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Research article

Some results on free boundary problems of incompressible ideal magnetohydrodynamics equations

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Abstract: We survey some recent results related to free boundary problems of incompressible ideal magnetohydrodynamics equations, and present the main ideas in the proofs of the ill-posedness in 2D when the Taylor sign condition is violated given [1], and the well-posedness of a linearized problem given in [2] in general n-dimensions ($n \geqslant 2$) when the Taylor sign condition is satisfied and the free boundaries are closed.

Keywords: free boundary problems; incompressible ideal magnetohydrodynamics equations; ill-posedness; linearized problem; well-posedness

1. Introduction

The free boundary problem for the incompressible ideal magnetohydrodynamics (MHD) equations can be described by

$$\begin{cases} v_t + v \cdot \nabla v + \nabla p = \mu H \cdot \nabla H, & \text{in } \mathcal{D}, & (1.1)\\ H_t + v \cdot \nabla H = H \cdot \nabla v, & \text{in } \mathcal{D}, & (1.2)\\ \operatorname{div} v = 0, \quad \operatorname{div} H = 0, & \text{in } \mathcal{D}, & (1.3) \end{cases}$$

where v is the velocity field, H is the magnetic field, p is the total pressure including the fluid pressure and the magnetic pressure, and $\mu > 0$ is the vacuum permeability, $\mathcal{D} = \cup_{0 \leqslant t \leqslant T}(\{t\} \times \Omega_t)$, $\Omega_t \subset \mathbb{R}^n$ ($n \geqslant 2$) is the domain occupied by the fluid at time t.

The conditions on the free boundary $\partial\mathcal{D}$ are

$$H \cdot \mathcal{N} = 0, \quad p = 0, \quad \text{on } \partial\mathcal{D}, \tag{1.4}$$

$$(\partial_t + v^k \partial_k)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \tag{1.5}$$

where $\mathcal{N}$ is the exterior unit normal to $\Gamma_t := \partial\Omega_t$. The condition $p = 0$ indicates that the total pressure vanishes outside the domain. Roughly speaking, the motion of the free boundary which is the level set of the total pressure is determined by the fluid velocity, the acceleration of the free boundary is determined by the total pressure and the magnetic tension. With the assumption that the boundary Γ_t is a perfect conductor, one has that $H \cdot \mathcal{N} = 0$, which holds true for all $t \in [0, T]$ if it holds initially as showed in [3]. Therefore, it is a constraint on the initial data.

Given a domain $\Omega \subset \mathbb{R}^n$, and initial data (v_0, H_0) satisfying the constraint (1.3), the free boundary problem is to seek a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ and vector fields (v, H) solving (1.1)–(1.5) with initial conditions

$$\{x : (0, x) \in \mathcal{D}\} = \Omega; \quad v = v_0, \ H = H_0, \ \text{on } \{0\} \times \Omega. \tag{1.6}$$

Set $\Omega_t = \{x : (t, x) \in \mathcal{D}\}$. Motivated by the Taylor sign condition on the fluid pressure for the Euler equations, the following condition for the total pressure was raised in [3]:

$$\nabla_{\mathcal{N}} p \leqslant -c_0 < 0 \text{ on } \partial\mathcal{D}, \tag{1.7}$$

where $\nabla_{\mathcal{N}} = \mathcal{N}^i \partial_{x^i}$. Here the summation convection over repeated upper and lower indices has been used. In [3], a priori estimates of Sobolev norms for υ and H and geometric quantities of the evolving free boundary $\partial\mathcal{D}$ for the problem (1.1)–(1.6) were derived under the condition (1.7). We also showed in [1] that the above free boundary problem (1.1)–(1.6) under consideration would be ill-posed at least for the case $n = 2$ if the condition (1.7) was violated. Thus, it will be much reasonable and necessary to require this condition (1.7) in the studies of well-posedness of the considering free boundary problem of incompressible ideal MHD equations.

Due to their importance both in practice and nonlinear PDE theory, fluids free boundary problems arising from physical, engineering and medical models have received extensive attention. Typical examples include water waves, evolution of boundaries of stars, vortex sheets, multi-phase flow, reacting flow, shock waves, biomedical modeling such as tumor growth, cell deformation, etc. In the most fundamental and simplest setting, important progress has been made on the free boundary problem of the incompressible Euler equations. For this problem, with the gravity modeling water waves, the local well-posedness in Sobolev spaces for inviscid irrotational flow was obtained first by Wu in [4, 5] for 2D and 3D, respectively. For the cases without the irrotational assumption, finite depth water waves, lower regularity, uniform estimates with respect to surface tension, etc., substantial progresses have been made, one may refer to [6–19] for these results. For the compressible inviscid flow, the local-in-time well-posedness of smooth solutions was established for liquids in [20, 21] (see also [22] for higher order energy estimates and large sound speed limit and [23] for the zero surface tension limits), the effects of heat-conductivity to fluid free surface of highly subsonic flow which is in between the compressible and the incompressible are studied in [24].

In many important physical situations, magnetic fields are essential (cf. [25–27]). Examples include solar flares in astrophysics [27]. In the study of the ideal MHD free boundary problems with a bounded initial domain homeomorphic to a ball, under the condition (1.7), the authors derived a priori estimates in [3] for the case when the size of the magnetic field is invariant on the free boundary. Luo and Zhang [28] obtained a priori estimates for the low regularity solution in the case when the domain has small volume. For the two-dimensional case, if the condition (1.7) is violated, we showed the ill-posedness in [1]. For the problem (1.1)–(1.6) in three dimensions, a local existence result was established in [29], for which the detailed proof is given in an initial flat domain of the form $\mathbb{T}^2 \times (0,1)$, for a two-dimensional period box $\mathbb{T}^2$ in x_1 and x_2. With the same set-up of the initial domain, the local well-posedness is obtained in [30] by Gu, Luo and Zhang for the case with the surface tension. For a general free boundary not restricted to the case of a graph, it might be feasible to reduce the problem to solving several free boundary problems of free boundaries being graphs simultaneously to use several coordinate charts, which should be quite technically involved, however, and this approach seems very difficult to be applied to study the long time evolution problem for general free domains. With the motivation of contributing to the study of the ideal MHD free surface problem with free closed curved surface with large curvature, the well-posedness of a linearized problem was proved by the authors in [2], via geometric approaches motivated by [11], [31] and [21] for the Euler equations of fluids, and developed in [3] for MHD equations and [28] for an inviscid highly subsonic heat-conductive fluid.

We survey here some related results of MHD free boundary problems. For the case where the magnetic field is zero on the free boundary and in vacuum, in the three-dimensional space with infinite and finite depth settings, Lee proved the local existence and uniqueness of the free boundary problem of incompressible viscous-diffusive MHD flow in [32], he also proved in [33] a local unique solution for the free boundary MHD without kinetic viscosity and magnetic diffusivity via zero kinetic viscosity-magnetic diffusivity limit. For the case when the magnetic field is constant on the free surface and outside an unbounded domain, the convergence rates of inviscid limits for the free boundary problems of the three-dimensional incompressible MHD with or without surface tension were studied in [34]. For the free boundary problems of compressible invisicd MHD, please refer to the recent results of Trakhinin and Wang in [35, 36].

The plasma-vacuum interface problem is a problem with an interface separating the plasma and vacuum regions. In the plasma region, the motion is governed by MHD equations, and in the vacuum region, the magnetic field satisfies the pre-Maxwell system that $\operatorname{div} H = \operatorname{curl} H = 0$. On the interface, the normal component of the magnetic fields and the total pressure are continuous across the interface. The a priori estimates were derived by Hao in [37] for the plasma-vacuum interface problem in a 3D bounded domain under the Taylor sign condition.

For a linearized plasma-vacuum interface problem of compressible MHD in 3D, the stability condition that the magnetic fields on two sides of the interface are noncollinear was proposed by Trakhinin [38]. Under this condition, for the plasma-vacuum interface problem, the local-in-time well-posedness was established by Secchi-Trakhinin [39] for compressible MHD, by Morando-Trakhinin-Trebeschi [40] for a linearized problem of incompressible MHD, and finally by Sun-Wang-Zhang [18] for the nonlinear problem.

The current-vortex sheet problem is a problem with an interface across which the tangential velocity and the tangential magnetic fields may be discontinuous, but the normal velocity, normal magnetic fields and the total pressure are continuous. In the regions separating by the interfaces, the motions are governed by MHD equations. For the current-vortex sheet problem, the neutral stability condition of planner compressible current-vortex sheets was identified by Trakhinin [41]. Using the Nash-Moser iteration, Chen-Wang [42] and Trakhinin [38] proved the existence of the compressible current-vortex sheet. In the two-dimensional case, the linear stability of current-vortex sheet was analyzed by Wang and Yu [43]. In 3D, the nonlinear stability of the current-vortex sheet to the incompressible MHD equations was proved by Sun, Wang and Zhang [44] under the Syrovatskij stability condition, which first justifies rigorously the stabilizing effect of the magnetic field on Kelvin-Helmholtz instability. When the magnetic diffusion (resistivity) is taken into account, the global-in-time well-posedness of a free interface problem for the incompressible inviscid resistive MHD was proved by Wang and Xin in [45].

In the rest of the paper, we will present the main results and the key ideas of the proofs on the ill-posedness of the free boundary problem of the ideal incompressible MHD in 2D when the Taylor sign condition is violated [1] and the well-posedness for a linearized problem when the free boundary is a closed hyper-surface in $\mathbb{R}^n$ and the Taylor sign condition holds [2].

2. Ill-posedness of free boundary problem of the incompressible ideal MHD

In this section, we present the main results and the key ideas of the proofs in [1] for the ill-posedness of the free boundary problem of ideal incompressible MHD in 2D when the Taylor sign condition is violated. Details for some derivations can be found in [1].

We rewrite the problem as the free boundary problem for the incompressible ideal MHD equations

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla q = \frac{1}{\mu_0}\tilde{H}\cdot\nabla\tilde{H}, & \text{in } \Omega_t, \quad (2.1\text{a})\\ \partial_t\tilde{H} + v\cdot\nabla\tilde{H} = \tilde{H}\cdot\nabla v, & \text{in } \Omega_t, \quad (2.1\text{b})\\ \operatorname{div} v = 0, \quad \operatorname{div}\tilde{H} = 0, & \text{in } \Omega_t, \quad (2.1\text{c})\\ \partial_t\Gamma(t) = v\cdot\mathcal{N}, \quad \tilde{H}\cdot\mathcal{N} = 0, \quad q = 0, & \text{on } \Gamma_t, \quad (2.1\text{d})\\ v(0,x) = v_0(x), \quad \tilde{H}(0,x) = \tilde{H}_0(x), & x\in\Omega := \Omega_0, \quad (2.1\text{e}) \end{cases}$$

where v is the velocity field, $\tilde{H}$ is the magnetic field, q is the total pressure and $\mu_0 > 0$ is the vacuum permeability; $\mathcal{N}$ is the exterior unit normal vector on the boundary Γ_t, Ω_0 is the bounded initial domain. In [3], we identified a stability condition,

$$\nabla_{\mathcal{N}} q < 0, \quad \text{on } \partial\Omega_t \tag{2.2}$$

under which the a priori estimates are derived.

In 2-spatial dimension case, there is a particular steady solution to (2.1) with rigid rotation: For $t \geqslant 0$, $\Omega_t = \{x\in\mathbb{R}^2 : |x| \leqslant 1\} = B_1(0)$, the unit disk, $v(t,x) = (-x_2,\ x_1) = x^\perp$, $\tilde{H}(t,x) = bv(t,\ x)$ (for $x = (x_1,\ x_2)$ and $b\in\mathbb{R}$). In this case, q is solved by the following Dirichlet problem:

$$\Delta q = 2\left(1 - \frac{b^2}{\mu_0}\right), \text{ in } B_1(0); \ q = 0, \text{ on } \partial B_1(0).$$

For this solution, it is easy to verify that, on $\partial B_1(0)$,

$$\nabla_{\mathcal{N}} q = \left(\frac{b^2}{\mu_0} - 1\right)(v\cdot\nabla v)\cdot x = 1 - \frac{b^2}{\mu_0}, \tag{2.3}$$

by noting that $\mathcal{N} = x$ on $\partial B_1(0)$. Therefore, the Taylor sign condition (2.2) fails in this case when $\mu_0 \geqslant b^2$. A natural question arises: For $\mu_0 \geqslant b^2$, is the free boundary problem (2.1) still well-posed? We prove in [1] that this is not the case when $\mu_0 \gg b^2$ in the following sense:

A family of initial data for the problem (2.1) *can be constructed which converges to the above particular steady rotation solution. However, as long as* $t > 0$*, the family of solutions to* (2.1) *with those initial data diverges in some Sobolev* H^μ*-norm for* $\mu \geqslant 2$*.*

This construction is strongly motivated by that of Ebin [13] for incompressible Euler equations.

2.1. Lagrangian description

For simplicity, we denote $H = \mu_0^{-1/2}\tilde{H}$. Then, the problem (2.1) reduces to the case $\mu_0 = 1$, i.e.,

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla q = H \cdot \nabla H, & \text{in } \Omega_t, \quad (2.4a)\\ \partial_t H + v \cdot \nabla H = H \cdot \nabla v, & \text{in } \Omega_t, \quad (2.4b)\\ \operatorname{div} v = 0, \quad \operatorname{div} H = 0, & \text{in } \Omega_t, \quad (2.4c)\\ \partial_t \Gamma(t) = v \cdot \mathcal{N}, \quad H \cdot \mathcal{N} = 0, \quad q = 0, & \text{on } \Gamma_t, \quad (2.4d)\\ v(0,x) = v_0(x), \quad H(0,x) = H_0(x), & x \in \Omega := \Omega_0. \quad (2.4e) \end{cases}$$

Denote $\eta = \eta(t,a) = (\eta^1, \eta^2)$ the position of a fluid element or parcel at time t with $a = (a^1, a^2)$ being the fluid element label, defined to be the position of the fluid element at the initial time, $a = \eta(0,a)$. Let Ω_t be the domain occupied by the fluid at time t, then $\eta : \Omega_0 \to \Omega_t$ is assumed to be 1-1 and onto, at each fixed time t.

Let $\partial\eta^i/\partial a^j =: \eta^i_{,j}$ be the deformation matrix, and $J := \det(\eta^i_{,j})$ be Jacobian determinant, i.e.,

$$J = \frac{1}{2}\varepsilon_{kj}\varepsilon^{il}\eta^k_{,i}\eta^j_{,l},$$

where $\varepsilon_{ij} = \varepsilon^{ij}$ is the two-dimensional unit, purely antisymmetric, Levi-Civita tensor density (repeated indices are summed). In this notation,

$$d\eta = J da,$$

and components of an area form map according to

$$(dS(\eta))_i = Ja^j_{,i}(dS(a))_j, \tag{2.5}$$

where $Ja^j_{,i}$ is the transpose of the cofactor matrix of $\eta^j_{,i}$, given by

$$Ja^j_{,i} = \varepsilon_{ik}\varepsilon^{jl}\eta^k_{,l}.$$

Clearly, $\dot{\eta}(t,a) = v(t,x)$, where $\dot{\eta}(t,a) = \frac{\partial\eta(t,a)}{\partial t}$. The label of the element will be given by $a = \eta^{-1}(t,x) =: a(t,x)$. For an incompressible fluid, $J = 1$, so that $\eta = \eta(t,a)$ can be inverted to obtain $a = a(t,\eta)$,

$$\eta^i_{,k}a^k_{,j} = a^i_{,k}\eta^k_{,j} = \delta^i_j,$$

where $a^k_{,j} = \partial a^k / \partial \eta^j$. Moreover,

$$\frac{\partial}{\partial x^k} = a^i_{,k} \frac{\partial}{\partial a^i}\bigg|_{a=a(t,x)}.$$

Hence, for $f(t,a) = \tilde{f}(t,x) = \tilde{f}(t,\eta(t,a))$ and for $x = \eta(t,a)$,

$$\dot{f}|_{a=a(t,x)} = \frac{\partial \tilde{f}}{\partial t} + \dot{\eta}(t,a)\left.\frac{\partial \tilde{f}}{\partial x^i}\right|_{a=a(t,x)} = \frac{\partial \tilde{f}}{\partial t} + \upsilon \cdot \nabla \tilde{f}(t,x).$$

Then, the magnetic field reduces to

$$H^i = \eta^i_{,j} H^j_0, \tag{2.6}$$

which can be also obtained from the equation (2.4b). The details of the above derivation can be found in [1].

2.2. The equation of pressure and position

In view of (2.4a), (2.4c) and $q = 0$ on $\Gamma_t = \eta(\Gamma)$, one has

$$\begin{cases} \Delta q = \mathrm{tr}(DH)^2 - \mathrm{tr}(D\upsilon)^2, & \text{in } \Omega_t, \quad (2.7a) \\ q = 0, & \text{on } \Gamma_t, \quad (2.7b) \end{cases}$$

where $\mathrm{tr}(D\upsilon)^2 := \partial_i \upsilon^j \partial_j \upsilon^i$ and $\mathrm{tr}(DH)^2 := \partial_i H^j \partial_j H^i$ are the trace of the square of the matrices of differentiations of υ and H, respectively. This elliptic Dirichlet boundary value problem admits a unique solution, so Δ^{-1} is well-defined. Here $\Delta^{-1} g = f$ in a domain means $\Delta f = g$ in this domain and $f = 0$ on the boundary of this domain. Hence,

$$q = \Delta^{-1}(\mathrm{tr}(DH)^2 - \mathrm{tr}(D\upsilon)^2).$$

With this, we rewrite (2.4a) as

$$\ddot{\eta} = (\nabla(\Delta^{-1}(\mathrm{tr}(D\upsilon)^2 - \mathrm{tr}(DH)^2))) \circ \eta + (H \cdot \nabla H) \circ \eta. \tag{2.8}$$

Since $\upsilon(t,x) = \dot{\eta}(t, a(t,x))$ and (2.6), (2.8) is of the form

$$\ddot{\eta} = Z(\eta, \dot{\eta}). \tag{2.9}$$

The initial value problem for (2.9) with the initial data $\eta(0) = \eta_0$ and $\dot{\eta}(0)$ will then be studied.

2.3. A steady state solution of MHD flow spinning at constant angular velocity

Let Ω be the unit disc in $\mathbb{R}^2$. We identify the points in Ω with the complex numbers $\mathbb{C}$. Then $\eta(0) = \eta_0$ is the inclusion of Ω in $\mathbb{C}$. $\dot{\eta}(0)$ is chosen to be a $\pi/2$ rotation. For $z \in \Omega$ being a complex variable, $\eta(0,z) = z$ and $\dot{\eta}(0,z) = \mathrm{i}z$. We set $H_0(\eta(z)) = \mathrm{i}bz$, in view of the fact $H_0 \cdot \mathcal{N} = 0$ on the boundary.

Therefore, $(\eta(t,z) = e^{\mathrm{i}t}z,\ H(t,z) = \mathrm{i}bz)$ is the solution to (2.8) with these initial data, and $\dot{\eta}(t,z) = \mathrm{i}\eta(t,z) = \mathrm{i}e^{\mathrm{i}t}z,$, $a(t,z) = e^{-\mathrm{i}t}z$, $\upsilon(t,z) = \mathrm{i}z$. Moreover, $D\eta = e^{\mathrm{i}t}I$ where I is the 2×2 unit matrix (δ^i_j).

By virtue of (2.6), one has $H(t,z) = \mathrm{i}bz$. The vector fields$v = (-x_2, x_1)$ and $H = b(-x_2, x_1)$ are both divergence-free, and $Dv = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $DH = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$. Thus, $q = (b^2-1)\Delta^{-1}(-2) = \frac{1-x_1^2-x_2^2}{2}(b^2-1)$ is the solution to Dirichlet problem (2.7), and $\nabla q = (1-b^2)z$. Moreover, $H \cdot \nabla H = -b^2 z$.

Since $\dot{\eta}(t,z) = \mathrm{i}\eta(t,z)$ and $\ddot{\eta}(t,z) = -\eta(t,z)$,

$$\ddot{\eta}(t,z) + \nabla q(t, \eta(t,z)) = (H \cdot \nabla H)(\eta(t,z)),$$

i.e., $\eta(t,z)$ satisfies (2.8).

2.4. Reformulation in Lagrangian variables

The following operators are defined in [13]: $\mathfrak{R}_\eta : C^\infty(\eta(\Omega)) \to C^\infty(\Omega)$ defined by $\mathfrak{R}_\eta(f) = f \circ \eta = f(\eta)$. The inverse of $\mathfrak{R}_\eta$ is $\mathfrak{R}_{\eta^{-1}}$ is given by $\mathfrak{R}_{\eta^{-1}}(g) = g \circ \eta^{-1} = g(\eta^{-1})$ where $\eta^{-1}(t,x) = a(t,x)$.

For a differential operator P, we set $P_\eta =: \mathfrak{R}_\eta P \mathfrak{R}_{\eta^{-1}}$, e.g., $D_\eta = \mathfrak{R}_\eta D \mathfrak{R}_{\eta^{-1}} : C^\infty(\Omega) \to C^\infty(\Omega)$, with D being the total derivative, and the operator $\nabla_\eta = \mathfrak{R}_\eta \nabla \mathfrak{R}_{\eta^{-1}}$ where ∇ is the gradient. Let

$$K(\eta) = \mathfrak{R}_\eta \Delta \mathfrak{R}_{\eta^{-1}} : C_0^\infty(\Omega) \to C^\infty(\Omega).$$

Namely, $K(\eta) = \Delta_\eta$, and

$$K(\eta)^{-1} = \mathfrak{R}_\eta \Delta^{-1} \mathfrak{R}_{\eta^{-1}}.$$

Hence, (2.9) can be written as

$$\begin{aligned} \ddot{\eta} =& \nabla_\eta K(\eta)^{-1} \mathrm{tr}(D_\eta \dot{\eta})^2 - \nabla_\eta K(\eta)^{-1} \mathrm{tr}(\eta^j_{,kl} \mathfrak{R}_\eta((\eta^{-1})^l_{,i} H_0^k))^2 \\ & - \nabla_\eta K(\eta)^{-1} \mathrm{tr}(\eta^j_{,kl} \mathfrak{R}_\eta(H_0^k H_0^l)_{,i}) - \nabla_\eta K(\eta)^{-1} \mathrm{tr}(\eta^j_{,k} \mathfrak{R}_\eta H^k_{0,i})^2 \\ & + \eta_{,nm} \mathfrak{R}_\eta((\eta^{-1})^m_{,l} H_0^l H_0^n) + \eta_{,n} \mathfrak{R}_\eta(H^n_{0,l} H_0^l). \end{aligned} \tag{2.10}$$

The details of derivation of (2.10) can be found in [1].

2.5. Linearization

Consider a family of solutions to (2.10) parameterized by s, call it $\zeta(t,s)$. Suppose that ζ is differentiable in s and set

$$w(t) = \partial_s \zeta(t,s)|_{s=0},$$

the tangent of $\zeta(t,s)$ at $s = 0$. Denote $\zeta(t,0)$ by $\zeta(t)$. Then one has

$$\ddot{w}(t) = DZ(\zeta(t), \dot{\zeta}(t))(w(t), \dot{w}(t)). \tag{2.11}$$

Set $u = \mathfrak{R}_{\zeta^{-1}} w$, $w = u \circ \zeta$. Clearly $K(\zeta)^{-1}$ is inverse to $K(\zeta)$ if the domain of $K(\zeta)$ is $C_0^\infty(\Omega)$. If this domain is enlarged to $C^\infty(\Omega)$, $K(\zeta)^{-1}$ is only a right inverse. That is $K(\zeta)K(\zeta)^{-1} = I$, but $K(\zeta)^{-1}K(\zeta) = I - \mathcal{H}(\zeta)$, where $\mathcal{H}(\zeta)$ is defined as follows. For $\zeta = Id$, the identity, $\mathcal{H}(\zeta)$ projects a function onto its harmonic part. Thus, if $\mathcal{H}(Id)f = g$ then $f = g$ on $\partial\Omega$ and $\Delta g = 0 = K(Id)g$. For arbitrary ζ, $\mathcal{H}(\zeta)f = g$ if $f = g$ on $\partial\Omega$ and $K(\zeta)g = 0$. It yields

$$\ddot{w} = [u \cdot \nabla, \nabla\Delta^{-1}]_\zeta \mathrm{tr}(D_\zeta \dot{\zeta})^2 - \nabla_\zeta (\mathcal{H}(u \cdot \nabla)\Delta^{-1})_\zeta \mathrm{tr}(D_\zeta \dot{\zeta})^2$$

$$
\begin{aligned}
&+ \nabla_\zeta \Delta_\zeta^{-1} \left(2\mathrm{tr}\left(-(Du\circ\zeta)((D\zeta)^{-1}D\dot\zeta)^2 + (D\zeta)^{-1}D\dot w(D\zeta)^{-1}D\dot\zeta\right)\right)\\
&- [u\cdot\nabla, \nabla\Delta^{-1}]_\zeta \mathrm{tr}(\zeta^j_{,kl}\mathfrak{R}_\zeta((\zeta^{-1})^l_{,i}H_0^k))^2\\
&+ \nabla_\zeta(\mathcal{H}(u\cdot\nabla)\Delta^{-1})_\zeta \mathrm{tr}(\zeta^j_{,kl}\mathfrak{R}_\zeta((\zeta^{-1})^l_{,i}H_0^k))^2\\
&- \nabla_\zeta\Delta_\zeta^{-1}(2\zeta^i_{,k'l'}\mathfrak{R}_\zeta((\zeta^{-1})^{l'}_{,j}H_0^{k'})w^j_{,kl}\mathfrak{R}_\zeta((\zeta^{-1})^l_{,i}H_0^k))\\
&+ \nabla_\zeta\Delta_\zeta^{-1}(2\zeta^i_{,k'l'}\zeta^j_{,kl}\mathfrak{R}_\zeta((\zeta^{-1})^{l'}_{,j}H_0^{k'}(\zeta^{-1})^l_{,n}u^n_{,i}H_0^k))\\
&- \nabla_\zeta\Delta_\zeta^{-1}(2\zeta^i_{,k'l'}\zeta^j_{,kl}\mathfrak{R}_\zeta((\zeta^{-1})^{l'}_{,j}H_0^{k'}(\zeta^{-1})^l_{,i}H^k_{0,m}u^m))\\
&- [u\cdot\nabla, \nabla\Delta^{-1}]_\zeta(\zeta^j_{,kl}\mathfrak{R}_\zeta(H_0^kH_0^l)_{,j}) + \nabla_\zeta(\mathcal{H}(u\cdot\nabla)\Delta^{-1})_\zeta(\zeta^j_{,kl}\mathfrak{R}_\zeta(H_0^kH_0^l)_{,j})\\
&- \nabla_\zeta\Delta_\zeta^{-1}(w^i_{,kl}\mathfrak{R}_\zeta(H_0^kH_0^l)_{,i}) - \nabla_\zeta\Delta_\zeta^{-1}(\zeta^i_{,kl}\mathfrak{R}_\zeta((H_0^kH_0^l)_{,im}u^m))\\
&- [u\cdot\nabla, \nabla\Delta^{-1}]_\zeta\mathrm{tr}(\zeta^j_{,k}\mathfrak{R}_\zeta H^k_{0,i})^2 + \nabla_\zeta(\mathcal{H}(u\cdot\nabla)\Delta^{-1})_\zeta\mathrm{tr}(\zeta^j_{,k}\mathfrak{R}_\zeta H^k_{0,i})^2\\
&- \nabla_\zeta\Delta_\zeta^{-1}(2\zeta^i_{,k'}w^j_{,k}\mathfrak{R}_\zeta(H^{k'}_{0,j}H^k_{0,i})) - \nabla_\zeta\Delta_\zeta^{-1}(2\zeta^i_{,k'}\zeta^j_{,k}\mathfrak{R}_\zeta(H^{k'}_{0,j}H^k_{0,il}u^l))\\
&+ \zeta_{,nm}\mathfrak{R}_\zeta((\zeta^{-1})^m_{,l}H_0^lH_0^n) + \zeta_{,n}\mathfrak{R}_\zeta(H^n_{0,l}H_0^l).
\end{aligned}
$$

Therefore, it turns out

$$
\begin{aligned}
\ddot w =&(1-b^2)([u\cdot\nabla,\nabla\Delta^{-1}]_\eta(-2) - \nabla_\eta(\mathcal{H}(u\cdot\nabla)\Delta^{-1})_\eta(-2)) &\text{(2.12a)}\\
&+ \nabla_\eta\Delta_\eta^{-1}\left(2\mathrm{tr}\left(Du\circ\eta + \mathrm{i}e^{-\mathrm{i}t}D\dot w\right)\right) &\text{(2.12b)}\\
&- \nabla_\eta\Delta_\eta^{-1}(w^i_{,kl}\mathfrak{R}_\eta(H_0^kH_0^l)_{,i}) &\text{(2.12c)}\\
&- e^{\mathrm{i}t}\nabla_\eta\Delta_\eta^{-1}(2w^j_{,k}\mathfrak{R}_\eta(H^i_{0,j}H^k_{0,i})) &\text{(2.12d)}\\
&+ b^2e^{\mathrm{i}t}\eta. &\text{(2.12e)}
\end{aligned}
$$

2.6. Special solutions of (2.12).

Assume that $w = \nabla f(t)\circ\eta$ for a harmonic function $f(t)$ on Ω so that $u = \nabla f$ a harmonic gradient. Thus, $Dw = D\eta(D\nabla f)\circ\eta$ and $\mathrm{tr}(\mathrm{i}e^{-\mathrm{i}t}D\dot w) = 0$, so (2.12b) , (2.12c) and (2.12d) vanish. Thus,

$$(\nabla f\circ\eta)^{\cdot\cdot} = (1-b^2)x\cdot\nabla(\nabla f\circ\eta) + b^2e^{\mathrm{i}t}\eta, \tag{2.13}$$

where $x\cdot\nabla$ is the radial derivative. The following lemma in [13] is crucial.

Lemma 2.1 ([13]). *If f is harmonic, and $\eta(t,z) = e^{\mathrm{i}t}z$, then there exists a harmonic function g such that $\nabla f\circ\eta = \nabla g$.*

In view of this, we may rewrite (2.13) as

$$(\nabla g)^{\cdot\cdot} = (1-b^2)x\cdot\nabla(\nabla g) + b^2e^{2\mathrm{i}t}z. \tag{2.14}$$

Set $\mathcal{A}$ by $\mathcal{A}\nabla g = (x\cdot\nabla)\nabla g$. Let $g(z) = \mathrm{Re}\, z^n$ $(n\geqslant 1)$, $\nabla g = n\bar z^{n-1}$. One then has $\mathcal{A}\nabla g = nz\cdot\nabla\bar z^{n-1} = n(n-1)\bar z^{n-1} = (n-1)\nabla g$. Similarly, $\mathcal{A}\nabla g = (n-1)\nabla g$ if $g(z) = \mathrm{Re}\,\mathrm{i}z^n$ $(n\geqslant 1)$, $\nabla g = -\mathrm{i}n\bar z^{n-1}$. It is worth noting that $E := \{n\bar z^{n-1}, \mathrm{i}n\bar z^{n-1}\}_{n=1}^\infty$ forms a basis of the set of Harmonic gradients on Ω . Therefore, $\mathcal{A}$ has this set as a complete set of eigenfunctions and has double eigenvalues $0, 1, 2, \cdots$. By separating variables of the form $\nabla g(t,z) = \sigma(t)h(z)$ for $h\in E$, we write (2.14) as

$$\ddot\sigma(t) = (1-b^2)(n-1)\sigma(t) + b^2e^{2\mathrm{i}t}z/h(z). \tag{2.15}$$

Let $B = \sqrt{(1-b^2)(n-1)}$. The usual solution to (2.15) is of the form

$$\sigma(t) = C_1 e^{Bt} + C_2 e^{-Bt} - \frac{b^2 z}{2Bh(z)}\left(\frac{2B}{B^2+4}e^{2it} - \frac{e^{-Bt}}{B+2i} - \frac{e^{Bt}}{B-2i}\right),$$

so that the solution of (2.14) is of

$$w_n(t,z) = C_1 e^{Bt} n\bar{z}^{n-1} + C_2 e^{-Bt} n\bar{z}^{n-1} - \frac{b^2 z}{2B}\left(\frac{2B}{B^2+4}e^{2it} - \frac{e^{-Bt}}{B+2i} - \frac{e^{Bt}}{B-2i}\right).$$

Assuming the initial conditions $w_n(0) = 0$ and $\dot{w}_n(0) = e^{-n^{1/4}}\bar{z}^n$, then for $n \geqslant 2$,

$$w_n(t) = \frac{1}{B}e^{-n^{1/4}}\sinh(Bt)\bar{z}^n + \frac{b^2 z}{B^2+4}\left(\cosh(Bt) + \frac{2i}{B}\sinh(Bt) - e^{2it}\right)$$

is a sequence of solutions to (2.12). When $b^2 < 1$, this sequence is as useful as for the Euler equation, discussed in [13], the initial data go to zero in $C^\infty(\Omega)$, but for any $t > 0$, $\{w_n(t)\}_{n=2}^\infty$ is unbounded in $C^\infty(\Omega)$.

2.7. Construction of the sequences of initial data and solutions

Let $\eta(t,z) = e^{it}z$, the solution to (2.8) given above, and set $\zeta_n(0,z) = \eta(0,z) = z$ and $\dot{\zeta}_n(0,z) = \dot{\eta}(0,z) + e^{-n^{1/4}}\bar{z}^n$. Then, $(\zeta_n(0,z), \dot{\zeta}_n(0,z)) \to (\eta(0,z), \dot{\eta}(0,z))$ in $C^\infty(\Omega)\times C^\infty(\Omega)$ as $n \to \infty$. suppose that there exists some positive T such that for all n, $\zeta_n(t)$ is the unique solution of (2.8) for $0 \leqslant t \leqslant T$, the goal is to show that $\zeta_n(t)$ does not converge to $\eta(t)$, not in $C^\infty(\Omega)$ for any positive $t \leqslant T$. Set

$$y_n(t) = \zeta_n(t) - \eta(t). \tag{2.16}$$

In view of (2.9), one has

$$\begin{aligned}\ddot{y}(t) =& Z_{,j}(\eta,\dot{\eta})(y,\dot{y})^j + \int_0^1 (1-s)\left(\int_0^s Z_{,jk}(\zeta(\sigma),\dot{\zeta}(\sigma))(y,\dot{y})^k d\sigma\right)(y,\dot{y})^j ds \\ =:& DZ(\eta,\dot{\eta})(y,\dot{y}) + \int_0^1 (1-s)\left(\int_0^s D^2 Z(\zeta(\sigma),\dot{\zeta}(\sigma))((y,\dot{y}),(y,\dot{y}))d\sigma\right)ds,\end{aligned} \tag{2.17}$$

where $\zeta(\sigma) = \zeta + \sigma(\eta - \zeta)$, and we suppress the subscript “n” in ζ for simplicity.

2.8. The estimates of the integrand in (2.17)

The following Proposition and Lemmas are proved in [1].

Proposition 2.2. *Let $s \geqslant 1$ and $H_0 \in H^{s+2}$. Then,*

$$\begin{aligned}\|D^2 Z(\zeta,\dot{\zeta})((y,\dot{y}),(y,\dot{y}))\|_s \leqslant & C\|(y,\dot{y})\|_1^{(2s-1)/2s}\|D^{s+1}(y,\dot{y})\|_0^{(2s+1)/2s} \\ & + C\|(y,\dot{y})\|_s\|D^{s+1}(y,\dot{y})\|_0 + Cb^2\|y\|_{s+2},\end{aligned}$$

where C is uniform for all $(\zeta,\dot{\zeta})$ in a H^{s+3} neighborhood of the curve $(\eta(t),\dot{\eta}(t))$.

Lemma 2.3. *[1] Let $m > l \geqslant 1$ be an integer satisfying $\|D^{m+1}y\|_0 \leqslant C_m\|Dy\|_0$. Then, it holds*

$$\|y\|_{l+1} \leqslant C\|y\|_1.$$

Corollary 2.4. *Let $m > s$ be an integer satisfying $\|D^m(y,\dot{y})\|_0 \leqslant C_m\|(y,\dot{y})\|_0$. Then, it holds*

$$\|D^2 Z(\zeta,\dot{\zeta})((y,\dot{y}),(y,\dot{y}))\|_s \leqslant C\|(y,\dot{y})\|_0^{2-1/m}\|D^m(y,\dot{y})\|_0^{1/m} + Cb^2\|y\|_{s+2}.$$

2.9. Decomposition of solutions

In [1], we decompose $y = y_n$ into three parts as follows. Let $q := y - \nabla\Delta^{-1}\mathrm{div}\, y$, the divergence free part of y, and h be a harmonic function satisfying

$$\langle \nabla h, \nu\rangle = \langle q, \nu\rangle, \text{ on } \partial\Omega.$$

Define

$$N = y - \nabla h.$$

Since h is a harmonic function, there exists a holomorphic function $\varphi(z)$ such that h is the real part of $\varphi(z)$. Hence, $h = \mathrm{Re}\, \sum_{j=0}^{\infty} a_j z^j$. Set $g = \mathrm{Re}\, \sum_{j=0}^{n-1} a_j z^j$ and $f = h - g = \mathrm{Re}\, \sum_{j=n}^{\infty} a_j z^j$. Then y can be decomposed as

$$y = \nabla f + \nabla g + N. \tag{2.18}$$

The projection onto the i-th summand of (2.18) is denoted by $\mathbb{P}_i$, $i = 1, 2, 3$.

Set

$$Q = Q(y, \dot{y}) := \int_0^1 (1 - s)\left(\int_0^s D^2 Z(\zeta(\sigma), \dot{\zeta}(\sigma))((y, \dot{y}), (y, \dot{y}))d\sigma\right) ds,$$

and $Q_i = \mathbb{P}_i Q$ for $i = 1, 2, 3$. Then (2.17) can be written as

$$\ddot{y} = DZ(\eta, \dot{\eta})(y, \dot{y}) + Q. \tag{2.19}$$

It follows from (2.12),

$$DZ(\eta, \dot{\eta})(y, \dot{y}) = -(1 - b^2)y + \tilde{\mathcal{A}}y + (\nabla\Delta^{-1})_\eta \mathrm{tr} M + b^2 e^{\mathrm{i}t}\eta,$$

where $\tilde{\mathcal{A}}y = (\nabla\mathcal{H})_\eta(\langle y, \eta\rangle)$ which depends on η, and

$$M = 2D_\eta N + 2\mathrm{i}e^{-\mathrm{i}t} D\dot{N} - N^i_{,kl}\mathfrak{R}_\eta(H_0^k H_0^l)_{,i'} - 2e^{\mathrm{i}t} N^j_{,k}\mathfrak{R}_\eta(H^i_{0,j'} H^k_{0,i}). \tag{2.20}$$

Hence,

$$\ddot{y} = (b^2 - 1)y + \tilde{\mathcal{A}}y + (\nabla\Delta^{-1})_\eta \mathrm{tr} M + b^2 e^{\mathrm{i}t}\eta + Q. \tag{2.21}$$

Apply $\mathbb{P}_3$ to (2.21) to obtain, by noticing that $\mathbb{P}_3\tilde{A}y = 0$, $\mathbb{P}_3(\nabla\mathcal{H})_\eta = 0$, $(\mathbb{P}_1 + \mathbb{P}_2)(\nabla\Delta^{-1})_\eta = 0$ and $\mathbb{P}_3\eta = \eta$, in view of (2.20),

$$\begin{aligned}\ddot{N} =&(b^2 - 1)N + 2(\nabla\Delta^{-1})_\eta \mathrm{div}_{\,\eta} N + 2\mathrm{i}e^{-\mathrm{i}t}(\nabla\Delta^{-1})_\eta \mathrm{tr}(D\dot{N}) + b^2 e^{\mathrm{i}t}\eta + Q_3\\ &- (\nabla\Delta^{-1})_\eta (N^i_{,kl}\mathfrak{R}_\eta(H_0^k H_0^l)_{,i}) - 2e^{\mathrm{i}t}(\nabla\Delta^{-1})_\eta (N^j_{,k}\mathfrak{R}_\eta(H^i_{0,j} H^k_{0,i}))\\ =:&(b^2 - 1)N + B_1 N + B_2\dot{N} + b^2 e^{\mathrm{i}t}\eta + Q_3 + B_3 N + B_4 N,\end{aligned} \tag{2.22}$$

Applying $\mathbb{P}_1 + \mathbb{P}_2$ to (2.21) yields

$$\ddot{\nabla} f + \ddot{\nabla} g =(b^2 - 1)(\nabla f + \nabla g) + \tilde{\mathcal{A}}(\nabla f + \nabla g) + \tilde{\mathcal{A}}N + Q_1 + Q_2.$$

Set $\tilde{\mathcal{A}}_j = \mathbb{P}_j\tilde{\mathcal{A}}$ for $j = 1$ or 2. The fact $\mathbb{P}_j\eta = \eta$ for $j = 1$ or 2 due to $\mathbb{P}_3\eta = \eta$ yields

$$\ddot{\nabla} f =(1 - b^2)\mathcal{A}\nabla f + \tilde{\mathcal{A}}_1 N + Q_1, \tag{2.23}$$

$$\ddot{\nabla} g =(1 - b^2)\mathcal{A}\nabla g + \tilde{\mathcal{A}}_2 N + Q_2. \tag{2.24}$$

In this way, we have decomposed (2.19) into (2.22), (2.23) and (2.24).

2.10. Estimates of ∇f, ∇g and N

Let $y_n = \nabla f_n + \nabla g_n + N_n$ be the sequence of solutions with the initial data $y_n(0) = 0$, $\dot{y}_n(0) = e^{-n^{1/4}}\bar{z}^n$. For any harmonic function h and any real s, let

$$\|\nabla h\|_s = (\mathcal{A}^s\nabla h, \mathcal{A}^s\nabla h)^{\frac{1}{2}}, \tag{2.25}$$

where $\mathcal{A}^s$ is defined by $\mathcal{A}^s\bar{z}^k = (k-1)^s\bar{z}^k$. Then, for any s,

$$\|\mathcal{A}\nabla g\|_s \leqslant (n-1)\|\nabla g\|_s. \tag{2.26}$$

For $\mu, \nu \geqslant 1$ and $\sigma \geqslant 2$, set

$$\begin{aligned} E^{\pm}_{\mu,b} =& \|\dot{\nabla} f \pm \sqrt{1-b^2}\mathcal{A}^{\frac{1}{2}}\nabla f\|^2_\mu = \|\mathcal{A}^\mu(\dot{\nabla} f \pm \sqrt{1-b^2}\mathcal{A}^{\frac{1}{2}}\nabla f)\|^2_0, \\ E_{\mu,b} =& E^+_{\mu,b} + E^-_{\mu,b}, \\ F_\sigma =& n\|N\|^2_\sigma + \|\dot{N}\|^2_\sigma, \\ G_\nu =& \|\dot{\nabla} g\|^2_\nu + \|\mathcal{A}^{\frac{1}{2}}\nabla g\|^2_\nu. \end{aligned} \tag{2.27}$$

Then we have

$$E^{\pm}_{\mu,b} \geqslant n^{2(\mu-\nu)}E^{\pm}_{\nu,b}, \quad \text{for } \mu \geqslant \nu. \tag{2.28}$$

The following results which lead to the ill-posedness was proved in [1].

Proposition 2.5. *Let $\mu \geqslant 2$. For sufficiently large n, the set $E^+_{\mu,b} \geqslant E^-_{\mu,b}$, $E^+_{\mu,b} \geqslant \sqrt{n}F_{\mu+1}$, and $E^+_{\mu+\frac{1}{4},b} \geqslant \frac{2\sqrt{1-b^2}}{(2-b^2)}n^{3/4}G_\mu$ is invariant under the evolution defined by* (2.22), (2.23) *and* (2.24). *Of course $E_{\mu,b} \leqslant 2E^+_{\mu,b}$.*

Theorem 2.6 ([1]). *Let $\mu \geqslant 2$ and $|b| \ll 1$. For large n, $E^+_{\mu,b}(t) \geqslant E^+_{\mu,b}(0)e^{\sqrt{1-b^2}\sqrt{nt}}$ for E, F and G in the invariant set of Proposition 2.5.*

2.11. Ill-posedness

Theorem 2.7 ([1]). *Suppose $|b| \ll 1$. For the initial data $\Omega_0 = \{z \in \mathbb{C}, |z| \leqslant 1\}$, $\zeta_n(0) = z$, $\dot{\zeta}_n(0) = e^{-n^{1/4}}\bar{z}^n + \mathrm{i}z$ $(n \geqslant 2)$ and $H_0(z) = \mathrm{i}bz$ for $z \in \Omega_0$, $\zeta_n(t)$ $(n \geqslant 2)$ be the solution to problem* (2.10) *in some time interval* $[0, T]$. *Then, for $\mu \geqslant 2$, $\|(y_n(0), \dot{y}_n(0))\|_\mu \to 0$ as $n \to \infty$, but for any $t > 0$,*

$$\|(y_n(t), \dot{y}_n(t))\|_\mu \to \infty, \ \ \textit{as} \ \ n \to \infty,$$

where $y_n(t) = \zeta_n(t) - \eta(t)$ and $\eta(t, z) = e^{\mathrm{i}t}z$ is a special solution to (2.10).

Main idea of the proof in [1]: Decompose y_n into $\nabla f + \nabla g + N$, then we define $E^{\pm}_{\mu,b}$, F_μ and G_μ which are in the invariant set of Proposition 2.5 at time $t = 0$. Hence, $E^+_{\mu,b}(t) \geqslant E^+_{\mu,b}(0)e^{\sqrt{1-b^2}\sqrt{nt}}$, by Theorem 2.6. However,

$$E^+_{\mu,b}(0) = \frac{2\pi(n-1)^{2\mu}}{2n+1}e^{-2n^{1/4}}.$$

Thus,

$$E^{+}_{\mu,b}(t) \geqslant \frac{2\pi(n-1)^{2\mu}}{2n+1} e^{\sqrt{1-b^2}\sqrt{n}t - 2n^{1/4}},$$

which tends to ∞ for any $t > 0$ as $n \to \infty$. Therefore,

$$\|(y_n(t), \dot{y}_n(t))\|_\mu \to \infty$$

for $t > 0$.

3. Well-posedness for the linearized problem with Taylor sign condition

In this section, we present the main results and the key ideas of the proofs in [2] for the well-posedness of the free boundary problem of ideal incompressible MHD in general n-dimensions ($n \geq 2$) when the Taylor sign condition is satisfied. Details for some derivations can be found in [2].

3.1. Lagrangian coordinates

For the velocity field v, Lagrangian coordinates $x = x(t, y) = f_t(y)$ are given by

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega. \tag{3.1}$$

In this setting, $f_t : \Omega \to \Omega_t$ is a volume-preserving diffeomorphism because of $\operatorname{div} v = 0$, and the free boundary becomes fixed in the new y-coordinates. For simplicity, we take f_0 the identity operator, that is, $x(0, y) = y$ and Ω is just the unit ball. For convenience, the letters a, b, c, d, e, and f will refer to quantities in the Lagrangian frame, whereas the letters i, j, k, l, m, and n will refer to ones in the Eulerian frame, e.g., $\partial_a = \partial/\partial y^a$ and $\partial_i = \partial/\partial x^i$.

Let

$$D_t = \partial_t + v^k \partial_k, \quad \partial_k = \frac{\partial}{\partial x^k} = \frac{\partial y^a}{\partial x^k}\frac{\partial}{\partial y^a}. \tag{3.2}$$

Similar to (2.6), we have

$$H^j(t, x(t, y)) = \bar{H}^a_0(y)\frac{\partial x^j(t, y)}{\partial y^a}, \tag{3.3}$$

where $\bar{H}^a_0(y) = H^a_0(x(0, y))$. Therefore,

$$H^k \partial_k H^i = \bar{H}^a_0 \frac{\partial x^k}{\partial y^a}\frac{\partial y^c}{\partial x^k}\partial_c(\bar{H}^b_0 \frac{\partial x^i}{\partial y^b}) = \bar{H}^a_0 \partial_a(\bar{H}^b_0 \partial_b x^i),$$

For convenience, we set

$$B := B^a(y)\frac{\partial}{\partial y^a}, \text{ with } B^a(y) := \sqrt{\mu}\bar{H}^a_0(y),$$

then we may write (1.1)–(1.5) as

$$\begin{cases} D_t^2 x^i + \partial_i P = B^2 x^i, & \text{in } [0, T] \times \Omega, \\ \kappa := \det\left(\frac{\partial x}{\partial y}\right) = 1, & \text{in } [0, T] \times \Omega, \\ P = 0, & \text{on } \Gamma, \end{cases} \tag{3.4}$$

where $P = P(t,y) = p(t, x(t,y))$, ∂_i is thought of as the differential operator in y given in (3.2) and D_t is the time derivative. The initial conditions become

$$x|_{t=0} = y, \quad D_t x|_{t=0} = v_0, \tag{3.5}$$

satisfying the constraint div $v_0 = 0$. Taking the divergence of (3.4) to obtain

$$\Delta P = -(\partial_i D_t x^k)(\partial_k D_t x^i) + \partial_i(B^2 x^i). \tag{3.6}$$

Condition (1.7) becomes

$$\nabla_N P \leqslant -c_0 < 0, \text{ on } \Gamma, \tag{3.7}$$

where N is the exterior unit normal to Γ_t parameterized by $x(t,y)$.

3.2. Linearization

Denote δ the variation w.r.t. certain parameter r in the Lagrangian coordinates:

$$\delta = \frac{\partial}{\partial r}\bigg|_{(t,y)=\text{const}}. \tag{3.8}$$

Think of $x(t,y)$ and $P(t,y)$ as depending on r and differentiating with respect to r, say, $\bar{x}(t,y,r)$ and $\bar{P}(t,y,r)$ respectively. Namely, $(\bar{x},\bar{P})|_{r=0} = (x,P)$. Differentiating (3.2) and using the formula for the derivative of the inverse of a matrix, $\delta M^{-1} = -M^{-1}(\delta M)M^{-1}$, one has the commutator

$$[\delta, \partial_i] = -(\partial_i \delta x^k)\partial_k. \tag{3.9}$$

Let

$$(\delta x, \delta P) = \left(\frac{\partial \bar{x}}{\partial r}, \frac{\partial \bar{P}}{\partial r}\right)\bigg|_{r=0}, \tag{3.10}$$

satisfying div $\delta x = 0$ and $\delta P|_\Gamma = 0$.

Thus,

$$D_t^2 \delta x^i + \partial_i \delta P - \partial_i(\delta x^k \partial_k P) - \delta x^k(\partial_k D_t v^i - \partial_k(B^2 x^i)) - B^2 \delta x^i = 0. \tag{3.11}$$

Set

$$W^a = \delta x^i \frac{\partial y^a}{\partial x^i}, \quad \delta x^i = W^b \frac{\partial x^i}{\partial y^b}, \quad q = \delta P. \tag{3.12}$$

Let g be the metric δ_{ij} expressed in the Lagrangian coordinates, i.e.,

$$g_{ab} = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}, \tag{3.13}$$

and g^{ab} be the inverse of g_{ab}. Then the time derivatives of the metric and the vorticity in the Lagrangian coordinates, respectively, are given by

$$\dot{g}_{ab} = D_t g_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}(\partial_k v_i + \partial_i v_k), \text{ and } \omega_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}(\partial_i v_k - \partial_k v_i). \tag{3.14}$$

One therefore has

$$\begin{aligned} D_t^2 W^d &+ g^{da}(\dot{g}_{ab} - \omega_{ab})D_t W^b - g^{da}\partial_a(W^c\partial_c P) + g^{da}\partial_a q \\ &+ g^{da}\delta_{il}\partial_a x^l[W^c\partial_c(B^2 x^i) - B^2(W^c\partial_c x^i)] = 0. \end{aligned} \tag{3.15}$$

The vector field B can be regarded as a tangential derivative since $B = B^a\partial_a$ is time independent and $\partial_a B^a = 0$. Thus, there is an advantage to use the Lie derivative corresponding to B given by

$$\mathcal{L}_B W^a = BW^a - \partial_b B^a W^b, \tag{3.16}$$

which is divergence-free since $\operatorname{div} \mathcal{L}_B W = \partial_a(B^b\partial_b W^a - \partial_b B^a W^b) = 0$ if $\operatorname{div} W = 0$. One also has

$$\mathcal{L}_B \partial_c x^i = B\partial_c x^i + \partial_c B^d \partial_d x^i. \tag{3.17}$$

Denote

$$\dot{W}^a(t, y) := D_t W^a(t, y), \quad \ddot{W}^a := D_t^2 W^a. \tag{3.18}$$

Since $q = \delta P$, one has $q|_\Gamma = 0$. One thus has the following system, in view of (3.15) and (1.3),

$$\begin{cases} \ddot{W}^d - \mathcal{L}_B^2 W^d + g^{da}\partial_a q - g^{da}\partial_a(W^c\partial_c P) + g^{da}(\dot{g}_{ab} - \omega_{ab})\dot{W}^b \\ \qquad - 2g^{da}\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c = 0, \\ \operatorname{div} W = \kappa^{-1}\partial_a(\kappa W^a) = 0, \\ q|_\Gamma = 0, \\ W|_{t=0} = W_0, \ \dot{W}|_{t=0} = W_1, \end{cases} \tag{3.19}$$

where $\operatorname{div} W_0 = \operatorname{div} W_1 = 0$.

3.3. The equation of Δq

(3.19) can be expressed in one equation, because $q = \delta P$ is determined as a functional of W and $\dot{W}$. In this setting, one can have an elliptic equation for q. For this, one has to derive $\operatorname{div} \ddot{W}$ first. Denote

$$u^a := \frac{\partial y^a}{\partial x^i} v^i, \text{ and } u_a = g_{ab} u^b.$$

From $\operatorname{div} W = 0$, one has that $\operatorname{div} \ddot{W} = 0$. Take the divergence of (3.19) to obtain

$$\begin{cases} \Delta q = \partial_d(g^{da}\partial_a(W^c\partial_c P) - g^{da}(\dot{g}_{ab} - \omega_{ab})\dot{W}^b + 2g^{da}\delta_{il}\partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c), \\ q|_\Gamma = 0, \end{cases} \tag{3.20}$$

since div $\mathcal{L}_B^2 W = 0$. We separate q into four parts:

$$q = \sum_{i=1}^{4} q_i,$$

where q_i's satisfy the following Dirichlet problems of Poisson equations:

$$\begin{cases} \Delta q_1 = \Delta(W^c \partial_c P), & q_1|_\Gamma = 0, \\ \Delta q_2 = -\partial_d(g^{da} \dot{g}_{ab} \dot{W}^b), & q_2|_\Gamma = 0, \\ \Delta q_3 = \partial_d(g^{da} \omega_{ab} \dot{W}^b), & q_3|_\Gamma = 0, \\ \Delta q_4 = 2\partial_d(g^{da} \delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c), & q_4|_\Gamma = 0. \end{cases}$$

In this setting, (3.19) becomes

$$L_1 W := \ddot{W} - \mathcal{L}_B^2 W + \mathcal{A} W + \dot{\mathcal{G}} \dot{W} - C \dot{W} + \mathcal{X} \mathcal{L}_B W = 0,$$

where

$$\begin{aligned} \mathcal{A} W^d &:= -g^{da} \partial_a (\partial_c P W^c - q_1), \\ \dot{\mathcal{G}} \dot{W}^d &:= g^{da} (\dot{g}_{ab} \dot{W}^b + \partial_a q_2), \\ C \dot{W}^d &:= g^{da} (\omega_{ab} \dot{W}^b - \partial_a q_3), \\ \mathcal{X} \mathcal{L}_B W^d &:= -2 g^{da} \delta_{il} \partial_a x^l \mathcal{L}_B \partial_c x^i \mathcal{L}_B W^c + g^{da} \partial_a q_4. \end{aligned}$$

3.4. Lie derivatives

Lie derivative of the vector field W with respect to the vector field T is given by

$$\mathcal{L}_T W^a = T W^a - (\partial_c T^a) W^c. \tag{3.21}$$

For those vector fields, it holds div $T = 0$, so div $W = 0$ implies that

$$\operatorname{div} \mathcal{L}_T W = T \operatorname{div} W - W \operatorname{div} T = 0.$$

The Lie derivative of a 1-form:

$$\mathcal{L}_T \alpha_a = T \alpha_a + (\partial_a T^c) \alpha_c.$$

An advantage to use Lie derivatives is that they also commute with the exterior differentiation, $[\mathcal{L}_T, d] = 0$, and

$$\mathcal{L}_T \partial_a q = \partial_a T q, \tag{3.22}$$

for any function q.

For a 2-form β, the Lie derivative is given by

$$\mathcal{L}_T \beta_{ab} = T \beta_{ab} + (\partial_a T^c) \beta_{cb} + (\partial_b T^c) \beta_{ac}. \tag{3.23}$$

In local coordinate notation, the Lie derivative of a (r, s) tensor field β along T is given by

$$\begin{aligned}\mathcal{L}_T\beta^{a_1\ldots a_r}{}_{b_1\ldots b_s} =& T\beta^{a_1\ldots a_r}{}_{b_1\ldots b_s}\\ &- (\partial_c T^{a_1})\beta^{ca_2\ldots a_r}{}_{b_1\ldots b_s} - \ldots - (\partial_c T^{a_r})\beta^{a_1\ldots a_{r-1}c}{}_{b_1\ldots b_s}\\ &+ (\partial_{b_1} T^{c})\beta^{a_1\ldots a_r}{}_{cb_2\ldots b_s} + \ldots + (\partial_{b_s} T^{c})\beta^{a_1\ldots a_r}{}_{b_1\ldots b_{s-1}c}.\end{aligned} \tag{3.24}$$

If w is a 1-form and $\operatorname{curl} w_{ab} = dw_{ab} = \partial_a w_b - \partial_b w_a$, then

$$\mathcal{L}_T \operatorname{curl} w_{ab} = \operatorname{curl} \mathcal{L}_T w_{ab}. \tag{3.25}$$

Relation on the commutator of two Lie derivatives is

$$[\mathcal{L}_T, \mathcal{L}_B]W^a = \mathcal{L}_{[T,B]}W^a. \tag{3.26}$$

Commutator of Lie derivative and ∂_a satisfies

$$[\mathcal{L}_T, \partial_a]W^b = W^d\partial_d\partial_a T^b. \tag{3.27}$$

Furthermore, we set

$$\mathcal{L}_{D_t} = D_t. \tag{3.28}$$

We also have

$$[\mathcal{L}_{D_t}, \mathcal{L}_T] = 0. \tag{3.29}$$

3.5. Tangential vector fields and the div-curl decomposition

Definition 3.1. Let c_1 be a constant satisfying

$$\sum_{a,b}(|g_{ab}| + |g^{ab}|) \leqslant c_1^2, \quad \left|\frac{\partial x}{\partial y}\right|^2 + \left|\frac{\partial y}{\partial x}\right|^2 \leqslant c_1^2,$$

and let K_1 denote a continuous function of c_1.

Since Ω is the unit ball in $\mathbb{R}^n$, we can express the vector fields explicitly. The rotation vector fields

$$y^a\partial_b - y^b\partial_a$$

span the tangent space of the boundary and are divergence-free in the interior. Clearly $B = B^a\partial_a$ belongs to this space. They also span the tangent space of the level sets of the distance function from the boundary in the Lagrangian coordinates $d(y) = \operatorname{dist}(y, \Gamma) = 1 - |y|$ for $y \neq 0$ away from the origin. This set of vector fields is denoted by $\mathcal{S}_0$. Thus, $B \in \mathcal{S}_0$. We define several Vector fields as follows:
$\mathcal{S}_1$: span the tangential space when $d \geqslant d_0$ and are compactly supported in the set where $d \geqslant d_0/2$.
$\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$: the family of space tangential vector fields.
$\mathcal{T} = \mathcal{S} \cup \{D_t\}$: the family of space-time tangential vector fields.
$R = y^a\partial_a$: radial vector field.
$\mathcal{R} = \mathcal{S} \cup \{R\}$: spans the full tangent space of the space everywhere.
$\mathcal{U} = \mathcal{S} \cup \{R\} \cup \{D_t\}$: the family of all vector fields.

Note that

$$[R, S] = 0, \quad S \in \mathcal{S}_0.$$

The following fact is important: the commutators of two vector fields in $\mathcal{S}_0$ is another vector field in $\mathcal{S}_0$. For $i = 0, 1$, let $\mathcal{R}_i = \mathcal{S}_i \cup \{R\}$, $\mathcal{T}_i = \mathcal{S}_i \cup \{D_t\}$ and $\mathcal{U}_i = \mathcal{T}_i \cup \{R\}$.

3.6. Several important estimates

The following estimates derived in [31] play important roles:

Lemma 3.2 ([31, Lemma 11.3]). *In the Lagrangian frame, with $\underline{W}_a = g_{ab}W^b$, we have*

$$|\mathcal{L}_U W| \leqslant K_1 \left(|\mathrm{curl}\, \underline{W}| + |\mathrm{div}\, W| + \sum_{S \in \mathcal{S}} |\mathcal{L}_S W| + [g]_1 |W| \right), \quad U \in \mathcal{R}, \tag{3.30}$$

$$|\mathcal{L}_U W| \leqslant K_1 \left(|\mathrm{curl}\, \underline{W}| + |\mathrm{div}\, W| + \sum_{T \in \mathcal{T}} |\mathcal{L}_T W| + [g]_1 |W| \right), \quad U \in \mathcal{U}, \tag{3.31}$$

where $[g]_1 = 1 + |\partial g|$. Furthermore,

$$|\partial W| \leqslant K_1 \left(|\mathcal{L}_R W| + \sum_{S \in \mathcal{S}} |\mathcal{L}_S W| + |W| \right). \tag{3.32}$$

When $d(y) \leqslant d_0$, we may replace the sums over $\mathcal{S}$ by the sums over $\mathcal{S}_0$ and the sum over $\mathcal{T}$ by the sum over $\mathcal{T}_0$.

One needs to apply the lemma to W replaced by $\mathcal{L}_U^J W$, and the divergence term will vanish in applications. This makes it possible to control the curl of $(\mathcal{L}_U^J \underline{W})_a = \mathcal{L}_U^J (g_{ab} W^b)$.

Definition 3.3. Let β be a function, a 1- or 2-form, or vector field, and let $\mathcal{V}$ be any of our families of vector fields. Set

$$|\beta|_s^{\mathcal{V}} = \sum_{|J| \leqslant s, J \in \mathcal{V}} \left| \mathcal{L}_S^J \beta \right|,$$

$$[\beta]_\mu^{\mathcal{V}} = \sum_{s_1 + \cdots + s_k \leqslant \mu, s_i \geqslant 1} |\beta|_{s_1}^{\mathcal{V}} \cdots |\beta|_{s_k}^{\mathcal{V}}, \quad [\beta]_0^{\mathcal{V}} = 1.$$

In particular, $|\beta|_r^{\mathcal{R}}$ and $|\beta|_r^{\mathcal{U}}$ are equivalent to $\sum_{|\alpha| \leqslant r} \left| \partial_y^\alpha \beta \right|$ and $\sum_{|\alpha| + k \leqslant r} \left| D_t^k \partial_y^\alpha \beta \right|$, respectively.

Lemma 3.4 ([31, Lemma 11.5]). *With the convention that* $|\mathrm{curl}\, \underline{W}|_{-1}^{\mathcal{V}} = |\mathrm{div}\, W|_{-1}^{\mathcal{V}} = 0$*, we have*

$$|W|_r^{\mathcal{R}} \leqslant K_1 \left(|\mathrm{curl}\, \underline{W}|_{r-1}^{\mathcal{R}} + |\mathrm{div}\, W|_{r-1}^{\mathcal{R}} + |W|_r^{\mathcal{S}} + \sum_{s=1}^{r} |g|_s^{\mathcal{R}} |W|_{r-s}^{\mathcal{R}} \right),$$

$$|W|_r^{\mathcal{R}} \leqslant K_1 \sum_{s=1}^{r} [g]_s^{\mathcal{R}} \left(|\mathrm{curl}\, \underline{W}|_{r-1-s}^{\mathcal{R}} + |\mathrm{div}\, W|_{r-1-s}^{\mathcal{R}} + |W|_{r-s}^{\mathcal{S}} \right).$$

The same inequalities also hold with $\mathcal{R}$ replaced by $\mathcal{U}$ everywhere and $\mathcal{S}$ replaced by $\mathcal{T}$:

$$|W|_r^{\mathcal{U}} \leqslant K_1 \left(|\mathrm{curl}\, \underline{W}|_{r-1}^{\mathcal{U}} + |\mathrm{div}\, W|_{r-1}^{\mathcal{U}} + |W|_r^{\mathcal{T}} + \sum_{s=1}^{r} |g|_s^{\mathcal{U}} |W|_{r-s}^{\mathcal{U}} \right),$$

$$|W|_r^{\mathcal{U}} \leqslant K_1 \sum_{s=1}^{r} [g]_s^{\mathcal{U}} \left(|\mathrm{curl}\, \underline{W}|_{r-1-s}^{\mathcal{U}} + |\mathrm{div}\, W|_{r-1-s}^{\mathcal{U}} + |W|_{r-s}^{\mathcal{T}} \right).$$

3.7. Main estimates and theorem

Proposition 3.5 ([2]). *Suppose that $x, P \in C^{r+2}([0,T]\times\Omega)$, $B \in C^{r+2}(\Omega)$, $P|_\Gamma = 0$, $\nabla_N P|_\Gamma \leqslant -c_0 < 0$, $B^a N_a|_\Gamma = 0$ and* $\operatorname{div} V = 0$, *where $V = D_t x$. Let W be the solution of the linearized problem with the inhomogeneous term F divergence-free. Then, there is a constant C depending only on the norm of (x, P, B), a lower bound for the constant c_0, and an upper bound for T, such that, for $s \leqslant r$, we have*

$$\begin{aligned}&\|W(t)\|_r + \|\dot W(t)\|_r + \|\mathcal{L}_B W(t)\|_r + \langle W(t)\rangle_{\mathcal{A},r}\\ \leqslant& C\Big(\|W(0)\|_r + \|\dot W(0)\|_r + \|\mathcal{L}_B W(0)\|_r + \langle W(0)\rangle_{\mathcal{A},r} + \int_0^t \|F\|_r d\tau\Big),\end{aligned}$$

where

$$\begin{aligned}\|W(t)\|_r &= \sum_{|I|\leqslant r, I\in\mathcal{R}} \|\mathcal{L}_U^I W(t)\|_{L^2(\Omega)},\\ \langle W(t)\rangle_{\mathcal{A},r} &= \sum_{|I|\leqslant r, I\in\mathcal{S}} \langle \mathcal{L}_S^I W(t), \mathcal{A}\mathcal{L}_S^I W(t)\rangle^{1/2}.\end{aligned}$$

Let $H^r(\Omega)$ be the completion of $C^\infty(\Omega)$ in the norm $\|W(t)\|_r$ and $N^r(\Omega)$ be the completion of the $C^\infty(\Omega)$ divergence-free vector fields in the norm $\|W\|_{N^r} = \|W(t)\|_r + \langle W(t)\rangle_{\mathcal{A},r}$. Note that the projection $\mathbb{P}$ is continuous in the H^r norm, which implies that H^r is also the completion of the $C^\infty(\Omega)$ divergence-free vector fields in the H^r norm. The main result in [2] is as follows.

Theorem 3.6 ([2]). *Suppose that $x, P \in C^{r+2}([0,T]\times\Omega)$, $B \in C^{r+2}(\Omega)$, $P|_\Gamma = 0$, $\nabla_N P|_\Gamma \leqslant -c_0 < 0$, $B^a N_a|_\Gamma = 0$ and* $\operatorname{div} D_t x = 0$. *Then, if initial data and the inhomogeneous term are divergence-free and satisfy*

$$(W_0, W_1, \mathcal{L}_B W_0) \in N^r(\Omega)\times H^r(\Omega)\times H^r(\Omega), \quad F \in L^1\left([0,T],\, H^r(\Omega)\right),$$

the linearized problem has a solution

$$(W, \dot W, \mathcal{L}_B W) \in C\left([0,T],\, N^r(\Omega)\times H^r(\Omega)\times H^r(\Omega)\right). \tag{3.33}$$

The proof of the main estimates and this theorem involves: The projection onto divergence-free vector field, the smoothed-out equation and existence of weak solutions, regularity estimates, etc.

Finally, we give some remarks. A key idea in [2] is to use the Lie derivative of the magnetic field, taking the advantage that the magnetic field is tangential to the free boundary and divergence free, which provides extensive advantages when one commutes the magnetic vector field with other vector fields used in [31]. Due to the magnetic tension force, a term involving the coupling of the perturbation of the velocity field and the initial magnetic field appears in the linearized equation.

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Conflict of interest

The authors declare there is no conflicts of interest.

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