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Maximal L^p-L^q regularity for two-phase fluid motion in the linearized Oberbeck-Boussinesq approximation

Chengchun Hao [a,b,c], Wei Zhang [a,c,*]

[a] *Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

[b] *Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, China*

[c] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

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Abstract

This paper is concerned with the generalized resolvent estimate and the maximal L^p-L^q regularity of the linearized Oberbeck-Boussinesq approximation for unsteady motion of a drop in another fluid without surface tension, which is indispensable for establishing the well-posedness of the Oberbeck-Boussinesq approximation for the two incompressible liquids separated by a closed interface. We prove the existence of $\mathcal{R}$-bounded solution operators for the model problems and the maximal L^p-L^q regularity for the system. The key step is to prove the maximal L^p-L^q regularity theorem for the linearized heat equation with the help of the $\mathcal{R}$-bounded solution operators for the corresponding resolvent problem and the Weis operator-valued Fourier multiplier theorem.

Keywords: Oberbeck-Boussinesq approximation; Free boundary problem; Maximal L^p-L^q regularity; Resolvent estimate

* Corresponding author at: Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.

E-mail addresses: hcc@amss.ac.cn (C. Hao), zhangwei16@mails.ucas.edu.cn (W. Zhang).

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1. Introduction

We consider the unsteady motion of a drop of one incompressible viscous fluid inside another one in the Oberbeck-Boussinesq approximation. The liquids are separated by a closed unknown interface on which we take into account the surface tension. Let Ω be a bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geqslant 2$) with solid boundary Γ_-, Ω_+ be a subdomain of Ω with a closed surface Γ, and $\Omega_- = \Omega \backslash \overline{\Omega_+}$. We assume that $\mathrm{dist}(\Gamma, \Gamma_-) = \inf\{|x-y| : x \in \Gamma, y \in \Gamma_-\} \geqslant 2d$ for some constant $d > 0$. Let Ω_{t+} and Γ_t be the evolution of Ω_+ and Γ, respectively, both of which depend on the time $t > 0$, and set $\Omega_{t-} = \Omega \backslash (\Omega_{t+} \cup \Gamma_t)$, with $\Omega_{0\pm} = \Omega_\pm$ and $\Gamma_0 = \Gamma$. Let $\mathbf{n}_t$ be the normal to Γ_t oriented from Ω_{t+} into Ω_{t-}, $\mathbf{n} = \mathbf{n}_0$, and $\mathbf{n}_-$ be the unit outward normal to Γ_-. Denote $\dot{\Omega}_t = \Omega_{t+} \cup \Omega_{t-}$ and $\dot{\Omega} = \dot{\Omega}_0$. For any $t > 0$, we consider the following two-phase fluid motion in the Oberbeck-Boussinesq approximation:

$$
\begin{cases}
\rho\,(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mathrm{Div}(\mu\mathbf{D}(\mathbf{v}) - p\mathbf{I}) = \mathbf{a}(x,t) - \alpha\mathbf{g}\theta' & \text{in } \dot{\Omega}_t, \quad \text{(a)}\\
\partial_t\theta' + \mathrm{div}(\mathbf{v}\theta' - \kappa\nabla\theta') = 0 & \text{in } \dot{\Omega}_t, \quad \text{(b)}\\
\mathrm{div}\,\mathbf{v} = 0 & \text{in } \dot{\Omega}_t, \quad \text{(c)}\\
[\![(\mu\mathbf{D}(\mathbf{v}) - p\mathbf{I})\mathbf{n}_t]\!] = 0, \quad [\![\mathbf{v}]\!] = 0 & \text{on } \Gamma_t, \quad \text{(d)}\\
[\![\kappa\nabla\theta'\cdot\mathbf{n}_t]\!] = 0, \quad [\![\theta']\!] = 0 & \text{on } \Gamma_t, \quad \text{(e)}\\
V_n = \mathbf{v}\cdot\mathbf{n}_t & \text{on } \Gamma_t, \quad \text{(f)}\\
\mathbf{v} = 0, \quad \nabla\theta'_-\cdot\mathbf{n}_- + \beta\theta'_- = b(x,t) & \text{on } \Gamma_-, \quad \text{(g)}\\
\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \theta'\big|_{t=0} = \theta'_0 & \text{in } \dot{\Omega}, \quad \text{(h)}
\end{cases}
\tag{1.1}
$$

where $\mathbf{v} = (v_1(x,t), \dots, v_N(x,t))$ is the velocity vector field, $p = p(x,t)$ is the pressure, $\theta'(x,t)$ is the deviation from the average temperature, $\mathbf{a}$ is a given vector function of mass forces, and $\mathbf{g} = g(0,0,1)$ is a constant vector with the gravity constant g. The piece-wise positive constants ρ, μ, α and κ correspond to the mass density, the kinematic viscosity, the temperature expansion coefficient and the thermal conductivity, respectively. Here, both the above functions $\mathbf{v}, p, \theta', \mathbf{a}$ and the constants $\rho, \mu, \alpha, \kappa$ are piece-wisely defined, for instance, $\mathbf{v} = \mathbf{v}_+\chi_{\Omega_+} + \mathbf{v}_-\chi_{\Omega_-}$, $\rho = \rho_+\chi_{\Omega_+} + \rho_-\chi_{\Omega_-}$, etc., where $\chi_{\Omega_\pm}$ are the characteristic function of $\Omega_\pm$. $\mathbf{D}(\mathbf{v})$ is the doubled deformation tensor with the $(i,j)^{\text{th}}$ component $\partial_i v_j + \partial_j v_i$, and $\mathbf{I}$ is the $N \times N$ identity matrix. $b(x,t)$ is a given function on the fixed boundary Γ_-, and $\beta \geqslant 0$ is a constant. $\dot{\Omega}$, $\mathbf{v}_0$ and θ'_0 are the prescribed initial data for $\dot{\Omega}_t$, $\mathbf{v}$ and θ', respectively. V_n is the evolution velocity of Γ_t along $\mathbf{n}_t$. Moreover, for any function $f(x,t) = f_\pm(x,t)$ for $x \in \Omega_{t\pm}$ and $t \geqslant 0$, we denote the jump of f across Γ_t by

$$
[\![f]\!]\,(x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega_{t+}}} f_+(x) - \lim_{\substack{x\to x_0\\ x\in\Omega_{t-}}} f_-(x)
$$

for every point $x_0 \in \Gamma_t$. Finally, for any matrix field $\mathbf{K} = (K_{ij})$, the quantity $\mathrm{Div}\,\mathbf{K}$ is an N-vector whose i^{th} component is $\sum_{j=1}^N \partial_j K_{ij}$, and for any vector functions $\mathbf{u} = (u_1, \dots, u_N)$, $\mathrm{div}\,\mathbf{u} = \sum_{j=1}^N \partial_j u_j$, and $\mathbf{u}\cdot\nabla\mathbf{u}$ is an N-vector whose i^{th} component is $\sum_{j=1}^N u_j\partial_j u_i$.

In order to transform the time-dependent domain $\dot{\Omega}_t$ to the fixed domain $\dot{\Omega}$, we introduce the Lagrangian coordinates. Let $\varphi(\xi)$ be a $C^\infty(\mathbb{R}^N)$ function which equals 1 if $\mathrm{dist}\,(\xi, \Gamma_-) \geqslant 2d$ and equals 0 if $\mathrm{dist}\,(\xi, \Gamma_-) \leqslant d$. Let $\mathbf{u}(\xi,t)$, $q(\xi,t)$, $\mathbf{a}(\xi,t)$ and $\theta(\xi,t)$ be the velocity field, the pressure, the vector of mass forces and the deviation from the average temperature via $x = x(\xi,t)$, respectively, in the Lagrangian coordinates. The connection between the Eulerian coordinates x and the Lagrangian coordinates ξ is given by

$$x(\xi,t)=\xi+\int_0^t \varphi(\xi)\mathbf{u}(\xi,s)ds.$$

In the present paper, we will prove the maximal regularity for the following linearized equations in the Lagrangian coordinates:

$$\begin{cases} \rho\partial_t\mathbf{u}-\mathrm{Div}(\mu\mathbf{D}(\mathbf{u})-q\mathbf{I})=\mathbf{N}_1(\mathbf{u},\mathbf{a},\theta)+\mathbf{a}(\xi,t)+\alpha\mathbf{g}\theta & \text{in } \dot\Omega, & \text{(a)}\\ \mathrm{div}\,\mathbf{u}=N_2(\mathbf{u})=\mathrm{div}\,\mathbf{N}_3(\mathbf{u}) & \text{in } \dot\Omega, & \text{(b)}\\ [\![(\mu\mathbf{D}(\mathbf{u})-q\mathbf{I})\mathbf{n}]\!]=\mathbf{N}_4(\mathbf{u}), \quad [\![\mathbf{u}]\!]=0 & \text{on } \Gamma, & \text{(c)}\\ \partial_t\theta-\kappa\Delta\theta=N_5(\mathbf{u},\theta) & \text{in } \dot\Omega, & \text{(d)}\\ [\![\kappa\nabla\theta\cdot\mathbf{n}]\!]=N_6(\mathbf{u},\theta), \quad [\![\theta]\!]=0 & \text{on } \Gamma, & \text{(e)}\\ \mathbf{u}=0, \quad \nabla\theta_-\cdot\mathbf{n}_-+\beta\theta_-=b(\xi,t) & \text{on } \Gamma_-, & \text{(f)}\\ \mathbf{u}|_{t=0}=\mathbf{u}_0, \quad \theta|_{t=0}=\theta_0 & \text{in } \dot\Omega, & \text{(g)} \end{cases} \tag{1.2}$$

for $t\in(0,\infty)$. Here, noting that $x=\xi$ near Γ_-, we have $\mathbf{n}_-\cdot\nabla\theta_-+\beta\theta_-=b(\xi,t)$ on Γ_-, and $\mathbf{N}_1(\mathbf{u},\mathbf{a},\theta),\dots,N_6(\mathbf{u},\theta)$ are some nonlinear terms generated by the coordinate transforms.

The Oberbeck-Boussinesq approximation has implications for a wide variety of flows within the context of astrophysical, geophysical and oceanographic fluid dynamics (e.g., see [10]). The approximate equations were first derived by Oberbeck [13,14] and independently derived by Boussinesq [1], to describe the thermo-mechanical response of linearly viscous fluids that are mechanically incompressible but thermally compressible. Numerous attempts have been made to provide a rigorous justification for this approximation such as [9,15,16,18] and so on.

The free boundary problems of two-phase problems of two incompressible viscous fluids have been studied by many mathematicians in recent decades. Tanaka proved the local solvability of the problem with general data [21] and global solvability for data close enough to an equilibrium state [22]. The global weak solutions were constructed for the two phase Stokes system by Giga and Takahashi [8] and for Navier-Stokes equations by Takahashi [25]. Shibata has proved local and global well-posedness for incompressible-incompressible two-phase problem in [31]. In the Hölder spaces for all $t>0$, the local existence theorem for the problem in the Oberbeck-Boussinesq approximation was established in [2]. Denisova and Solonnikov [3] proved the global existence of classical solutions for capillary fluids in the framework of Hölder spaces. For the resolvent problems, Shibata and his collaborators have done a lot of work, e.g., [11,26,28–30,33], with the help of the $\mathcal{R}$-bounded solution operators. By $\mathcal{R}$-bounded solution operator theory, Frolova and Shibata proved the maximal L^p-L^q regularity and local well-posedness for the MHD in [6,7]. Recently, Saito, Shibata and Zhang established some local and global solutions for the two-phase incompressible flows with moving interfaces in L^p-L^q maximal regularity class in [17].

Now, we state the main result as the following theorem.

Theorem 1.1. *Let* $1<p,q,r<\infty$ *with* $2/p+1/q\notin\{1,2\}$ *and* $t>0$. *Assume that* Ω *is a uniform* $W_r^{2-1/r}$ *domain,* $\mathbf{u}_0\in B_{q,p}^{2(1-1/p)}(\dot\Omega)$ *and* $\theta_0\in B_{q,p}^{2(1-1/p)}(\dot\Omega)$ *are initial data* *for the equations* (1.2), $e^{-\gamma t}\mathbf{a}(\xi,t) \in L^p(\mathbb{R}, L^q(\dot{\Omega}))$ *and* $e^{-\gamma t}b(\xi,t) \in L^p\left(\mathbb{R}, W_q^1(\Gamma_-)\right) \cap H_p^{1/2}(\mathbb{R}, L^q(\Gamma_-))$. *Let* $\mathbf{N}_1(\mathbf{u},\mathbf{a},\theta), \ldots, N_6(\mathbf{u},\theta_-)$ *be functions appearing on the right side of the equations* (1.2) *which satisfy the conditions:*

$$\begin{aligned} &e^{-\gamma t}(\mathbf{N}_1, N_5) \in L^p\left(\mathbb{R}, L^q(\dot{\Omega})\right), \qquad e^{-\gamma t}\mathbf{N}_3 \in H_p^1\left(\mathbb{R}, L^q(\dot{\Omega})\right), \\ &e^{-\gamma t}(N_2, \mathbf{N}_4, N_6) \in L^p\left(\mathbb{R}, H_q^1(\dot{\Omega})\right) \cap H_p^{1/2}\left(\mathbb{R}, L^q(\dot{\Omega})\right), \end{aligned} \tag{1.3}$$

for any $\gamma \geqslant \gamma_0$ *with some* γ_0. *Assume that the compatibility conditions hold:*

$$\operatorname{div}\mathbf{u}_0 = N_2|_{t=0} \quad in \quad \dot{\Omega}, \quad \mathbf{u}_0 - \mathbf{N}_3|_{t=0} \in \mathcal{D}(\Omega), \tag{1.4}$$

where $\mathcal{D}(\Omega)$ *can be found in the appendix, and for* $2/p + 1/q < 1$

$$\begin{aligned} &[\![(\mu \mathbf{D}(\mathbf{u}_0)\mathbf{n})_\tau]\!] = [\![(\mathbf{N}_4)_\tau]\!]\big|_{t=0}, \quad [\![\kappa \nabla\theta_0 \cdot \mathbf{n}]\!] = N_6|_{t=0} \quad on\ \Gamma, \\ &\nabla\theta_{0-} \cdot \mathbf{n}_- + \beta\theta_{0-} = b|_{t=0} \qquad\qquad\qquad\qquad\qquad on\ \Gamma_-, \end{aligned} \tag{1.5}$$

for $2/p + 1/q < 2$

$$[\![\mathbf{u}_0]\!] = 0, \quad [\![\theta_0]\!] = 0 \quad on\ \Gamma, \quad \mathbf{u}_0 = 0 \quad on\ \Gamma_-. \tag{1.6}$$

Then, the problem (1.2) *admits a unique solution* $(\mathbf{u},\theta)$ *in*

$$L^p\left((0,\infty), H_q^2(\dot{\Omega})\right) \cap H_p^1\left((0,\infty), L^q(\dot{\Omega})\right),$$

possessing the estimate

$$\begin{aligned} &\left\|e^{-\gamma t}\partial_t(\mathbf{u},\theta)\right\|_{L^p((0,\infty),L^q(\dot{\Omega}))} + \left\|e^{-\gamma t}(\mathbf{u},\theta)\right\|_{L^p\left((0,\infty),H_q^2(\dot{\Omega})\right)} \\ &\lesssim \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\dot{\Omega})} + \|\theta_0\|_{B_{q,p}^{2(1-1/p)}(\dot{\Omega})} + \left\|e^{-\gamma t}(\mathbf{a}, \mathbf{N}_1, N_5)\right\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))} \\ &\quad + \left\|e^{-\gamma t}\partial_t \mathbf{N}_3\right\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))} + \left\|e^{-\gamma t}(N_2,\mathbf{N}_4,N_6)\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)} + \left\|e^{-\gamma t}b\right\|_{L^p\left(\mathbb{R},H_q^1(\Omega_-)\right)} \\ &\quad + \left(1+\gamma^{1/2}\right)\left(\left\|e^{-\gamma t}(N_2,\mathbf{N}_4,N_6)\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot{\Omega}))} + \left\|e^{-\gamma t}b\right\|_{H_p^{1/2}(\mathbb{R},L^q(\Omega_-))}\right), \end{aligned}$$

where the symbol "$\lesssim$" denotes "$\leqslant C$" for some constant $C > 0$.

Remark 1.2. We do not impose any compatibility conditions if $2/p + 1/q > 2$ in the theorem.

Remark 1.3. In this paper, $\mathbf{N}_1(\mathbf{u},\mathbf{a},\theta), \ldots, N_6(\mathbf{u},\theta_-)$ are regarded as some given functions, we just consider the model problem which is indispensable for establishing the local well-posedness of the Oberbeck-Boussinesq approximation for the two incompressible liquids separated by a closed interface. The specific definition will be given in proof of the local well-posedness of the Oberbeck-Boussinesq approximation in the future. Here we will not analyze these nonlinear terms. In addition, on the fixed boundary Γ_-, we consider a mixed boundary condition for θ in (1.2f) instead of the Neumann or Dirichlet boundary condition.

Remark 1.4. We can also consider the two-phase fluid motion of the Oberbeck-Boussinesq approximation (1.1) with surface tension on the free interface, that is, the first equation in (1.1d) is replaced by

$$[\![(\mu\mathbf{D}(\mathbf{v}) - p\mathbf{I})\mathbf{n}_t]\!] = \sigma H(\Gamma_t)\,\mathbf{n}_t,$$

where σ is a positive constant describing the coefficient of the surface tension and $H(\Gamma_t)$ is $N-1$ times the mean curvature of Γ_t given by the relation $H(\Gamma_t)\,\mathbf{n}_t = \Delta_{\Gamma_t}x$, with the Laplace-Beltrami operator Δ_{Γ_t} on Γ_t. For this case, the surface tension can be represented by the Laplace-Beltrami equations on the fixed interface by using the Hanzawa transformation (cf. [32]).

The rest of this paper is structured as follows. Firstly, we use the fact that the maximal L^p-L^q regularity theorem for the Stokes equations with interface and non-slip boundary conditions in [11] and Theorem 2.1 to obtain Theorem 1.1 in section 2. The existence of $\mathcal{R}$-bounded solution operators for the model problems is proved in section 3. Section 4 is devoted to prove Theorem 3.1. In section 5, Theorem 2.1 is proved with the help of $\mathcal{R}$-bounded solution operators given in Theorem 3.1 and the Weis operator- valued Fourier multiplier theorem [24]. Then, we will prove the maximal L^p-L^q regularity theorem for the problem (1.2) in a finite time interval in section 6. Finally, we will recall some notations and useful results in Appendix.

2. Linear theory

We consider two linearized equations. One is the Stokes equations with transmission conditions on Γ and non-slip conditions on Γ_-, the related results of which have been recalled in A.2. The other is the system of the heat equations with interface and boundary conditions on Γ and Γ_-, respectively.

We discuss the maximal L^p-L^q regularity for the system of heat equations with interface conditions has the form: for $t > 0$

$$\begin{cases} \partial_t\theta - \kappa\Delta\theta = f & \text{in } \dot{\Omega}, \quad \text{(a)} \\ [\![\kappa\nabla\theta\cdot\mathbf{n}]\!] = g, \quad [\![\theta]\!] = 0 & \text{on } \Gamma, \quad \text{(b)} \\ \nabla\theta_-\cdot\mathbf{n}_- + \beta\theta_- = h & \text{on } \Gamma_-, \quad \text{(c)} \\ \theta|_{t=0} = \theta_0 & \text{in } \dot{\Omega}. \quad \text{(d)} \end{cases} \tag{2.1}$$

We can prove the following key result of the present paper.

Theorem 2.1. *Let $1 < p, q, r < \infty$ with $2/p + 1/q \notin \{1, 2\}$, and $t > 0$. Assume that Ω is a uniform $W_r^{2-1/r}$ domain. Let $\theta_0 \in B_{q,p}^{2(1-1/p)}(\dot{\Omega})$ be initial data for equations (2.1). Let f, g and h be functions appearing in the right side of equations (2.1) which satisfy the conditions:*

$$e^{-\gamma t}f \in L^p\left(\mathbb{R}, L^q(\dot{\Omega})\right), \quad e^{-\gamma t}g \in L^p\left(\mathbb{R}, H_q^1(\dot{\Omega})\right) \cap H_p^{1/2}\left(\mathbb{R}, L^q(\dot{\Omega})\right),$$
$$e^{-\gamma t}h \in L^p\left(\mathbb{R}, H_q^1(\Omega_-)\right) \cap H_p^{1/2}\left(\mathbb{R}, L^q(\Omega_-)\right),$$

for any $\gamma \geqslant \gamma_0$ with some γ_0. Assume that the compatibility conditions:

$$\begin{cases} [\![\kappa\nabla\theta_0\cdot\mathbf{n}]\!] = g|_{t=0} & \text{on } \Gamma, \\ \nabla\theta_{0-}\cdot\mathbf{n}_- + \beta\theta_{0-} = h|_{t=0} & \text{on } \Gamma_-, \end{cases} \quad \text{if } 2/p+1/q<1;$$
$$[\![\theta_0]\!] = 0 \ \ \text{on } \Gamma, \qquad \text{if } 2/p+1/q<2. \tag{2.2}$$

Then, problem (2.1) *admits a unique solution* θ *with*

$$\theta \in L^p\left((0,\infty), H^2_q(\dot\Omega)\right) \cap H^1_p\left((0,\infty), L^q(\dot\Omega)\right)$$

possessing the estimate

$$\begin{aligned} &\left\|e^{-\gamma t}\partial_t\theta\right\|_{L^p((0,\infty),L^q(\dot\Omega))} + \left\|e^{-\gamma t}\theta\right\|_{L^p\left((0,\infty),H^2_q(\dot\Omega)\right)} \\ &\lesssim \|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \left\|e^{-\gamma t} f\right\|_{L^p(\mathbb{R},L^q(\dot\Omega))} \\ &\quad + \left\|e^{-\gamma t} g\right\|_{L^p\left(\mathbb{R},H^1_q(\dot\Omega)\right)} + \left\|e^{-\gamma t} h\right\|_{L^p\left(\mathbb{R},H^1_q(\Omega_-)\right)} \\ &\quad + \left(1+\gamma^{1/2}\right)\left(\left\|e^{-\gamma t} g\right\|_{H^{1/2}_p(\mathbb{R},L^q(\dot\Omega))} + \left\|e^{-\gamma t} h\right\|_{H^{1/2}_p(\mathbb{R},L^q(\Omega_-))}\right). \end{aligned} \tag{2.3}$$

To prove Theorem 2.1, we use an $\mathcal{R}$-bounded solution operator associated with the following generalized resolvent equations corresponding to (2.1):

$$\begin{cases} \lambda\theta - \kappa\Delta\theta = f & \text{in } \dot\Omega, \\ [\![\kappa\nabla\theta\cdot\mathbf{n}]\!] = g, \quad [\![\theta]\!] = 0 & \text{on } \Gamma, \\ \nabla\theta_-\cdot\mathbf{n}_- + \beta\mathbf{n}_- = h & \text{on } \Gamma_-, \end{cases} \tag{2.4}$$

with $\lambda \in \mathbb{C}$. The proof will be given in section 5.

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Theorem A.4 and (1.2), for $\mathbf{f} = \mathbf{N}_1(\mathbf{u}) + \mathbf{a} + \alpha\mathbf{g}\theta$, we have

$$\left\|e^{-\gamma t}\mathbf{f}\right\|_{L^p(\mathbb{R},L^q(\dot\Omega))} \lesssim \left\|e^{-\gamma t}\mathbf{a}\right\|_{L^p(\mathbb{R},L^q(\dot\Omega))} + \left\|e^{-\gamma t}\mathbf{N}_1\right\|_{L^p(\mathbb{R},L^q(\dot\Omega))} + \left\|e^{-\gamma t}\alpha\mathbf{g}\theta\right\|_{L^p(\mathbb{R},L^q(\dot\Omega))}.$$

Then, by Theorem 2.1 and the fact that $\theta = 0$ for $t < 0$ (proved in next section), we get

$$\left\|e^{-\gamma t}\alpha\mathbf{g}\theta\right\|_{L^p(\mathbb{R},L^q(\dot\Omega))} \lesssim \left\|e^{-\gamma t}\theta\right\|_{L^p\left((0,\infty),H^2_q(\dot\Omega)\right)}. \tag{2.5}$$

From (2.3), (A.4) and (2.5), we complete the proof of Theorem 1.1. □

3. Model problem

Before proving Theorem 2.1, we need to prove the following theorem:

Theorem 3.1. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $N < r < \infty$. Assume that Ω is a uniform $W_r^{2-1/r}$ domain in $\mathbb{R}^N$. Let*

$$E_q(\dot\Omega) = \left\{\mathbf{F} = (f, g, h) : f \in L^q(\dot\Omega), \quad g \in H_q^1(\dot\Omega), \quad h \in H_q^1(\Omega_-)\right\},$$
$$\mathcal{E}_q(\dot\Omega) = \left\{(F_0, F_1, \dots, F_4) : F_0, F_1 \in L^q(\dot\Omega),\ F_2 \in H_q^1(\dot\Omega),\ F_3 \in L^q(\Omega_-),\ F_4 \in H_q^1(\Omega_-)\right\}.$$

Then, there exist a constant $\lambda_1 \geqslant 1$ and an operator family

$$\mathcal{Z}(\lambda)(\dot\Omega) \in \mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}\left(\mathcal{E}_q(\dot\Omega), H_q^2(\dot\Omega)\right)\right)$$

such that for any $\lambda \in \Sigma_{\epsilon,\lambda_1}$, $\mathbf{F} \in E_q(\dot\Omega)$, the unique solution of (2.4) is given by $\theta = \mathcal{Z}(\lambda)F_\lambda\mathbf{F}$, where

$$F_\lambda\mathbf{F} = \left(f, \lambda^{1/2}g, g, \lambda^{1/2}h, h\right) \in \mathcal{E}_q(\dot\Omega).$$

The estimate

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{E}_q(\dot\Omega), H_q^{2-k}(\dot\Omega)\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{k/2}\mathcal{Z}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_1}\right\}\right) \leqslant \gamma,\ \tau = \mathrm{Im}\,\lambda,$$

is valid for $\ell = 0, 1$ and $k = 0, 1, 2$. Here, γ is a positive constant depending on $\kappa_\pm$, ϵ, β, q and N.

Remark 3.2. (1) The functions F_0, F_1, F_2, F_3 and F_4 correspond to f, $\lambda^{1/2}g$, g, $\lambda^{1/2}h$ and h, respectively.

(2) The norms of $E_q(\dot\Omega)$ and $\mathcal{E}_q(\dot\Omega)$ are defined by

$$\|\mathbf{F}\|_{E_q(\dot\Omega)} = \|f\|_{L^q(\dot\Omega)} + \|g\|_{H_q^1(\dot\Omega)} + \|h\|_{H_q^1(\Omega_-)},$$
$$\|(F_0, F_1, \dots, F_4)\|_{\mathcal{E}_q(\dot\Omega)} = \|(F_0, F_1)\|_{L^q(\dot\Omega)} + \|F_2\|_{H_q^1(\dot\Omega)} + \|F_3\|_{L^q(\Omega_-)} + \|F_4\|_{H_q^1(\Omega_-)}.$$

3.1. Model problem in the whole space

In this subsection, we consider the equation in whole space:

$$\lambda\theta - \kappa\Delta\theta = f \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

We define the operators $\mathcal{A}_\pm$ acting on $f \in L_q(\mathbb{R}^N)$ by the formula

$$\mathcal{A}_\pm(\lambda)f = \mathcal{F}_\xi^{-1}\left[\frac{\mathcal{F}[f](\xi)}{\lambda + \kappa_\pm|\xi|^2}\right], \tag{3.2}$$

where $\mathcal{F}[f]$ and $\mathcal{F}_\xi^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively, given by

$$\mathcal{F}[f](\xi)=\int_{\mathbb{R}^N} e^{-ix\cdot\xi}f(x)dx, \quad \mathcal{F}_\xi^{-1}[g(\xi)](x)=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}g(\xi)d\xi.$$

Concerning the spectrum, we can get the following result.

Lemma 3.3. *Let $0<\epsilon<\pi/2$ and Σ_ϵ be defined in (A.1). For any $\lambda\in\Sigma_\epsilon$ and $\xi\in\mathbb{R}^N$, we have*

$$\left|\kappa^{-1}\lambda+|\xi|^2\right|\geqslant C_{\epsilon,\kappa}\left(|\lambda|+|\xi|^2\right),$$

with $C_{\epsilon,\kappa}$ depending on Σ_ϵ and κ.

Proof. Without loss of generality, it suffices to prove $|\lambda+x|\geqslant C_\epsilon(|\lambda|+x)(x\geqslant 0)$. Let $\lambda=|\lambda|e^{i\theta}$ and $\theta\leqslant\pi-\epsilon$, we have

$$|\lambda+x|^2=|\lambda|^2+2x|\lambda|\cos\theta+x^2\geqslant|\lambda|^2+2x|\lambda|\cos\epsilon+x^2\geqslant\cos\epsilon\,(|\lambda|+x)^2.$$

Thus, we complete the proof of the lemma. □

Finally, we have the following result about the $\mathcal{R}$-boundedness of $\mathcal{A}_\pm(\lambda)$.

Theorem 3.4. *Let $1<q<\infty$, $0<\epsilon<\pi/2$, and $\lambda_0>0$. Let Σ_ϵ and $\Sigma_{\epsilon,\lambda_0}$ be the sets defined in (A.1), $\mathcal{A}_\pm$ be the operators defined in (3.2). Then $\theta_\pm=\mathcal{A}_\pm f\in H_q^2(\mathbb{R}^N)$ is a unique solution of (3.1) for any $\lambda\in\Sigma_{\epsilon,\lambda_0}$ and $f\in L^q(\mathbb{R}^N)$. Moreover, $\mathcal{A}_\pm\in\mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_0},\mathcal{L}(L^q(\mathbb{R}^N),H_q^2(\mathbb{R}^N))\right)$ and the estimate*

$$\mathcal{R}_{\mathcal{L}\left(L^q(\mathbb{R}^N),H_q^{2-j}(\mathbb{R}^N)\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{j/2}\mathcal{A}_\pm(\lambda)\right):\lambda\in\Sigma_{\epsilon,\lambda_0}\right\}\right)\leqslant\gamma_{\lambda_0},\ \tau=\mathrm{Im}\,\lambda,$$

holds for $\ell=0,1$ and $j=0,1,2$, where the constant γ_{λ_0} depends on λ_0 in such a way that $\gamma_{\lambda_0}\to\infty$ as $\lambda_0\to 0$.

Proof. The proof is based on the theory of L^p-multipliers in Fourier integrals initiated by Mikhalin [12]. Indeed, from Lemmas A.5, A.6, A.7 and A.8, it follows the conclusion. □

3.2. Model interface problem

Let

$$\mathbb{R}_+^N=\left\{x=(x_1,\ldots,x_N)\in\mathbb{R}^N:x_N>0\right\},$$
$$\mathbb{R}_-^N=\left\{x=(x_1,\ldots,x_N)\in\mathbb{R}^N:x_N<0\right\},$$
$$\mathbb{R}_0^N=\left\{x=(x_1,\ldots,x_N)\in\mathbb{R}^N:x_N=0\right\},$$

and $\dot{\mathbb{R}}^N=\mathbb{R}_+^N\cup\mathbb{R}_-^N$.

We consider the following equations

$$\begin{cases} \lambda\theta_\pm - \kappa\Delta\theta_\pm = f_\pm & \text{in } \mathbb{R}^N_\pm, \\ [\![\kappa\nabla\theta\cdot\mathbf{n}]\!] = g, \quad [\![\theta]\!] = 0 & \text{on } \mathbb{R}^N_0, \end{cases} \tag{3.3}$$

with $\mathbf{n} = (0, \dots, 0, -1)$.

Let $F_\pm$ be the zero extension of $f_\pm$ to $\mathbb{R}^N_\mp$. We can reduce (3.3) to the case $f = 0$ by using $\mathcal{A}_\pm(\lambda)F_\pm$ in (3.2). Thus, we can consider

$$\begin{cases} \lambda\theta - \kappa\Delta\theta = 0 & \text{in } \dot{\mathbb{R}}^N, & \text{(a)} \\ -\kappa_+\partial_N\theta_+ + \kappa_-\partial_N\theta_- = g & \text{on } \mathbb{R}^N_0, & \text{(b)} \\ \theta_+ - \theta_- = 0 & \text{on } \mathbb{R}^N_0. & \text{(c)} \end{cases} \tag{3.4}$$

The aim of this subsection is to prove the existence of $\mathcal{R}$-bounded solution operators of (3.4). Let $\mathcal{F}'$ and $\mathcal{F}^{-1}_{\xi'}$ be the partial Fourier transform with respect to $x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$ and the inverse partial Fourier transform with respect to $\xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$, respectively, defined by

$$\widehat{f} = \mathcal{F}'[f] = \int\limits_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f\left(x', x_N\right) dx',$$

$$\mathcal{F}^{-1}_{\xi'}\left[g\left(\xi', x_N\right)\right] = \frac{1}{(2\pi)^{N-1}} \int\limits_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} g\left(\xi', x_N\right) d\xi'.$$

Applying the partial Fourier transform to (3.4a), we have

$$\begin{cases} \lambda\widehat{\theta}_\pm + \kappa_\pm\left|\xi'\right|^2\widehat{\theta}_\pm - \kappa_\pm\partial_N^2\widehat{\theta}_\pm = 0 & \text{for } \pm x_N > 0, & \text{(a)} \\ -\kappa_+\partial_N\widehat{\theta}_+ + \kappa_-\partial_N\widehat{\theta}_- = \widehat{g} & \text{on } \{x_N = 0\}, & \text{(b)} \\ \widehat{\theta}_+ - \widehat{\theta}_- = 0 & \text{on } \{x_N = 0\}. & \text{(c)} \end{cases} \tag{3.5}$$

Let $A_\pm = \sqrt{\frac{\lambda}{\kappa_\pm} + |\xi'|^2}$, $\operatorname{Re} A_\pm > 0$. Then, the solutions to (3.5a) are of the form $\widehat{\theta}_\pm = \alpha_\pm e^{\mp A_\pm x_N}$. Inserting them into (3.5b) and (3.5c) yields

$$\begin{cases} \kappa_+\alpha_+A_+ + \kappa_-\alpha_-A_- = \widehat{g}, \\ \alpha_+ - \alpha_- = 0. \end{cases} \tag{3.6}$$

Thus, we can get

$$\alpha = \alpha_- = \alpha_+ = \frac{\widehat{g}}{A_+\kappa_+ + A_-\kappa_-}.$$

By the Volevich method (cf. [23]), we have for $\pm x_N > 0$

$$\widehat{\theta}_\pm(x) = -\int_0^\infty e^{\mp A_\pm(x_N+y_N)} \partial_N \alpha\left(\xi', y_N\right) dy_N + \int_0^\infty A_\pm e^{\mp A_\pm(x_N+y_N)} \alpha\left(\xi', y_N\right) dy_N.$$

Using the identities $A_\pm = \frac{\kappa_\pm^{-1}\lambda}{A_\pm} + \frac{|\xi'|^2}{A_\pm}$, we obtain

$$\begin{aligned}
&\widehat{\theta}_\pm\left(\xi', x_N\right) \\
&= -\int_0^{\pm\infty} \left\{ \lambda^{1/2} e^{\mp A_\pm(x_N+y_N)} \frac{\kappa_\pm^{-1}\lambda^{1/2}}{A_\pm^2} + \left|\xi'\right| e^{\mp A_\pm(x_N+y_N)} \frac{\left|\xi'\right|}{A_\pm^2} \right\} \partial_N \alpha\left(\xi', y_N\right) dy_N \\
&\quad \pm \int_0^{\pm\infty} \left\{ \lambda^{1/2} e^{\mp A_\pm(x_N+y_N)} \frac{\kappa_\pm^{-1}\lambda^{1/2}}{A_\pm} + \left|\xi'\right| e^{\mp A_\pm(x_N+y_N)} \frac{\left|\xi'\right|}{A_\pm} \right\} \alpha\left(\xi', y_N\right) dy_N.
\end{aligned}$$

It follows that

$$\begin{aligned}
\widehat{\theta}_\pm\left(\xi', x_N\right) = &\pm \int_0^\infty \lambda^{1/2} e^{\mp A_\pm(x_N+y_N)} \frac{1}{\kappa_\pm A_\pm} \left(\frac{1}{A_+\kappa_+ + A_-\kappa_-} \mathcal{F}'\left[\lambda^{1/2} g\right]\left(\xi', y_N\right) \right) dy_N \\
&- \sum_{j=1}^{N-1} \int_0^\infty \left|\xi'\right| e^{\mp A_\pm(x_N+y_N)} \frac{i\xi_j}{|\xi'|\, A_+} \left(\frac{1}{A_+\kappa_+ + A_\pm\kappa_-} \mathcal{F}'\left[\partial_j g\right]\left(\xi', y_N\right) \right) dy_N \\
&+ \int_0^\infty \left(\lambda^{1/2} e^{\mp A_\pm(x_N+y_N)} \frac{\lambda^{1/2}}{\kappa_\pm A_\pm^2} + \left|\xi'\right| e^{\mp A_\pm(x_N+y_N)} \frac{\left|\xi'\right|}{A_\pm^2} \right) \\
&\qquad \times \left(\frac{1}{A_+\kappa_+ + A_-\kappa_-} \mathcal{F}'\left[\partial_N g\right]\left(\xi', y_N\right) \right) dy_N.
\end{aligned}$$

Define the operators $\mathcal{B}_\pm$ by

$$\begin{aligned}
&\mathcal{B}_\pm(\lambda)\left(F_1, F_2\right) \\
&= \pm \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\lambda^{1/2} e^{\mp A_\pm(x_N+y_N)} \frac{1}{\kappa_\pm A_\pm} \frac{1}{A_+\kappa_+ + A_-\kappa_-} \mathcal{F}'\left[F_1\right]\left(\xi', y_N\right) \right] dy_N \\
&\quad - \sum_{j=1}^{N-1} \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\left|\xi'\right| e^{\mp A_\pm(x_N+y_N)} \frac{i\xi_j}{|\xi'|\, A_\pm} \frac{1}{A_+\kappa_+ + A_-\kappa_-} \mathcal{F}'\left[\partial_j F_2\right]\left(\xi', y_N\right) \right] dy_N \\
&\quad + \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[\left(\lambda^{1/2} e^{\mp A_\pm(x_N+y_N)} \frac{\lambda^{1/2}}{\kappa_\pm A_\pm^2} + \left|\xi'\right| e^{-A_\pm(x_N+y_N)} \frac{\left|\xi'\right|}{A_\pm^2} \right) \right.
\end{aligned}$$

$$\times \left(\frac{1}{A_+\kappa_+ + A_-\kappa_-} \mathcal{F}' \left[\partial_N F_2 \right] \left(\xi', y_N \right) \right) \Bigg] \, dy_N .$$

Obviously,

$$\theta_\pm(x) = \mathcal{F}^{-1}_{\xi'} \left[\widehat{\theta}_\pm \left(\xi', x_N \right) \right] \left(x' \right) = \mathcal{B}_\pm(\lambda) \left(\lambda^{1/2} g, g \right).$$

Thus, we should show the $\mathcal{R}$-bound of the operator $\mathcal{B}_\pm$.

In what follows, we denote the set of all multipliers defined on $\mathbb{R}^{N-1} \backslash \{0\} \times \Sigma_{\epsilon,\lambda_0}$ of order s with type i $(i = 1, 2)$ (defined in Definition A.9) by $\mathbf{M}_{s,i}$. We also introduce the following two fundamental lemmas.

Lemma 3.5. *Let $s_1, s_2 \in \mathbb{R}$, then the following assertions hold:*

(1) *Given $m_i \in \mathbf{M}_{s_i,1} (i = 1, 2)$, we have $m_1 m_2 \in \mathbf{M}_{s_1+s_2,1}$.*
(2) *Given $l_i \in \mathbf{M}_{s_i,i} (i = 1, 2)$, we have $l_1 l_2 \in \mathbf{M}_{s_1+s_2,2}$.*
(3) *Given $n_i \in \mathbf{M}_{s_i,2} (i = 1, 2)$, we have $n_1 n_2 \in \mathbf{M}_{s_1+s_2,2}$.*

Proof. It immediately follows from the inequality

$$\left(|\lambda|^{1/2} + \left|\xi'\right| \right)^{-|\alpha'|} \leqslant \left|\xi'\right|^{-|\alpha'|}$$

and the Leibniz rule. □

Lemma 3.6. *Let $s \in \mathbb{R}$ and $0 < \epsilon < \pi/2$. Then the following assertions hold:*

$$\left| \partial^{\alpha'}_{\xi'} A^s_\pm \right| \leqslant C_{\alpha',s,\kappa_\pm} \left(|\lambda|^{1/2} + \left|\xi'\right| \right)^{s-|\alpha'|}, \tag{3.7}$$

$$\left| \partial^{\alpha'}_{\xi'} |\xi'|^s \right| \leqslant C_{\alpha',s} |\xi'|^{s-|\alpha'|}. \tag{3.8}$$

Proof. By Lemma 3.3, we have

$$c_1 \left(|\lambda|^{1/2} + \left|\xi'\right| \right) \leqslant |A_\pm| \leqslant c_2 \left(|\lambda|^{1/2} + \left|\xi'\right| \right). \tag{3.9}$$

Let $f(t) = t^{s/2}$, we observe that

$$\begin{aligned} \left| \partial^{\alpha'}_{\xi'} A^s_\pm \right| &\leqslant C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} \left| f^{(\ell)} \left(A^2_\pm \right) \right| \sum_{\alpha'_1+\cdots+\alpha'_\ell=\alpha'} \left| \partial^{\alpha'_1}_{\xi'} A^2_\pm \cdots \partial^{\alpha'_\ell}_{\xi'} A^2_\pm \right| \\ &\leqslant \sum_{|\alpha'|/2 \leqslant \ell \leqslant |\alpha'|} C_{s,\ell} \left| A^2_\pm \right|^{(s/2)-\ell} |\xi'|^{2\ell-|\alpha'|}. \end{aligned}$$

Since $\left|\xi'\right|^{2\ell-|\alpha'|} \leqslant \left(|\lambda|^{1/2} + \left|\xi'\right| \right)^{2\ell-|\alpha'|}$ provided that $\left|\alpha'\right|/2 \leqslant \ell \leqslant \left|\alpha'\right|$, by (3.9), we have (3.7). The estimate (3.8) can be proved similarly. □

By the Leibniz rule and Definition A.9, we can obtain directly

$$\left|\partial_{\xi'}^{\alpha'}\left(i\xi_j\left|\xi'\right|^{-1}\right)\right| \leqslant C_{\alpha'}\left|\xi'\right|^{-|\alpha'|}, \tag{3.10}$$

thus $i\xi_j\left|\xi'\right|^{-1} \in \mathbf{M}_{0,2}$. By Lemmas 3.5 and 3.6, and (3.10), we have

$$\lambda^{1/2}A_{\pm}^{-1} \in \mathbf{M}_{0,1}, \quad \frac{1}{\kappa_+A_+}\left(\frac{1}{A_+\kappa_+ + A_-\kappa_-}\right), \quad \frac{\lambda^{1/2}}{\kappa_\pm A_\pm^2}\frac{1}{A_+\kappa_+ + A_-\kappa_-} \in \mathbf{M}_{-2,1},$$

and

$$\frac{i\xi_j}{|\xi'|\,A_+}\left(\frac{1}{A_+\kappa_+ + A_-\kappa_-}\right), \quad \frac{\left|\xi'\right|}{A_+^2}\frac{1}{A_+\kappa_+ + A_-\kappa_-} \in \mathbf{M}_{-2,2}.$$

By Lemma A.10, we can get the $\mathcal{R}$-boundedness of the operator $\mathcal{B}_\pm$, then we obtain that $\theta_\pm = \mathcal{B}_\pm(\lambda)\left(\lambda^{1/2}g, g\right)$ is the solution of problem (3.4). Moreover, we have the following theorem.

Theorem 3.7. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, and $\lambda_0 > 0$. Let*

$$X_q(\dot{\mathbb{R}}^N) = \left\{(f, g) : f \in L^q(\dot{\mathbb{R}}^N),\, g \in H_q^1(\dot{\mathbb{R}}^N)\right\},$$
$$\mathcal{X}_q(\dot{\mathbb{R}}^N) = \left\{(F_0, F_1, F_2) : F_0, F_1 \in L^q(\dot{\mathbb{R}}^N),\, F_2 \in H_q^1(\dot{\mathbb{R}}^N)\right\}.$$

Then, there exists an operator family

$$\mathcal{B}_\pm(\lambda) \in \mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}\left(\mathcal{X}_q(\dot{\mathbb{R}}^N), H_q^2(\dot{\mathbb{R}}^N)\right)\right),$$

such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $(f, g) \in X_q(\dot{\mathbb{R}}^N)$, the unique solution of (3.3) is given by $\theta_\pm = \mathcal{B}_\pm(\lambda)F_\lambda^1(f, g)$, where

$$F_\lambda^1(f, g) = \left(f, \lambda^{1/2}g, g\right) \in \mathcal{X}_q(\dot{\mathbb{R}}^N).$$

Moreover,

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{X}_q(\dot{\mathbb{R}}^N), H_q^{2-j}(\dot{\mathbb{R}}^N)\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{j/2}\mathcal{B}_\pm(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}\right) \leqslant \gamma_{\lambda_0}, \quad \tau = \mathrm{Im}\,\lambda, \tag{3.11}$$

holds for $\ell = 0, 1$ and $j = 0, 1, 2$, where the constant γ_{λ_0} depends on λ_0 in such a way that $\gamma_{\lambda_0} \to \infty$ as $\lambda_0 \to 0$.

Proof. By Lemma A.10, we can get (3.11), and the existence has been completed. Thus, we only need to prove the uniqueness, let $\theta_\pm \in H_q^2(\dot{\mathbb{R}}^N)$ satisfy the homogeneous equations

$$\begin{cases} \lambda\theta - \kappa\Delta\theta = 0 & \text{in } \dot{\mathbb{R}}^N, \\ [\![\kappa\nabla\theta\cdot\mathbf{n}]\!] = 0, \quad [\![\theta]\!] = 0 & \text{on } \mathbb{R}_0^N. \end{cases} \tag{3.12}$$

Since the $\mathcal{R}$-boundedness implies the usual boundedness, by (3.11), we have

$$\sum_{j=0}^{2} |\lambda|^{j/2} \|\theta_\pm\|_{H_q^{2-j}(\dot{\mathbb{R}}^N)} \leqslant C \left\| F_\lambda^1(f,g) \right\|_{\mathcal{X}_q(\dot{\mathbb{R}}^N)},$$

where $\|\cdot\|_{\mathcal{X}_q(\dot{\mathbb{R}}^N)}$ is similarly defined as in Remark 3.2. Obviously, when $\theta_\pm$ satisfy homogeneous equations, $\|\theta_\pm\|_{H_q^{2-k}(\dot{\mathbb{R}}^N)} = 0$ yields $\theta_\pm = 0$, which shows the uniqueness. □

3.3. Model problem in a half space

In this subsection, we consider the model problem in the half space

$$\begin{cases} \lambda\theta_- - \kappa_-\Delta\theta_- = f_- & \text{in } \mathbb{R}_-^N, \\ \nabla\theta_- \cdot \mathbf{n} + \beta\theta_- = h & \text{on } \mathbb{R}_0^N, \end{cases} \tag{3.13}$$

where $\mathbf{n} = (0, \dots, -1)$. We can directly use similar methods of model interface problem. Thus, we get $\widehat{\theta_-} = \alpha_- e^{A_- x_N}$ with $A_-\alpha_- + \beta\alpha_- = \widehat{h}$, i.e., $\alpha_- = \frac{\widehat{h}}{\beta + A_-}$. By Lemma 3.6, we have

$$\left| \partial_{\xi'}^{\alpha'} (\beta + A_-)^{-1} \right| \leqslant C_{\alpha',\epsilon} \left(|\lambda|^{1/2} + |\xi'| \right)^{-1-|\alpha'|}. \tag{3.14}$$

Then, by (3.14) and Lemma 3.5, we obtain

$$\frac{\kappa_+^{-1}}{A_+} \left(\frac{1}{\beta + A_-} \right), \quad \frac{\kappa_\pm^{-1}\lambda^{1/2}}{A_+^2} \frac{1}{\beta + A_-} \in \mathbf{M}_{-2,1},$$

and

$$\frac{i\xi_j}{|\xi'| A_+} \left(\frac{1}{\beta + A_-} \right), \quad \frac{|\xi'|}{A_+^2} \frac{1}{\beta + A_-} \in \mathbf{M}_{-2,2}.$$

Finally, by Lemma A.10, we have the following theorem.

Theorem 3.8. *Let* $1 < q < \infty$, $0 < \epsilon < \pi/2$, *and* $\lambda_0 > 0$. *Let*

$$Y_q(\mathbb{R}_-^N) = \left\{ (f_-, h) : f_- \in L^q(\mathbb{R}_-^N), h \in H_q^1\left(\mathbb{R}_-^N\right) \right\},$$
$$\mathcal{Y}_q(\mathbb{R}_-^N) = \left\{ F_{0-}, F_3, F_4 : F_{0-}, F_3 \in L^q(\mathbb{R}_-^N), F_4 \in H_q^1(\mathbb{R}_-^N) \right\}.$$

Then, there exists an operator family

$$\mathcal{C}(\lambda) \in \mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}\left(\mathcal{Y}_q(\mathbb{R}_-^N), H_q^2(\mathbb{R}_-^N) \right) \right),$$

such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *and* $(f_-, h) \in Y_q\left(R_-^N\right)$, *the unique solution of* (3.13) *is given by* $\theta_- = \mathcal{C}(\lambda) F_\lambda^2(f_-, h)$, *where*

$$F_\lambda^2(f_-, h) = \left(f_-, \lambda^{1/2}h, h\right) \in \mathcal{Y}_q(\mathbb{R}^N_-).$$

Moreover, the estimate

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\mathbb{R}^N_-), H_q^{2-j}(\mathbb{R}^N_-)\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{j/2}\mathcal{C}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}\right) \leqslant \gamma_{\lambda_0}, \quad \tau = \operatorname{Im}\lambda, \tag{3.15}$$

holds for $\ell = 0, 1$ *and* $j = 0, 1, 2$*, where the constant* γ_{λ_0} *depends on* λ_0 *in such a way that* $\gamma_{\lambda_0} \to \infty$ *as* $\lambda_0 \to 0$.

3.4. Model problem in a bent space

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a bijection of C^1 class and Φ^{-1} be its inverse map. We write

$$(\nabla\Phi)_{ij} = \left[\frac{\partial y_i}{\partial x_j}\right] = \mathcal{M} + M(x), \quad (\nabla\Phi^{-1})_{ij} = \left[\frac{\partial x_i}{\partial y_j}\right] = \mathcal{M}_{-1} + M_{-1}(x),$$

where $\mathcal{M}$ and $\mathcal{M}_{-1}$ are orthonormal matrices with constant coefficients, and $M(x)$ and $M_{-1}(x)$ are matrices of functions in $H_r^2(\mathbb{R}^N)$ with $N < r < \infty$ such that

$$\|(M, M_{-1})\|_{L^\infty(\mathbb{R}^N)} \leqslant K_1, \quad \|\nabla(M, M_{-1})\|_{H_r^1(\mathbb{R}^N)} \leqslant K_2. \tag{3.16}$$

We will choose K_1 small enough eventually, so that in the sequel, we may assume that $0 < K_1 \leqslant 1 \leqslant K_2$. Let $\Omega_+ = \Phi\left(\mathbb{R}^N_+\right)$, $\Omega_- = \Phi\left(\mathbb{R}^N_-\right)$ and $\Gamma = \Phi\left(\mathbb{R}^N_0\right)$. Let $\mathbf{n}$ be the unit normal to Γ, which points from Ω_+ to Ω_-. Similarly, we assume that Ψ meets the same requirements as Φ, $\Omega_+ = \Psi\left(\mathbb{R}^N_+\right)$, $\Omega_- = \Psi\left(\mathbb{R}^N_-\right)$ and $\Gamma_- = \Psi\left(\mathbb{R}^N_0\right)$. Let $\mathbf{n}_-$ be the unit normal to Γ_-. In what follows, we consider the two problems:

$$\begin{cases} \partial_t\theta - \kappa\Delta\theta = f & \text{in } \dot\Omega, \\ [\![\kappa\nabla\theta\cdot\mathbf{n}]\!] = g, \quad [\![\theta]\!] = 0 & \text{on } \Gamma, \end{cases} \tag{3.17}$$

and

$$\begin{cases} \partial_t\theta_- - \kappa_-\Delta\theta_- = f_- & \text{in } \Omega_-, \\ \nabla\theta_-\cdot\mathbf{n}_- + \beta\theta_- = h & \text{on } \Gamma_-. \end{cases} \tag{3.18}$$

Theorem 3.9. *Let* $1 < q < \infty$, $0 < \epsilon < \pi/2$, $\lambda_0 \geqslant 1$. *Assume* (3.16) *holds for some* $0 < K_1 \leqslant 1 \leqslant K_2$. *Then there exists a* λ_0 *depending only on* ϵ, $\kappa_\pm$, q, N *and* K_2.

(1) Let $X_q(\dot\Omega)$, $\mathcal{X}_q(\dot\Omega)$ *be defined in Theorem* 3.7*. Then, there exists an operator family*

$$\mathcal{B}_\pm(\lambda) \in \operatorname{Hol}\left(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}\left(\mathcal{X}_q(\dot\Omega), H_q^2(\dot\Omega)\right)\right),$$

such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *and* $(f, g) \in X_q(\dot\Omega)$*, the unique solution of* (3.17) *is given by* $\theta_\pm = \mathcal{B}_\pm(\lambda)F_\lambda^1(f, g)$*, where*

$$F_\lambda^1(f, g) = \left(f, \lambda^{1/2}g, g\right) \in \mathcal{X}_q(\dot\Omega).$$

Moreover,

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{X}_q(\dot{\Omega}), H_q^{2-j}(\dot{\Omega})\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{j/2}\mathcal{B}_\pm(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}\right) \leqslant \gamma, \quad \tau = \operatorname{Im}\lambda, \tag{3.19}$$

holds for $\ell = 0, 1$ *and* $j = 0, 1, 2$*, where* γ *depends only on* $\epsilon, \kappa_\pm, K_2, q$ *and* N*.*

(2) Let $Y_q(\Omega_-)$*,* $\mathcal{Y}_q(\Omega_-)$ *be defined in Theorem 3.8. Then, there exists an operator family*

$$\mathcal{C}(\lambda) \in \operatorname{Hol}\left(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}\left(\mathcal{Y}_q(\Omega_-), H_q^2(\Omega_-)\right)\right),$$

such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *and* $(f_-, h) \in Y_q(\Omega_-)$*, the unique solution of (3.18) is given by* $\theta_- = \mathcal{C}(\lambda)F_\lambda^2(f_-, h)$*, where*

$$F_\lambda^2(f_-, h) = \left(f_-, \lambda^{1/2}h, h\right) \in \mathcal{Y}_q(\Omega_-).$$

Moreover,

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\Omega_-), H_q^{2-j}(\Omega_-)\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{j/2}\mathcal{C}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}\right) \leqslant \gamma, \quad \tau = \operatorname{Im}\lambda, \tag{3.20}$$

holds for $\ell = 0, 1$ *and* $j = 0, 1, 2$*, where the constant* γ *depends on* $\epsilon, \kappa_\pm, \beta, K_2, q$ *and* N*.*

Since the proofs are similar to those of the Stokes equations with free boundary conditions given in Shibata [28,29], we omit the details and only give a sketch of the proof here. We first convert (3.17) and (3.18) to problems in whole space and half-space problems, respectively. Then, we can use Theorems 3.7 and 3.8 to prove Theorem 3.9. We only consider (3.18) as an example. Let us reformulate the functions in (3.18) by the change $y = \Psi(x)$. Let $\theta_-(x) = \tilde{\theta}_-(\Psi(x))$ and $h(x) = \tilde{h}(\Psi(x))$ satisfy (3.18). Let $\mathcal{M}_{-1}$ and M_{-1} be the $N \times N$ matrices appearing in the condition (3.16) and let $\mathcal{M}_{jk}$ and $M_{jk}(x)$ be the $(j,k)^{\text{th}}$ component of $\mathcal{M}_{-1}$ and $M_{-1}(\Phi(x))$, respectively. For simplicity, we set $Q_{jk} = \mathcal{M}_{jk} + M_{jk}$. By (3.16), we get

$$\left|\mathcal{M}_{jk}\right| \leqslant 1, \quad \left\|M_{jk}\right\|_{L^\infty(\mathbb{R}^N)} \leqslant K_1, \quad \left\|M_{jk}\right\|_{H_r^2(\mathbb{R}^N)} \leqslant K_2.$$

In this case, we have

$$\frac{\partial}{\partial y_j} = \sum_{k=1}^N \frac{\partial x_k}{\partial y_j}\frac{\partial}{\partial x_k} = \sum_{k=1}^N Q_{kj}\frac{\partial}{\partial x_k}.$$

For the equation $\partial_t\tilde{\theta}_- - \kappa_-\Delta\tilde{\theta}_- = \tilde{f}_-$ in Ω_-, if we set $\Theta_- = \mathcal{M}_{-1}\theta_-$, then we have the equation in $\mathbb{R}_-^N$

$$\lambda\Theta_- - \kappa_-\Delta\Theta_- = \mathcal{M}_{-1}f_- + \mathbf{V}(\Theta_-, \mathcal{M}_{-1}, M_{-1})$$

with a nonlinear term $\mathbf{V}(\Theta_-, \mathcal{M}_{-1}, M_{-1})$. Since $\Psi^{-1}(\Gamma_-) = \mathbb{R}_0^N$, we may assume that

$$\mathbf{n}_- = \frac{\left(\frac{\partial x_N}{\partial y_1}, \dots, \frac{\partial x_N}{\partial y_N}\right)}{\left(\sum_{j=1}^N \left|\frac{\partial x_N}{\partial y_j}\right|^2\right)^{1/2}} \quad \text{on } \Gamma_-.$$

Recalling that $\left(\partial x_j/\partial y_\ell\right)(\Phi(x)) = Q_{j\ell}(x)$ and setting $\mathbb{Q}(x) = \sqrt{\sum_{j=1}^N |Q_{Nj}(x)|^2}$, we have

$$\mathbf{n}(\Phi(x)) = (Q_{N1}(x), \dots, Q_{NN}(x))/\mathbb{Q}(x) \quad \text{for } x \in \mathbb{R}_0^N.$$

Thus, we can get new equations in the half space

$$\begin{cases} \lambda\Theta_- - \kappa_-\Delta\Theta_- = \mathcal{M}_{-1}f_- + \mathbf{V}(\Theta_-, \mathcal{M}_{-1}, M_{-1}) & \text{in } \mathbb{R}_-^N, \\ \partial_N\Theta_- + \beta\Theta_- = \mathcal{M}_{-1}h + \mathbf{V}_1(\Theta_-, \mathcal{M}_{-1}, M_{-1}) & \text{on } \mathbb{R}_0^N, \end{cases}$$

with nonlinear terms $\mathbf{V}$ and $\mathbf{V}_1$. Then, we can apply Theorem 3.8 to obtain Theorem 3.9 with the help of Lemma A.3.

4. Proof of Theorem 3.1

4.1. Local solutions

We set $\Phi_j^1(\mathbb{R}_+^N) = \mathcal{H}_{+j}^1$, $\Phi_j^1(\mathbb{R}_-^N) = \mathcal{H}_{-j}^1$, $\Phi_j^1(\mathbb{R}_0^N) = \Gamma_j^1$, $\Phi_j^2(\mathbb{R}_-^N) = \mathcal{H}_{-j}^2$, $\Phi_j^2(\mathbb{R}_0^N) = \Gamma_{-j}^2$ and $\mathcal{H}_{j\pm}^0 = \mathbb{R}^N$. Let $\mathbf{n}_j^1$ and $\mathbf{n}_{-j}^2$ be the unit normal to Γ_j^1 oriented from $\mathcal{H}_{+j}^1$ into $\mathcal{H}_{-j}^1$ and the unit outer normal to Γ_{-j}^2, respectively. Let $\dot{\mathcal{H}}_j^1 = \mathcal{H}_{+j}^1 \cup \mathcal{H}_{-j}^1$. We consider the following problems

$$\begin{cases} \lambda\theta_{\pm j}^0 - \kappa\Delta\theta_{\pm j}^0 = \tilde{\zeta}_{\pm j}^0 f & \text{in } \mathcal{H}_{j\pm}^0, \\ \lambda\theta_j^1 - \kappa\Delta\theta_j^1 = \tilde{\zeta}_j^1 f & \text{in } \dot{\mathcal{H}}_j^1, \\ [\![\kappa\nabla\theta_j^1 \cdot \mathbf{n}_j^1]\!] = \tilde{\zeta}_j^1 g, \quad [\![\theta_j^1]\!] = 0 & \text{on } \Gamma_j^1, \\ \theta_{-j}^2 - \kappa\Delta\theta_{-j}^2 = \tilde{\zeta}_j^1 f_- & \text{in } \mathcal{H}_{-j}^2, \\ \nabla\theta_{-j}^2 \cdot \mathbf{n}_{-j}^2 + \beta\theta_{-j}^2 = \tilde{\zeta}_j^2 h & \text{on } \Gamma_{-j}^2. \end{cases} \tag{4.1}$$

Obviously, by Proposition A.11, $\left\|(\mathbf{n}_j^1, \mathbf{n}_{-j}^2)\right\|_{H_r^2(B_j^i)} \leqslant C_N K_2$. From Theorems 3.4, 3.7 and 3.8, there exists a constant $\lambda_0 \geqslant 1$ and operator families

$$\begin{aligned} \mathcal{T}_{\pm j}^0(\lambda) &\in \mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}\left(L^q(\mathcal{H}_{j\pm}^0), H_q^2(\mathcal{H}_{j\pm}^0)\right)\right), \\ \mathcal{T}_j^1(\lambda) &\in \mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}\left(\mathcal{X}_q^1(\dot{\mathcal{H}}_j^1), H_q^2(\dot{\mathcal{H}}_j^1)\right)\right), \\ \mathcal{T}_j^2(\lambda) &\in \mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}\left(\mathcal{Y}_q^2(\mathcal{H}_{-j}^2), H_q^2(\mathcal{H}_{-j}^2)\right)\right), \end{aligned} \tag{4.2}$$

such that

$$\theta^0_{\pm j} = \mathcal{T}^0_{\pm j}(\lambda)\widetilde{\zeta}^0_{\pm} f, \quad \theta^1_j = \mathcal{T}^1_j(\lambda) F^1_\lambda\left(\widetilde{\zeta}^1_j f, \widetilde{\zeta}^1_j g\right), \quad \theta^2_{-j} = \mathcal{T}^2_j(\lambda) F^2_\lambda\left(\widetilde{\zeta}^2_j f_-, \widetilde{\zeta}^2_j h\right), \tag{4.3}$$

where

$$F^1_\lambda\left(\widetilde{\zeta}^1_j f, \widetilde{\zeta}^1_j g\right) = \left(\widetilde{\zeta}^1_j f, \lambda^{1/2}\widetilde{\zeta}^1_j g, \widetilde{\zeta}^1_j g\right), \quad F^2_\lambda\left(\widetilde{\zeta}^2_j f_-, \widetilde{\zeta}^2_j h\right) = \left(\widetilde{\zeta}^2_j f_-, \lambda^{1/2}\widetilde{\zeta}^2_j h, \widetilde{\zeta}^2_j h\right)$$

is the unique solution to (4.1). Moreover, we have

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}\left(L^q(\mathcal{H}^0_{j\pm}), H^{2-k}_q(\mathcal{H}^0_{j\pm})\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{k/2}\mathcal{T}^0_{\pm j}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}\right) &\leqslant \nu, \\
\mathcal{R}_{\mathcal{L}\left(\mathcal{X}_q(\dot{\mathcal{H}}^1_j), H^{2-k}_q(\dot{\mathcal{H}}^1_j)\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{k/2}\mathcal{T}^1_j(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}\right) &\leqslant \nu, \\
\mathcal{R}_{\mathcal{L}\left(\mathcal{Y}_q(\mathcal{H}^2_{-j}), H^{2-k}_q(\mathcal{H}^2_{-j})\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{k/2}\mathcal{T}^2_j(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}\right) &\leqslant \nu,
\end{aligned} \tag{4.4}$$

with $k = 0, 1, 2$ and some constant ν independent of $j \in \mathbb{N}$. Since the $\mathcal{R}$-boundedness implies the usual boundedness, by (4.4), we have

$$\begin{aligned}
\sum_{k=0}^{2} |\lambda|^{k/2}\left\|\theta^0_{\pm j}\right\|_{H^{2-k}_q(\mathcal{H}^0_{j\pm})} &\leqslant \nu\left\|\widetilde{\zeta}^0_{j\pm} f\right\|_{L^q(\mathcal{H}^0_{j\pm})}, \\
\sum_{k=0}^{2} |\lambda|^{k/2}\left\|\theta^1_j\right\|_{H^{2-k}_q(\dot{\mathcal{H}}^1_j)} &\leqslant \nu\left\|F^1_\lambda\left(\widetilde{\zeta}^1_j f, \widetilde{\zeta}^1_j g\right)\right\|_{\mathcal{X}_q(\dot{\mathcal{H}}^1_j)}, \\
\sum_{k=0}^{2} |\lambda|^{k/2}\left\|\theta^2_{-j}\right\|_{H^{2-k}_q(\mathcal{H}^2_j)} &\leqslant \nu\left\|F^2_\lambda\left(\widetilde{\zeta}^2_j f_-, \widetilde{\zeta}^2_j h\right)\right\|_{\mathcal{Y}_q(\mathcal{H}^2_j)},
\end{aligned} \tag{4.5}$$

with the same constant ν as in (4.4) and $j \in \mathbb{N}$.

4.2. Construction of a parametrix

We define the parametrix $\mathbf{P}(\lambda)\mathbf{F}$ by

$$\begin{aligned}
\mathbf{P}(\lambda)\mathbf{F} &= \sum_{\pm}\sum_{j=1}^{\infty}\zeta^0_{\pm j}\theta^0_{\pm j} + \sum_{j=1}^{\infty}\zeta^1_j\theta^1_j + \sum_{j=1}^{\infty}\zeta^2_j\theta^2_{-j} \\
&= \sum_{j=1}^{\infty}\zeta^0_{+j}\mathcal{T}^0_{+j}(\lambda)\widetilde{\zeta}^0_{+j} f + \sum_{j=1}^{\infty}\zeta^0_{-j}\mathcal{T}^0_{-j}(\lambda)\widetilde{\zeta}^0_{-j} f + \sum_{j=1}^{\infty}\zeta^1_j\mathcal{T}^1_j(\lambda)F^1_\lambda\left(\widetilde{\zeta}^1_j f, \widetilde{\zeta}^1_j g\right) \\
&\quad + \sum_{j=1}^{\infty}\zeta^2_j\mathcal{T}^2_j(\lambda)F^2_\lambda\left(\widetilde{\zeta}^2_j f_-, \widetilde{\zeta}^2_j h\right),
\end{aligned} \tag{4.6}$$

for $\mathbf{F} = (f, g, h) \in E_q(\dot{\Omega})$. Since there are jump quantities on the boundary, we set $F^1_\lambda = F^1_\lambda\left(\widetilde{\zeta}^1_j f, \widetilde{\zeta}^1_j g\right)$, let $E_\pm\left[\mathcal{T}^1_j(\lambda)\Big|_{\mathcal{H}^1_{\pm j}}\right] F^1_\lambda$ be the Lions extension of Γ^1_j into $\mathcal{H}^1_{\pm j}$ such that

$$
\begin{aligned}
&\sum_{k=0}^{2} |\lambda|^{k/2} \left\| E_{\pm}\left[\mathcal{T}_j^1(\lambda)\Big|_{\mathcal{H}^1_{\pm j}} \right] F^1_\lambda \right\|_{H^{2-k}_q\left(\dot{\mathcal{H}}^1_j\right)} \leqslant C\nu \| F^1_\lambda \|_{\mathcal{X}_q\left(\dot{\mathcal{H}}^1_j\right)}, \\
&\partial_x^\alpha \left(E_{\pm}\left[\mathcal{T}_j^1(\lambda)\Big|_{\mathcal{H}^1_{\pm j}} \right] F^1 \right)\Big|_{\Gamma^1_j} = \partial_x^\alpha \left(\mathcal{T}_j^1(\lambda) F^1 \right)\Big|_{\Gamma^1_j},
\end{aligned}
\tag{4.7}
$$

where

$$
f|_{\Gamma^1_j}(x_0) = \lim_{\substack{x \in \mathcal{H}^1_{\pm j} \\ x \to x_0}} f(x) \qquad \text{for } x_0 \in \Gamma^1_j.
$$

Let $F^2_\lambda = F^2_\lambda\left(\widetilde{\zeta}^2_j f_-, \widetilde{\zeta}^2_j h\right)$. Then using these notations we obtain

$$
\begin{aligned}
&[\![\kappa \nabla \left(\zeta^1_j \mathcal{T}^1_j(\lambda) F^1_\lambda \right)]\!] \cdot \mathbf{n}^1_j = \zeta^1_j [\![\kappa \nabla \left(\mathcal{T}^1_j(\lambda) F^1_\lambda \right) \cdot \mathbf{n}^1_j]\!] + R^1_j(\lambda) F^1_\lambda, \\
&\nabla \left(\zeta^2_j \mathcal{T}^2_j(\lambda) F^2_\lambda \right) \cdot \mathbf{n}^2_{-j} = \zeta^2_j \nabla (\mathcal{T}^2_j(\lambda) F^2_\lambda) \cdot \mathbf{n}^2_{-j} + R^2_j(\lambda) F^2_\lambda,
\end{aligned}
\tag{4.8}
$$

where

$$
R^1_j(\lambda) F^1_\lambda = \nabla \left(\zeta^1_j \right) \left\{ \kappa_+ E_+ \left[\mathcal{T}^1_j(\lambda)\Big|_{\mathcal{H}^1_{+j}} \right] F^1_\lambda - \kappa_- E_- \left[\mathcal{T}^1_j(\lambda)\Big|_{\mathcal{H}^1_{-j}} \right] F^1_\lambda \right\}\Big|_{\Gamma^1_j} \cdot \mathbf{n}^1_j,
$$

$$
R^2_j(\lambda) F^2_\lambda = \nabla \left(\zeta^2_j \right) \mathcal{T}^2_j(\lambda) F^2_\lambda \cdot \mathbf{n}^2_{-j}.
$$

We can get $\mathbf{P}(\lambda)\mathbf{F} \in H^2_q(\dot{\Omega})$ by Proposition A.13 and (4.5). Inserting $\theta = \mathbf{P}(\lambda)\mathbf{F}$ into (2.4), and taking into account that $\mathbf{n} = \mathbf{n}^1_j$ on supp $\zeta^1_j \cap \Gamma$ and $\mathbf{n}_- = \mathbf{n}^2_{-j}$ on supp $\zeta^2_j \cap \Gamma_-$, we have

$$
\begin{cases}
\lambda\theta - \kappa \Delta \theta = f + \mathbf{V}^1(\lambda)\mathbf{F} & \text{in } \dot{\Omega}, \\
[\![\kappa \nabla \theta \cdot \mathbf{n}]\!] = g + \mathbf{V}^2(\lambda)\mathbf{F}, \quad [\![\theta]\!] = 0 & \text{on } \Gamma, \\
\nabla \theta_- \cdot \mathbf{n}_- + \beta \theta_- = h + \mathbf{V}^3(\lambda)\mathbf{F} & \text{on } \Gamma_-,
\end{cases}
\tag{4.9}
$$

where

$$
\begin{aligned}
\mathbf{V}^1(\lambda)\mathbf{F} = \kappa \sum_{\pm} \sum_{j=1}^{\infty} &\left[2\left(\nabla \zeta^0_{\pm j}\right) \cdot \left(\nabla \left(\mathcal{T}^0_{\pm j}(\lambda) \widetilde{\zeta}^0_{\pm j} f \right) \right) + \left(\Delta \zeta^0_{\pm j} \right) \mathcal{T}^0_{\pm j}(\lambda) \widetilde{\zeta}^0_{\pm j} f \right] \\
&+ \kappa \left\{ \sum_{j=1}^{\infty} \left[2\left(\nabla \zeta^1_j\right) \cdot \left(\nabla \left(\mathcal{T}^1_j(\lambda) F^1_\lambda \right) \right) + \left(\Delta \zeta^1_j \right) \mathcal{T}^1_j(\lambda) F^1_\lambda \right] \right\} \\
&+ \kappa \left\{ \sum_{j=1}^{\infty} \left[2\left(\nabla \zeta^2_j\right) \cdot \left(\nabla \left(\mathcal{T}^2_j(\lambda) F^2_\lambda \right) \right) + \left(\Delta \zeta^2_j \right) \mathcal{T}^2_j(\lambda) F^2_\lambda \right] \right\},
\end{aligned}
$$

and

$$\mathbf{V}^2(\lambda)\mathbf{F} = \sum_{j=1}^{\infty} R_j^1(\lambda) F_\lambda^1, \quad \mathbf{V}^3(\lambda)\mathbf{F} = \sum_{j=1}^{\infty} R_j^2(\lambda) F_\lambda^2.$$

Let

$$\mathbf{V}(\lambda)\mathbf{F} = \left(\mathbf{V}^1(\lambda)\mathbf{F}, \mathbf{V}^2(\lambda)\mathbf{F}, \mathbf{V}^3(\lambda)\mathbf{F}\right).$$

Then, Proposition A.13 and the estimate (4.5) imply $\mathbf{V}(\lambda) \in E_q(\dot{\Omega})$, and

$$\|F_\lambda \mathbf{V}(\lambda)\mathbf{F}\|_{\mathcal{E}_q(\dot{\Omega})} \leqslant C\lambda_1^{-1/2} \|F_\lambda \mathbf{F}\|_{\mathcal{E}_q(\dot{\Omega})}$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_1}, \lambda_1 \geqslant \lambda_0 \geqslant 1$, where F_λ is the operator given in Theorem 3.1. Since $\|F_\lambda \mathbf{F}\|_{\mathcal{E}_q(\Omega)}, \lambda \neq 0$ are equivalent norms of $E_q(\dot{\Omega})$, we can choose $\lambda_1 \geqslant \lambda_0$ so large that $C\lambda_1^{-1/2} \leqslant 1/2$. We see that $(\mathbf{I} - \mathbf{V}(\lambda))^{-1}$ exists and $\theta = \mathbf{P}(\lambda)(\mathbf{I} - \mathbf{V}(\lambda))^{-1}\mathbf{F}$ is a solution of (2.4). The uniqueness follows from the existence theorem of dual problem.

4.3. Construction of solution operators

For $F = (F_0, F_1, \ldots, F_4) \in \mathcal{E}_q(\dot{\Omega})$, $F^1 = (F_0, F_1, F_2) \in \mathcal{X}_q(\dot{\Omega})$, $F^2 = \left(F_0|_{\Omega_-}, F_3, F_4\right) \in \mathcal{Y}_q(\dot{\Omega})$, we define the following operators

$$\mathcal{P}(\lambda)\mathbf{F} = \sum_{\pm}\sum_{j=1}^{\infty} \zeta_{\pm j}^0 \mathcal{T}_{\pm j}^0(\lambda) F_0 + \sum_{i=1}^{2}\sum_{j=1}^{\infty} \zeta_j^i \mathcal{T}_j^i(\lambda) F^i,$$

$$\begin{aligned}
\mathcal{V}^1(\lambda)\mathbf{F} = \kappa \sum_{\pm}\sum_{j=1}^{\infty} &\left[2\left(\nabla\zeta_{\pm j}^0\right)\cdot\left(\nabla\left(\mathcal{T}_{\pm j}^0(\lambda)\widetilde{\zeta}_{\pm j}^0 F_0\right)\right) + \left(\Delta\zeta_{\pm j}^0\right)\mathcal{T}_{\pm j}^0(\lambda)\widetilde{\zeta}_{\pm j}^0 F_0\right] \\
&+ \kappa\left\{\sum_{j=1}^{\infty}\left[2\left(\nabla\zeta_j^1\right)\cdot\left(\nabla\left(\mathcal{T}_j^1(\lambda)\widetilde{\zeta}_j^1 F^1\right)\right) + \left(\Delta\zeta_j^1\right)\mathcal{T}_j^1(\lambda)\widetilde{\zeta}_j^1 F^1\right]\right\} \\
&+ \kappa\left\{\sum_{j=1}^{\infty}\left[2\left(\nabla\zeta_j^2\right)\cdot\left(\nabla\left(\mathcal{T}_j^2(\lambda)\widetilde{\zeta}_j^2 F^2\right)\right) + \left(\Delta\zeta_j^2\right)\mathcal{T}_j^2(\lambda)\widetilde{\zeta}_j^2 F^2\right]\right\},
\end{aligned}$$

$$\mathcal{V}^2(\lambda)\mathbf{F} = \sum_{j=1}^{\infty} R_j^1(\lambda)\widetilde{\zeta}_j^1 F^1, \quad \mathcal{V}^3(\lambda)\mathbf{F} = \sum_{j=1}^{\infty} R_j^2(\lambda)\widetilde{\zeta}_j^2 F^2,$$

$$\mathcal{V}(\lambda)\mathbf{F} = \left(\mathcal{V}^1(\lambda)\mathbf{F}, \mathcal{V}^2(\lambda)\mathbf{F}, \mathcal{V}^3(\lambda)\mathbf{F}\right).$$

Obviously, $\mathbf{P}(\lambda)\mathbf{F} = \mathcal{P}(\lambda)F_\lambda\mathbf{F}$ and $\mathbf{V}(\lambda)\mathbf{F} = \mathcal{V}(\lambda)F_\lambda\mathbf{F}$. By (4.3), (4.4) and Proposition A.13, we see that

$$\mathcal{P}(\lambda) \in \mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}\left(\mathcal{E}_q(\dot{\Omega}), H_q^2(\dot{\Omega})\right)\right),$$
$$\mathcal{V}(\lambda) \in \mathrm{Hol}\left(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}\left(\mathcal{E}_q(\dot{\Omega}), E_q(\dot{\Omega})\right)\right).$$

Moreover, by (4.4) and Proposition A.13 we have

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}\left(\mathcal{E}_q(\dot{\Omega}), H_q^{2-j}(\dot{\Omega})\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{j/2}\mathcal{P}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_2}\right\}\right) \leqslant C\nu, \quad (j=0,1,2), \\
&\mathcal{R}_{\mathcal{L}(\mathcal{E}_q(\dot{\Omega}), \mathcal{E}_q(\dot{\Omega}))}\left(\left\{(\tau\partial_\tau)^\ell F_\lambda \mathcal{V}(\lambda) : \lambda \in \Sigma_{\epsilon,\lambda_2}\right\}\right) \leqslant C\lambda_2^{-1/2}\nu, \quad (\ell=0,1),
\end{aligned} \tag{4.10}$$

for any $\lambda_2 \geqslant \lambda_1$. By (4.10), $\mathcal{Z}(\lambda)\mathbf{F} = \mathcal{P}(\lambda)(\mathbf{I} - F_\lambda \mathcal{V}(\lambda))^{-1}\mathbf{F}$ exists and

$$\mathcal{R}_{\mathcal{L}\left(\mathcal{E}_q(\Omega), H_q^{2-j}(\dot{\Omega})\right)}\left(\left\{(\tau\partial_\tau)^\ell\left(\lambda^{j/2}\mathcal{Z}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_2}\right\}\right) \leqslant C\nu,$$

for $\ell = 0, 1$ and $j = 0, 1, 2$. Since $\mathcal{V}(\lambda)F_\lambda \mathbf{F} = \mathbf{V}(\lambda)\mathbf{F}$, we have

$$\begin{aligned}
F_\lambda(\mathbf{I} - \mathbf{V}(\lambda))^{-1} &= \sum_{j=0}^{\infty} F_\lambda \mathbf{V}(\lambda)^j = \sum_{j=0}^{\infty} F_\lambda\left(\mathcal{V}(\lambda)F_\lambda\right)^j = \sum_{j=0}^{\infty}\left(F_\lambda \mathcal{V}(\lambda)\right)^j F_\lambda \\
&= \left(\mathbf{I} - F_\lambda \mathcal{V}(\lambda)\right)^{-1} F_\lambda.
\end{aligned}$$

Thus,

$$\begin{aligned}
\theta &= \mathbf{P}(\lambda)(\mathbf{I} - \mathbf{V}(\lambda))^{-1}\mathbf{F} = \mathcal{P}(\lambda)F_\lambda(\mathbf{I} - \mathbf{V}(\lambda))^{-1}\mathbf{F} \\
&= \mathcal{P}(\lambda)\left(\mathbf{I} - F_\lambda \mathcal{V}(\lambda)\right)^{-1} F_\lambda \mathbf{F} = \mathcal{Z}(\lambda)F_\lambda \mathbf{F}.
\end{aligned}$$

Therefore, we complete the proof of Theorem 3.1.

5. Proof of Theorem 2.1

Now, we prove Theorem 2.1 with the help of Theorem 3.1. The key tool in the proof of the maximal regularity results is the Weis operator-valued Fourier multiplier theorem, i.e., Theorem A.15.

We first consider the equations of $\theta_1 = \sum_\pm \theta_{1\pm}\chi_{\Omega_\pm}$:

$$\begin{cases}
\partial_t\theta_1 - \kappa\Delta\theta_1 = f & \text{in } \dot{\Omega} \times \mathbb{R}, \\
[\![\kappa\nabla\theta_1 \cdot \mathbf{n}]\!] = g, \quad [\![\theta_1]\!] = 0 & \text{on } \Gamma \times \mathbb{R}, \\
\nabla\theta_{1-} \cdot \mathbf{n}_- + \beta\theta_{1-} = h & \text{on } \Gamma_- \times \mathbb{R}.
\end{cases} \tag{5.1}$$

Let $\mathcal{F}_L$ and $\mathcal{F}_L^{-1}$ be the Laplace transform and the inverse Laplace transform, respectively, defined by

$$\hat{f}(\lambda) = \mathcal{F}_L[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)dt, \quad \mathcal{F}_L^{-1}[g(\lambda)](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\tau)d\tau,$$

for $\lambda = \gamma + i\tau \in \mathbb{C}$. Obviously,

$$\mathcal{F}_L[f](\lambda) = \int_{-\infty}^{\infty} e^{-i\tau t} e^{-\gamma t} f(t)dt = \mathcal{F}\left[e^{-\gamma t} f\right](\tau),$$

$$\mathcal{F}_L^{-1}[g](t) = e^{\gamma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\tau t} g(\tau)d\tau = e^{\gamma t}\mathcal{F}^{-1}[g](t), \quad \mathcal{F}_L\mathcal{F}_L^{-1} = \mathcal{F}_L^{-1}\mathcal{F}_L = \mathbf{I}.$$

Applying the Laplace transform to (5.1) gives

$$\begin{cases} \lambda\hat{\theta}_1 - \kappa\Delta\hat{\theta}_1 = \hat{f} & \text{in } \dot{\Omega} \times \mathbb{R}, \\ [\![\kappa\nabla\hat{\theta}_1 \cdot \mathbf{n}]\!] = \hat{g}, \quad [\![\hat{\theta}_1]\!] = 0 & \text{on } \Gamma \times \mathbb{R}, \\ \nabla\hat{\theta}_{1-} \cdot \mathbf{n}_- + \beta\hat{\theta}_{1-} = \hat{h} & \text{on } \Gamma_- \times \mathbb{R}. \end{cases} \tag{5.2}$$

By Theorem 3.1, we have $\hat{\theta}_1 = \mathcal{Z}(\lambda)F_\lambda\mathbf{G}$ for $\lambda \in \Sigma_{\epsilon,\lambda_1}$, where

$$F_\lambda\mathbf{G} = \left(\hat{f}(\lambda), \lambda^{1/2}\hat{g}(\lambda), \hat{g}(\lambda), \lambda^{1/2}\hat{h}(\lambda), \hat{h}(\lambda)\right).$$

Let

$$\Lambda_\gamma^{1/2} f = \mathcal{F}_L^{-1}\left[\lambda^{1/2}\mathcal{F}_L[f]\right] = e^{\gamma t}\mathcal{F}_\tau^{-1}\left[\lambda^{1/2}\mathcal{F}\left[e^{-\gamma t} f\right]\right], \quad \lambda^{1/2}\hat{f}(\lambda) = \mathcal{F}\left[e^{-\gamma t}\Lambda_\gamma^{1/2} f\right],$$

and define

$$\theta_1(\cdot, t) = \mathcal{F}_L^{-1}\left[\mathcal{Z}(\lambda)F_\lambda\mathbf{G}\right] = e^{\gamma t}\mathcal{F}_\tau^{-1}\left[\mathcal{Z}(\lambda)\mathcal{F}\left[e^{-\gamma t}G(t)\right](\tau)\right],$$

with $G(t) = \left(f, \Lambda_\gamma^{1/2} g, g, \Lambda_\gamma^{1/2} h, h\right)$, where γ is chosen such that $\gamma > \lambda_1$, and so $\gamma + i\tau \in \Sigma_{\epsilon,\lambda_1}$ for any $\tau \in \mathbb{R}$. By Cauchy's theorem in the theory of one complex variable, θ_1 is independent of choice of γ whenever $\gamma > \lambda_1$. Noting that

$$\partial_t\theta_1 = \mathcal{F}_L^{-1}\left[\lambda\mathcal{Z}(\lambda)F_\lambda\mathbf{G}\right] = e^{\gamma t}\mathcal{F}_\tau^{-1}\left[\lambda\mathcal{Z}(\lambda)\mathcal{F}\left[e^{-\gamma t}G(t)\right](\tau)\right]$$

and all Lebesgue spaces and Sobolev spaces on $\mathbb{R}^N$ are UMD spaces, by applying Theorems 3.1 and A.15, we have

$$\begin{aligned} &\|e^{-\gamma t}\partial_t\theta_1\|_{L^p(\mathbb{R}, L^q(\dot{\Omega}))} + \|e^{-\gamma t}\theta_1\|_{L^p\left(\mathbb{R}, H_q^2(\dot{\Omega})\right)} \\ &\quad \leqslant C\Big\{\|e^{-\gamma t} f\|_{L^p(\mathbb{R}, L^q(\dot{\Omega}))} + \|e^{-\gamma t} g\|_{L^p\left(\mathbb{R}, H_q^1(\dot{\Omega})\right)} + \left\|e^{-\gamma t}\Lambda_\gamma^{1/2} g\right\|_{L^p(\mathbb{R}, L^q(\dot{\Omega}))} \\ &\qquad + \|e^{-\gamma t} h\|_{L^p\left(\mathbb{R}, H_q^1(\Omega_-)\right)} + \left\|e^{-\gamma t}\Lambda_\gamma^{1/2} h\right\|_{L^p(\mathbb{R}, L^q(\Omega_-))}\Big\}. \end{aligned} \tag{5.3}$$

Since $\left|\lambda^{1/2}\left(1+\tau^2\right)^{-1/4}\right| \leqslant C\left(1+\gamma^{1/2}\right)$, we get

$$\begin{aligned}&\left\|e^{-\gamma t}\Lambda_\gamma^{1/2} g\right\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))}+\left\|e^{-\gamma t}\Lambda_\gamma^{1/2} h\right\|_{L^p\left(\mathbb{R},L^q(\Omega_-)\right)}\\&\qquad\leqslant C\left(1+\gamma^{1/2}\right)\left(\left\|e^{-\gamma t} g\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot{\Omega}))}+\left\|e^{-\gamma t} h\right\|_{H_p^{1/2}(\mathbb{R},L^q(\Omega_-))}\right).\end{aligned}$$

We now prove that $\theta_1=0$ for $t<0$. Since $|\gamma/\lambda| \leqslant 1$, we have

$$\gamma\left\|e^{-\gamma t}\theta_1\right\|_{L^p\left(\mathbb{R},L^q(\dot{\Omega})\right)} \leqslant\left\|e^{-\gamma t}\partial_t\theta_1\right\|_{L^p\left(\mathbb{R},L^q(\dot{\Omega})\right)},$$

and by (5.3) we obtain

$$\begin{aligned}\|\theta_1\|_{L^p\left((-\infty,0),L^q(\dot{\Omega})\right)} &\leqslant\left\|e^{-\gamma t}\theta_1\right\|_{L^p\left(\mathbb{R},L^q(\dot{\Omega})\right)} \leqslant \gamma^{-1}\left\|e^{-\gamma t}\partial_t\theta_1\right\|_{L^p\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\&\leqslant \gamma^{-1} C\left\{\left\|e^{-\gamma t} f\right\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))}+\left\|e^{-\gamma t} g\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}+\left\|e^{-\gamma t} h\right\|_{L^p\left(\mathbb{R},H_q^1(\Omega_-)\right)}\right.\\&\left.+\left(1+\gamma^{1/2}\right)\left(\left\|e^{-\gamma t} g\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot{\Omega}))}+\left\|e^{-\gamma t} h\right\|_{H_p^{1/2}(\mathbb{R},L^q(\Omega_-))}\right)\right\}.\end{aligned}$$

Thus, letting $\gamma\to\infty$, we have $\|\theta_1\|_{L^p\left((-\infty,0),L^q(\dot{\Omega})\right)}=0$, which leads to $\theta_1=0$ for $t<0$.

Next, we consider the initial value problem of θ_2

$$\begin{cases}\partial_t\theta_2-\kappa\Delta\theta_2=0 & \text{in } \dot{\Omega}\times(0,\infty),\\ [\![\kappa\nabla\theta_2\cdot\mathbf{n}]\!]=0, \quad [\![\theta_2]\!]=0 & \text{on } \Gamma\times(0,\infty),\\ \nabla\theta_{2-}\cdot\mathbf{n}_-+\beta\theta_{2-}=0 & \text{on } \Gamma_-\times(0,\infty).\\ \theta_2|_{t=0}=\theta_0-\theta_1|_{t=0} & \text{in } \dot{\Omega}.\end{cases}\tag{5.4}$$

Define

$$J_q(\dot{\Omega})=\left\{\theta\in H_q^2(\dot{\Omega}):[\![\kappa\nabla\theta\cdot\mathbf{n}]\!]=0,\quad [\![\theta]\!]=0 \text{ on } \Gamma,\quad \nabla\theta_-\cdot\mathbf{n}_-=0 \text{ on } \Gamma_-\right\}.$$

We consider

$$\lambda\theta_2-\kappa\Delta\theta_2=f,\quad \text{for}\quad \theta_2\in J_q.$$

Since the $\mathcal{R}$-boundedness implies the usual boundedness, by Theorem 3.1, we have $\rho(\kappa\Delta)\supset\Sigma_{\epsilon,\lambda_1}$ and

$$|\lambda|\left\|(\lambda-\kappa\Delta)^{-1} f\right\|_{L^q(\dot{\Omega})}+\left\|(\lambda-\kappa\Delta)^{-1} f\right\|_{H_q^2(\dot{\Omega})} \leqslant C\|f\|_{L^q(\dot{\Omega})}.$$

By the theory of analytic semigroups, we see that $\kappa\Delta$ generates C^0 analytic semigroups $T(t)$ on $J_q(\dot{\Omega})$. Let $\theta_2=T(t)(\theta_0-\theta_1|_{t=0})$. Then, by a standard real-interpolation method (cf. [34]), we have the following theorem.

Theorem 5.1. *Let* $1 < p, q < \infty$*, we set*

$$\mathcal{D}_{q,p}(\dot{\Omega}) = \left[L^q(\dot{\Omega}), J_q(\dot{\Omega})\right]_{1-1/p,p},$$

where $[\cdot,\cdot]_{1-1/p,p}$ *denotes a real interpolation functor. Then for any* $\theta_2|_{t=0} \in \mathcal{D}_{q,p}(\dot{\Omega})$ *and* $\gamma \geqslant \lambda_1$ *with some constant* $C > 0$*, where* λ_1 *is the same constant as in Theorem 3.1, the equation (5.4) has a unique solution* θ_2 *with*

$$e^{-\gamma t}\theta_2 \in L^p\left((0,\infty), H^2_q(\dot{\Omega})\right) \cap H^1_p\left((0,\infty), L^q(\dot{\Omega})\right),$$

possessing the estimate

$$\left\|e^{-\gamma t}\partial_t\theta_2\right\|_{L^p((0,\infty),L^q(\Omega))} + \left\|e^{-\gamma t}\theta_2\right\|_{L^p\left((0,\infty),H^2_q(\dot{\Omega})\right)} \leqslant C\|\theta_2|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})}. \tag{5.5}$$

Remark 5.2. Let us define the Besov space $B^{2(1-1/p)}_{q,p}(\dot{\Omega})$ by the real interpolation

$$B^{2(1-1/p)}_{q,p}(\dot{\Omega}) = \left[L^q(\dot{\Omega}), H^2_q(\dot{\Omega})\right]_{1-1/p,p}.$$

From [19, Remark 2.3], we have

$$\mathcal{D}_{q,p}(\dot{\Omega}) = \begin{cases} \left\{\theta \in B^{2(1-1/p)}_{q,p}(\dot{\Omega}) : \theta \in J_p(\dot{\Omega})\right\} & \text{when } 2(1-1/p) > 1+1/q, \\ \left\{\theta \in B^{2(1-1/p)}_{q,p}(\dot{\Omega}) : [\![\theta]\!]|_\Gamma = 0\right\} & \text{when } 1/q < 2(1-1/p) < 1+1/q, \\ B^{2(1-1/p)}_{q,p}(\Omega) & \text{when } 2(1-1/p) < 1/q. \end{cases}$$

By Remark 5.2 and the compatibility condition (2.2), we have $\theta_0 - \theta_1|_{t=0} \in \mathcal{D}_{q,p}(\dot{\Omega})$, then

$$\|\theta_2|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})} < \|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})} + \|\theta_1|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})}.$$

By the interpolation theory in [20], we know that

$$\|\theta_1|_{t=0}\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})} \lesssim \left\|e^{-\gamma t}\partial_t\theta_1\right\|_{L^p((0,\infty),L^q(\dot{\Omega}))} + \left\|e^{-\gamma t}\theta_1\right\|_{L^p\left((0,\infty),H^2_q(\dot{\Omega})\right)}.$$

Thus, $\theta = \theta_1 + \theta_2$ is a solution of (2.1).

We finally prove the uniqueness. For $t > 0$, the uniqueness of θ_1 follows from Theorem 3.1 and the uniqueness of θ_2 follows from the existence of analytic semigroups. This proves the uniqueness, which completes the proof of Theorem 2.1.

6. Maximal regularity theorem in a finite time interval

First, we introduce the extension map $\iota : L_{1,\mathrm{loc}}(\Omega) \to L_{1,\mathrm{loc}}\left(\mathbb{R}^N\right)$ possessing the following properties:

(i) For any $1 < q < \infty$ and $f \in H_q^1(\Omega)$, $\iota f \in H_q^1\left(\mathbb{R}^N\right)$, $\iota f = f$ in Ω and $\|\iota f\|_{H_q^i(\mathbb{R}^N)} \leqslant C_q \|f\|_{H_q^i(\Omega)}$ for $i = 0, 1$ with some constant C_q depending on q and Ω.

(ii) For any $1 < q < \infty$ and $f \in H_q^1(\Omega)$, $\left\|(1-\Delta)^{-1/2}\iota(\nabla f)\right\|_{L^q(\mathbb{R}^N)} \leqslant C_q\|f\|_{L^q(\Omega)}$ with some constant C_q depending on q and Ω. Here, $(1-\Delta)^s$ is the operator defined by $(1-\Delta)^s f = \mathcal{F}_\xi^{-1}\left[\left(1+|\xi|^2\right)^{s/2}\mathcal{F}[f]\right]$ for $s \in \mathbb{R}$.

In the following, such extension map ι is fixed. We define $H_q^{-1}(\Omega)$ by

$$H_q^{-1}(\Omega) = \left\{f \in L_{1,\mathrm{loc}}(\Omega) : (1-\Delta)^{-1/2}\iota f \in L^q(\Omega)\right\}.$$

As proved in [27], we have

$$\begin{aligned}\left\|e^{-\gamma t}\Lambda_\gamma^{1/2} f\right\|_{L^p(\mathbb{R},L^q(\Omega))} \leqslant C\Big\{&\left\|e^{-\gamma t}\partial_t\left[(1-\Delta)^{-1/2}(\iota f)\right]\right\|_{L^p(\mathbb{R},L^q(\mathbb{R}^N))} \\ &+ \left\|e^{-\gamma t} f\right\|_{L^p\left(\mathbb{R},H_q^1(\Omega)\right)}\Big\},\end{aligned} \tag{6.1}$$

for any $\gamma \geqslant \gamma_0$. We can set $\iota = \sum_\pm \iota_\pm\chi_{\Omega_\pm}$ and $\iota_\pm : L_{1,\mathrm{loc}}(\Omega_\pm) \to L_{1,\mathrm{loc}}(\mathbb{R}^N)$. Combining Theorem 1.1 with (6.1), we have the following theorem.

Theorem 6.1. *Let $1 < p, q < \infty$ with $2/p + 1/q \notin \{1, 2\}$, T be any positive number with $t \in (0, T]$. Assume that Ω is a uniform $W_r^{2-1/r}$ domain. Let $\mathbf{u}_0 \in B_{q,p}^{2(1-1/p)}(\dot\Omega)$ and $\theta_0 \in B_{q,p}^{2(1-1/p)}(\dot\Omega)$ be initial data for (1.2), $\mathbf{a}(\xi, t) \in L^p((0, T), L^q(\dot\Omega))$ and $b(\xi, t) \in L^p\Big((0, T), H_q^1(\Omega_-)\Big) \cap H_p^1\Big((0, T), H_q^{-1}(\Omega_-)\Big)$. Let $\mathbf{N}_1(\mathbf{u}, \mathbf{a}, \theta), \ldots, N_6(\mathbf{u}, \theta_-)$ be functions appearing on the right side of (1.2) satisfying the conditions:*

$$\begin{aligned}&\mathbf{N}_1, N_5 \in L^p\left((0, T), L^q(\dot\Omega)\right), \quad \mathbf{N}_3 \in H_p^1\left((0, T), L^q(\dot\Omega)\right),\\ &N_2, \mathbf{N}_4, N_6 \in L^p\left((0, T), H_q^1(\dot\Omega)\right) \cap H_p^1\left((0, T), H_q^{-1}(\dot\Omega)\right).\end{aligned}$$

Assume that the compatibility conditions $(N_2, \mathbf{N}_3, \mathbf{N}_4, N_6, b)\,|_{t=0} = 0$, (1.4), (1.5) and (1.6) hold. Then, problem (1.2) admits a unique solution $(\mathbf{u}, \theta)$ with

$$\begin{aligned}&\mathbf{u} \in L^p\left((0, T), H_q^2(\dot\Omega)\right) \cap H_p^1\left((0, T), L^q(\dot\Omega)\right),\\ &\theta \in L^p\left((0, T), H_q^2(\dot\Omega)\right) \cap H_p^1\left((0, T), L^q(\dot\Omega)\right),\end{aligned}$$

possessing the estimate for any $t \in (0, T]$

$$
\begin{aligned}
&\|\partial_t(\mathbf{u},\theta)\|_{L^p((0,t),L^q(\dot\Omega))}+\|(\mathbf{u},\theta)\|_{L^p\left((0,t),H^2_q(\dot\Omega)\right)}\\
&\lesssim e^{\gamma t}\Big\{\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}+\|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}+\left\|e^{-\gamma t}\,(\mathbf{a},\mathbf{N}_1,N_5)\right\|_{L^p((0,t),L^q(\dot\Omega))}\\
&\quad+\|\partial_t\mathbf{N}_3\|_{L^p((0,t),L^q(\dot\Omega))}+\|(N_2,\mathbf{N}_4,N_6)\|_{L^p\left((0,t),H^1_q(\dot\Omega)\right)}+\|b\|_{L^p\left((0,t),H^1_q(\Omega_-)\right)}\\
&\quad+\left\|\partial_s\left[(1-\Delta)^{-1/2}(\iota N_2,\iota\mathbf{N}_4,\iota N_6)\right]\right\|_{L^p((0,t),L^q(\mathbb{R}^N))}\\
&\quad+\left\|\partial_s\left[(1-\Delta)^{-1/2}\iota_- b\right]\right\|_{L^p((0,t),L^q(\mathbb{R}^N))}\Big\},
\end{aligned}
$$

for any $\gamma\geqslant\gamma_0$ and some positive number γ_0.

Proof. Given any function $f(\cdot,t)$ defined on $(0,\infty)$, let f_0 denote the zero extension of f to $(-\infty,0)$, namely $f_0(\cdot,t)=f(\cdot,t)$ for $t\in(0,\infty)$ and $f_0(\cdot,t)=0$ for $t\in(-\infty,0)$. Let t be any number in $(0,T]$ and e_t be an operator defined by

$$
[e_t f](\cdot,s)=\begin{cases} f_0(\cdot,s) & \text{for } s\leqslant t,\\ f_0(\cdot,2t-s) & \text{for } s\geqslant t.\end{cases}
$$

Obviously, $[e_t f](\cdot,s)=0$ for $s\notin(0,2t)$. Moreover, if $f|_{t=0}=0$, then we have

$$
\partial_s[e_t f](\cdot,s)=\begin{cases} 0 & \text{for } s\notin(0,2t),\\ (\partial_s f)(\cdot,s) & \text{for } s\in(0,t),\\ -(\partial_s f)(\cdot,2t-s) & \text{for } s\in(t,2t).\end{cases} \tag{6.2}
$$

Let $\mathbf{U}(\cdot,s)=[e_t\mathbf{u}](\cdot,s)$ and $\Theta(\cdot,s)=[e_t\theta](\cdot,s)$ be solutions to the equations: for $s\in(0,\infty)$

$$
\begin{cases}
\rho\partial_s\mathbf{U}-\mathrm{Div}(\mu\mathbf{D}(\mathbf{U})-e_t q\mathbf{I})=e_t\mathbf{N}_1+e_t\mathbf{a}+\beta\mathbf{g}\Theta & \text{in } \dot\Omega,\\
\mathrm{div}\,\mathbf{U}=e_t N_2=\mathrm{div}(e_t\mathbf{N}_3) & \text{in } \dot\Omega,\\
[\![(\mu\mathbf{D}(\mathbf{U})-e_t q\mathbf{I})\mathbf{n}]\!]=e_t\mathbf{N}_4,\quad [\![\mathbf{U}]\!]=0 & \text{on } \Gamma,\\
\partial_s\Theta-\kappa\Delta\Theta=e_t N_5 & \text{in } \dot\Omega,\\
[\![\kappa\nabla\Theta\cdot\mathbf{n}]\!]=e_t N_6,\quad [\![\Theta]\!]=0 & \text{on } \Gamma,\\
\mathbf{U}=0,\quad \nabla\Theta_-\cdot\mathbf{n}_-+\beta\Theta_-=e_t b & \text{on } \Gamma_-,\\
\mathbf{U}|_{s=0}=\mathbf{u}_0,\quad \Theta|_{s=0}=\theta_0 & \text{in } \dot\Omega.
\end{cases} \tag{6.3}
$$

Since $e_{t_1}f=e_{t_2}f$ for $0<t_1,t_2\leqslant T$, the uniqueness of solutions yields that

$$
e_{t_1}(\mathbf{u},\theta)(\cdot,s)=e_{t_2}(\mathbf{u},\theta)(\cdot,s)
$$

for $s\in[0,t_1]$ with $0<t_1<t_2\leqslant T$. By (5.3) and (5.5), we easily have

$$
\begin{aligned}
&\left\|e^{-\gamma s}\partial_s\Theta\right\|_{L^p\left((0,\infty),L^q(\dot\Omega)\right)}+\left\|e^{-\gamma s}\Theta\right\|_{L^p\left((0,\infty),H^2_q(\dot\Omega)\right)}\\
\lesssim{}&\|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}+\left\|e^{-\gamma s}e_tN_5\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}+\left\|e^{-\gamma s}e_tN_6\right\|_{L^p\left(\mathbb{R},H^1_q(\dot\Omega)\right)}\\
&+\left\|e^{-\gamma s}\Lambda_\gamma^{1/2}e_tN_6\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}+\left\|e^{-\gamma s}e_tb\right\|_{L^p\left(\mathbb{R},H^1_q(\Omega_-)\right)}+\left\|e^{-\gamma s}\Lambda_\gamma^{1/2}e_tb\right\|_{L^p\left(\mathbb{R},L^q(\Omega_-)\right)}.
\end{aligned}
$$

By the compatibility conditions, (6.1) and (6.2), we have

$$
\begin{aligned}
&\left\|e^{-\gamma s}\Lambda_\gamma^{1/2}e_tb\right\|_{L^p\left(\mathbb{R},L^q(\Omega_-)\right)}\\
\lesssim{}&\left\|e^{-\gamma s}b\right\|_{L^p\left((0,t),H^1_q(\Omega_-)\right)}+\left\|e^{-\gamma s}\partial_s\left[(1-\Delta)^{-1/2}(\iota_-b)\right]\right\|_{L^p\left((0,t),L^q\left(\mathbb{R}^N\right)\right)}.
\end{aligned}
$$

Other terms can be estimated as similarly as in [27]. Setting $\mathbf{u}=e_T\mathbf{u}$ and $\theta=e_T\theta$, and noting that $(\mathbf{u}(\cdot,s),\theta(\cdot,s))=(e_t\mathbf{u}(\cdot,s),e_t\theta(\cdot,s))$ for $0<s<t$, we complete the proof of Theorem 6.1. □

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Appendix A. Notations and useful results

A.1. Further notations

For any scalar function $f=f(x)$ and N-vector function $\mathbf{g}=(g_1(x),\dots,g_N(x))$, we write

$$
\begin{aligned}
&\nabla f=(\partial_1 f(x),\dots,\partial_N f(x)),\quad \nabla\mathbf{g}=(\nabla g_1(x),\dots,\nabla g_N(x)),\\
&\operatorname{div}\mathbf{g}=\sum_{j=1}^N\partial_j g_j(x),\quad \nabla^2 f=\left(\partial_i\partial_j f\right)_{i,j=1}^N,\quad \nabla^2\mathbf{g}=(\nabla^2 g_1,\dots,\nabla^2 g_N).
\end{aligned}
$$

For an open set Ω of $\mathbb{R}^N$, $p,q\in[1,\infty]$ and $s\in\mathbb{R}$, let $L^q(\Omega)$, $H^s_q(\Omega)$ and $B^s_{q,p}(\Omega)$ denote Lebesgue spaces, Sobolev spaces and Besov spaces on Ω, with norms $\|\cdot\|_{L^q(\Omega)}$, $\|\cdot\|_{H^s_q(\Omega)}$ and $\|\cdot\|_{B^s_{q,p}(\Omega)}$, respectively. Let

$$
X(\dot\Omega)=\left\{f: f|_{\Omega_\pm}\in X(\Omega_\pm)\right\},\quad \|f\|_{X(\dot\Omega)}=\left\|f|_{\Omega_+}\right\|_{X(\Omega_+)}+\left\|f|_{\Omega_-}\right\|_{X(\Omega_-)},
$$

for $X\in\left\{L^q,H^s_q,B^s_{q,p}\right\}$. For simplicity, we write $\|\mathbf{g}\|_{X(\dot\Omega)^N}=\|\mathbf{g}\|_{X(\dot\Omega)}$. For a Banach space X and any time interval (a,b), $L^p((a,b),X)$ and $H^s_q((a,b),X)$ denote the standard X-valued Lebesgue spaces and X-valued Sobolev spaces with norms $\|\cdot\|_{L^p((a,b),X)}$ and $\|\cdot\|_{H^s_q((a,b),X)}$, respectively. For any two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y. Let $\mathcal{D}(\mathbb{R},X)$ denote the space of X-valued distributions. $\mathcal{S}(\mathbb{R},X)$ denotes the space of X-valued Schwartz functions and $\mathcal{S}'(\mathbb{R},X)=\mathcal{L}(\mathcal{S}(\mathbb{R},X)$ is the space of

X-valued tempered distributions. For a domain U in $\mathbb{C}$, let $\mathrm{Hol}(U, \mathcal{L}(X, Y))$ be the set of all $\mathcal{L}(X, Y)$-valued holomorphic functions defined on U, where $\mathbb{C}$ denotes the set of all complex numbers. Set

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| \leqslant \pi - \epsilon\}, \quad \Sigma_{\epsilon,\lambda_0} = \{\lambda \in \Sigma_\epsilon : |\lambda| \geqslant \lambda_0\}. \tag{A.1}$$

For any N-vectors $\mathbf{a}$ and $\mathbf{b}$, we set $\mathbf{a}\cdot\mathbf{b} = (\mathbf{a}, \mathbf{b}) = \sum_{j=1}^N a_i b_i$, and the tangential component $\mathbf{a}_\tau$ of $\mathbf{a}$ with respect to the normal $\mathbf{n}$ is defined by $\mathbf{a}_\tau = \mathbf{a} - (\mathbf{a}, \mathbf{n})\,\mathbf{n}$. For complex-valued functions f and g, we set $(f, g)_\Omega = \int_\Omega f(x)\overline{g(x)}dx$ where $\overline{g(x)}$ is the complex conjugate of $g(x)$, and for any two N-vector functions $\mathbf{f}$ and $\mathbf{g}$, denote $(\mathbf{f}, \mathbf{g})_\Omega = \sum_{j=1}^N \left(f_j, g_j\right)_\Omega$. Let $1 < q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$, we introduce the following spaces

$$\hat{H}_q^1(\Omega) := \left\{u \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla u \in L_q(\Omega)\right\},$$
$$\mathcal{D}(\Omega) := \left\{\mathbf{f} \in L_q(\Omega) \mid (\mathbf{f}, \nabla\varphi)_\Omega = 0 \quad \text{for any} \quad \varphi \in \hat{H}_{q'}^1(\Omega)\right\}.$$

Since $C_0^\infty(\Omega) \in \hat{H}_q^1(\Omega)$, we see that $\operatorname{div}\mathbf{f} = 0$ in Ω provided $\mathbf{f} \in \mathcal{D}(\Omega)$. But, the opposite direction does not hold in general.

Next, we recall two definitions.

Definition A.1. Let both X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called to be *$\mathcal{R}$-bounded* on $\mathcal{L}(X, Y)$, if there exist some constants $C > 0$ and $p \in [1, \infty)$ such that for each $n \in \mathbb{N}$, $T_j \in \mathcal{L}(X, Y)$ and $f_j \in X (j = 1, \ldots, n)$, we have

$$\left\|\sum_{j=1}^n r_j T_j f_j\right\|_{L^p((0,1),Y)} \leqslant C \left\|\sum_{j=1}^n r_j f_j\right\|_{L^p((0,1),X)}.$$

Here, the Rademacher functions $\left\{r_j\right\}_{j=1}^n$ are defined from $[0, 1]$ into $\{-1, 1\}$. The smallest of such C's is called the *$\mathcal{R}$-bound of $\mathcal{T}$* on $\mathcal{L}(X, Y)$, and denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}\mathcal{T}$.

Definition A.2. Let $1 < r < \infty$, and Ω be a domain in $\mathbb{R}^N$ with boundaries Γ and Γ_-. We say that Ω is a *uniform $W_r^{2-1/r}$ domain*, if there exist some positive constants α, β, γ and K such that

(1) for any $x_0 = (x_{01}, x_{02}, \ldots, x_{0N}) \in \Gamma$, there exist a coordinate number j and a $W_r^{2-1/r}$ function $h(x')$ (where $x' = \left(x_1, \ldots, \hat{x}_j, \ldots, x_N\right) = \left(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N\right)$) for $x' \in B'_\alpha\left(x'_0\right)$ with $x'_0 = \left(x_{01}, \ldots \hat{x}_{0j}, \ldots, x_{0N}\right)$ and $\|h\|_{W_r^{2-1/r}(B'_\alpha(x'_0))} \leqslant K$ such that

$$\Omega \cap B_\beta(x_0) = \left\{x \in \mathbb{R}^N : -\gamma + h\left(x'_j\right) < x_j < h\left(x'_j\right) + \gamma\right\} \cap B_\beta(x_0),$$
$$\Gamma \cap B_\beta(x_0) = \left\{x \in \mathbb{R}^N : x_j = h\left(x'_j\right)\right\} \cap B_\beta(x_0);$$

(2) for any $x_0 \in \Gamma_-$, there exist a coordinate number j and a $W_r^{2-1/r}$ function $h(x')$ for $x' \in B'_\alpha\left(x'_0\right)$ with $\|h\|_{W_r^{2-1/r}(B'_\alpha(x'_0))} \leqslant K$ such that

$$\Omega \cap B_\beta(x_0) = \left\{x \in \mathbb{R}^N : x_j > h\left(x'\right)\right\} \cap B_\beta(x_0),$$
$$\Gamma_- \cap B_\beta(x_0) = \left\{x \in \mathbb{R}^N : x_j = h\left(x'\right)\right\} \cap B_\beta(x_0).$$

Here, $B'_\alpha\left(x'_0\right) = \left\{x' \in \mathbb{R}^{N-1} : |x' - x'_0| < \alpha\right\}$, $B_\beta(x_0) = \left\{x \in \mathbb{R}^N : |x - x_0| < \beta\right\}$.

Lemma A.3 *([26]). Let $1 < q \leqslant r < \infty$ and $r > N$. Then there exists a constant $C_{N,r,q}$ such that for any $\sigma > 0$, $a \in L^r(\mathbb{R}^N_+)$ and $b \in H^1_q(\mathbb{R}^N_+)$, the following estimate*

$$\|ab\|_{L^q(\mathbb{R}^N_+)} \leqslant \sigma \|\nabla b\|_{L^q(\mathbb{R}^N_+)} + C_{N,r,q}\sigma^{-\frac{N}{r-N}} \|a\|^{\frac{r}{r-N}}_{L^r(\mathbb{R}^N_+)} \|b\|_{L^q(\mathbb{R}^N_+)}$$

holds.

A.2. Two phase problem for the Stokes equations with free boundary conditions

We recall the L^p-L^q maximal regularity for the two-phase problem of the Stokes equations with free boundary conditions and the non-slip condition given as follows: for $t > 0$

$$\begin{cases} \partial_t \mathbf{u} - \mathrm{Div}(\mu \mathbf{D}(\mathbf{u}) - q\mathbf{I}) = \mathbf{f} & \text{in } \dot\Omega, \quad (a) \\ \mathrm{div}\, \mathbf{u} = g = \mathrm{div}\, \mathbf{g} & \text{in } \dot\Omega, \quad (b) \\ [\![(\mu \mathbf{D}(\mathbf{u}) - q\mathbf{I})\mathbf{n}]\!] = \mathbf{h}, \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma, \quad (c) \\ \mathbf{u} = 0 & \text{on } \Gamma_-, \quad (d) \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \dot\Omega. \quad (e) \end{cases} \tag{A.2}$$

Theorem A.4 *(cf. [32]). Let $1 < p, q, r < \infty$ with $2/p + 1/q \notin \{1, 2\}$, and $t > 0$. Assume that Ω is a uniform $W^{2-1/r}_r$ domain and $\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\dot\Omega)$ is the initial data for equations (A.2). Let $\mathbf{f}$, $\mathbf{g}$, g and $\mathbf{h}$ be functions appearing on the right side of equations (A.2a)-(A.2c) satisfying the conditions:*

$$e^{-\gamma t}\mathbf{f} \in L^p\left(\mathbb{R}, L^q(\dot\Omega)\right), \quad e^{-\gamma t} g \in L^p\left(\mathbb{R}, H^1_q(\dot\Omega)\right) \cap H^{1/2}_p\left(\mathbb{R}, L^q(\dot\Omega)\right),$$
$$e^{-\gamma t}\mathbf{g} \in H^1_p\left(\mathbb{R}, L^q(\dot\Omega)\right), \quad e^{-\gamma t}\mathbf{h} \in L^p\left(\mathbb{R}, H^1_q(\dot\Omega)\right) \cap H^{1/2}_p\left(\mathbb{R}, L^q(\dot\Omega)\right),$$

for any $\gamma \geqslant \gamma_0$ with some γ_0. Assume that the compatibility conditions hold:

$$\begin{aligned} &\mathrm{div}\, \mathbf{u}_0 = N_2|_{t=0} \quad \text{in} \quad \dot\Omega, \quad \mathbf{u}_0 - \mathbf{N}_3|_{t=0} \in \mathcal{D}(\Omega), \\ &[\![\mu \mathbf{D}(\mathbf{u}_0)\mathbf{n}]\!]_\tau = \mathbf{h}_\tau|_{t=0} \qquad \text{on } \Gamma, \quad \text{if } 2/p + 1/q < 1, \\ &[\![\mathbf{u}_0]\!] = 0 \quad \text{on } \Gamma, \quad \mathbf{u}_0 = 0 \quad \text{on } \Gamma_-, \quad \text{if } 2/p + 1/q < 2. \end{aligned} \tag{A.3}$$

Then, problem (A.2) admits a unique solution $\mathbf{u}$ and q with

$$\mathbf{u} \in L^p\left((0,\infty), H^2_q(\dot\Omega)\right) \cap H^1_p\left((0,\infty), L^q(\dot\Omega)\right),$$
$$q \in L^p\left((0,\infty), H^1_q(\dot\Omega) + \hat H^1_q(\dot\Omega)\right),$$

possessing the estimate

$$\begin{aligned}
&\left\|e^{-\gamma t}\partial_t \mathbf{u}\right\|_{L^p\left((0,\infty),L^q(\dot{\Omega})\right)}+\left\|e^{-\gamma t}\mathbf{u}\right\|_{L^p\left((0,\infty),H^2_q(\dot{\Omega})\right)}\\
\lesssim&\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})}+\left\|e^{-\gamma t}\mathbf{f}\right\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))}\\
&+\left\|e^{-\gamma t}\partial_t \mathbf{g}\right\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))}+\left\|e^{-\gamma t}\left(g,\mathbf{h}\right)\right\|_{L^p\left(\mathbb{R},H^1_q(\dot{\Omega})\right)}\\
&+\left(1+\gamma^{1/2}\right)\left(\left\|e^{-\gamma t}g\right\|_{H^{1/2}_p(\mathbb{R},L^q(\dot{\Omega}))}+\left\|e^{-\gamma t}\mathbf{h}\right\|_{H^{1/2}_p(\mathbb{R},L^q(\dot{\Omega}))}\right).
\end{aligned}\tag{A.4}$$

A.3. Some fundamental properties of the $\mathcal{R}$-bounded operators

We recall the following technical lemma in order to prove the $\mathcal{R}$-boundedness of $\mathcal{A}_\pm(\lambda)$ given in (3.2).

Lemma A.5 *([5, Theorem 3.3]). Let $1<q<\infty$ and $\Lambda\subset\mathbb{C}$, $m=m(\lambda,\xi)$ be a function defined on $\Lambda\times(\mathbb{R}^N\backslash\{0\})$ which is infinitely differentiable with respect to $\alpha\in(\mathbb{N}\cup\{0\})^N$ for each $\lambda\in\Lambda$. Assume that for any multi-index α there exists a constant C_α depending on α and Λ such that*

$$\left|\partial^\alpha_\xi m(\lambda,\xi)\right|\leqslant C_\alpha|\xi|^{-|\alpha|}\tag{A.5}$$

for any $(\lambda,\xi)\in\Lambda\times\left(\mathbb{R}^N\backslash\{0\}\right)$. Let K_λ be an operator defined by $K_\lambda=\mathcal{F}^{-1}_\xi[m(\lambda,\xi)\mathcal{F}f(\xi)]$. Then, the family of operators $\{K_\lambda:\lambda\in\Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}\left(L^q(\mathbb{R}^N),L^q(\mathbb{R}^N)\right)$ and

$$\mathcal{R}_{\mathcal{L}\left(L^q(\mathbb{R}^N),L^q(\mathbb{R}^N)\right)}(\{K_\lambda:\lambda\in\Lambda\})\leqslant C_{q,N}\max_{|\alpha|\leqslant N+2}C_\alpha,\tag{A.6}$$

with some constant $C_{q,N}$ depending only on q and N.

We recall the fundamental properties of the $\mathcal{R}$-bounded operators as follows.

Lemma A.6 *(cf. [4, Proposition 3.4]). Let X and Y be Banach spaces, $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families on $\mathcal{L}(X,Y)$. Then, $\mathcal{T}+\mathcal{S}=\{T+S:T\in\mathcal{T},S\in\mathcal{S}\}$ is $\mathcal{R}$-bounded and $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}+\mathcal{S})\leqslant\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})+\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S})$.*

Let X,Y,Z be Banach spaces, and $\mathcal{T}\in\mathcal{L}(X,Y)$ and $\mathcal{S}\in\mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded, then $\mathcal{T}\mathcal{S}=\{TS:T\in\mathcal{T},S\in\mathcal{S}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(X,Z)$, and $\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{T}\mathcal{S})\leqslant\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S})$.

Lemma A.7 *(cf. [11, Proposition 2.5]). Let $1\leqslant q<\infty$ and Ω be a domain in $\mathbb{R}^N$. Let $m=m(\lambda)$ be a bounded function defined on a subset $\Lambda\subset\mathbb{C}$ and let $M_m(\lambda)$ be a map defined by $M_m(\lambda)f=m(\lambda)f$ for any $f\in L^q(\Omega)$. Then, we get*

$$\mathcal{R}_{\mathcal{L}(L^q(\Omega),L^q(\Omega))}(\{M_m(\lambda):\lambda\in\Lambda\})\leqslant C_{N,q,\Omega}\|m\|_{L^\infty(\Lambda)}.$$

Lemma A.8 *(cf. [34, Proposition 2.3]). Let $1<p,q<\infty$ and Ω be a domain in $\mathbb{R}^N$. Let $n=n(\tau)$ be a C^1-function defined on $\mathbb{R}\backslash\{0\}$ which satisfies the conditions $|n(\tau)|\leqslant\gamma$ and $|\tau n'(\tau)|\leqslant\gamma$ with some constant $\gamma>0$ for any $\tau\in\mathbb{R}\backslash\{0\}$. Let T_n be the operator-valued Fourier multiplier* *defined by* $T_n f = \mathcal{F}^{-1}[n\mathcal{F}[f]]$ *for any* f *with* $\mathcal{F}[f] \in \mathcal{D}(\mathbb{R}, L^q(\Omega))$. *Then,* T_n *can be extended to a bounded linear operator from* $L^p(\mathbb{R}, L^q(\Omega))$ *into itself. Moreover, denoting this extension also by* T_n*, we have*

$$\|T_n\|_{\mathcal{L}(L^p(\mathbb{R}, L^q(\Omega)))} \leqslant C_{\Omega,p,q}\gamma.$$

Here, $\mathcal{D}(\mathbb{R}, L^q(\Omega))$ *denotes the set of all* $L^q(\Omega)$*-valued* C^∞*-functions on* $\mathbb{R}$ *with compact support.*

We introduce two classes of multipliers.

Definition A.9. Let $0 < \epsilon < \pi/2$ and $\lambda_0 \geqslant 0$, and $m(\xi', \lambda)$ be a function defined on $\mathbb{R}^{N-1}\backslash\{0\} \times \Sigma_{\epsilon,\lambda_0}$, which is infinitely differentiable with respect to ξ' and holomorphic with respect to λ. If for any multi-index $\alpha' = (\alpha_1, \ldots, \alpha_{N-1})$, there hold the estimates:

$$\begin{aligned} \left|\partial_{\xi'}^{\alpha'} m\left(\lambda, \xi'\right)\right| &\leqslant C_{s,\alpha',\epsilon,\lambda_0}\left(|\lambda|^{1/2} + \left|\xi'\right|\right)^{s-|\alpha'|}, \\ \left|\partial_{\xi'}^{\alpha'}\left(\tau\partial_\tau m\left(\lambda, \xi'\right)\right)\right| &\leqslant C_{s,\alpha',\epsilon,\lambda_0}\left(|\lambda|^{1/2} + \left|\xi'\right|\right)^{s-|\alpha'|}, \end{aligned} \tag{A.7}$$

with $\tau = \mathrm{Im}\lambda$ and some positive constant $C_{s,\alpha',\epsilon,\lambda_0}$, then $m(\xi', \lambda)$ is called a *multiplier of order s with type* 1.

Similarly, if there hold the estimates:

$$\begin{aligned} \left|\partial_{\xi'}^{\alpha'} m\left(\lambda, \xi'\right)\right| &\leqslant C_{s,\alpha',\epsilon,\lambda_0}\left(|\lambda|^{1/2} + \left|\xi'\right|\right)^{s}\left|\xi'\right|^{-|\alpha'|}, \\ \left|\partial_{\xi'}^{\alpha'}\left(\tau\partial_\tau m\left(\lambda, \xi'\right)\right)\right| &\leqslant C_{s,\alpha',\epsilon,\lambda_0}\left(|\lambda|^{1/2} + \left|\xi'\right|\right)^{s}\left|\xi'\right|^{-|\alpha'|}, \end{aligned} \tag{A.8}$$

with $\tau = Im\lambda$ and some positive constant $C_{s,\alpha',\epsilon,\lambda_0}$, then $m(\xi', \lambda)$ is called a *multiplier of order s with type* 2.

Lemma A.10 *([33]). Let* $0 < \epsilon < \pi/2$, $1 < q < \infty$ *and* $\lambda_0 \geqslant 0$. *Let* $m_{1,\pm} \in \mathbf{M}_{-2,1}$, $m_{2,\pm} \in \mathbf{M}_{-2,2}$, *we define the operators* $K_{j,\pm}(\lambda)(j = 1, 2)$ *for* $\lambda \in \Sigma_{\epsilon,\lambda_0}$ *by the formulas:*

$$\begin{aligned} \left[K_{1,\pm}(\lambda)h\right](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m_{1,\pm}\left(\lambda, \xi'\right)\lambda^{1/2} e^{\mp A_\pm(x_N+y_N)}\mathcal{F}'[h]\left(\xi', y_N\right)\right]\left(x'\right)dy_N, \\ \left[K_{2,\pm}(\lambda)h\right](x) &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[m_{2,\pm}\left(\lambda, \xi'\right)\left|\xi'\right| e^{\mp A_\pm(x_N+y_N)}\mathcal{F}'[h]\left(\xi', y_N\right)\right]\left(x'\right)dy_N. \end{aligned}$$

Then, for $\ell = 0, 1$ *and* $j = 1, 2$*, the following sets*

$$\left\{(\tau\partial_\tau)^\ell\left(\lambda K_{j,\pm}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}, \quad \left\{(\tau\partial_\tau)^\ell\left(|\lambda|^{1/2}\nabla K_{j,\pm}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\},$$
$$\left\{(\tau\partial_\tau)^\ell\left(\nabla^2 K_{j,\pm}(\lambda)\right) : \lambda \in \Sigma_{\epsilon,\lambda_0}\right\}$$

are $\mathcal{R}$-bounded families in $\mathcal{L}(L^q(\mathbb{R}^N))$, whose $\mathcal{R}$-bounds do not exceed some constant $C_{N,q,\epsilon,\lambda_0,\kappa_\pm}$ depending only on $N, q, \epsilon, \lambda_0$ and $\kappa_\pm$.

A.4. Several properties of uniform $W_2^{2-1/r}$ domains

We recall several properties of uniform $W_2^{2-1/r}$ domains given in Enomoto and Shibata [5].

Proposition A.11. *Let $N < r < \infty$ and Ω be a uniform $W_r^{2-1/r}$ domain in $\dot{\mathbb{R}}^N$. Let K_1 be any small number in $(0, 1)$ and K_2 be given in (3.16). Then, there exist some constants $\sigma^0, \sigma^1, \sigma^2 \in (0, 1)$, at most countably many N-vector functions $\Phi_j^i \in H_r^2(\mathbb{R}^N)$, and points $x_{\pm j}^0 \in \Omega_\pm$, $x_j^1 \in \Gamma$ and $x_j^2 \in \Gamma_-$ such that the following assertions hold:*

(1) *The maps: $\mathbb{R}^N \ni x \mapsto \Phi_j^i(x) \in \mathbb{R}^N$ $(i = 1, 2;\ j \in \mathbb{N})$ are bijective.*
(2) $\Omega = \left(\bigcup_{j=1}^\infty \left(B_{\sigma^0}(x_{+j}^0) \cup B_{\sigma^0}(x_{-j}^0)\right)\right) \bigcup \left(\bigcup_{i=1}^2 \bigcup_{j=1}^\infty \left(\Phi_j^i\left(H^i\right) \cap B_{\sigma^i}(x_j^i)\right)\right)$*, with* $H^1 = \mathbb{R}^N$ *and* $H^2 = \mathbb{R}_-^N$*, where* $B_{\sigma^0}(x_{\pm j}^0) \subset \Omega_\pm$, $\Phi_j^1\left(\mathbb{R}_\pm^N\right) \cap B_{\sigma^1}(x_j^1) = \Omega_\pm \cap B_{\sigma^1}(x_j^1)$, $\Phi_j^2\left(\mathbb{R}_-^N\right) \cap B_{\sigma^2}(x_j^2) = \Omega_- \cap B_{\sigma^2}(x_j^2)$ *and* $\Phi_j^1(\mathbb{R}_0^N) \cap B_{\sigma^i}(x_j^i) = \Gamma^i \cap B_{\sigma^1}(x_j^1) (i = 1, 2)$ *with* $\Gamma^1 = \Gamma$ *and* $\Gamma^2 = \Gamma_-$.
(3) *There exist C^∞ functions $\zeta_{\pm j}^0, \tilde{\zeta}_{\pm j}^0$ and $\zeta_j^k, \tilde{\zeta}_j^k$ $(k = 1, 2;\ j \in \mathbb{N})$, such that*

$$0 \leqslant \zeta_{\pm j}^0, \tilde{\zeta}_{\pm j}^0, \zeta_j^i, \tilde{\zeta}_j^i \leqslant 1,$$

$$\operatorname{supp} \zeta_j^i,\ \operatorname{supp} \tilde{\zeta}_j^i \subset B_{\sigma^i}\left(x_j^i\right), \quad \operatorname{supp} \zeta_{\pm j}^0,\ \operatorname{supp} \tilde{\zeta}_{\pm j}^0 \subset B_{\sigma^0}\left(x_{\pm j}^0\right),$$

$$\left\|\left(\zeta_j^i, \zeta_{\pm j}^2, \tilde{\zeta}_j^i, \tilde{\zeta}_{\pm j}^2\right)\right\|_{H_\infty^2(\mathbb{R}^N)} \leqslant c_0, \quad \tilde{\zeta}_j^i = 1 \text{ on } \operatorname{supp} \zeta_j^i, \quad \tilde{\zeta}_{\pm j}^0 = 1 \text{ on } \operatorname{supp} \zeta_{\pm j}^0,$$

$$\sum_\pm \sum_{j=1}^\infty \zeta_{\pm j}^0 + \sum_{i=1}^2 \sum_{j=1}^\infty \zeta_j^i = 1 \text{ on } \bar{\Omega}, \quad \sum_{j=1}^\infty \zeta_j^1 = 1 \text{ on } \Gamma, \quad \sum_{j=1}^\infty \zeta_j^2 = 1 \text{ on } \Gamma_-.$$

Here, c_0 is a constant depending on K_2, N, q and r, but independent of $j \in \mathbb{N}$.
(4) $\nabla\Phi_j^i = \mathcal{M}_j^i + M_j^i$, $\nabla\left(\Phi_j^i\right)^{-1} = \mathcal{M}_{j,-}^i + M_{j,-}^i$*, where $\mathcal{M}_j^i$ and $\mathcal{M}_{j,-}^i$ are $N \times N$ constant orthonormal matrices, and M_j^i and $M_{j,-}^i$ are $N \times N$ matrices of $H_r^1(\mathbb{R}^N)$ functions defined on $\mathbb{R}^N$ which satisfy the conditions:* $\left\|M_j^i\right\|_{L^\infty(\mathbb{R}^N)} \leqslant K_1$, $\left\|M_{j,-}^i\right\|_{L^\infty(\mathbb{R}^N)} \leqslant K_1$, $\left\|\nabla M_j^i\right\|_{L^r(\mathbb{R}^N)} \leqslant K_2$ *and* $\left\|\nabla M_{j,-}^i\right\|_{L^r(\mathbb{R}^N)} \leqslant K_2$ *for $i = 0, 1$ and $j \in \mathbb{N}$.*
(5) *There exists a natural number $L \geqslant 2$ such that any $L + 1$ distinct sets of $\left\{B_{\sigma^i}(x_j^i)\right\} \cup \left\{B_{\sigma^0}(x_{\pm j}^0)\right\}$ $(i = 1, 2;\ j \in \mathbb{N})$ have an empty intersection.*

Let $B_{\pm j}^0 = B_{\sigma^0}\left(x_{\pm j}^0\right)$, $B_j^i = B_{d^i}\left(x_j^i\right)$ with $i = 1, 2$ and $j \in \mathbb{N}$ for short. Then, by the finite intersection property stated in Proposition A.11 (5), we see that, for any $r \in [1, \infty)$, there is a positive constant $C_{r,L}$ such that, for any $f \in L^r(\Omega)$, we have

$$\left(\sum_{j=1}^{\infty}\|f\|^r_{L^r\left(\Omega\cap A^i_j\right)}\right)^{1/r} \leqslant C_{r,L}\|f\|_{L^r(\Omega)}. \tag{A.9}$$

In fact, for $1 \leqslant r < \infty$,

$$\begin{aligned}\sum_{j=1}^{\infty}\|f\|^r_{L^s\left(\Omega\cap A^i_j\right)} &= \int_{\Omega}\sum_{j=1}^{\infty}\chi_{A^i_j}(x)|f(x)|^r dx \\ &\leqslant \left\|\sum_{j=1}^{\infty}\chi_{A^i_j}\right\|_{L^\infty(\mathbb{R}^N)}\|f\|^r_{L^r(\Omega)} \leqslant L\|f\|^r_{L^s(\Omega)},\end{aligned}$$

where $A^i_j \in \left\{B^0_{\pm j}, B^1_j, B^2_j\right\}$. Then, we have the following lemma to construct parametrizes.

Lemma A.12 *([26]). Let X be a Banach space and X^* be its dual space. Let $\left\{f_j\right\}_{j=1}^{\infty}$ be a sequence in X^* such that there exists a constant $M > 0$ such that*

$$\sum_{j=1}^{\infty}\left|(f_j, \varphi)\right| \leqslant M\|\varphi\|_X \quad \text{for any } \varphi \in X,$$

then $\sum_{j=1}^{\infty} f_j$ exists and

$$\left\|\sum_{j=1}^{\infty} f_j\right\|_{X^*} \leqslant M.$$

With the help of (A.9) and Lemma A.12, we have the following proposition.

Proposition A.13. *Let $1 < q < \infty$ and $q' = q/(q-1)$. Let $A_j \in \left\{B^0_{\pm j}, B^1_j, B^2_j\right\}$. Let m be a non-negative integer. Let $\left\{f_j\right\}_{j=1}^{\infty}$ be a sequence in $H^m_q(\Omega)$ and $\left\{g^{(\ell)}_j\right\}_{j=1}^{\infty}$ $(\ell = 0, 1, \ldots, m)$ be a sequence of positive real numbers. Assume that for any $\varphi \in L^{q'}(\Omega)$,*

$$\sum_{j=1}^{\infty}\left(g^{(\ell)}_j\right)^q < \infty, \quad \left|\left(\nabla^\ell f_j, \varphi\right)_{\Omega}\right| \leqslant M g^{(\ell)}_j \|\varphi\|_{L^{q'}(\Omega\cap A_j)},$$

with $\ell = 0, 1, \ldots, m$ and some constant M independent of $j \in \mathbb{N}$. Then, $f = \sum_{j=1}^{\infty} f_j$ exists in the strong topology of $H^m_q(\Omega)$ and

$$\left\|\nabla^\ell f\right\|_{L^q(\Omega)} \leqslant C_{q',L} M \left(\sum_{j=1}^{\infty}\left(g^{(\ell)}_j\right)^q\right)^{\frac{1}{q}}.$$

A.5. Weis operator-valued Fourier multiplier theorem

Let X and Y be Banach spaces. Then, given $m \in L_{1,\mathrm{loc}}(\mathbb{R}, \mathcal{L}(X,Y))$, we define an operator $T_m : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_m\phi = \mathcal{F}^{-1}[m\mathcal{F}[\phi]] \quad \text{for all } \phi \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X). \tag{A.10}$$

Definition A.14. A Banach space X is said to be a UMD *Banach space*, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for $p \in (1, \infty)$. Here, the Hilbert transform H operating on $f \in \mathcal{S}(\mathbb{R}, X)$ is defined by

$$[Hf](t) = \frac{1}{\pi} \lim_{\epsilon \to 0+} \int\limits_{|t-s|>\epsilon} \frac{f(s)}{t-s} ds \quad (t \in \mathbb{R}).$$

We recall the Weis operator-valued Fourier multiplier theorem as follows:

Theorem A.15 *([24]). Let X and Y be two* UMD *Banach spaces and $1 < p < \infty$. Let m be a function in $C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X,Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{m(\lambda) : \tau \in \mathbb{R}\backslash\{0\}\}) = \kappa_0 < \infty,$$
$$\mathcal{R}_{\mathcal{L}(X,Y)}\left(\left\{\lambda m'(\tau) : \lambda \in \mathbb{R}\backslash\{0\}\right\}\right) = \kappa_1 < \infty.$$

Then, the operator T_m defined in (A.10) is extended to a bounded linear operator from $L^p(\mathbb{R}, X)$ into $L^p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_m, we have

$$\|T_m f\|_{L^p(\mathbb{R},Y)} \leqslant C\,(\kappa_0 + \kappa_1)\,\|f\|_{L^p(\mathbb{R},X)} \quad \textit{for } f \in L^p(\mathbb{R}, X)$$

with some positive constant C depending on p.

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