A.3), we get

$$\begin{aligned}
&-\langle \mathcal{L}_T\mathcal{L}_B^2 W, \dot{W}_T\rangle + \langle \mathcal{X}\mathcal{L}_T\mathcal{L}_B W, \dot{W}_T\rangle \\
=&-2\int_\Omega g_{ad}\mathcal{L}_{[T,B]}\mathcal{L}_B W^d\dot{W}_T^a dy + \int_\Omega g_{ad}\mathcal{L}_{[[T,B],B]}W^d\dot{W}_T^a dy + \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{L}_B W_T|^2 dy \\
&-\frac{1}{2}\int_\Omega \dot{g}_{ab}\mathcal{L}_B W_T^a\mathcal{L}_B W_T^b dy - 2\int_\Omega \delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i\mathcal{L}_{[T,B]}W^c\dot{W}_T^a dy \\
&+\int_\Omega \delta_{il}(\mathcal{L}_B\partial_a x^l\partial_c x^i - \partial_a x^l\mathcal{L}_B\partial_c x^i)\mathcal{L}_B W_T^c\dot{W}_T^a dy.
\end{aligned}$$

From the antisymmetry of $\dot{\omega}$, one has

$$\dot{E}_T = 2\langle F_T + \mathcal{G}_T F, \dot{W}_T\rangle - 2\langle \dot{\mathcal{G}}_T\dot{W}, \dot{W}_T\rangle + 2\langle \mathcal{C}_T\dot{W}, \dot{W}_T\rangle - 2\langle \mathcal{G}_T\ddot{W}, \dot{W}_T\rangle \tag{6.26a}$$
$$-4\langle (\delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i)_T\mathcal{L}_B W^c, \dot{W}_T^a\rangle$$
$$-2\int_\Omega \delta_{il}(\mathcal{L}_B\partial_a x^l\partial_c x^i - \partial_a x^l\mathcal{L}_B\partial_c x^i)\mathcal{L}_B W_T^c\dot{W}_T^a dy \tag{6.26b}$$
$$+4\int_\Omega \delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i\mathcal{L}_{[T,B]}W^c\dot{W}_T^a dy \tag{6.26c}$$
$$+4\int_\Omega g_{ad}\mathcal{L}_{[T,B]}\mathcal{L}_B W^d\dot{W}_T^a dy \tag{6.26d}$$
$$-2\int_\Omega g_{ad}\mathcal{L}_{[[T,B],B]}W^d\dot{W}_T^a dy \tag{6.26e}$$
$$-\langle \dot{\mathcal{G}}\dot{W}_T, \dot{W}_T\rangle + \langle \dot{\mathcal{G}}W_T, W_T\rangle + \langle W_T, \dot{\mathcal{A}}W_T\rangle + \langle \dot{\mathcal{G}}W_T, \mathcal{A}W_T\rangle \tag{6.26f}$$
$$+\int_\Omega \dot{g}_{ab}\mathcal{L}_B W_T^a\mathcal{L}_B W_T^b dy + 2\langle W_T, \dot{W}_T\rangle \tag{6.26g}$$
$$-2\langle \mathcal{A}_T W, \dot{W}_T\rangle. \tag{6.26h}$$

Now, we deal with the term $\langle \mathcal{A}_T W, \dot{W}_T\rangle$. As the same argument as in the estimates of $E_1(t)$, we have to handle it in an indirect way, by including it in the energies. Let

$$D_T = 2\langle \mathcal{A}_T W, W_T\rangle,$$

then

$$\dot{D}_T = 2\langle \dot{\mathcal{A}}_T W, W_T\rangle + 2\langle \mathcal{A}_T \dot{W}, W_T\rangle + 2\langle \mathcal{A}_T W, \dot{W}_T\rangle.$$

Therefore, we obtain

$$\begin{aligned}\dot{E}_T + \dot{D}_T =& (6.26a) + (6.26b) + (6.26c) + (6.26d) + (6.26e) + (6.26f) + (6.26g) \\ &+ 2\langle \dot{\mathcal{A}}_T W, W_T\rangle + 2\langle \mathcal{A}_T \dot{W}, W_T\rangle. \end{aligned} \tag{6.27}$$

From (3.7), (3.8) and (3.6), it yields

$$\begin{aligned} |(6.26a)| \leqslant& 2\big(\|F_T\| + \|g^T\|_{L^\infty(\Omega)}\|F\| + \|\dot{g}^T\|_{L^\infty(\Omega)}E_0 \\ &+ \|\omega^T\|_{L^\infty(\Omega)}E_0 + \|g^T\|_{L^\infty(\Omega)}E_1\big)E_T^{1/2}, \\ |(6.26b)| \leqslant& 4\|(\delta_{il}\partial x^l \mathcal{L}_B \partial x^i)_T\|_{L^\infty(\Omega)}E_0 E_T^{1/2} + 2\|\delta_{il}\partial x^l \mathcal{L}_B \partial x^i\|_{L^\infty(\Omega)}E_T, \\ |(6.26f) + (6.26g)| \leqslant& \left(1 + \|\dot{g}\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_N \dot{P}}{\nabla_N P}\right\|\right)E_T, \\ |(6.27)| \leqslant& 2\left(\left\|\frac{\nabla_N T\dot{P}}{\nabla_N P}\right\|E_0 + \left\|\frac{\nabla_N TP}{\nabla_N P}\right\|E_1\right)E_T^{1/2}, \\ |(6.26c)| \leqslant& 4\|\delta_{il}\partial x^l \mathcal{L}_B \partial x^i\|_{L^\infty(\Omega)}\|\mathcal{L}_{[T,B]}W\|E_T^{1/2}, \\ |(6.26e)| \leqslant& 2\|g\|_{L^\infty(\Omega)}\|\mathcal{L}_{[[T,B],B]}W\|E_T^{1/2}, \end{aligned}$$

and

$$|(6.26d)| \leqslant 4\|g\|_{L^\infty(\Omega)}\|\mathcal{L}_{[T,B]}\mathcal{L}_B W\|E_T^{1/2}.$$

Since $B \in \mathcal{S}$ and for $T \in \mathcal{S}$,

$$\operatorname{div}[T,B] = \partial_b(T^a\partial_a B^b - B^a\partial_a T^b) = \partial_b T^a \partial_a B^b - \partial_b B^a \partial_a T^b = 0,$$

we get $[T,B] \in \mathcal{S}$. Similarly, $[[T,B],B] \in \mathcal{S}$. Thus, from the above estimates and observation, we see that the energies should include E_T for any $T \in \mathcal{T}$ in order to deal with the commutators. Thus, we define the energy as

$$E_1^T := E_T^{1/2} \text{ for } T \in \mathcal{T}, \quad E_1^{\mathcal{T}} = \sum_{T\in\mathcal{T}} E_1^T.$$

Let

$$\begin{aligned} \bar{n}_1^T(t) =& \left\|\frac{\nabla_N TP}{\nabla_N P}\right\|_{L^\infty(\Omega)} E_0, \\ n_1(t) =& \frac{1}{2}\left(1 + \|\dot{g}\|_{L^\infty(\Omega)} + 3\|g\|_{L^\infty(\Omega)} + \left\|\frac{\nabla_N \dot{P}}{\nabla_N P}\right\|\right) + 2\|\delta_{il}\partial x^l \mathcal{L}_B \partial x^i\|_{L^\infty(\Omega)}, \end{aligned}$$

$$
\begin{aligned}
\tilde{n}_1^T(t) = & \|\dot{g}^T\|_{L^\infty(\Omega)} E_0 + \|\omega^T\|_{L^\infty(\Omega)} E_0 + \|g^T\|_{L^\infty(\Omega)} E_1 \\
& + 2\|(\delta_{il}\partial x^l \mathcal{L}_B \partial x^i)_T\|_{L^\infty(\Omega)} E_0 + \left\| \frac{\nabla_N T \dot{P}}{\nabla_N P} \right\| E_0 + \left\| \frac{\nabla_N T P}{\nabla_N P} \right\| E_1, \\
f_1^T(t) = & \|F_T\| + \|g^T\|_{L^\infty(\Omega)} \|F\|, \\
\tilde{n}_1^{\mathcal{T}}(t) = & 2 \sum_{T \in \mathcal{T}} \sup_{[0,t]} \bar{n}_1^T + 2 \int_0^t \sum_{T \in \mathcal{T}} \tilde{n}_1^T d\tau, \\
f_1^{\mathcal{T}}(t) = & 2 \int_0^t \sum_{T \in \mathcal{T}} f_1^T d\tau.
\end{aligned}
$$

Then, we have the following estimates.

Proposition 6.9. *It holds*

$$
E_1^{\mathcal{T}} \leqslant \tilde{n}_1^{\mathcal{T}}(t) + f_1^{\mathcal{T}}(t) + \int_0^t \left(\tilde{n}_1^{\mathcal{T}}(s) + f_1^{\mathcal{T}}(s) \right) n_1(s) \exp \left(\int_s^t n_1(\tau) d\tau \right) ds.
$$

Proof. From the above argument, we have obtained

$$
\begin{aligned}
& \dot{E}_T + \dot{D}_T \\
\leqslant & 2E_1^T \Bigg\{ \|F_T\| + \|g^T\|_{L^\infty(\Omega)} \|F\| + \|\dot{g}^T\|_{L^\infty(\Omega)} E_0 + \|\omega^T\|_{L^\infty(\Omega)} E_0 + \|g^T\|_{L^\infty(\Omega)} E_1 \\
& + 2\|(\delta_{il}\partial x^l \mathcal{L}_B \partial x^i)_T\|_{L^\infty(\Omega)} E_0 + \left(\left\| \frac{\nabla_N T \dot{P}}{\nabla_N P} \right\| E_0 + \left\| \frac{\nabla_N T P}{\nabla_N P} \right\| E_1 \right) \\
& + \frac{1}{2} \left(1 + \|\dot{g}\|_{L^\infty(\Omega)} + 3\|g\|_{L^\infty(\Omega)} + \left\| \frac{\nabla_N \dot{P}}{\nabla_N P} \right\| + 4\|\delta_{il}\partial x^l \mathcal{L}_B \partial x^i\|_{L^\infty(\Omega)} \right) E_1^T \Bigg\}.
\end{aligned}
$$

Since $E_T(0) = D_T(0) = 0$, the integration over $[0, t]$ in time gives

$$
E_T \leqslant 2E_1^T \bar{n}_1^T + 2 \int_0^t E_1^T \big[n_1 E_1^T + \tilde{n}_1^T + f_1^T \big] d\tau.
$$

Taking the supremum on $[0, t]$ in time and dividing by $\sup_{[0,t]} E_1^T$, we sum over $T \in \mathcal{T}$ to get

$$
E_1^{\mathcal{T}} \leqslant 2 \sum_{T \in \mathcal{T}} \sup_{[0,t]} \bar{n}_1^T + 2 \int_0^t \Big[n_1 E_1^{\mathcal{T}} + \sum_{T \in \mathcal{T}} \tilde{n}_1^T \Big] d\tau + f_1^{\mathcal{T}}
$$

$$\leqslant \int_0^t n_1 E_1^{\mathcal{T}} d\tau + \tilde{n}_1^{\mathcal{T}} + f_1^{\mathcal{T}}. \tag{6.28}$$

By the Gronwall inequality, we can obtain the desired estimates. □

6.7. Estimates for the curl and the full derivatives of the first order

We will derive the estimates of normal derivatives close to the boundary. This is done, in view of Lemma 6.6, by using the estimates of the curl and the estimates of the tangential derivatives. Thus, the estimates of the curl and the time derivatives of the curl are needed to derive. Due to this, the 1-form of W and $\dot{W}$, denoted by w and $\dot{w}$, respectively are needed, i.e., $w_a = g_{ab}W^b$ and $\dot{w}_a = g_{ab}\dot{W}^b$, in which the latter notation is slightly confusing and $\dot{w}$ is not equal to $D_t w$, but we only try to indicate that $\dot{w}$ is the corresponding 1-form obtained by lowering the indices of the vector field $\dot{W}$.

Let

$$\operatorname{curl} w_{ab} = \partial_a w_b - \partial_b w_a, \quad \underline{F}_a = g_{ab}F^b.$$

Since $D_t w_a = D_t(g_{ab}W^b) = \dot{g}_{ab}W^b + g_{ab}\dot{W}^b$, we have

$$\begin{aligned} D_t \operatorname{curl} w_{ab} =& \operatorname{curl} \dot{w}_{ab} + \partial_c \omega_{ab} W^c + \dot{g}_{bc}\partial_a W^c - \dot{g}_{ac}\partial_b W^c \\ &+ [(\dot{g}_{eb} - \omega_{eb})\partial_a\partial_c x^k - (\dot{g}_{ea} - \omega_{ea})\partial_b\partial_c x^k]\frac{\partial y^e}{\partial x^k} W^c, \end{aligned} \tag{6.29}$$

since from (2.16) we have $2\partial_b v_i = (\dot{g}_{cb} - \omega_{cb})\frac{\partial y^c}{\partial x^i}$ and

$$\begin{aligned} \partial_a \dot{g}_{db} - \partial_d \dot{g}_{ab} =& \partial_a[\partial_d x^i \partial_b x^k(\partial_k v_i + \partial_i v_k)] - \partial_d[\partial_a x^i \partial_b x^k(\partial_k v_i + \partial_i v_k)] \\ =& \partial_d x^i \partial_b x^k \partial_a x^l \partial_l \partial_k v_i - \partial_a x^i \partial_b x^k \partial_d x^l \partial_l \partial_k v_i \\ &+ (\partial_d x^i \partial_a \partial_b x^k - \partial_a x^k \partial_d \partial_b x^i)(\partial_k v_i + \partial_i v_k) \\ =& \partial_d x^i \partial_a x^k \partial_b(\partial_k v_i - \partial_i v_k) + (\partial_d x^i \partial_a \partial_b x^k + \partial_a x^k \partial_d \partial_b x^i)(\partial_k v_i - \partial_i v_k) \\ &+ 2\partial_d x^i \partial_a \partial_b x^k \partial_i v_k - 2\partial_a x^k \partial_d \partial_b x^i \partial_k v_i \\ =& \partial_b \omega_{ad} + 2(\partial_d v_k \partial_a \partial_b x^k - \partial_a v_k \partial_d \partial_b x^k) \\ =& \partial_b \omega_{ad} + [(\dot{g}_{cd} - \omega_{cd})\partial_a \partial_b x^k - (\dot{g}_{ca} - \omega_{ca})\partial_d \partial_b x^k]\frac{\partial y^c}{\partial x^k}. \end{aligned}$$

Due to div $W = 0$, we can get from Lemma 6.4 and (6.29) that

$$\begin{aligned} |D_t \operatorname{curl} w| &\leqslant |\operatorname{curl} \dot{w}| + |\partial\omega|\,|W| + 2\,|\dot{g}|\,|\partial W| + (|\dot{g}| + |\omega|)\left|\partial^2 x\right|\left|\frac{\partial y}{\partial x}\right||W| \\ &\leqslant |\operatorname{curl} \dot{w}| + K_1\,|\dot{g}|\Big(|\operatorname{curl} w| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S W| + [g]_1|W|\Big) \end{aligned}$$

$$+\left[|\partial\omega|+(|\dot{g}|+|\omega|)\left|\partial^2 x\right|\left|\frac{\partial y}{\partial x}\right|\right]|W|.$$

Thus, we have to derive the estimates of curl $\dot{w}$. From (5.3), we get

$$D_t\dot{w}_a = g_{ab}\mathcal{L}_B^2 W^b - \partial_a q + \partial_a(W^c\partial_c P) + \omega_{ab}\dot{W}^b + 2\delta_{il}\partial_a x^l \mathcal{L}_B\partial_c x^i \mathcal{L}_B W^c + g_{ab}F^b.$$

Note that the above equation can be also formulated as

$$D_t\dot{w}_a - g_{ab}\mathcal{L}_B^2 W^b + g_{ab}(\mathcal{A}W^b - \mathcal{C}\dot{W}^b + \mathcal{X}\mathcal{L}_B W^b) = \underline{F}_a. \tag{6.30}$$

Then, we have

$$\begin{aligned}
D_t \mathrm{curl}\,\dot{w}_{ad} =& \partial_a(g_{db}\mathcal{L}_B^2 W^b - \partial_d q + \partial_d(W^c\partial_c P) + \omega_{db}\dot{W}^b + 2\delta_{il}\partial_d x^l\mathcal{L}_B\partial_c x^i\mathcal{L}_B W^c \\
&+ \underline{F}_d) - \partial_d(g_{ab}\mathcal{L}_B^2 W^b - \partial_a q + \partial_a(W^c\partial_c P) \\
&+ \omega_{ab}\dot{W}^b + 2\delta_{il}\partial_a x^l\mathcal{L}_B\partial_c x^i\mathcal{L}_B W^c + \underline{F}_a) \\
=& \mathrm{curl}\,\underline{\mathcal{L}_B^2 W}_{ad} + \partial_b\omega_{da}\dot{W}^b + \omega_{db}\partial_a\dot{W}^b - \omega_{ab}\partial_d\dot{W}^b + \mathrm{curl}\,\underline{F}_{ad} \\
&+ 2\delta_{il}\partial_d x^l\partial_a(\mathcal{L}_B\partial_c x^i\mathcal{L}_B W^c) - 2\delta_{il}\partial_a x^l\partial_d(\mathcal{L}_B\partial_c x^i\mathcal{L}_B W^c),
\end{aligned} \tag{6.31}$$

where we have used the identity $\partial_a\omega_{db} - \partial_d\omega_{ab} = \partial_b\omega_{da}$ which can be verified by (2.15). In fact,

$$\begin{aligned}
\partial_a\omega_{db} - \partial_d\omega_{ab} =& \partial_a[\partial_d x^i\partial_b x^k(\partial_i v_k - \partial_k v_i)] - \partial_d[\partial_a x^i\partial_b x^k(\partial_i v_k - \partial_k v_i)] \\
=& \partial_d x^i\partial_a\partial_b x^k(\partial_i v_k - \partial_k v_i) - \partial_a x^i\partial_d\partial_b x^k(\partial_i v_k - \partial_k v_i) \\
&+ \partial_d x^i\partial_b x^k\partial_a(\partial_i v_k - \partial_k v_i) - \partial_a x^i\partial_b x^k\partial_d(\partial_i v_k - \partial_k v_i) \\
=& \partial_b[(\partial_d x^i\partial_a x^k)(\partial_i v_k - \partial_k v_i)] - (\partial_d x^i\partial_a x^k)\partial_b x^l\partial_l(\partial_i v_k - \partial_k v_i) \\
&+ \partial_d x^i\partial_b x^k\partial_a x^l\partial_l(\partial_i v_k - \partial_k v_i) - \partial_a x^i\partial_b x^k\partial_d x^l\partial_l(\partial_i v_k - \partial_k v_i) \\
=& \partial_b\omega_{da} - \partial_l\partial_i v_k(\partial_d x^i\partial_a x^k\partial_b x^l - \partial_d x^i\partial_b x^k\partial_a x^l + \partial_a x^i\partial_b x^k\partial_d x^l) \\
&+ \partial_l\partial_k v_i(\partial_d x^i\partial_a x^k\partial_b x^l - \partial_d x^i\partial_b x^k\partial_a x^l + \partial_a x^i\partial_b x^k\partial_d x^l) \\
=& \partial_b\omega_{da} - \partial_l\partial_i v_k\partial_d x^i\partial_a x^k\partial_b x^l + \partial_l\partial_k v_i\partial_a x^i\partial_b x^k\partial_d x^l \\
=& \partial_b\omega_{da}.
\end{aligned}$$

From (A.5), we get

$$\underline{\mathcal{L}_B^2 W}_a = g_{ea}\mathcal{L}_B^2 W^e = \mathcal{L}_B^2 w_a - 2\mathcal{L}_B^2 g_{ea}W^e - \mathcal{L}_B g_{ea}\mathcal{L}_B W^e,$$

and then

$$
\begin{aligned}
\operatorname{curl} \underline{\mathcal{L}_B^2 W}_{ad} =& \operatorname{curl} \mathcal{L}_B^2 w_{ad} - 2[\partial_a(\mathcal{L}_B^2 g_{ed} W^e) - \partial_d(\mathcal{L}_B^2 g_{ea} W^e)] \\
& - [\partial_a(\mathcal{L}_B g_{ed} \mathcal{L}_B W^e) - \partial_d(\mathcal{L}_B g_{ea} \mathcal{L}_B W^e)] \\
=& \operatorname{curl} \mathcal{L}_B^2 w_{ad} - 2(\operatorname{curl} \mathcal{L}_B^2 g_{e\cdot})_{ad} W^e - 2[\mathcal{L}_B^2 g_{ed} \partial_a W^e - \mathcal{L}_B^2 g_{ea} \partial_d W^e] \\
& - (\operatorname{curl} \mathcal{L}_B g_{e\cdot})_{ad} \mathcal{L}_B W^e - [\mathcal{L}_B g_{ed} \partial_a \mathcal{L}_B W^e - \mathcal{L}_B g_{ea} \partial_d \mathcal{L}_B W^e]. \qquad (6.32)
\end{aligned}
$$

From (6.31) and (6.32), it follows that

$$
\begin{aligned}
D_t \operatorname{curl} \dot{w}_{ad} =& \operatorname{curl} \mathcal{L}_B^2 w_{ad} - 2(\operatorname{curl} \mathcal{L}_B^2 g_{e\cdot})_{ad} W^e - (\operatorname{curl} \mathcal{L}_B g_{e\cdot})_{ad} \mathcal{L}_B W^e + \partial_b \omega_{da} \dot{W}^b \\
& - 2[\mathcal{L}_B^2 g_{ed} \partial_a W^e - \mathcal{L}_B^2 g_{ea} \partial_d W^e] + [\omega_{db} \partial_a \dot{W}^b - \omega_{ab} \partial_d \dot{W}^b] \\
& + 2\partial_c B^b [\partial_a g_{db} - \partial_d g_{ab}] \mathcal{L}_B W^c + 2[g_{db} \partial_a \partial_c B^b - g_{ab} \partial_d \partial_c B^b] \mathcal{L}_B W^c \\
& + 2[\delta_{il} \partial_d x^l \partial_a B \partial_c x^i - \delta_{il} \partial_a x^l \partial_d B \partial_c x^i] \mathcal{L}_B W^c \\
& - [\mathcal{L}_B g_{cd} \partial_a \mathcal{L}_B W^c - \mathcal{L}_B g_{ca} \partial_d \mathcal{L}_B W^c] \\
& + 2\partial_c B^b [g_{db} \partial_a \mathcal{L}_B W^c - g_{ab} \partial_d \mathcal{L}_B W^c] \\
& + 2\delta_{il} B \partial_c x^i [\partial_d x^l \partial_a \mathcal{L}_B W^c - \partial_a x^l \partial_d \mathcal{L}_B W^c] + \operatorname{curl} \underline{F}_{ad}. \qquad (6.33)
\end{aligned}
$$

With the help of (A.11) and (A.7), we have

$$
\begin{aligned}
& \langle \operatorname{curl} \underline{\mathcal{L}_B^2 W}, \operatorname{curl} \dot{w} \rangle \\
=& \int_\Omega g^{ab} g^{cd} \operatorname{curl} \underline{\mathcal{L}_B^2 W}_{ad} \operatorname{curl} \dot{w}_{bc} dy \\
=& -\frac{1}{2} D_t \langle \operatorname{curl} \mathcal{L}_B w, \operatorname{curl} \mathcal{L}_B w \rangle + \int_\Omega \dot{g}^{ab} g^{cd} \operatorname{curl} \mathcal{L}_B w_{ad} \operatorname{curl} \mathcal{L}_B w_{bc} dy \\
& - 2\int_\Omega (\mathcal{L}_B g^{ab}) g^{cd} \operatorname{curl} \mathcal{L}_B w_{ad} \operatorname{curl} \dot{w}_{bc} dy \\
& - \int_\Omega g^{ab} g^{cd} \operatorname{curl} \mathcal{L}_B w_{ad} \mathcal{L}_B \Big\{ \partial_e \omega_{cb} W^e + \dot{g}_{be} \partial_c W^e - \dot{g}_{ce} \partial_b W^e \\
& + [(\dot{g}_{eb} - \omega_{eb}) \partial_c \partial_f x^k - (\dot{g}_{ec} - \omega_{ec}) \partial_b \partial_f x^k] \frac{\partial y^e}{\partial x^k} W^f \Big\} dy \\
& - \int_\Omega g^{ab} g^{cd} \big\{ 2(\operatorname{curl} \mathcal{L}_B^2 g_{e\cdot})_{ad} W^e + 2[\mathcal{L}_B^2 g_{ed} \partial_a W^e - \mathcal{L}_B^2 g_{ea} \partial_d W^e] \\
& + (\operatorname{curl} \mathcal{L}_B g_{e\cdot})_{ad} \mathcal{L}_B W^e + [\mathcal{L}_B g_{ed} \partial_a \mathcal{L}_B W^e - \mathcal{L}_B g_{ea} \partial_d \mathcal{L}_B W^e] \big\} \operatorname{curl} \dot{w}_{bc} dy.
\end{aligned}
$$

Let

$$
E_{\text{curl}}(t) = \langle \operatorname{curl} w, \operatorname{curl} w \rangle + \langle \operatorname{curl} \dot{w}, \operatorname{curl} \dot{w} \rangle + \langle \operatorname{curl} \mathcal{L}_B w, \operatorname{curl} \mathcal{L}_B w \rangle.
$$

Taking the inner product of (6.33) with curl $\dot{w}$, we obtain, with the help of (A.9), that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}E_{\text{curl}}(t) \leqslant\ & (2\|\dot{g}\|_{L^\infty(\Omega)} + \|\mathcal{L}_B g\|_{L^\infty(\Omega)})E_{\text{curl}} \\
& + E_{\text{curl}}^{1/2}[\|\mathcal{L}_B\partial\omega\|_{L^\infty(\Omega)} + \|\partial\omega\|_{L^\infty(\Omega)}]E_0 \\
& + E_{\text{curl}}^{1/2}[\|\mathcal{L}_B\dot{g}\|_{L^\infty(\Omega)} + \|\dot{g}\|_{L^\infty(\Omega)}] \\
& \cdot [\|\partial W\| + \|\partial\mathcal{L}_B W\| + \|\partial^2 B\|_{L^\infty(\Omega)}E_0] \\
& + E_{\text{curl}}^{1/2}[\|\mathcal{L}_B\dot{g}\|_{L^\infty(\Omega)} + \|\mathcal{L}_B\omega\|_{L^\infty(\Omega)}]\|\partial^2 x\|_{L^\infty(\Omega)}\|\frac{\partial y}{\partial x}\|_{L^\infty(\Omega)}E_0 \\
& + E_{\text{curl}}^{1/2}[\|\dot{g}\|_{L^\infty(\Omega)} + \|\omega\|_{L^\infty(\Omega)}]\big[\|\mathcal{L}_B\partial^2 x\|_{L^\infty(\Omega)}\|\frac{\partial y}{\partial x}\|_{L^\infty(\Omega)} \\
& \quad + \|\partial^2 x\|_{L^\infty(\Omega)}\|\mathcal{L}_B\frac{\partial y}{\partial x}\|_{L^\infty(\Omega)} + 2\|\partial^2 x\|_{L^\infty(\Omega)}\|\frac{\partial y}{\partial x}\|_{L^\infty(\Omega)}\big]E_0 \\
& + E_{\text{curl}}^{1/2}[2\|\text{curl}\,\mathcal{L}_B^2 g\|_{L^\infty(\Omega)} + \|\text{curl}\,\mathcal{L}_B g\|_{L^\infty(\Omega)}]E_0 \\
& + E_{\text{curl}}^{1/2}[4\|\mathcal{L}_B^2 g\|_{L^\infty(\Omega)}\|\partial W\| + 2\|\mathcal{L}_B g\|_{L^\infty(\Omega)}\|\partial\mathcal{L}_B W\|] \\
& + 4E_{\text{curl}}^{1/2}[\|\delta_{il}\partial_d x^l\partial_a(B\partial_c x^i) + \partial_a(g_{db}\partial_c B^b)\|_{L^\infty(\Omega)}E_0 \\
& \quad + \|\delta_{il}\partial_d x^l B\partial_c x^i + g_{db}\partial_c B^b\|_{L^\infty(\Omega)}\|\partial\mathcal{L}_B W\|] \\
& + 2E_{\text{curl}}^{1/2}\|\omega\|_{L^\infty(\Omega)}\|\partial\dot{W}\| + E_{\text{curl}}^{1/2}\|\text{curl}\,\underline{F}\|.
\end{aligned} \tag{6.34}$$

From (6.8) and (6.6), it follows that

$$\begin{aligned}
\|\partial W\| &\leqslant K_1\left(\|\text{curl}\,w\| + \sum_{S\in\mathcal{S}}\|\mathcal{L}_S W\| + \|[g]_1\|_{L^\infty(\Omega)}\|W\|\right) \\
&\leqslant K_1\|\text{curl}\,w\| + K_1 E_1^{\mathcal{S}} + \|[g]_1\|_{L^\infty(\Omega)}E_0,
\end{aligned} \tag{6.35}$$

$$\begin{aligned}
\|\partial \dot{W}\| &\leqslant K_1\left(\|\text{curl}\,\dot{w}\| + \sum_{S\in\mathcal{S}}\|\mathcal{L}_S \dot{W}\| + \|[g]_1\|_{L^\infty(\Omega)}\|\dot{W}\|\right) \\
&\leqslant K_1\|\text{curl}\,\dot{w}\| + K_1 E_1^{\mathcal{S}} + \|[g]_1\|_{L^\infty(\Omega)}E_0.
\end{aligned} \tag{6.36}$$

Then, from

$$\underline{\mathcal{L}_B W}_a = g_{ab}\mathcal{L}_B W^b = \mathcal{L}_B w_a - (\mathcal{L}_B g_{ab})W^b,$$

we have

$$\begin{aligned}
\|\partial\mathcal{L}_B W\| &\leqslant K_1\left(\|\text{curl}\,\underline{\mathcal{L}_B W}\| + \sum_{S\in\mathcal{S}}\|[\mathcal{L}_S,\mathcal{L}_B]W\| + E_1^{\mathcal{S}} + \|[g]_1\|_{L^\infty(\Omega)}E_0\right) \\
&\leqslant K_1\Big(\|\text{curl}\,\mathcal{L}_B w\| + (\|B\|_1^{\mathcal{S}} + \|\mathcal{L}_B g\|_{L^\infty(\Omega)})\|\text{curl}\,w\| + E_1^{\mathcal{S}}
\end{aligned}$$

$$+ (\|[g]_1\|_{L^\infty(\Omega)} + \|\partial \mathcal{L}_B g\|_{L^\infty(\Omega)}) E_0\Big). \tag{6.37}$$

Combining (6.34), (6.35), (6.36) and (6.37), we obtain

$$\frac{d}{dt}(E_{\text{curl}}^{1/2}(t)) \leqslant n_{1,\text{curl}}(E_{\text{curl}}^{1/2} + E_1^{\mathcal{S}}) + \tilde{n}_{1,\text{curl}} E_0 + \|\text{curl}\, \underline{F}\|,$$

where

$$\begin{aligned}
n_{1,\text{curl}} =& K_1(1 + \|B\|_1^{\mathcal{S}} + \|\mathcal{L}_B g\|_{L^\infty(\Omega)})\Big(\|\mathcal{L}_B \dot{g}\|_{L^\infty(\Omega)} + \|\dot{g}\|_{L^\infty(\Omega)} \\
&+ \|\mathcal{L}_B^2 g\|_{L^\infty(\Omega)} + \|\mathcal{L}_B g\|_{L^\infty(\Omega)} + \|\delta_{il}\partial_d x^l B \partial_c x^i \\
&+ g_{db}\partial_c B^b\|_{L^\infty(\Omega)} + \|\omega\|_{L^\infty(\Omega)}\Big), \\
\tilde{n}_{1,\text{curl}} =& n_{1,\text{curl}}(\|[g]_1\|_{L^\infty(\Omega)} + \|\partial \mathcal{L}_B g\|_{L^\infty(\Omega)}) \\
&+ K_1\Big(\|\mathcal{L}_B \partial\omega\|_{L^\infty(\Omega)} + \|\partial\omega\|_{L^\infty(\Omega)} + [\|\mathcal{L}_B \dot{g}\|_{L^\infty(\Omega)} \\
&+ \|\dot{g}\|_{L^\infty(\Omega)}]\|\partial^2 B\|_{L^\infty(\Omega)} \\
&+ [\|\mathcal{L}_B \dot{g}\|_{L^\infty(\Omega)} + \|\mathcal{L}_B \omega\|_{L^\infty(\Omega)}]\|\partial^2 x\|_{L^\infty(\Omega)} \left\|\frac{\partial y}{\partial x}\right\|_{L^\infty(\Omega)} \\
&+ [\|\dot{g}\|_{L^\infty(\Omega)} + \|\omega\|_{L^\infty(\Omega)}]\Big[\|\mathcal{L}_B \partial^2 x\|_{L^\infty(\Omega)} \left\|\frac{\partial y}{\partial x}\right\|_{L^\infty(\Omega)} \\
&+ \|\partial^2 x\|_{L^\infty(\Omega)} \left\|\mathcal{L}_B \frac{\partial y}{\partial x}\right\|_{L^\infty(\Omega)} + 2\|\partial^2 x\|_{L^\infty(\Omega)} \left\|\frac{\partial y}{\partial x}\right\|_{L^\infty(\Omega)}\Big] \\
&+ \|\text{curl}\, \mathcal{L}_B^2 g\|_{L^\infty(\Omega)} + \|\text{curl}\, \mathcal{L}_B g\|_{L^\infty(\Omega)} \\
&+ \|\delta_{il}\partial_d x^l \partial_a(B\partial_c x^i) + \partial_a(g_{db}\partial_c B^b)\|_{L^\infty(\Omega)}\Big).
\end{aligned}$$

Due to $E_{\text{curl}}(0) = 0$, the integration over $[0, t]$ in time gives

$$E_{\text{curl}}^{1/2}(t) \leqslant \int_0^t [n_{1,\text{curl}}(E_{\text{curl}}^{1/2} + E_1^{\mathcal{S}}) + \tilde{n}_{1,\text{curl}} E_0 + \|\text{curl}\, \underline{F}\|] d\tau. \tag{6.38}$$

From (6.28) and (6.38), we have

$$E_1^{\mathcal{S}} + E_{\text{curl}}^{1/2} \leqslant \int_0^t (n_{1,\text{curl}} + n_1^{\mathcal{S}} + \bar{n}_{1,\text{curl}})(E_{\text{curl}}^{1/2} + E_1^{\mathcal{S}}) d\tau + \tilde{f}_1,$$

where

$$\bar{n}_{1,\text{curl}} = K_1 \|g\|_{L^\infty(\Omega)} \|B\|_1^{\mathcal{S}} (1 + \|B\|_1^{\mathcal{S}} + \|B\|_{W^{1,\infty}(\Omega)} + \|\delta_{il} \partial x^l B \partial x^i\|_{L^\infty(\Omega)}),$$

$$\tilde{f}_1 = \tilde{n}_1^{\mathcal{S}} + \int_0^t (\tilde{n}_{1,\text{curl}} + \bar{n}_{1,\text{curl}} \|\text{curl}\, \mathcal{L}_B g\|_{L^\infty(\Omega)}) E_0 d\tau$$

$$+ 2 \int_0^t \Big[\sum_{T \in \mathcal{S}} f_1^T + \|\text{curl}\, \underline{F}\| \Big] d\tau.$$

Therefore, by the Gronwall inequality, we have obtained the following estimates for both the first order tangential derivatives and the curl.

Proposition 6.10. *It holds*

$$E_1^{\mathcal{S}}(t) + E_{\text{curl}}^{1/2}(t) \leqslant \tilde{f}_1(t) + \int_0^t \tilde{f}_1(s) \Big(n_{1,\text{curl}}(s) + n_1^{\mathcal{S}}(s) + \bar{n}_{1,\text{curl}}(s) \Big)$$

$$\cdot \exp\left(\int_s^t (n_{1,\text{curl}}(\tau) + n_1^{\mathcal{S}}(\tau) + \bar{n}_{1,\text{curl}}(\tau)) d\tau \right) ds.$$

Remark 6.11. By Lemma 6.4, we have the estimates for the first-order derivative of W.

6.8. The higher-order estimates for the curl and the normal derivatives

We derive the equations for the curl of higher order derivatives. Since the Lie derivative commutes with D_t and the curl, applying $\mathcal{L}_U^J$ to (6.33) and (6.29) gives

$$\begin{aligned}
D_t \text{curl}\, \mathcal{L}_U^J \dot{w}_{ad} =& \text{curl}\, \mathcal{L}_U^J \mathcal{L}_B^2 w_{ad} + \text{curl}\, \mathcal{L}_U^J \underline{F}_{ad} - 2c_{J_1 J_2} (\text{curl}\, \mathcal{L}_U^{J_1} \mathcal{L}_B^2 g_{e\cdot})_{ad} \mathcal{L}_U^{J_2} W^e \\
& - c_{J_1 J_2} (\text{curl}\, \mathcal{L}_U^{J_1} \mathcal{L}_B g_{e\cdot})_{ad} \mathcal{L}_U^{J_2} \mathcal{L}_B W^e + c_{J_1 J_2} \mathcal{L}_U^{J_1} \partial_b \omega_{da} \mathcal{L}_U^{J_2} \dot{W}^b \\
& - 2c_{J_1 J_2} [\mathcal{L}_U^{J_1} \mathcal{L}_B^2 g_{ed} [\mathcal{L}_U^{J_2}, \partial_a] W^e - \mathcal{L}_U^{J_1} \mathcal{L}_B^2 g_{ea} [\mathcal{L}_U^{J_2}, \partial_d] W^e] \\
& - 2c_{J_1 J_2} [\mathcal{L}_U^{J_1} \mathcal{L}_B^2 g_{ed} \partial_a \mathcal{L}_U^{J_2} W^e - \mathcal{L}_U^{J_1} \mathcal{L}_B^2 g_{ea} \partial_d \mathcal{L}_U^{J_2} W^e] \\
& + c_{J_1 J_2} [\mathcal{L}_U^{J_1} \omega_{db} [\mathcal{L}_U^{J_2}, \partial_a] \dot{W}^b - \mathcal{L}_U^{J_1} \omega_{ab} [\mathcal{L}_U^{J_2}, \partial_d] \dot{W}^b] \\
& + c_{J_1 J_2} [\mathcal{L}_U^{J_1} \omega_{db} \partial_a \mathcal{L}_U^{J_2} \dot{W}^b - \mathcal{L}_U^{J_1} \omega_{ab} \partial_d \mathcal{L}_U^{J_2} \dot{W}^b] \\
& + 2c_{J_1 J_2} \mathcal{L}_U^{J_1} [\partial_c B^b (\partial_a g_{db} - \partial_d g_{ab})] \mathcal{L}_U^{J_2} \mathcal{L}_B W^c \\
& + 2c_{J_1 J_2} \mathcal{L}_U^{J_1} [g_{db} \partial_a \partial_c B^b - g_{ab} \partial_d \partial_c B^b] \mathcal{L}_U^{J_2} \mathcal{L}_B W^c \\
& + 2c_{J_1 J_2} \mathcal{L}_U^{J_1} [\delta_{il} \partial_d x^l \partial_a B \partial_c x^i - \delta_{il} \partial_a x^l \partial_d B \partial_c x^i] \mathcal{L}_U^{J_2} \mathcal{L}_B W^c \\
& - c_{J_1 J_2} [\mathcal{L}_U^{J_1} \mathcal{L}_B g_{cd} [\mathcal{L}_U^{J_2}, \partial_a] \mathcal{L}_B W^c - \mathcal{L}_U^{J_1} \mathcal{L}_B g_{ca} [\mathcal{L}_U^{J_2}, \partial_d] \mathcal{L}_B W^c] \\
& - c_{J_1 J_2} [\mathcal{L}_U^{J_1} \mathcal{L}_B g_{cd} \partial_a \mathcal{L}_U^{J_2} \mathcal{L}_B W^c - \mathcal{L}_U^{J_1} \mathcal{L}_B g_{ca} \partial_d \mathcal{L}_U^{J_2} \mathcal{L}_B W^c] \\
& + 2c_{J_1 J_2} \delta_{il} [\mathcal{L}_U^{J_1} (B \partial_c x^i \partial_d x^l) [\mathcal{L}_U^{J_2}, \partial_a] \mathcal{L}_B W^c
\end{aligned}$$

$$
\begin{aligned}
&- \mathcal{L}_U^{J_1}(B\partial_c x^i \partial_a x^l)[\mathcal{L}_U^{J_2}, \partial_d]\mathcal{L}_B W^c] \\
&+ 2c_{J_1 J_2}\delta_{il}[\mathcal{L}_U^{J_1}(B\partial_c x^i \partial_d x^l)\partial_a \mathcal{L}_U^{J_2}\mathcal{L}_B W^c \\
&- \mathcal{L}_U^{J_1}(B\partial_c x^i \partial_a x^l)\partial_d \mathcal{L}_U^{J_2}\mathcal{L}_B W^c] \\
&+ 2c_{J_1 J_2}[\mathcal{L}_U^{J_1}(g_{db}\partial_c B^b)[\mathcal{L}_U^{J_2}, \partial_a]\mathcal{L}_B W^c \\
&- \mathcal{L}_U^{J_1}(g_{ab}\partial_c B^b)[\mathcal{L}_U^{J_2}, \partial_d]\mathcal{L}_B W^c] \\
&+ 2c_{J_1 J_2}[\mathcal{L}_U^{J_1}(g_{db}\partial_c B^b)\partial_a \mathcal{L}_U^{J_2}\mathcal{L}_B W^c \\
&- \mathcal{L}_U^{J_1}(g_{ab}\partial_c B^b)\partial_d \mathcal{L}_U^{J_2}\mathcal{L}_B W^c],
\end{aligned} \tag{6.39}
$$

and

$$
\begin{aligned}
&D_t \mathrm{curl}\, \mathcal{L}_U^J w_{ab} \\
=&\mathrm{curl}\, \mathcal{L}_U^J \dot{w}_{ab} + c_{J_1 J_2}\mathcal{L}_U^{J_1}\partial_c \omega_{ab}\mathcal{L}_U^{J_2} W^c \\
&+ c_{J_1 J_2}(\mathcal{L}_U^{J_1}\dot{g}_{bc}[\mathcal{L}_U^{J_2}, \partial_a]W^c - \mathcal{L}_U^{J_1}\dot{g}_{ac}[\mathcal{L}_U^{J_2}, \partial_b]W^c) \\
&+ c_{J_1 J_2}(\mathcal{L}_U^{J_1}\dot{g}_{bc}\partial_a \mathcal{L}_U^{J_2} W^c - \mathcal{L}_U^{J_1}\dot{g}_{ac}\partial_b \mathcal{L}_U^{J_2} W^c) \\
&+ c_{J_1 J_2}\mathcal{L}_U^{J_1}\left\{[(\dot{g}_{eb} - \omega_{eb})\partial_a \partial_c x^k - (\dot{g}_{ea} - \omega_{ea})\partial_b \partial_c x^k]\frac{\partial y^e}{\partial x^k}\right\}\mathcal{L}_U^{J_2} W^c.
\end{aligned} \tag{6.40}
$$

At this point, the commutator $[\mathcal{L}_U^J, \partial]$ needs to be derived. If $|J| = 1$, it is the identity (A.9). For $|J| \geqslant 2$, we have the following identity.

Lemma 6.12. *For $|J| = r \geqslant 1$ and $|J_r| = 1$, it holds*

$$
\begin{aligned}
[\mathcal{L}_U^{J-J_r}\mathcal{L}_U^{J_r}, \partial_a]W^b =&W^c \mathcal{L}_U^{J-J_r}\partial_c \partial_a U_{J_r}^b \\
&+ \sum_{\substack{J=I_1+I_2+I_3 \\ |I_3|=1}} \mathrm{sgn}(|I_1|)\mathcal{L}_U^{I_1} W^c \mathcal{L}_U^{I_2}\partial_c \partial_a U_{I_3}^b.
\end{aligned} \tag{6.41}
$$

Proof. For $r = 1$, it follows from (A.4)

$$
[\mathcal{L}_U, \partial_a]W^b = \partial_c \partial_a U^b W^c. \tag{6.42}
$$

For $r \geqslant 2$, we prove it by induction argument. For $r = 2$, we have

$$
\begin{aligned}
[\mathcal{L}_{U_1}\mathcal{L}_{U_2}, \partial_a]W^b =&\mathcal{L}_{U_1}[\mathcal{L}_{U_2}, \partial_a]W^b + [\mathcal{L}_{U_1}, \partial_a]\mathcal{L}_{U_2} W^b \\
=&\mathcal{L}_{U_1}(W^d \partial_d \partial_a U_2^b) + (\mathcal{L}_{U_2} W^d)\partial_d \partial_a U_1^b \\
=&(\mathcal{L}_{U_1} W^d)\partial_d \partial_a U_2^b + (\mathcal{L}_{U_2} W^d)\partial_d \partial_a U_1^b + W^d \mathcal{L}_{U_1}\partial_d \partial_a U_2^b,
\end{aligned}
$$

which satisfies (6.41).

Now, we assume that (6.41) holds for $r = s$. Then, we derive the case $r = s + 1$. For $|J| = s + 1$ and $|J_r| = 1$, one gets by using (A.9)

$$\begin{aligned}
&[\mathcal{L}_U^{J-J_{s+1}}\mathcal{L}_U^{J_{s+1}},\partial_a]W^b\\
=&\mathcal{L}_U^{J-J_{s+1}}[\mathcal{L}_U^{J_{s+1}},\partial_a]W^b+[\mathcal{L}_U^{J-J_{s+1}},\partial_a]\mathcal{L}_U^{J_{s+1}}W^b\\
=&\mathcal{L}_U^{J-J_{s+1}}(W^d\partial_d\partial_a U^b_{J_{s+1}})+\mathcal{L}_U^{J_{s+1}}W^d\mathcal{L}_U^{J-J_s-J_{s+1}}\partial_d\partial_a U^b_{J_s}\\
&+\sum_{\substack{J-J_{s+1}=J_1+J_2+J_3\\|J_1|\geqslant 1,\,|J_3|=1}}\mathcal{L}_U^{J_1}\mathcal{L}_U^{J_{s+1}}W^d\mathcal{L}_U^{J_2}\partial_d\partial_a U^b_{J_3}\\
=&\sum_{J-J_{s+1}=J_1+J_2}\mathcal{L}_U^{J_1}W^d\mathcal{L}_U^{J_2}\partial_d\partial_a U^b_{J_{s+1}}+\mathcal{L}_U^{J_{s+1}}W^d\mathcal{L}_U^{J-J_s-J_{s+1}}\partial_d\partial_a U^b_{J_s}\\
&+\sum_{\substack{J-J_{s+1}=J_1+J_2+J_3\\|J_1|\geqslant 1,\,|J_3|=1}}\mathcal{L}_U^{J_1}\mathcal{L}_U^{J_{s+1}}W^d\mathcal{L}_U^{J_2}\partial_d\partial_a U^b_{J_3}\\
=&W^d\mathcal{L}_U^{J-J_{s+1}}\partial_d\partial_a U^b_{J_{s+1}}+\sum_{\substack{J=J_1+J_2+J_3\\|J_1|\geqslant 1,\,|J_3|=1}}\mathcal{L}_U^{J_1}W^d\mathcal{L}_U^{J_2}\partial_d\partial_a U^b_{J_3},
\end{aligned}$$

which is of the form in (6.41) with $r=s+1$. Thus, we proved the identity by induction. □

For $U\in\mathcal{U}$ and $|J|=r-1$, let

$$\begin{aligned}
E_{r-1,\mathrm{curl}}(t)=&\langle\mathrm{curl}\,\mathcal{L}_U^J w,\mathrm{curl}\,\mathcal{L}_U^J w\rangle+\langle\mathrm{curl}\,\mathcal{L}_U^J\dot{w},\mathrm{curl}\,\mathcal{L}_U^J\dot{w}\rangle\\
&+\langle\mathrm{curl}\,\mathcal{L}_B\mathcal{L}_U^J w,\mathrm{curl}\,\mathcal{L}_B\mathcal{L}_U^J w\rangle.
\end{aligned}$$

Then, from (6.40) it follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E_{r,\mathrm{curl}}(t)\\
=&\int_\Omega \dot{g}^{ab}g^{cd}(\mathrm{curl}\,\mathcal{L}_U^J w_{ad}\mathrm{curl}\,\mathcal{L}_U^J w_{bc}+\mathrm{curl}\,\mathcal{L}_U^J\dot{w}_{ad}\mathrm{curl}\,\mathcal{L}_U^J\dot{w}_{bc}\\
&\qquad+\mathrm{curl}\,\mathcal{L}_B\mathcal{L}_U^J w_{ad}\mathrm{curl}\,\mathcal{L}_B\mathcal{L}_U^J w_{bc})dy\\
&-\int_\Omega \mathcal{L}_B(g^{ab}g^{cd})\mathrm{curl}\,\mathcal{L}_B\mathcal{L}_U^J w_{ad}\mathrm{curl}\,\mathcal{L}_U^J\dot{w}_{bc}dy\\
&+\langle\mathrm{curl}\,\mathcal{L}_U^J\dot{w},\mathrm{curl}\,\mathcal{L}_U^J w\rangle+\langle\mathrm{curl}\,\mathcal{L}_U^J\dot{w},D_t\mathrm{curl}\,\mathcal{L}_U^J\dot{w}-\mathrm{curl}\,\mathcal{L}_B^2\mathcal{L}_U^J w\rangle\\
&+\Big\langle\mathrm{curl}\,\mathcal{L}_U^J w,c_{J_1J_2}\Big(\mathcal{L}_U^{J_1}\partial_c\omega_{ab}\mathcal{L}_U^{J_2}W^c+(\mathcal{L}_U^{J_1}\dot{g}_{bc}[\mathcal{L}_U^{J_2},\partial_a]W^c\\
&\quad-\mathcal{L}_U^{J_1}\dot{g}_{ac}[\mathcal{L}_U^{J_2},\partial_b]W^c)+(\mathcal{L}_U^{J_1}\dot{g}_{bc}\partial_a\mathcal{L}_U^{J_2}W^c-\mathcal{L}_U^{J_1}\dot{g}_{ac}\partial_b\mathcal{L}_U^{J_2}W^c)\\
&\quad+\mathcal{L}_U^{J_1}\Big\{[(\dot{g}_{eb}-\omega_{eb})\partial_a\partial_c x^k-(\dot{g}_{ea}-\omega_{ea})\partial_b\partial_c x^k]\frac{\partial y^e}{\partial x^k}\Big\}\mathcal{L}_U^{J_2}W^c\Big)\Big\rangle\\
&+\Big\langle\mathrm{curl}\,\mathcal{L}_B\mathcal{L}_U^J w,c_{J_1J_2}\mathcal{L}_B\Big(\mathcal{L}_U^{J_1}\partial_c\omega_{ab}\mathcal{L}_U^{J_2}W^c+(\mathcal{L}_U^{J_1}\dot{g}_{bc}[\mathcal{L}_U^{J_2},\partial_a]W^c
\end{aligned}$$

$$- \mathcal{L}_U^{J_1} \dot{g}_{ac} [\mathcal{L}_U^{J_2}, \partial_b] W^c) + (\mathcal{L}_U^{J_1} \dot{g}_{bc} \partial_a \mathcal{L}_U^{J_2} W^c - \mathcal{L}_U^{J_1} \dot{g}_{ac} \partial_b \mathcal{L}_U^{J_2} W^c)$$
$$+ \mathcal{L}_U^{J_1} \left\{ [(\dot{g}_{eb} - \omega_{eb}) \partial_a \partial_c x^k - (\dot{g}_{ea} - \omega_{ea}) \partial_b \partial_c x^k] \frac{\partial y^e}{\partial x^k} \right\} \mathcal{L}_U^{J_2} W^c \Big) \Big\rangle.$$

Thus, by (6.39) and Lemma 6.4, we get

$$\begin{aligned}
&\frac{d}{dt} E_{r-1,\mathrm{curl}}^{1/2}(t) \\
&\leqslant (1 + \|\dot{g}\|_{L^\infty(\Omega)} + \|\mathcal{L}_B g\|_{L^\infty(\Omega)}) E_{r,\mathrm{curl}}^{1/2} + \|\mathrm{curl}\, \mathcal{L}_U^J \underline{F}\| \\
&\quad + \|\mathrm{curl}\, [\mathcal{L}_U^J, \mathcal{L}_B^2] w\| + C c_{J_1 J_2} \Big(\|\mathrm{curl}\, \mathcal{L}_U^{J_1} \mathcal{L}_B^2 g\|_{L^\infty(\Omega)} \|\mathcal{L}_U^{J_2} W\| \qquad (6.43) \\
&\quad + \|\mathcal{L}_U^{J_1} \partial\omega\|_{L^\infty(\Omega)} \|\mathcal{L}_U^{J_2} \dot{W}\| + \|\mathrm{curl}\, \mathcal{L}_U^{J_1} \mathcal{L}_B g\|_{L^\infty(\Omega)} (\|\mathcal{L}_B \mathcal{L}_U^{J_2} W\| \\
&\quad + K_1 \tilde{c}_{J_2}^{J_{21} J_{22}} [\|B_{J_{21}}\|_{L^\infty(\Omega)} (\|\mathrm{curl}\, \underline{W_{J_{22}}}\| + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S W_{J_{22}}\|) + \|W_{J_{22}}\| \qquad (6.44) \\
&\quad \cdot (\|\mathrm{curl}\, \underline{B_{J_{21}}}\|_{L^\infty(\Omega)} + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S B_{J_{21}}\|_{L^\infty(\Omega)} + \|[g]_1\|_{L^\infty(\Omega)} \|B_{J_{21}}\|_{L^\infty(\Omega)})]) \\
&\quad + (\|\mathcal{L}_U^{J_1} \mathcal{L}_B^2 g\|_{L^\infty(\Omega)} + \|\mathcal{L}_U^{J_1} \dot{g}\|_{L^\infty(\Omega)} + \|\mathcal{L}_B \mathcal{L}_U^{J_1} \dot{g}\|_{L^\infty(\Omega)}) \\
&\qquad \cdot (\|[\mathcal{L}_U^{J_2}, \partial] W\| + K_1 (\|\mathrm{curl}\, \underline{\mathcal{L}_U^{J_2} W}\| + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S \mathcal{L}_U^{J_2} W\| \qquad (6.45) \\
&\quad + \|[g]_1\|_{L^\infty(\Omega)} \|\mathcal{L}_U^{J_2} W\|)) + \|\mathcal{L}_U^{J_1} \omega\|_{L^\infty(\Omega)} (\|[\mathcal{L}_U^{J_2}, \partial] \dot{W}\| + K_1 (\|\mathrm{curl}\, \underline{\mathcal{L}_U^{J_2} \dot{W}}\| \qquad (6.46) \\
&\qquad\qquad + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S \mathcal{L}_U^{J_2} \dot{W}\| + \|[g]_1\|_{L^\infty(\Omega)} \|\mathcal{L}_U^{J_2} \dot{W}\|)) \\
&\quad + [\|\mathcal{L}_U^{J_1} [\partial B^b (\partial_a g_{db} - \partial_d g_{ab}) + g_{db} \partial_a \partial B^b - g_{ab} \partial_d \partial B^b]\|_{L^\infty(\Omega)} \\
&\qquad + \|\mathcal{L}_U^{J_1} [\delta_{il} \partial_d x^l \partial_a B \partial_c x^i - \delta_{il} \partial_a x^l \partial_d B \partial_c x^i]\|_{L^\infty(\Omega)}] (\|\mathcal{L}_B \mathcal{L}_U^{J_2} W\| \\
&\quad + K_1 \tilde{c}_{J_2}^{J_{21} J_{22}} [\|B_{J_{21}}\|_{L^\infty(\Omega)} (\|\mathrm{curl}\, \underline{W_{J_{22}}}\| + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S W_{J_{22}}\|) + (\|\mathrm{curl}\, \underline{B_{J_{21}}}\|_{L^\infty(\Omega)} \\
&\quad + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S B_{J_{21}}\|_{L^\infty(\Omega)} + \|[g]_1\|_{L^\infty(\Omega)} \|B_{J_{21}}\|_{L^\infty(\Omega)}) \|W_{J_{22}}\|]) \\
&\quad + (\|\mathcal{L}_U^{J_1} \mathcal{L}_B g\|_{L^\infty(\Omega)} + \|\delta_{il} \mathcal{L}_U^{J_1} (B \partial x^i \partial x^l)\|_{L^\infty(\Omega)} + \|\mathcal{L}_U^{J_1} (g_{db} \partial_c B^b)\|_{L^\infty(\Omega)}) \\
&\qquad \cdot (\|[\mathcal{L}_U^{J_2}, \partial] \mathcal{L}_B W\| + K_1 (\|\mathrm{curl}\, \underline{\mathcal{L}_U^{J_2} \mathcal{L}_B W}\| + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S \mathcal{L}_U^{J_2} \mathcal{L}_B W\| \qquad (6.47) \\
&\qquad + \|[g]_1\|_{L^\infty(\Omega)} (\|\mathcal{L}_B \mathcal{L}_U^{J_2} W\| + K_1 \tilde{c}_{J_2}^{J_{21} J_{22}} [\|B_{J_{21}}\|_{L^\infty(\Omega)} (\|\mathrm{curl}\, \underline{W_{J_{22}}}\| \\
&\qquad + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S W_{J_{22}}\|) + (\|\mathrm{curl}\, \underline{B_{J_{21}}}\|_{L^\infty(\Omega)} + \sum_{S \in \mathcal{S}} \|\mathcal{L}_S B_{J_{21}}\|_{L^\infty(\Omega)} \\
&\qquad + \|[g]_1\|_{L^\infty(\Omega)} \|B_{J_{21}}\|_{L^\infty(\Omega)}) \|W_{J_{22}}\|]))) \\
&\quad + (\|\mathcal{L}_U^{J_1} \partial\omega\|_{L^\infty(\Omega)} + \|\mathcal{L}_B \mathcal{L}_U^{J_1} \partial\omega\|_{L^\infty(\Omega)}) (\|\mathcal{L}_U^{J_2} W\| + \|\mathcal{L}_B \mathcal{L}_U^{J_2} W\|)
\end{aligned}$$

$$
\begin{aligned}
&+\|\mathcal{L}_U^{J_1}\Big\{[(\dot g_{eb}-\omega_{eb})\partial_a\partial_c x^k-(\dot g_{ea}-\omega_{ea})\partial_b\partial x^k]\frac{\partial y^e}{\partial x^k}\Big\}\|_{L^\infty(\Omega)}\\
&\cdot(\|\mathcal{L}_U^{J_2}W\|+\|\mathcal{L}_B\mathcal{L}_U^{J_2}W\|)+\|\mathcal{L}_U^{J_2}W\|\\
&\cdot\left\|\mathcal{L}_B\mathcal{L}_U^{J_1}\Big\{[(\dot g_{eb}-\omega_{eb})\partial_a\partial_c x^k-(\dot g_{ea}-\omega_{ea})\partial_b\partial x^k]\frac{\partial y^e}{\partial x^k}\Big\}\right\|_{L^\infty(\Omega)}\Big).
\end{aligned}
$$

We first consider the term $|\mathrm{curl}\,[\mathcal{L}_U^I,\mathcal{L}_B^2]w|$ in the line labeled (6.43). It holds

$$
\begin{aligned}
[\mathcal{L}_U^I,\mathcal{L}_B^2]w_a=&[\mathcal{L}_U^I,\mathcal{L}_B]\mathcal{L}_Bw_a+\mathcal{L}_B[\mathcal{L}_U^I,\mathcal{L}_B]w_a\\
=&\tilde c_I^{I_1I_2}[B_{I_1}^b\partial_b\mathcal{L}_U^{I_2}\mathcal{L}_Bw_a+(\partial_aB_{I_1}^c)\mathcal{L}_U^{I_2}\mathcal{L}_Bw_c\\
&+\mathcal{L}_B[B_{I_1}^b\partial_b\mathcal{L}_U^{I_2}w_a+(\partial_aB_{I_1}^c)\mathcal{L}_U^{I_2}w_c]]\\
=&\tilde c_I^{I_1I_2}[B_{I_1}^b\partial_b\mathcal{L}_U^{I_2}\mathcal{L}_Bw_a+(\partial_aB_{I_1}^c)\mathcal{L}_U^{I_2}\mathcal{L}_Bw_c+\mathcal{L}_BB_{I_1}^b\partial_b\mathcal{L}_U^{I_2}w_a\\
&+B_{I_1}^b\mathcal{L}_B\partial_b\mathcal{L}_U^{I_2}w_a+\mathcal{L}_B(\partial_aB_{I_1}^c)\mathcal{L}_U^{I_2}w_c+(\partial_aB_{I_1}^c)\mathcal{L}_B\mathcal{L}_U^{I_2}w_c],
\end{aligned}
$$

which yields

$$
\begin{aligned}
&(\mathrm{curl}\,[\mathcal{L}_U^I,\mathcal{L}_B^2]w)_{ad}\\
=&\tilde c_I^{I_1I_2}[\partial_aB_{I_1}^b\partial_b\mathcal{L}_U^{I_2}\mathcal{L}_Bw_d-\partial_dB_{I_1}^b\partial_b\mathcal{L}_U^{I_2}\mathcal{L}_Bw_a+B_{I_1}^b\partial_b(\mathrm{curl}\,\mathcal{L}_U^{I_2}\mathcal{L}_Bw)_{ad}\\
&+(\partial_dB_{I_1}^c)\partial_a(\mathcal{L}_U^{I_2}\mathcal{L}_Bw_c+\mathcal{L}_B\mathcal{L}_U^{I_2}w_c)-(\partial_aB_{I_1}^c)\partial_d(\mathcal{L}_U^{I_2}\mathcal{L}_Bw_c+\mathcal{L}_B\mathcal{L}_U^{I_2}w_c)\\
&+\partial_a\mathcal{L}_BB_{I_1}^b\partial_b\mathcal{L}_U^{I_2}w_d-\partial_d\mathcal{L}_BB_{I_1}^b\partial_b\mathcal{L}_U^{I_2}w_a+\partial_aB_{I_1}^b\mathcal{L}_B\partial_b\mathcal{L}_U^{I_2}w_d\\
&-\partial_dB_{I_1}^b\mathcal{L}_B\partial_b\mathcal{L}_U^{I_2}w_a+\mathcal{L}_BB_{I_1}^b\partial_b(\mathrm{curl}\,\mathcal{L}_U^{I_2}w)_{ad}+B_{I_1}^b\partial_a\mathcal{L}_B\partial_b\mathcal{L}_U^{I_2}w_d\\
&-B_{I_1}^b\partial_d\mathcal{L}_B\partial_b\mathcal{L}_U^{I_2}w_a+(\partial_a\mathcal{L}_B\partial_dB_{I_1}^c-\partial_d\mathcal{L}_B\partial_aB_{I_1}^c)\mathcal{L}_U^{I_2}w_c+(\mathcal{L}_B\partial_dB_{I_1}^c)\partial_a\mathcal{L}_U^{I_2}w_c\\
&-(\mathcal{L}_B\partial_aB_{I_1}^c)\partial_d\mathcal{L}_U^{I_2}w_c].
\end{aligned}
$$

Since

$$
[\mathcal{L}_B,\partial_a]w_b=-\partial_a\partial_bB^cw_c,
$$

one gets

$$
\begin{aligned}
&\left|\mathrm{curl}\,[\mathcal{L}_U^I,\mathcal{L}_B^2]w\right|\\
\leqslant&K_1\tilde c_I^{I_1I_2}[\left|\partial B_{I_1}\right|(|\partial\mathcal{L}_U^{I_2}\mathcal{L}_Bw|+|\partial\mathcal{L}_B\mathcal{L}_U^{I_2}w|+|\partial^2B||\mathcal{L}_U^{I_2}w|)\\
&+|B_{I_1}||\partial(\mathrm{curl}\,\mathcal{L}_U^{I_2}\mathcal{L}_Bw)|+|\mathcal{L}_BB_{I_1}||\partial(\mathrm{curl}\,\mathcal{L}_U^{I_2}w)|+|B_{I_1}|(|\partial^2\mathcal{L}_B\mathcal{L}_U^{I_2}w|\\
&+|\partial^3B||\mathcal{L}_U^{I_2}w|+|\partial^2B||\partial\mathcal{L}_U^{I_2}w|)+(|\partial\mathcal{L}_BB_{I_1}|+|\partial^2B||B_{I_1}|)|\partial\mathcal{L}_U^{I_2}w|\\
&+(|\partial^2\mathcal{L}_BB_{I_1}|+|\partial^3B||B_{I_1}|+|\partial^2B||\partial B_{I_1}|)|\mathcal{L}_U^{I_2}w|].
\end{aligned}
$$

Due to

$$\mathcal{L}_U^J w_a = \mathcal{L}_U^J (g_{ab} W^b) = g_{ab}\mathcal{L}_U^J W^b + \tilde{c}^J_{J_1 J_2} g^{J_1}_{ab}\mathcal{L}_U^{J_2} W^b, \quad g^J_{ab} = \mathcal{L}_U^J g_{ab},$$

where the sum is over all $J_1 + J_2 = J$, and $\tilde{c}^J_{J_1 J_2} = 1$ for $|J_2| < |J|$, and $\tilde{c}^J_{J_1 J_2} = 0$ for $|J_2| = |J|$, from Lemma 6.4, it follows that

$$\begin{aligned}
|\mathcal{L}_U^{I_2} w| &\leqslant |g||\mathcal{L}_U^{I_2} W| + \tilde{c}^{I_2}_{I_{21} I_{22}} |g^{I_{21}}||\mathcal{L}_U^{I_{22}} W|, \\
|\partial\mathcal{L}_U^{I_2} w| &\leqslant |\partial g||\mathcal{L}_U^{I_2} W| + |g||\partial\mathcal{L}_U^{I_2} W| + \tilde{c}^{I_2}_{I_{21} I_{22}} |\partial g^{I_{21}}||\mathcal{L}_U^{I_{22}} W| \\
&\quad + \tilde{c}^{I_2}_{I_{21} I_{22}} |g^{I_{21}}||\partial\mathcal{L}_U^{I_{22}} W| \\
&\leqslant K_1[|g| + |\partial g|]\Big(|\mathrm{curl}\, \underline{\mathcal{L}_U^{I_2} W}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S\mathcal{L}_U^{I_2} W| + [g]_1|\mathcal{L}_U^{I_2} W|\Big) \\
&\quad + K_1 \tilde{c}^{I_2}_{I_{21} I_{22}} [|g^{I_{21}}| + |\partial g^{I_{21}}|] \\
&\qquad \cdot \Big(|\mathrm{curl}\, \underline{\mathcal{L}_U^{I_{22}} W}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S\mathcal{L}_U^{I_{22}} W| + [g]_1|\mathcal{L}_U^{I_{22}} W|\Big),
\end{aligned}$$

and

$$\begin{aligned}
&|\partial\mathcal{L}_U^{I_2}\mathcal{L}_B w_a| \\
=&|\partial\mathcal{L}_U^{I_2}(\mathcal{L}_B g_{ab} W^b + g_{ab}\mathcal{L}_B W^b)| \\
\leqslant& \tilde{c}^{I_2}_{I_{21} I_{22}} (|\partial\mathcal{L}_U^{I_{21}}\mathcal{L}_B g||\mathcal{L}_U^{I_{22}} W| + |\mathcal{L}_U^{I_{21}}\mathcal{L}_B g||\partial\mathcal{L}_U^{I_{22}} W| \\
&+ |\partial\mathcal{L}_U^{I_{21}} g||\mathcal{L}_U^{I_{22}}\mathcal{L}_B W| + |\mathcal{L}_U^{I_{21}} g||\partial\mathcal{L}_U^{I_{22}}\mathcal{L}_B W|) \\
\leqslant& K_1 \tilde{c}^{I_2}_{I_{21} I_{22}} (|\mathcal{L}_U^{I_{21}}\mathcal{L}_B g| + |\partial\mathcal{L}_U^{I_{21}}\mathcal{L}_B g|) \\
&\quad \cdot \Big(|\mathrm{curl}\, \underline{\mathcal{L}_U^{I_{22}} W}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S\mathcal{L}_U^{I_{22}} W| + [g]_1|\mathcal{L}_U^{I_{22}} W|\Big) \\
&+ K_1 \tilde{c}^{I_2}_{I_{21} I_{22}} (|\mathcal{L}_U^{I_{21}} g| + |\partial\mathcal{L}_U^{I_{21}} g|)\Big(|\mathrm{curl}\, \underline{\mathcal{L}_U^{I_{22}}\mathcal{L}_B W}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S\mathcal{L}_U^{I_{22}} W| \\
&+ [g]_1 \tilde{c}^{I_{22}}_{I_{221} I_{222}} \Big[|\mathcal{L}_U^{I_{221}} B|\Big(|\mathrm{curl}\, \underline{\mathcal{L}_U^{I_{222}} W}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S\mathcal{L}_U^{I_{222}} W| + [g]_1|\mathcal{L}_U^{I_{222}} W|\Big) \\
&\quad + \Big(|\mathrm{curl}\, \underline{\mathcal{L}_U^{I_{221}} B}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S\mathcal{L}_U^{I_{221}} B|\Big)|\mathcal{L}_U^{I_{222}} W|\Big]\Big),
\end{aligned}$$

since

$$\begin{aligned}
\Big|[\mathcal{L}_T^I, \mathcal{L}_B]W\Big| &\leqslant \tilde{c}^{I_1 I_2}_I [\big|B_{I_1}\big|\big|\partial W_{I_2}\big| + \big|\partial B_{I_1}\big|\big|W_{I_2}\big|] \\
&\leqslant \tilde{c}^{I_1 I_2}_I K_1 \Bigg[\big|B_{I_1}\big|\Bigg(\sum_{S\in\mathcal{R}} |\mathcal{L}_S W_{I_2}| + |W_{I_2}|\Bigg) + \big|\partial B_{I_1}\big|\big|W_{I_2}\big|\Bigg]. \qquad (6.48)
\end{aligned}$$

Now, we have to express the term, like $|\text{curl}\, \underline{W_I}| = |\text{curl}\, \underline{\mathcal{L}_U^I W}|$ in the above inequality and in the line labeled (6.44) and other lines, in term of w. By Lemma 6.4, we have

$$
\begin{aligned}
&|\text{curl}\, \underline{\mathcal{L}_U^I W}_a| = |\text{curl}\, g_{ab}\mathcal{L}_U^I W^b| \\
&\leqslant |\text{curl}\, \mathcal{L}_U^I w_a| + \tilde{c}_{I_1 I_2}^I |\text{curl}\,(g_{ab}^{I_1}\mathcal{L}_U^{I_2} W^b)| \\
&\leqslant |\text{curl}\, \mathcal{L}_U^I w_a| + \tilde{c}_{I_1 I_2}^I |\partial_d(g_{ab}^{I_1}\mathcal{L}_U^{I_2} W^b) - \partial_a(g_{db}^{I_1}\mathcal{L}_U^{I_2} W^b)| \\
&\leqslant |\text{curl}\, \mathcal{L}_U^I w_a| + 2\tilde{c}_{I_1 I_2}^I (|\partial g^{I_1}||\mathcal{L}_U^{I_2} W| + |g^{I_1}||\partial \mathcal{L}_U^{I_2} W|) \\
&\leqslant |\text{curl}\, \mathcal{L}_U^I w_a| \\
&\quad + K_1 \tilde{c}_{I_1 I_2}^I (|\partial g^{I_1}| + |g^{I_1}|)\Big(|\text{curl}\, \underline{\mathcal{L}_U^{I_2} W}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S \mathcal{L}_U^{I_2} W| + [g]_1 |\mathcal{L}_U^{I_2} W|\Big).
\end{aligned}
$$

The term $|\text{curl}\, \underline{\mathcal{L}_U^I \mathcal{L}_B W}|$ in (6.47) can be estimated as a similar argument as above by regarding B as a tangential vector field of form U. By (6.41), for the term $[\mathcal{L}_U^{J_2}, \partial]W$ in (6.45), we get

$$
|[\mathcal{L}_U^{J_2}, \partial]W| \leqslant |W||\mathcal{L}_U^{J_2 - J_{2,|J_2|}} \partial^2 U_{J_{2,|J_2|}}| + \sum_{\substack{J_2 = I_1 + I_2 + I_3 \\ |I_3| = 1}} \text{sgn}(|I_1|)|\mathcal{L}_U^{I_2}\partial^2 U_{I_3}||\mathcal{L}_U^{I_1} W|,
$$

and a similar estimate holds for the term $[\mathcal{L}_U^{J_2}, \partial]\dot{W}$ in (6.46). Similarly, for the term $[\mathcal{L}_U^{J_2}, \partial]\mathcal{L}_B W$ in (6.47), we have with the help of (6.48)

$$
\begin{aligned}
|[\mathcal{L}_U^{J_2}, \partial]\mathcal{L}_B W| \leqslant\ & |\mathcal{L}_B W||\mathcal{L}_U^{J_2 - J_{2,|J_2|}} \partial^2 U_{J_{2,|J_2|}}| + \sum_{\substack{J_2 = I_1 + I_2 + I_3 \\ |I_3| = 1}} \text{sgn}(|I_1|)|\mathcal{L}_U^{I_2}\partial^2 U_{I_3}| \\
&\cdot \Bigg[|\mathcal{L}_B \mathcal{L}_U^{I_1} W| + K_1 \tilde{c}_I^{I_{11} I_{12}} \left|B_{I_{11}}\right| \Big(|\text{curl}\, \underline{W_{I_{12}}}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S W_{I_{12}}| + [g]_1 |W_{I_{12}}|\Big) \\
&\quad + \Big(|\text{curl}\, \underline{B_{I_{11}}}| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S B_{I_{11}}|\Big) \left|W_{I_{12}}\right| \Bigg].
\end{aligned}
$$

For the term $|\mathcal{L}_S \mathcal{L}_U^{J_2} \mathcal{L}_B W|$, we can use (6.48) to get estimates.

For convenience, we introduce some new norms and notation.

Definition 6.13. For any family $\mathcal{V}$ of our families of vector fields, let

$$
\|W\|_r^{\mathcal{V}} = \|W(t)\|_{\mathcal{V}^r(\Omega)} = \sum_{|I|\leqslant r,\, T\in\mathcal{V}} \left(\int_\Omega |\mathcal{L}_U^I W(t,y)|^2 dy \right)^{1/2},
$$

$$
\|W\|_{r,B}^{\mathcal{V}} = \|W(t)\|_{\mathcal{V}_B^r(\Omega)} = \sum_{|I|\leqslant r,\, T\in\mathcal{V}} \left(\int_\Omega |\mathcal{L}_B \mathcal{L}_U^I W(t,y)|^2 dy \right)^{1/2},
$$

and

$$C_r^{\mathcal{V}} = \sum_{|J|\leqslant r-1,\, J\in\mathcal{V}} \left(\int_\Omega (|\operatorname{curl} \mathcal{L}_U^J \dot{w}|^2 + |\operatorname{curl} \mathcal{L}_B \mathcal{L}_U^J w|^2 + |\operatorname{curl} \mathcal{L}_U^J w|^2) dy \right)^{1/2}, \tag{6.49}$$
$$C_0^{\mathcal{V}} = 0.$$

Note that the norm $\|W(t)\|_{\mathcal{R}^r(\Omega)}$ is equivalent to the usual Sobolev norm in the Lagrangian coordinates.

Definition 6.14. For $\mathcal{V}$ any of our families of vector fields and β a function, a 1-form, a 2-form, or a vector field, let $|\beta|_s^{\mathcal{V}}$ be as in Definition 6.5 and set

$$\|\beta\|_{s,\infty}^{\mathcal{V}} = \| |\beta|_s^{\mathcal{V}} \|_{L^\infty(\Omega)},$$
$$[[g]]_{s,\infty}^{\mathcal{V}} = \sum_{s_1+\cdots+s_k\leqslant s,\, s_i\geqslant 1} \|g\|_{s_1,\infty}^{\mathcal{V}} \cdots \|g\|_{s_k,\infty}^{\mathcal{V}}, \quad [[g]]_{0,\infty}^{\mathcal{V}} = 1,$$

where the sum is over all combinations with $s_i \geqslant 1$. Furthermore, let

$$\begin{aligned} m_r^{\mathcal{V}} &= [[g]]_{r,\infty}^{\mathcal{V}}, \\ \dot{m}_r^{\mathcal{V}} &= \sum_{s+u\leqslant r} [[g]]_{s,\infty}^{\mathcal{V}} (\|\dot{g}\|_{u,\infty}^{\mathcal{V}} + \|\mathcal{L}_B g\|_{u,\infty}^{\mathcal{V}} + \|\mathcal{L}_B^2 g\|_{u,\infty}^{\mathcal{V}} + \|\omega\|_{u,\infty}^{\mathcal{V}}), \\ \bar{m}_r^{\mathcal{V}} &= \sum_{s\leqslant r} \left(\|B\|_{s+2,\infty}^{\mathcal{V}} + \|\partial x\|_{s,\infty}^{\mathcal{V}} + \|\partial^2 x\|_{s,\infty}^{\mathcal{V}} + \left\| \frac{\partial y}{\partial x} \right\|_{s,\infty}^{\mathcal{V}} \right). \end{aligned} \tag{6.50}$$

Let $F_{r,\mathrm{curl}}^{\mathcal{U}} = \|\operatorname{curl} \underline{F}\|_{\mathcal{U}^{r-1}(\Omega)}$. Then, it follows from the above arguments in this subsection that

$$\left| \frac{d}{dt} C_r^{\mathcal{U}} \right| \leqslant K_1 \sum_{s=0}^{r} (\dot{m}_{r-s}^{\mathcal{U}} + \bar{m}_{r-s}^{\mathcal{U}})(C_s^{\mathcal{U}} + E_s^{\mathcal{T}}) + F_{r,\mathrm{curl}}^{\mathcal{U}}, \tag{6.51}$$

where $E_s^{\mathcal{T}}$ is the energy of the tangential derivatives defined in (6.17). Here, we note that the same inequalities hold with $\mathcal{U}$ and $\mathcal{T}$ replaced by $\mathcal{R}$ and $\mathcal{S}$, respectively. Thus, by the Gronwall inequality, we obtain for $r \geqslant 1$

$$\begin{aligned} C_r^{\mathcal{U}} \leqslant K_1 e^{\int_0^t K_1(\dot{m}_0^{\mathcal{U}} + \bar{m}_0^{\mathcal{U}}) d\tau} \int_0^t \Big(&\operatorname{sgn}(r-1) \sum_{s=1}^{r-1} (\dot{m}_{r-s}^{\mathcal{U}} + \bar{m}_{r-s}^{\mathcal{U}}) C_s^{\mathcal{U}} \\ &+ \sum_{s=0}^{r} (\dot{m}_{r-s}^{\mathcal{U}} + \bar{m}_{r-s}^{\mathcal{U}}) E_s^{\mathcal{T}} + F_{r,\mathrm{curl}}^{\mathcal{U}} \Big) d\tau. \end{aligned}$$

Since we have already proved a bound for $E_s^{\mathcal{T}}$ in Proposition 6.8, it inductively follows that $C_r^{\mathcal{U}}$ is bounded. By Lemma 6.6, we obtain

$$\|W(t)\|_{\mathcal{U}^r(\Omega)} + \|\dot{W}(t)\|_{\mathcal{U}^r(\Omega)} + \|\mathcal{L}_B W(t)\|_{\mathcal{U}^r(\Omega)} \leqslant K_1 \sum_{s=0}^{r} m_{r-s}^{\mathcal{U}} (C_s^{\mathcal{U}} + E_s^{\mathcal{T}}). \tag{6.52}$$

Therefore, we have the following estimates.

Proposition 6.15. *Suppose that $x, P \in C^{r+2}([0,T]\times\Omega)$, $B \in C^{r+2}(\Omega)$, $P|_\Gamma = 0$, $\nabla_N P|_\Gamma \leqslant -c_0 < 0$, $B^a N_a|_\Gamma = 0$ and $\operatorname{div} V = 0$, where $V = D_t x$. Then, there is a constant $C = C(x, P, B)$ depending only on the norm of (x, P, B), a lower bound for c_0, and an upper bound for T such that if $E_s^{\mathcal{T}}(0) = C_s^{\mathcal{U}}(0) = 0$ for $s \leqslant r$, then*

$$\|W\|_r^{\mathcal{U}} + \|\dot{W}\|_r^{\mathcal{U}} + \|\mathcal{L}_B W\|_r^{\mathcal{U}} + E_r^{\mathcal{T}} \leqslant C \int_0^t \|F\|_r^{\mathcal{U}} d\tau, \quad \text{for } t \in [0,T].$$

7. The smoothed-out equation and existence of weak solutions

7.1. The smoothed-out normal operator

In order to prove the existence of solutions, the normal operator $\mathcal{A}$ should be replaced by a sequence $\mathcal{A}^\varepsilon$ of bounded symmetric and positive operators that converge to $\mathcal{A}$ as $\varepsilon \to 0$.

Let $\rho = \rho(d)$ be a smooth function of the distance $d = d(y) = \operatorname{dist}(y, \Gamma)$ such that

$$\rho' \geqslant 0, \quad \rho(d) = d \text{ for } d \leqslant \frac{1}{4}, \quad \text{and } \rho(d) = \frac{1}{2} \text{ for } d \geqslant \frac{3}{4}.$$

Let $\chi(\rho)$ be a smooth function such that

$$\chi'(\rho) \geqslant 0, \quad \chi(\rho) = 0 \text{ for } \rho \leqslant \frac{1}{4}, \quad \text{and } \chi(\rho) = 1 \text{ for } \rho \geqslant \frac{3}{4}.$$

For a function f vanishing on the boundary, we define

$$\mathcal{A}_f^\varepsilon W^a = \mathbb{P}(-g^{ab}\chi_\varepsilon(\rho)\partial_b(f\rho^{-1}(\partial_c\rho)W^c)),$$

where $\chi_\varepsilon(\rho) = \chi(\rho/\varepsilon)$. The integration by parts gives

$$\langle U, \mathcal{A}_f^\varepsilon W\rangle = \int_\Omega f\rho^{-1}\chi_\varepsilon'(\rho)(U^a\partial_a\rho)(W^b\partial_b\rho)dy, \tag{7.1}$$

which yields the symmetry of $\mathcal{A}_f^\varepsilon$. In particular, $\mathcal{A}^\varepsilon = \mathcal{A}_P^\varepsilon$ is positive if $P \geqslant 0$, at least close to the boundary, i.e.,

$$\langle W, \mathcal{A}^\varepsilon W\rangle \geqslant 0.$$

We have another expression for $\mathcal{A}^\varepsilon_f$:

$$\mathcal{A}^\varepsilon_f W^a = \mathbb{P}(g^{ab}\chi'_\varepsilon(\rho)(\partial_b\rho)f\rho^{-1}(\partial_c\rho)W^c).$$

Since the projection is continuous on $H^r(\Omega)$, if the metric and pressure are sufficiently regular, one gets, as in [21,23], that

$$\sum_{j=0}^{k}\|D_t^j\mathcal{A}^\varepsilon W\|_{H^r(\Omega)} \leqslant C_{\varepsilon,r,k}\sum_{j=0}^{k}\|D_t^j W\|_{H^r(\Omega)}. \tag{7.2}$$

Moreover, we have

$$\begin{aligned}
\|\mathcal{A}^\varepsilon W-\mathcal{A}W\|^2 =&\langle\mathcal{A}^\varepsilon W-\mathcal{A}W,\mathcal{A}^\varepsilon W-\mathcal{A}W\rangle\\
=&\langle\mathcal{A}^\varepsilon W-\mathcal{A}W,\mathcal{A}^\varepsilon W\rangle-\langle\mathcal{A}^\varepsilon W-\mathcal{A}W,\mathcal{A}W\rangle\\
=&-\int_\Omega(\mathcal{A}^\varepsilon W^a-\mathcal{A}W^a)\chi_\varepsilon(\rho)\partial_a(P\rho^{-1}(\partial_c\rho)W^c)dy\\
&+\int_\Omega(\mathcal{A}^\varepsilon W^a-\mathcal{A}W^a)\partial_a(\partial_c PW^c)dy\\
=&-\int_\Omega(\mathcal{A}^\varepsilon W^a-\mathcal{A}W^a)\chi_\varepsilon(\rho)\partial_a[(P\rho^{-1}\partial_c\rho-\partial_c P)W^c]dy\\
&+\int_\Omega(\mathcal{A}^\varepsilon W^a-\mathcal{A}W^a)(1-\chi_\varepsilon(\rho))\partial_a(\partial_c PW^c)dy\\
=&\int_\Omega(\mathcal{A}^\varepsilon W^a-\mathcal{A}W^a)\chi'_\varepsilon(\rho)\partial_a\rho[(P\rho^{-1}\partial_c\rho-\partial_c P)W^c]dy\\
&+\int_\Omega(\mathcal{A}^\varepsilon W^a-\mathcal{A}W^a)(1-\chi_\varepsilon(\rho))\partial_a(\partial_c PW^c)dy
\end{aligned}$$

due to $P\rho^{-1}\partial_c\rho=\partial_c P$ on the boundary, which yields

$$\begin{aligned}
\|\mathcal{A}^\varepsilon W-\mathcal{A}W\| \leqslant& \|\chi'_\varepsilon(\rho)\|_{L^\infty(\Omega)}\|\partial_a\rho(P\rho^{-1}\partial_c\rho-\partial_c P)W^c\|\\
&+\|(1-\chi_\varepsilon(\rho))\|_{L^\infty(\Omega)}\|\partial_a(\partial_c PW^c)\|\to 0, \text{ as } \varepsilon\to 0,
\end{aligned}$$

since $\chi'_\varepsilon(\rho)\to 0$ and $\chi_\varepsilon(\rho)\to 1$ in $L^\infty(\Omega)$ as $\varepsilon\to 0$. Thus, we obtain

$$\mathcal{A}^\varepsilon U\to\mathcal{A}U \text{ in } L^2(\Omega), \quad \text{if } U\in H^1(\Omega). \tag{7.3}$$

As in (3.5), it holds

$$\left|\langle U, \mathcal{A}^{\varepsilon}_{fP} W\rangle\right| \leqslant \langle U, \mathcal{A}^{\varepsilon}_{fP} U\rangle^{1/2}\langle W, \mathcal{A}^{\varepsilon}_{fP} W\rangle^{1/2}$$
$$\leqslant \|f\|_{L^\infty(\Omega\backslash\Omega^\varepsilon)}\langle U, \mathcal{A}^{\varepsilon} U\rangle^{1/2}\langle W, \mathcal{A}^{\varepsilon} W\rangle^{1/2}, \tag{7.4}$$

where

$$\Omega^\varepsilon = \{y \in \Omega : \mathrm{dist}\,(y, \Gamma) > \varepsilon\}. \tag{7.5}$$

In fact, taking the supremum over the set where $d(y) \leqslant \varepsilon$ suffices since $\chi_\varepsilon' = 0$ when $d(y) \geqslant \varepsilon$. The only difference with (3.5) is that the supremum is over a small neighborhood of the boundary instead of on the boundary. One can see that $\dot{P} = D_t P$ vanishes on the boundary and $\dot{P}/P$ is a smooth function since P vanishes on the boundary, $P > 0$ in the interior, and $\nabla_N P \leqslant -c_0 < 0$ on the boundary. Let $\dot{\mathcal{A}}^\varepsilon = \mathcal{A}^\varepsilon_{\dot{P}}$ be the time derivative of the operator $\mathcal{A}^\varepsilon$, satisfying

$$\left|\langle W, \dot{\mathcal{A}}^\varepsilon W\rangle\right| \leqslant \|\dot{P}/P\|_{L^\infty(\Omega\backslash\Omega^\varepsilon)}\langle W, \mathcal{A}^\varepsilon W\rangle. \tag{7.6}$$

The commutators between $\mathcal{A}^\varepsilon_f$ and the Lie derivatives $\mathcal{L}_T$ with respect to tangential vector fields T are basically the same as for $\mathcal{A}$. Since $Td = 0$ for $T \in \mathcal{T}_0 = \mathcal{S}_0 \cup \{D_t\}$, we have

$$\mathbb{P}(g^{ca}\mathcal{L}_T(g_{ab}\mathcal{A}^\varepsilon_f W^b)) = \mathcal{A}^\varepsilon_f \mathcal{L}_T W^c + \mathcal{A}^\varepsilon_{Tf} W^c. \tag{7.7}$$

In order to get additional regularity in the interior, we include the vector fields $\mathcal{S}_1$ that span the tangent space in the interior. The vector fields in $\mathcal{S}_1$ satisfy $S\rho = \mathcal{L}_S\rho = 0$ when $d \leqslant d_0/2$. Due to $\chi_\varepsilon'(\rho) = 0$ when $d \geqslant \varepsilon$, the above relation (7.7) holds for these as well if we assume that $\varepsilon \leqslant d_0/2$.

It remains to estimate the curl of $\mathcal{A}^\varepsilon$ now. Although the curl of $\mathcal{A}$ vanishes, it is not the case for the curl of $\mathcal{A}^\varepsilon$. However, it vanishes away from the boundary. Let $(\underline{\mathcal{A}}^\varepsilon W)_a = g_{ab}\mathcal{A}^\varepsilon W^b$, we have

$$\begin{aligned}(\underline{\mathcal{A}}^\varepsilon W)_a =& g_{ab}\mathbb{P}(-g^{bd}\chi_\varepsilon(\rho)\partial_d(P\rho^{-1}(\partial_c\rho)W^c))\\ =& -\chi_\varepsilon(\rho)\partial_a(P\rho^{-1}(\partial_c\rho)W^c) - \partial_a q_1,\end{aligned}$$

for some function q_1 vanishing on the boundary and determined so that the divergence vanishes. Then, when $d(y) \geqslant \varepsilon$, we get $\chi_\varepsilon'(\rho) = 0$ and

$$\begin{aligned}(\mathrm{curl}\,\underline{\mathcal{A}}^\varepsilon W)_{ab} =& \partial_a(\underline{\mathcal{A}}^\varepsilon W)_b - \partial_b(\underline{\mathcal{A}}^\varepsilon W)_a\\ =& -\partial_a(\chi_\varepsilon(\rho)\partial_b(P\rho^{-1}(\partial_c\rho)W^c)) + \partial_b(\chi_\varepsilon(\rho)\partial_a(P\rho^{-1}(\partial_c\rho)W^c))\\ =& -\chi_\varepsilon'(\rho)[\partial_a\rho\partial_b(P\rho^{-1}(\partial_c\rho)W^c) - \partial_b\rho\partial_a(P\rho^{-1}(\partial_c\rho)W^c)]\\ =& 0.\end{aligned} \tag{7.8}$$

7.2. The smoothed-out equation and existence of weak solutions

We introduce the following ε smoothed-out linear equation

$$\ddot{W}_\varepsilon^a - \mathcal{L}_B^2 W_\varepsilon^a + \mathcal{A}^\varepsilon W_\varepsilon^a + \dot{\mathcal{G}} \dot{W}_\varepsilon^a - \mathcal{C}\dot{W}_\varepsilon^a + \mathcal{X}\mathcal{L}_B W_\varepsilon^a = F^a, \tag{7.9a}$$

$$W_\varepsilon|_{t=0} = 0, \quad \dot{W}_\varepsilon|_{t=0} = 0. \tag{7.9b}$$

This is a wave equation with variable coefficients, the existence of weak solutions in $H^1(\Omega)$ can be proved by standard methods and noticing that $B^a N_a = 0$ on the boundary, or in $H^r(\Omega)$ by (7.2), since all operators are bounded and $\mathcal{L}_B$ can be regarded as the first-order derivative with respect to spatial variables.

In order to obtain the additional regularity in time as well, more time derivatives need to apply using (7.2) and (3.1), the initial data for these vanish as well since we constructed F in (7.9) vanishing to any given order. If the initial data, encoded in F, are smooth, a smooth solution of the ε approximate linear equation is therefore obtained.

We shall prove that $W_\varepsilon \to W$ weakly in L^2, where $W \in H^r(\Omega)$ for some large r. It will follow that W is a weak solution. We can show that it is indeed a classical solution from the additional regularity of W; hence the a priori bounds in the earlier section hold.

The norm of $\mathcal{A}^\varepsilon$ tends to infinity as $\varepsilon \to 0$, but we can include it in the energy because it is a positive operator. The energy will be the same as before with $\mathcal{A}$ replaced by $\mathcal{A}^\varepsilon$, so (4.5) becomes

$$E^\varepsilon(t) = \langle \dot{W}_\varepsilon, \dot{W}_\varepsilon\rangle + \langle (\mathcal{A}^\varepsilon + I)W_\varepsilon, W_\varepsilon\rangle + \langle \mathcal{L}_B W_\varepsilon, \mathcal{L}_B W_\varepsilon\rangle. \tag{7.10}$$

Since $D_t d = 0$, it follows from taking the time derivative of (7.1), with $f = P$, that

$$\frac{d}{dt}\langle \mathcal{A}^\varepsilon W_\varepsilon, W_\varepsilon\rangle = 2\langle \mathcal{A}^\varepsilon W_\varepsilon, \dot{W}_\varepsilon\rangle + \langle \mathcal{A}^\varepsilon_{\dot{P}} W_\varepsilon, W_\varepsilon\rangle,$$

where the last term is bounded by (7.6). Thus, by (4.7), one has

$$|\dot{E}^\varepsilon| \leqslant \left(1 + \left\|\frac{\dot{P}}{P}\right\|_{L^\infty(\Omega)} + 2\|\dot{g}\|_{L^\infty(\Omega)} + 2\|g\|_{L^\infty(\Omega)}\|\partial B\|_{L^\infty(\Omega)}\right) E^\varepsilon + 2\sqrt{E^\varepsilon}\|F\|,$$

from which we obtain a uniform bound for $t \in [0, T]$ independent of ε, i.e., $E^\varepsilon(t) \leqslant C$.

A subsequence W_{ε_n} can be chosen such that $W_{\varepsilon_n} \rightharpoonup W$ weakly in the inner product since $\|W_\varepsilon\| \leqslant C$. We now show that the limit W is a weak solution of the equation. Multiplying (7.9a) by a smooth divergence-free vector field U that vanished for $t \geqslant T$ and integrating by parts, we have

$$\begin{aligned}
&\int_0^T \int_\Omega g_{ab} U^b F^a dy dt \\
&= \int_0^T \int_\Omega g_{ab}(\ddot{W}_\varepsilon^a - \mathcal{L}_B^2 W_\varepsilon^a + \mathcal{A}^\varepsilon W_\varepsilon^a + \dot{\mathcal{G}}\dot{W}_\varepsilon^a - \mathcal{C}\dot{W}_\varepsilon^a + \mathcal{X}\mathcal{L}_B W_\varepsilon^a) U^b dy dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_\Omega (D_t(g_{ab}\dot{W}_\varepsilon^a)U^b - g_{ab}\mathcal{L}_B^2 W_\varepsilon^a U^b + \chi_\varepsilon'(\rho)(\partial_b\rho)P\rho^{-1}(\partial_c\rho)W^c U^b \\
&\quad - \omega_{bc}\dot{W}_\varepsilon^c U^b - 2\delta_{il}\partial_b x^l B\partial_c x^i \mathcal{L}_B W_\varepsilon^c U^b - 2g_{ab}\partial_c B^a \mathcal{L}_B W_\varepsilon^c U^b)dydt \\
&= -\int_0^T \int_\Omega g_{ab}\dot{W}_\varepsilon^a \dot{U}^b dydt + \int_0^T \int_\Omega ((\mathcal{L}_B g_{ab})U^b + g_{ab}\mathcal{L}_B U^b)\mathcal{L}_B W_\varepsilon^a dydt \\
&\quad + \int_0^T \int_\Omega g_{ab}\mathcal{A}^\varepsilon U^b W_\varepsilon^a dydt + \int_0^T \int_\Omega (\dot{\omega}_{bc}U^b + \omega_{bc}\dot{U}^b)W_\varepsilon^c dydt \\
&\quad + 2\int_0^T \int_\Omega \delta_{il}\partial_b x^l B\partial_c x^i W_\varepsilon^c \mathcal{L}_B U^b dydt + 2\int_0^T \int_\Omega \mathcal{L}_B(\delta_{il}\partial_b x^l B\partial_c x^i)W_\varepsilon^c U^b dydt \\
&\quad + 2\int_0^T \int_\Omega \mathcal{L}_B(g_{ab}\partial_c B^a)U^b W_\varepsilon^c dydt + 2\int_0^T \int_\Omega g_{ab}\partial_c B^a \mathcal{L}_B U^b W_\varepsilon^c dydt \\
&= \int_0^T \int_\Omega g_{ab}(\ddot{U}^b - \mathcal{L}_B^2 U^b + \mathcal{A}^\varepsilon U^b + \dot{\mathcal{G}}\dot{U}^b - \mathcal{C}\dot{U}^b + \mathcal{X}\mathcal{L}_B U^b)W_\varepsilon^a dydt \\
&\quad + \int_0^T \int_\Omega \dot{\omega}_{bc}U^b W_\varepsilon^c dydt + \int_0^T \int_\Omega \mathcal{L}_B(\delta_{il}\partial_b x^l B\partial_c x^i - \delta_{il}B\partial_b x^l \partial_c x^i)U^b W_\varepsilon^c dydt \\
&\quad + \int_0^T \int_\Omega \mathcal{L}_B(g_{ab}\partial_c B^a - g_{ac}\partial_b B^a)U^b W_\varepsilon^c dydt.
\end{aligned}
$$

From (7.3), we know that $\mathcal{A}^\varepsilon U$ converges to $\mathcal{A}U$ strongly in the norm if $U \in H^1$. Because $W_{\varepsilon_n} \rightharpoonup W$ weakly, this proves that we have a weak solution W of the equation

$$
\begin{aligned}
&\int_0^T \int_\Omega g_{ab}(\ddot{U}^b - \mathcal{L}_B^2 U^b + \mathcal{A}U^b + \dot{\mathcal{G}}\dot{U}^b - \mathcal{C}\dot{U}^b + \mathcal{X}\mathcal{L}_B U^b)dydt \\
&\quad + \int_0^T \int_\Omega \dot{\omega}_{bc}U^b W^c dydt + \int_0^T \int_\Omega \mathcal{L}_B(\delta_{il}\partial_b x^l B\partial_c x^i - \delta_{il}B\partial_b x^l \partial_c x^i)U^b W^c dydt \\
&\quad + \int_0^T \int_\Omega \mathcal{L}_B(g_{ab}\partial_c B^a - g_{ac}\partial_b B^a)U^b W^c dydt = \int_0^T \int_\Omega g_{ab}U^b F^a dydt
\end{aligned}
$$

for any smooth divergence-free vector field U that vanishes for $t \geqslant T$. Moreover, due to $\operatorname{div} W_\varepsilon = 0$, we get

$$\int_0^T \int_\Omega (\partial_a q) W_\varepsilon^a \, dy dt = 0$$

for any smooth q that vanishes on the boundary and thus

$$\int_0^T \int_\Omega (\partial_a q) W^a \, dy dt = 0. \tag{7.11}$$

Therefore, W is weakly divergence-free.

8. Existence of smooth solutions for the linearized equation

In order to show that W is divergence-free classical solution, we need to prove the additional regularity, i.e., $W, \dot{W} \in H^r(\Omega)$ for any $r \geqslant 0$. Then, the integration by parts for (7.11) yields

$$\int_0^T \int_\Omega q \partial_a W^a \, dy dt = 0$$

for any smooth function q that vanishes on the boundary. Thus, W is divergence-free.

Moreover,

$$\int_0^T \int_\Omega g_{ab} U^b (\ddot{W}^a - \mathcal{L}_B^2 W^a + \mathcal{A} W^a + \dot{\mathcal{G}} \dot{W}^a - \mathcal{C} \dot{W}^a + \mathcal{X} \mathcal{L}_B W^a) dy dt$$
$$= \int_0^T \int_\Omega g_{ab} U^b F^a \, dy dt \tag{8.1}$$

for any smooth, divergence-free vector field U that vanished for $t \geqslant T$. Since W is divergence-free, it follows that $\ddot{W}^a - \mathcal{L}_B^2 W^a + \mathcal{A} W^a + \dot{\mathcal{G}} \dot{W}^a - \mathcal{C} \dot{W}^a + \mathcal{X} \mathcal{L}_B W^a$ is divergence-free. By construction, F is also divergence-free, it follows that (8.1) holds for any smooth vector field U that vanishes for $t \geqslant T$. Thus, we conclude that

$$\ddot{W}^a - \mathcal{L}_B^2 W^a + \mathcal{A} W^a + \dot{\mathcal{G}} \dot{W}^a - \mathcal{C} \dot{W}^a + \mathcal{X} \mathcal{L}_B W^a = F^a, \quad \operatorname{div} W = 0.$$

Therefore, it only remains to prove that $W \in H^r(\Omega)$. For this, uniform bounds for the ε smoothed-out equation similar to the a priori bounds for the linearized equation are needed. The uniform tangential bounds for the ε smoothed-out equation follow the proof of the a priori tangential bounds in Section 6.5, which is just a change of notation. Let

$$E_I^\varepsilon = \langle \dot{W}_{\varepsilon I}, \dot{W}_{\varepsilon I}\rangle + \langle W_{\varepsilon I}, (\mathcal{A}+I)W_{\varepsilon I}\rangle + \langle \mathcal{L}_B W_{\varepsilon I}, \mathcal{L}_B W_{\varepsilon I}\rangle, \quad W_{\varepsilon I} = \mathcal{L}_T^I W_\varepsilon.$$

If $\varepsilon < d_0$, then the commutator relation for $\mathcal{A}^\varepsilon$, (7.7), is the same as for $\mathcal{A}$, (6.13). Furthermore, the positivity property for $\mathcal{A}_f^\varepsilon$ differs only from the one for $\mathcal{A}_f$ in which the supremum over the boundary in (3.5) is replaced by the supremum over a neighborhood of the boundary where $d(y) < \varepsilon$ in (7.4). Thus, all the calculations and inequalities in Sections 6.5, 6.7 and 6.8 hold with $\mathcal{A}$ replaced by $\mathcal{A}^\varepsilon$ if we replace the supremum of $\nabla_N q/\nabla_N P$ over the boundary in (6.16) by the supremum of q/P over the domain $\Omega \setminus \Omega^\varepsilon$, where Ω^ε is given by (7.5). Hence, we will reach the energy bound (6.25) for $E_r^{\mathcal{T}}$ replaced by

$$E_r^{\mathcal{T},\varepsilon} = \sum_{|I|\leqslant r, I\in\mathcal{T}} \sqrt{E_I^\varepsilon}, \tag{8.2}$$

namely, Proposition 6.8 holds for $E_r^{\mathcal{T}}$ replaced by $E_r^{\mathcal{T},\varepsilon}$ with a constant independent of ε. It is where we require vanishing initial data and an inhomogeneous term which vanishes to higher order when $t = 0$ so that the higher-order time derivatives of the solution of (7.9a) also vanish when $t = 0$. If the initial data for higher-order time derivatives were obtained from the ε smoothed-out equation, then they would depend on ε, and so we would not have been able to get a uniform bound for the energy $E_r^{\mathcal{T},\varepsilon}$.

The estimate for the curl is simple since the curl of $\mathcal{A}_\varepsilon$ vanishes in Ω^ε by (7.8), it follows that all the formula in Sections 6.7 and 6.8 hold when $d(y) \geqslant \varepsilon$. This follows from replacing $\mathcal{A}$ in (6.30) by $\mathcal{A}^\varepsilon$ and vanishing of its curl for $d(y) \geqslant \varepsilon$. Let

$$C_r^{\mathcal{U},\varepsilon} = \sum_{|J|\leqslant r-1,\, J\in\mathcal{U}} \left(\int_{\Omega^\varepsilon} (|\mathrm{curl}\, \mathcal{L}_U^J w_\varepsilon|^2 + |\mathrm{curl}\, \mathcal{L}_U^J w_\varepsilon|^2) dy \right)^{1/2}, \tag{8.3}$$

$$\|W(t)\|_{\mathcal{U}^r(\Omega^\varepsilon)} = \sum_{|I|\leqslant r, I\in\mathcal{U}} \left(\int_{\Omega^\varepsilon} |\mathcal{L}_U^I W(t,y)|^2 dy \right)^{1/2}. \tag{8.4}$$

Because all the used estimates from Section 6.3 are pointwise estimates, we conclude that the inequality in Proposition 6.15 holds with a constant C independent of ε if we replace $C_s^{\mathcal{U}}$ by $C_s^{\mathcal{U},\varepsilon}$ and the norms by (8.4), as follows.

Proposition 8.1. *Suppose that $x, P \in C^{r+2}([0,T]\times\Omega)$, $B \in C^{r+2}(\Omega)$, $P|_\Gamma = 0$, $\nabla_N P|_\Gamma \leqslant -c_0 < 0$, $B^a N_a|_\Gamma = 0$ and* $\mathrm{div}\, V = 0$, *where $V = D_t x$. Suppose that W_ε is a solution of (7.9a) where F is divergence-free and vanishing to order r as $t \to 0$. Let $E_s^{\mathcal{T},\varepsilon}$ be defined by (8.2). Then, there is a constant $C = C(x,P,B)$ depending only on the norm of (x,P,B), a lower bound for c_0, and an upper bound for T, but independent of ε such that if $E_s^{\mathcal{T},\varepsilon}(0) = C_s^{\mathcal{U},\varepsilon}(0) = 0$ for $s \leqslant r$, then for $t \in [0,T]$*

$$\|W_\varepsilon\|_{\mathcal{U}^r(\Omega^\varepsilon)} + \|\dot{W}_\varepsilon\|_{\mathcal{U}^r(\Omega^\varepsilon)} + \|\mathcal{L}_B W_\varepsilon\|_{\mathcal{U}^r(\Omega^\varepsilon)} + E_r^{\mathcal{T},\varepsilon} \leqslant C \int_0^t \|F\|_r^{\mathcal{U}} d\tau.$$

Hence, it implies that the limit W satisfies the same bound with Ω^ε replaced by Ω, and so the weak solution in Section 7.2 is a smooth solution indeed.

9. The energy estimate with inhomogeneous initial data

In this section, we consider the original equations with inhomogeneous initial data and an inhomogeneous term:

$$\ddot{W}^a - \mathcal{L}_B^2 W^a + \mathcal{A}W^a + \dot{\mathcal{G}}\dot{W}^a - \mathcal{C}\dot{W}^a + \mathcal{X}\mathcal{L}_B W^a = F^a. \tag{9.1}$$

Some estimates of the commutators are needed with the operator $\mathcal{A}$, $\dot{\mathcal{G}}$, $\mathcal{C}$ and tangential vector fields. We recall them from [21],

$$\begin{aligned} &[\mathcal{L}_S, \mathcal{A}]W = (\mathcal{A}_S - \mathcal{G}_S\mathcal{A})W, && [\mathcal{L}_T, \mathcal{G}_S]W = (\mathcal{G}_{TS} - \mathcal{G}_T\mathcal{G}_S)W, \\ &[\mathcal{L}_S, \mathcal{C}]W = (\mathcal{C}_S - \mathcal{G}_S\mathcal{C})W, && [\mathcal{L}_S^I, \mathcal{A}]W = \tilde{d}_I^{I_1 I_k}\mathcal{G}_{I_1}\cdots\mathcal{G}_{I_{k-2}}\mathcal{A}_{I_{k-1}}W_{I_k}, \\ &[\mathcal{L}_S^I, \dot{\mathcal{G}}]W = \tilde{e}_I^{I_1 I_k}\mathcal{G}_{I_1}\cdots\mathcal{G}_{I_{k-2}}\dot{\mathcal{G}}_{I_{k-1}}W_{I_k}, && [\mathcal{L}_S^I, \mathcal{C}]W = \tilde{e}_I^{I_1 I_k}\mathcal{G}_{I_1}\cdots\mathcal{G}_{I_{k-2}}\mathcal{C}_{I_{k-1}}W_{I_k}, \end{aligned}$$

where $\mathcal{A}_S = \mathcal{A}_{SP}$, $\mathcal{G}_S = \mathcal{M}_{g^S}$ defined by $\mathcal{G}_S W^a = \mathbb{P}(g^{ac}g^S_{cb}W^b)$, $g^{TS}_{bc} = \mathcal{L}_T\mathcal{L}_S g_{bc}$, $\mathcal{G}_{TS}W^a = \mathbb{P}(g^{ab}g^{TS}_{bc}W^c)$, $\mathcal{C}_T W^a = \mathbb{P}(g^{ab}\omega^T_{bc}W^c)$, $\omega^T_{bc} = \mathcal{L}_T\omega_{bc}$, $\mathcal{G}_J W^a = \mathcal{M}_{g^J}W^a = \mathbb{P}(g^{ac}g^J_{cb}W^b)$, $g^J_{ac} = \mathcal{L}^J_S g_{ac}$, $\mathcal{A}_J = \mathcal{A}_{S^J P}$, and $W_J = \mathcal{L}^J_S W$, the sum is over all combinations with $I_1 + I_2 + \cdots + I_k = I$ in last three identities, with $k \geqslant 2$ and $|I_k| < |I|$, $\tilde{d}_I^{I_1 I_k}$ and $\tilde{e}_I^{I_1 I_k}$ are some constants. For the operator $\mathcal{X}$, we have similar equality for its commutator with tangential vector fields. Denote $\beta_{bc} = \delta_{il}\partial_b x^l \mathcal{L}_B \partial_c x^i$, we have

$$\begin{aligned} &\mathcal{L}_T\mathcal{X}W^a = \mathcal{L}_T(g^{ab}(-2\beta_{bc}W^c + \partial_b q)) \\ =&(\mathcal{L}_T g^{ab})(-2\beta_{bc}W^c + \partial_b q) - 2g^{ab}(\mathcal{L}_T\beta_{bc})W^c - 2g^{ab}\beta_{bc}\mathcal{L}_T W^c + g^{ab}\partial_b T q, \end{aligned}$$

where

$$\Delta q = 2\partial_a(g^{ab}\beta_{bc}W^c), \quad q|_\Gamma = 0.$$

Projecting each term onto the divergence-free vector fields, we obtain

$$[\mathcal{L}_T, \mathcal{X}]W = (\mathcal{X}_T - \mathcal{G}_T\mathcal{X})W,$$

where $\mathcal{X}_T W^a = \mathbb{P}(-2g^{ab}\mathcal{L}_T(\delta_{il}\partial_b x^l\mathcal{L}_B\partial_c x^i)W^c)$. Similarly, we have

$$[\mathcal{L}_S^I, \mathcal{X}]W = \tilde{e}_I^{I_1 I_k}\mathcal{G}_{I_1}\cdots\mathcal{G}_{I_{k-2}}\mathcal{X}_{I_{k-1}}W_{I_k}.$$

These commutators are bounded operator and lower order since $|I_k| < |I|$. In addition, $[\mathcal{L}_T^I, \mathcal{L}_B^2]$ is also a bounded operator since B is a tangential vector field. Thus, we obtain

$$L_I W = \ddot{W}_I - \mathcal{L}_B^2 W_I + \mathcal{A}W_I + \dot{\mathcal{G}}\dot{W}_I - \mathcal{C}\dot{W}_I + \mathcal{X}\mathcal{L}_B W_I = H_I,$$

with

$$H_I = F_I + \tilde{d}_I^{I_1 I_k} \mathcal{G}_{I_1} \cdots \mathcal{G}_{I_{k-2}} \mathcal{A}_{I_{k-1}} W_{I_k} \tag{9.2}$$

$$+ [\mathcal{L}_T^I, \mathcal{L}_B^2] W + \tilde{e}_I^{I_1 I_k} \mathcal{G}_{I_1} \cdots \mathcal{G}_{I_{k-2}} \mathcal{X}_{I_{k-1}} [\mathcal{L}_T^{I_k}, \mathcal{L}_B] W \tag{9.3}$$

$$+ \tilde{e}_I^{I_1 I_k} \mathcal{G}_{I_1} \cdots \mathcal{G}_{I_{k-2}} \dot{\mathcal{G}}_{I_{k-1}} \dot{W}_{I_k} + \tilde{e}_I^{I_1 I_k} \mathcal{G}_{I_1} \cdots \mathcal{G}_{I_{k-2}} \mathcal{C}_{I_{k-1}} \dot{W}_{I_k}, \tag{9.4}$$

where $|I_k| < |I|$ and $F_I = \mathcal{L}_T^I F$. We consider only $W_I = \mathcal{L}_S^I W$ with $S \in \mathcal{S}$, as before, let

$$E_I = \langle \dot{W}_I, \dot{W}_I \rangle + \langle W_I, (\mathcal{A} + I) W_I \rangle + \langle \mathcal{L}_B W_I, \mathcal{L}_B W_I \rangle.$$

The energy estimate is similar to that before, and we only need to estimate the L^2-norm of the H_I. It is obvious that (9.3) and (9.4) are bounded by E_J for some $|J| \leqslant |I|$. Since A_{I_k} is of order 1, it contains derivatives in any direction, the term thus has to be estimated by $\|\partial W_{I_k}\|_{L^2(\Omega)}$, and then it does not directly get an estimate for $\|\mathcal{L}_S W_{I_k}\|_{L^2(\Omega)}$ for all tangential derivatives S. However, we can combine the estimates for the curl to obtain the desired estimate.

Let $C_r^{\mathcal{R}}$ and $E_r^{\mathcal{S}}$ be defined as in (6.49) and (6.17), respectively. Let $m_r^{\mathcal{V}}$, $\dot{m}_r^{\mathcal{V}}$ and $\bar{m}_r^{\mathcal{V}}$ be defined as in (6.50). Then, we have by (6.52)

$$\|W\|_r + \|\dot{W}\|_r + \|\mathcal{L}_B W\|_r \leqslant K_1 \sum_{s=0}^{r} m_{r-s}^{\mathcal{R}} (C_s^{\mathcal{R}} + E_s^{\mathcal{S}}), \tag{9.5}$$

where $\|W\|_r = \|W(t)\|_{\mathcal{R}^r(\Omega)}$. Since the projection $\mathbb{P}$ has norm 1, and $\|G_J W\| \leqslant \|g^J\|_{L^\infty(\Omega)} \|W\|$, it follows that

$$\begin{aligned} \|\mathcal{G}_{I_1} \cdots \mathcal{G}_{I_{k-2}} \dot{\mathcal{G}}_{I_{k-1}} \dot{W}_{I_k}\| &\leqslant \|g^{I_1}\|_{L^\infty(\Omega)} \cdots \|g^{I_{k-2}}\|_{L^\infty(\Omega)} \|\dot{g}^{I_{k-1}}\|_{L^\infty(\Omega)} \|\dot{W}\|_s \\ &\leqslant \dot{m}_{r-s}^{\mathcal{R}} \|\dot{W}\|_s, \end{aligned} \tag{9.6}$$

$$\begin{aligned} \|\mathcal{G}_{I_1} \cdots \mathcal{G}_{I_{k-2}} \dot{\mathcal{C}}_{I_{k-1}} \dot{W}_{I_k}\| &\leqslant \|g^{I_1}\|_{L^\infty(\Omega)} \cdots \|g^{I_{k-2}}\|_{L^\infty(\Omega)} \|\omega^{I_{k-1}}\|_{L^\infty(\Omega)} \|\dot{W}\|_s \\ &\leqslant \dot{m}_{r-s}^{\mathcal{R}} \|\dot{W}\|_s, \end{aligned} \tag{9.7}$$

and

$$\begin{aligned} &\|\mathcal{G}_{I_1} \cdots \mathcal{G}_{I_{k-2}} \mathcal{X}_{I_{k-1}} [\mathcal{L}_T^{I_k}, \mathcal{L}_B] W\| \\ &\leqslant \|g^{I_1}\|_{L^\infty(\Omega)} \cdots \|g^{I_{k-2}}\|_{L^\infty(\Omega)} \|\mathcal{L}_T^{I_{k-1}} (\delta_{il} \partial_b x^l \mathcal{L}_B \partial_c x^i)\|_{L^\infty(\Omega)} \|[\mathcal{L}_T^{I_k}, \mathcal{L}_B] W\| \\ &\leqslant (\dot{m}_{r-s}^{\mathcal{R}} + \bar{m}_{r-s}^{\mathcal{R}}) \|W\|_{s,B}, \end{aligned}$$

where $s = |I_k| < r = |I|$. Denote

$$P_r^{\mathcal{R}} = \sum_{s=0}^{r} [[g]]_{r-s,\infty}^{\mathcal{R}} \sum_{|J| \leqslant s+1, J \in \mathcal{S}} \|\partial S^J P\|_{L^\infty(\partial\Omega)}.$$

Then, we have

$$\|\tilde{d}_I^{I_1 I_k}\mathcal{G}_{I_1}\cdots\mathcal{G}_{I_{k-2}}\mathcal{A}_{I_{k-1}}W_{I_k}\| \leqslant \|g^{I_1}\|_{L^\infty(\Omega)}\cdots\|g^{I_{k-2}}\|_{L^\infty(\Omega)}\|\mathcal{A}^{I_{k-1}}W_{I_k}\|_{L^\infty(\Omega)} \\ \leqslant P^{\mathcal{R}}_{r-s}\|W\|_s + P^{\mathcal{R}}_{r-s-1}\|W\|_{s+1}. \tag{9.8}$$

Similar to (4.7), we can get

$$|\dot{E}_I| \leqslant \left(1+2\|\dot{g}\|_{L^\infty(\Omega)} + \frac{\|\partial\dot{P}\|_{L^\infty(\Gamma)}}{c_0} + 2\|\partial x\|_{L^\infty(\Omega)}\|\mathcal{L}_B\partial x\|_{L^\infty(\Omega)}\right)E_I \\ + 2\sqrt{E_I}\|H_I\|, \tag{9.9}$$

where c_0 is the constant in the condition (1.7). From (9.6)-(9.8) and (9.5), we get

$$\|H_I\| \leqslant C\sum_{s=0}^{r-1}(\dot{m}^{\mathcal{R}}_{r-s}\|\dot{W}\|_s + (\dot{m}^{\mathcal{R}}_{r-s}+\bar{m}^{\mathcal{R}}_{r-s})\|W\|_{s,B} + P^{\mathcal{R}}_{r-s}\|W\|_s) \\ + P^{\mathcal{R}}_0\|W\|_r + \|F\|_r \\ \leqslant K_1\sum_{s=0}^{r-1}(\dot{m}^{\mathcal{R}}_{r-s}+\bar{m}^{\mathcal{R}}_{r-s}+P^{\mathcal{R}}_{r-s})(C^{\mathcal{R}}_s+E^{\mathcal{R}}_s) \\ + K_1P^{\mathcal{R}}_0(C^{\mathcal{R}}_r+E^{\mathcal{R}}_r) + \|F\|_r. \tag{9.10}$$

Summing (9.9) over all $I\in\mathcal{S}$ with $|I|=r$ and using (9.10), we have

$$\left|\frac{dE^{\mathcal{S}}_r}{dt}\right| \leqslant K_1\Bigg(1+2\|\dot{g}\|_{L^\infty(\Omega)} + \frac{\|\partial\dot{P}\|_{L^\infty(\Gamma)}}{c_0} + 2\|\partial x\|_{L^\infty(\Omega)}\|\mathcal{L}_B\partial x\|_{L^\infty(\Omega)} \\ + \sum_{S\in\mathcal{S}}\|\partial SP\|_{L^\infty(\Gamma)}\Bigg)(C^{\mathcal{R}}_r+E^{\mathcal{R}}_r) \\ + K_1\sum_{s=0}^{r-1}(\dot{m}^{\mathcal{R}}_{r-s}+\bar{m}^{\mathcal{R}}_{r-s}+P^{\mathcal{R}}_{r-s})(C^{\mathcal{R}}_s+E^{\mathcal{R}}_s) + \|F\|_r. \tag{9.11}$$

Since (6.51) holds with $\mathcal{U}$ and $\mathcal{T}$ replaced by $\mathcal{R}$ and $\mathcal{S}$, respectively, we get

$$\left|\frac{dC^{\mathcal{R}}_r}{dt}\right| \leqslant K_1(\dot{m}^{\mathcal{R}}_0+\bar{m}^{\mathcal{R}}_0)(C^{\mathcal{R}}_r+E^{\mathcal{S}}_r) \\ + K_1\sum_{s=0}^{r-1}(\dot{m}^{\mathcal{R}}_{r-s}+\bar{m}^{\mathcal{R}}_{r-s})(C^{\mathcal{R}}_s+E^{\mathcal{S}}_s) + \|F\|_r. \tag{9.12}$$

Thus, (9.11) and (9.12) yield a bound for $C^{\mathcal{R}}_r+E^{\mathcal{S}}_r$ in terms of $C^{\mathcal{R}}_s+E^{\mathcal{S}}_s$ for $s<r$, namely

$$C_r^{\mathcal{R}}(t) + E_r^{\mathcal{S}}(t) \leqslant K_1 e^{K_1 \int_0^t n d\tau} \Big(C_r^{\mathcal{R}}(0) + E_r^{\mathcal{S}}(0) + \int_0^t \Big(\sum_{s=0}^{r-1} (\dot{m}_{r-s}^{\mathcal{R}} + \bar{m}_{r-s}^{\mathcal{R}})(C_s^{\mathcal{R}} + E_s^{\mathcal{S}}) + \|F\|_r \Big) d\tau \Big),$$

where

$$\begin{aligned} n =& 1 + 2\|\dot{g}\|_{L^\infty(\Omega)} + \|\partial \dot{P}\|_{L^\infty(\Gamma)}/c_0 + 2\|\partial x\|_{L^\infty(\Omega)} \|\mathcal{L}_B \partial x\|_{L^\infty(\Omega)} \\ &+ \sum_{S\in\mathcal{S}} \|\partial S P\|_{L^\infty(\Gamma)} + \|\omega\|_{L^\infty(\Omega)} + \|\mathcal{L}_B g\|_{L^\infty(\Omega)} + \|\mathcal{L}_B^2 g\|_{L^\infty(\Omega)} + \|B\|_{2,\infty}^{\mathcal{R}} \\ &+ \|\partial x\|_{L^\infty(\Omega)} + \|\partial^2 x\|_{L^\infty(\Omega)} + \|\frac{\partial y}{\partial x}\|_{L^\infty(\Omega)}. \end{aligned}$$

Because the bound for $E_0^{\mathcal{S}} = E_0$ have been already proven, we can get the bound for $C_r^{\mathcal{R}} + E_r^{\mathcal{S}}$ inductively. Therefore, from (9.5), we obtain the following estimates.

Proposition 9.1. *Suppose that $x, P \in C^{r+2}([0,T] \times \Omega)$, $B \in C^{r+2}(\Omega)$, $P|_\Gamma = 0$, $\nabla_N P|_\Gamma \leqslant -c_0 < 0$, $B^a N_a|_\Gamma = 0$ and $\operatorname{div} V = 0$, where $V = D_t x$. Let W be the solution of (9.1) where F is divergence-free. Then, there is a constant C depending only on the norm of (x, P, B), a lower bound for the constant c_0, and an upper bound for T, such that, for $s \leqslant r$, we have*

$$\begin{aligned} &\|W(t)\|_r + \|\dot{W}(t)\|_r + \|\mathcal{L}_B W(t)\|_r + \langle W(t)\rangle_{\mathcal{A},r} \\ \leqslant& C\Big(\|W(0)\|_r + \|\dot{W}(0)\|_r + \|\mathcal{L}_B W(0)\|_r + \langle W(0)\rangle_{\mathcal{A},r} + \int_0^t \|F\|_r d\tau \Big), \end{aligned} \tag{9.13}$$

where

$$\|W(t)\|_r = \sum_{|I|\leqslant r, I\in\mathcal{R}} \|\mathcal{L}_U^I W(t)\|_{L^2(\Omega)},$$

$$\langle W(t)\rangle_{\mathcal{A},r} = \sum_{|I|\leqslant r, I\in\mathcal{S}} \langle \mathcal{L}_S^I W(t), \mathcal{A}\mathcal{L}_S^I W(t)\rangle^{1/2}.$$

10. The main result

As the same as in [21], $\|W(t)\|_r$ is equivalent to the usual time-independent Sobolev norm; $\langle W(t)\rangle_{\mathcal{A},r}$ is only a seminorm on divergence-free vector fields, which is not only equivalent to a time-independent seminorm given by (3.2) with f the distance function $d(y)$ due to $0 < c_0 \leqslant -\nabla_N P \leqslant C$, but also equivalent to the normal component of the vector field $W_N = N_a W^a$ being in $H^r(\Gamma)$ in view of (3.2), up to lower-order terms which can be controlled by $\|W(t)\|_r$, since we only apply tangential vector fields.

We define $H^r(\Omega)$ to be the completion of $C^\infty(\Omega)$ in the norm $\|W(t)\|_r$ and define $N^r(\Omega)$ to be the completion of the $C^\infty(\Omega)$ divergence-free vector fields in the norm $\|W\|_{N^r} = \|W(t)\|_r +$

$\langle W(t)\rangle_{\mathcal{A},r}$. Since the projection $\mathbb{P}$ is continuous in the H^r norm, H^r is also the completion of the $C^\infty(\Omega)$ divergence-free vector fields in the H^r norm. We state the main result as follows.

Theorem 10.1. *Suppose that* $x, P \in C^{r+2}([0,T]\times\Omega)$, $B \in C^{r+2}(\Omega)$, $P|_\Gamma = 0$, $\nabla_N P|_\Gamma \leqslant -c_0 < 0$, $B^a N_a|_\Gamma = 0$ *and* $\operatorname{div} D_t x = 0$. *Then, if initial data and the inhomogeneous term in* (5.1b) *are divergence-free and satisfy*

$$(W_0, W_1, \mathcal{L}_B W_0) \in N^r(\Omega)\times H^r(\Omega)\times H^r(\Omega), \quad F \in L^1\left([0,T],\ H^r(\Omega)\right),$$

the linearized equations (5.1) *have a solution*

$$(W, \dot{W}, \mathcal{L}_B W) \in C\left([0,T],\ N^r(\Omega)\times H^r(\Omega)\times H^r(\Omega)\right). \tag{10.1}$$

Proof. If W_0, W_1 and F are divergence-free and C^∞, and F is supported in $t>0$, then there exists a solution by the arguments in Section 8. It follows, by approximating W_0, W_1 and F with $C^\infty(\Omega)$ divergence-free vector fields and applying the estimate (9.13) to the differences, that we obtain a convergent sequence in (10.1), thus the limit must be in the same space. □

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Appendix A. Lie derivatives

Let us review the Lie derivative of the vector field W with respect to the vector field T constructed in the previous section,

$$\mathcal{L}_T W^a = TW^a - (\partial_c T^a)W^c. \tag{A.1}$$

For those vector fields, it holds that $\operatorname{div} T = 0$, so $\operatorname{div} W = 0$ implies that

$$\operatorname{div}\mathcal{L}_T W = T\operatorname{div} W - W\operatorname{div} T = 0.$$

The Lie derivative of a 1-form is defined by

$$\mathcal{L}_T\alpha_a = T\alpha_a + (\partial_a T^c)\alpha_c.$$

The Lie derivatives also commute with the exterior differentiation, $[\mathcal{L}_T, d] = 0$, so if q is a function, then

$$\mathcal{L}_T\partial_a q = \partial_a T q. \tag{A.2}$$

The Lie derivative of a 2-form is given by

$$\mathcal{L}_T\beta_{ab} = T\beta_{ab} + (\partial_a T^c)\beta_{cb} + (\partial_b T^c)\beta_{ac}. \tag{A.3}$$

In general, in local coordinate notation, for a type (r, s) tensor field β, the Lie derivative along T is given by

$$\begin{aligned}\mathcal{L}_T \beta^{a_1\ldots a_r}{}_{b_1\ldots b_s} =& T\beta^{a_1\ldots a_r}{}_{b_1\ldots b_s}\\ &- (\partial_c T^{a_1})\beta^{ca_2\ldots a_r}{}_{b_1\ldots b_s} - \ldots - (\partial_c T^{a_r})\beta^{a_1\ldots a_{r-1}c}{}_{b_1\ldots b_s}\\ &+ (\partial_{b_1} T^c)\beta^{a_1\ldots a_r}{}_{cb_2\ldots b_s} + \ldots + (\partial_{b_s} T^c)\beta^{a_1\ldots a_r}{}_{b_1\ldots b_{s-1}c}.\end{aligned} \tag{A.4}$$

It follows that the Lie derivative satisfies the Leibniz rule, e.g.

$$\begin{aligned}\mathcal{L}_T(\alpha_c W^c) =& (\mathcal{L}_T \alpha_c) W^c + \alpha_c \mathcal{L}_T W^c,\\ \mathcal{L}_T(\beta_{ac} W^c) =& (\mathcal{L}_T \beta_{ac}) W^c + \beta_{ac} \mathcal{L}_T W^c,\\ \mathcal{L}_T(g^{ab}\alpha_b) =& \mathcal{L}_T g^{ab}\alpha_b + g^{ab}\mathcal{L}_T \alpha_b,\end{aligned} \tag{A.5}$$

and

$$\mathcal{L}_T(W^c \partial_c \beta_{ab}) = \mathcal{L}_T W^c \partial_c \beta_{ab} + W^c \mathcal{L}_T \partial_c \beta_{ab}. \tag{A.6}$$

If w is a 1-form and $\operatorname{curl} w_{ab} = dw_{ab} = \partial_a w_b - \partial_b w_a$, then

$$\mathcal{L}_T \operatorname{curl} w_{ab} = \operatorname{curl} \mathcal{L}_T w_{ab}, \tag{A.7}$$

since the Lie derivative commutes with exterior differentiation.

From (A.1), we have the following relation on the commutator of two Lie derivatives

$$[\mathcal{L}_T, \mathcal{L}_B] W^a = \mathcal{L}_{[T,B]} W^a. \tag{A.8}$$

From (A.4), we get the commutator of Lie derivative and ∂_a

$$[\mathcal{L}_T, \partial_a] W^b = W^d \partial_d \partial_a T^b. \tag{A.9}$$

Furthermore, we also treat D_t as if it were a Lie derivative and we set

$$\mathcal{L}_{D_t} = D_t. \tag{A.10}$$

Of course, this is not a space Lie derivative but rather could be interpreted as a space-time Lie derivative in the domain $[0, T] \times \Omega$. What matters is that it satisfies all the properties of the other Lie derivatives considered, such as $\operatorname{div} W = 0$ implies that $\operatorname{div} D_t W = 0$ and $D_t \operatorname{curl} w = \operatorname{curl} D_t w$, since it commutes with partial differentiation with respect to the y coordinates. More efficient to use the same notation, since products of Lie derivatives and (A.10) will be applied. Furthermore

$$[\mathcal{L}_{D_t}, \mathcal{L}_T] = 0, \tag{A.11}$$

since this quantity is $\mathcal{L}_{[D_t, T]}$ and $[D_t, T] = 0$ for the vector fields we considered, or it follows from (A.1) and that $T^a = T^a(y)$ is independent of t.

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