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LOCAL WELL-POSEDNESS FOR TWO-PHASE FLUID MOTION IN THE OBERBECK-BOUSSINESQ APPROXIMATION

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Abstract. This paper is concerned with the local well-posedness of the Oberbeck Boussinesq approximation for the unsteady motion of a drop in another fluid separated by a closed interface with surface tension. We are devoted to obtaining the linearized Oberbeck-Boussinesq approximation in the fixed domain by using the Hanzawa transformation, and using maximal L^p-L^q regularities for the two-phase fluid motion of the linearized system obtained by the authors in [10] to establish the existence and uniqueness of the solutions of nonlinear problem with the help of the contraction mapping principle, in which the differences of nonlinear terms are estimated.

1. **Introduction.** We consider the following two-phase fluid motion in the Oberbeck Boussinesq approximation in which the liquids are separated by a closed moving interface with surface tension:

$$
\begin{cases}
\rho\left(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right) - \mathrm{Div}(\mu\mathbf{D}(\mathbf{v}) - p\mathbf{I}) = \mathbf{a}(x,t) - \alpha\mathbf{g}\theta' & \text{in } \dot\Omega_t,\\
\partial_t\theta' + \mathrm{div}\left(\mathbf{v}\theta' - \kappa\nabla\theta'\right) = 0 & \text{in } \dot\Omega_t,\\
\mathrm{div}\,\mathbf{v} = 0 & \text{in } \dot\Omega_t,\\
[\![(\mu\mathbf{D}(\mathbf{v}) - p\mathbf{I})\mathbf{n}_t]\!] = \sigma H\left(\Gamma_t\right), \quad [\![\mathbf{v}]\!] = 0 & \text{on } \Gamma_t,\\
[\![\kappa\nabla\theta'\cdot\mathbf{n}_t]\!] = 0, \quad [\![\theta']\!] = 0 & \text{on } \Gamma_t,\\
V_n = \mathbf{v}\cdot\mathbf{n}_t & \text{on } \Gamma_t,\\
\mathbf{v} = 0, \quad \nabla\theta'_-\cdot\mathbf{n}_- + \beta\theta'_- = b(x,t) & \text{on } \Gamma_-,\\
\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \theta'|_{t=0} = \theta'_0 & \text{in } \dot\Omega,
\end{cases}
\tag{1.1}
$$

where $\mathbf{v} = (v_1(x,t),\dots,v_N(x,t))^\top$ is the velocity vector field, $p = p(x,t)$ is the pressure, $\theta' = \theta'(x,t)$ is the deviation from the average temperature, $\mathbf{a}$ is a given

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vector function of mass forces, and $\mathbf{g} = g(0, \cdots, 0, 1)^\top$ is a constant vector with the gravity constant g. Let Ω be a bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geqslant 2$) with solid boundary Γ_- , Ω_+ be a subdomain of Ω with a closed surface Γ, and $\Omega_- = \Omega\backslash\overline{\Omega_+}$. Throughout this paper, we assume that the interface Γ is a compact hypersurface of C^3 class, Γ_- is a compact hypersurface of C^2 class and $\operatorname{dist}(\Gamma, \Gamma_-) = \inf\{|x-y| : x \in \Gamma, y \in \Gamma_-\} \geqslant 2d$ for some constant $d > 0$. Let Ω_{t+} and Γ_t be the evolution of Ω_+ and Γ, respectively, both of which depend on the time $t > 0$, and set $\Omega_{t-} = \Omega\backslash(\Omega_{t+} \cup \Gamma_t)$, with $\Omega_{0\pm} = \Omega_\pm$ and $\Gamma_0 = \Gamma$. Let $\mathbf{n}_t$ be the unit normal to Γ_t oriented from Ω_{t+} into Ω_{t-}, $\mathbf{n} = \mathbf{n}_0$, and $\mathbf{n}_-$ be the unit outward normal to Γ_-. Denote $\dot\Omega_t = \Omega_{t+} \cup \Omega_{t-}$ and $\dot\Omega = \dot\Omega_0$ for convenience. The piecewise positive constants ρ, μ, α and κ correspond to the mass density, kinematic viscosity, temperature expansion coefficient and thermal conductivity, respectively. Here, both the above functions $\mathbf{v}, p, \theta', \mathbf{a}$ and the constants ρ, μ, α, κ are piecewise defined, for instance, $\mathbf{v} = \mathbf{v}_+\chi_{\Omega_+} + \mathbf{v}_-\chi_{\Omega_-}$, $\rho = \rho_+\chi_{\Omega_+} + \rho_-\chi_{\Omega_-}$, etc., where $\chi_{\Omega_\pm}$ is the characteristic function of $\Omega_\pm$. $\mathbf{D}(\mathbf{v})$ is the doubled deformation tensor with the $(i,j)^{\text{th}}$ component $\partial_i v_j + \partial_j v_i$, and $\mathbf{I}$ is the $N \times N$ identity matrix. $b(x,t)$ is a given function on the fixed boundary Γ_-, and $\beta \geqslant 0$ is a constant. $\dot\Omega$, $\mathbf{v}_0$ and θ'_0 are the prescribed initial data for $\dot\Omega_t$, $\mathbf{v}$ and θ', respectively. V_n is the evolution velocity of Γ_t along $\mathbf{n}_t$. σ is a positive constant describing the coefficient of the surface tension and $H(\Gamma_t)$ is $N-1$ times the mean curvature of Γ_t given by the relation $H(\Gamma_t)\,\mathbf{n}_t = \Delta_{\Gamma_t} x$, with the Laplace-Beltrami operator Δ_{Γ_t} on Γ_t(more details in Appendix A.2). Moreover, for any function $f(x,t) = f_\pm(x,t)$ for $x \in \Omega_{t\pm}$ and $t \geqslant 0$, we denote the jump of f across Γ_t by

$$[\![f]\!](x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega_{t+}}} f_+(x) - \lim_{\substack{x\to x_0\\ x\in\Omega_{t-}}} f_-(x)$$

for every point $x_0 \in \Gamma_t$. Finally, for any matrix field $\mathbf{K} = (K_{ij})$, the quantity $\operatorname{Div}\mathbf{K}$ is an N-vector whose i^{th} component is $\sum_{j=1}^N \partial_j K_{ij}$, and for any vector function $\mathbf{u} = (u_1, \ldots, u_N)$, $\operatorname{div}\mathbf{u} = \sum_{j=1}^N \partial_j u_j$, and $\mathbf{u}\cdot\nabla\mathbf{u}$ is an N-vector whose i^{th} component is $\sum_{j=1}^N u_j \partial_j u_i$.

The Oberbeck-Boussinesq approximation has implications for a wide variety of flows within the context of astrophysical, geophysical and oceanographic fluid dynamics (e.g., see [12]). The approximate equations were first derived by Oberbeck [15, 16] and independently derived by Boussinesq [3], to describe the thermomechanical response of linearly viscous fluids that are mechanically incompressible but thermally compressible. Numerous attempts have been made to provide a rigorous justification for this approximation such as [11, 20, 21, 31].

The free boundary problems of two-phase problems of two incompressible viscous fluids have been studied by many mathematicians in recent decades. Shibata have proved local and global well-posedness for incompressible-incompressible two-phase problem in [26]. For the resolvent problems, Shibata and his collaborators have done a lot of work, e.g., [13, 22, 23, 25, 28, 30], with the help of the $\mathcal{R}$-bounded solution operators. Prüss and Simonett proved the existence of strong solutions of Navier-Stokes equations with small initial data where $\mathbb{R}^N$ is separated by a non-compact free surface, in addition, Prüss and Simonett [17–19] contributed to the L^p approach for two-phase problems, especially for the case of the surface tension.

For the 2D Boussinesq equations, the global regularity or global well-posedness was proved in [1,4] with partial viscosity and in [2] with variable kinematic viscosity.

The global well-posedness for the 3D Boussinesq system with horizontal dissipation was given in [14].

In fact, there are few results about two-phase fluid motion in the Oberbeck-Boussinesq approximation. In the Hölder spaces, the local existence for the problem was established in [5]. Denisova and Solonnikov [6] proved the global existence of classical solutions for capillary fluids in the framework of Hölder spaces. Hao and Zhang established the maximal L^p-L^q regularity for the two-phase fluid motion of the linearized Oberbeck-Boussinesq approximation in [10].

Remark 1.1. Different from the free boundary problem for the Navier-Stokes equations, on the fixed boundary Γ_-, we consider a mixed boundary condition for θ' in (1.1) instead of the Neumann or Dirichlet boundary condition.

Remark 1.2. One of the difficulties of this paper is that we have to deal with the estimates of those nonlinear terms in an anisotropic space $H_p^{1/2}(\mathbb{R}, L^q) \cap L^p\left(\mathbb{R}, H_q^1\right)$ given by Theorems 3.1 and 3.2. Thus, in order to prove the local well-posedness, we have to find a suitable extension method to extend the function defined on $(0, T)$ to the whole time space $\mathbb{R}$.

The rest of this paper is structured as follows. First, by using the Hanzawa transformation, we transform equations (1.1) to the fixed domain in Section 2. Then, in Section 3, we use the fact that the maximal L^p-L^q regularity theorem for the Stokes equations with interface and non-slip boundary conditions and heat equations with interface conditions that have been got in [13]. Section 4 is devoted to estimating nonlinear terms. In Section 5, we need to estimate the difference of nonlinear terms, then by the contraction mapping principle, we complete the proof of the local well-posedness. Finally, we will recall some notations and useful results in the Appendix.

2. Hanzawa transform and the main theorem.

In this section, we shall transform (1.1) to some problem formulated on a fixed domain by using the Hanzawa transform, then we give a statement of the local well-posedness theorem.

2.1. Hanzawa transform.

In order to transform the time dependent unknown domain $\dot{\Omega}_t$ to the fixed domain $\dot{\Omega}$, we introduce the Hanzawa transform. Let $\mathbf{n}$ be the unit normal to Γ oriented from Ω_+ into Ω_-. Since Γ is a hypersurface of C^3 class, we may assume that $\mathbf{n}$ is defined on $\mathbb{R}^N$ with $\|\mathbf{n}\|_{H^2_\infty(\mathbb{R}^N)} < \infty$. And Γ_- is hypersurface of C^2 class with $\|\mathbf{n}_-\|_{H^1_\infty(\mathbb{R}^N)} < \infty$. We assume that

$$\operatorname{dist}\left(\operatorname{supp}(\omega(y)\mathbf{n}(y)), \Gamma_-\right) \geqslant d,$$

where $\omega(y)$ is a C^∞ function which equals one near Γ and zero far from Γ. Let

$$\Gamma_t = \{x = y + h(y,t)\mathbf{n} \mid y \in \Gamma\}$$

with an unknown function $h(y,t)$. Let H_h be a suitable extension of $h(y,t)$ such that $H_h(y,t) = h(y,t)$ for $(y,t) \in \Gamma \times (0,T)$ and it possesses the estimates

$$\begin{aligned} C_1\|h(\cdot,t)\|_{W_q^{k-1/q}(\Gamma)} \leqslant \|H_h(\cdot,t)\|_{H_q^k(\Omega)} \leqslant C_2\|h(\cdot,t)\|_{W_q^{k-1/q}(\Gamma)}, \\ C_1\left\|\partial_t h(\cdot,t)\right\|_{W_q^{\ell-1/q}(\Gamma)} \leqslant \left\|\partial_t H_h(\cdot,t)\right\|_{H_q^\ell(\Omega)} \leqslant C_2\left\|\partial_t h(\cdot,t)\right\|_{W_q^{\ell-1/q}(\Gamma)} \end{aligned} \tag{2.1}$$

for $k = 1,2,3$ and $\ell = 1,2$. Let $\Psi_h(y,t) = \omega(y)H_h(y,t)\mathbf{n}(y)$, we use the Hanzawa transformation defined by

$$x = y + \Psi_h(y,t), \tag{2.2}$$

which was introduced by [19] to treat solutions of the Stokes problem. In the following, we assume that

$$\sup_{t\in(0,T)} \|\Psi_h(\cdot,t)\|_{H^1_\infty(\mathbb{R}^N)} \leqslant \lambda, \tag{2.3}$$

where λ is a small positive number in such a way that several conditions stated below will be satisfied. In fact, we choose $\lambda \in (0,1)$, and then the Hanzawa transform defined above is an injective mapping. For $x_i = y_i + \Psi_h(y_i, t)$ $(i = 1, 2)$,

$$\begin{aligned}|x_1 - x_2| &= |y_1 + \Psi_h(y_1,t) - (y_2 + \Psi_h(y_2,t))| \\ &\geqslant |y_1 - y_2| - \|\nabla(\Psi_h(\cdot,t))\|_{L^\infty(\mathbb{R}^N)} |y_1 - y_2| \geqslant (1-\lambda)|y_1 - y_2|,\end{aligned}$$

from which $y_1 \neq y_2$ implies $x_1 \neq x_2$. Let

$$\dot\Omega_t = \left\{x = y + \Psi_h(y,t) \mid y \in \dot\Omega\right\},$$

the Hanzawa transform maps $\dot\Omega$ onto $\dot\Omega_t$ injectively. Let $\mathbf{v}$, p and θ' be solutions of (1.1), and we set

$$\mathbf{u}(y,t) = \mathbf{v}(y+\Psi_h(y,t),t), \quad q(y,t) = p(y+\Psi_h(y,t),t), \quad \theta(y,t) = \theta'(y+\Psi_h(y,t),t).$$

Noting that $x = y$ near Γ_-, we have

$$\begin{aligned}&\mathbf{u} = 0, \quad \nabla\theta_- \cdot \mathbf{n}_- + \beta\theta_- = b(y,t) \quad \text{on } \Gamma_- \times (0,T), \\ &\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \dot\Omega, \quad H_h|_{t=0} = h_0 \quad \text{on } \Gamma.\end{aligned} \tag{2.4}$$

2.2. **Reformulation of equations and the divergence free condition.** Now, we reformulate the equations by using the Hanzawa transformation: $x = y + \Psi_h(y,t)$. Let $\partial x/\partial y$ be the Jacobi matrix of the transformation, that is,

$$\frac{\partial x}{\partial y} = \mathbf{I} + \nabla\Psi_h(y,t), \quad \nabla\Psi_h = \begin{pmatrix} \partial_1\Psi_1 & \partial_2\Psi_1 & \dots & \partial_N\Psi_1 \\ \partial_1\Psi_2 & \partial_2\Psi_2 & \dots & \partial_N\Psi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1\Psi_N & \partial_2\Psi_N & \dots & \partial_N\Psi_N \end{pmatrix},$$

where $\Psi_h(y,t) = (\Psi_1(y,t), \dots, \Psi_N(y,t))^\top$ and $\partial_i\Psi_j = \frac{\partial\Psi_j}{\partial y_i}$. Since λ is a small positive number, then by (2.3) and the chain rule, we have

$$\frac{\partial y}{\partial x} = \left(\frac{\partial x}{\partial y}\right)^{-1} = \mathbf{I} + \sum_{k=1}^{\infty}(-\nabla\Psi_h(y,t))^k = \mathbf{I} + \mathbf{V}_0(\nabla\Psi_h), \tag{2.5}$$

where $\mathbf{V}_0(\mathbf{k})$ is an $N \times N$ matrix of analytic functions defined on $|\mathbf{k}| < \lambda$ such that $\mathbf{V}_0(0) = 0$. Hereafter, $\mathbf{k} = (k_{ij})$ and k_{ij} are the variables corresponding to $\partial_i\Psi_j$. Then we have

$$\nabla_x = (\mathbf{I} + \mathbf{V}_0(\mathbf{k}))\nabla_y, \quad \frac{\partial}{\partial x_\ell} = \frac{\partial}{\partial y_\ell} + \sum_{j=1}^{N} V_{0\ell j}(\mathbf{k})\frac{\partial}{\partial y_j}, \tag{2.6}$$

where $\nabla_z = (\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_N})^\top$ for $z = x$, y. $V_{0\ell j}$ is the $(\ell, j)^{\text{th}}$ component of the $N \times N$ matrix $\mathbf{V}_0$. Since

$$\frac{\partial}{\partial t}\left[v_\ell(y + \Psi_h(y,t),t)\right] = \frac{\partial v_\ell}{\partial t}(x,t) + \sum_{j=1}^{N} \frac{\partial\Psi_j}{\partial t}(y,t)\frac{\partial v_\ell}{\partial x_j}(x,t),$$

in view of (2.6), we can directly get

$$\begin{aligned} \frac{\partial v_\ell}{\partial t} + \sum_{j=1}^N v_j \frac{\partial v_\ell}{\partial x_j} &= \frac{\partial u_\ell}{\partial t} + \sum_{j,k=1}^N \left(u_j - \frac{\partial \Psi_j}{\partial t} \right) (\delta_{jk} + V_{0jk}(\mathbf{k})) \frac{\partial u_\ell}{\partial y_k}, \\ \frac{\partial p}{\partial x_\ell} &= \frac{\partial q}{\partial y_\ell} + \sum_{j=1}^N V_{0\ell j} (\Psi_h) \frac{\partial q}{\partial y_j}. \end{aligned} \tag{2.7}$$

Then, by (2.6), we can write

$$\begin{aligned} &\sum_{j=1}^N \frac{\partial}{\partial x_j} (\mu \mathbf{D}(\mathbf{v})_{\ell j} - p\delta_{\ell j}) \\ &= \sum_{j,k=1}^N \mu (\delta_{jk} + V_{0jk}) \frac{\partial}{\partial y_k} \Big(\mathbf{D}(\mathbf{u})_{\ell j} + (\mathbf{V}_1(\mathbf{k})\nabla \mathbf{u})_{\ell j} \Big) - \sum_{j=1}^N (\delta_{\ell j} + V_{0\ell j}) \frac{\partial q}{\partial y_j} \end{aligned} \tag{2.8}$$

with

$$\mathbf{D}(\mathbf{u})_{\ell j} = \frac{\partial u_\ell}{\partial y_j} + \frac{\partial u_j}{\partial y_\ell}, \quad (\mathbf{V}_1(\mathbf{k})\nabla \mathbf{u})_{\ell j} = \sum_{k=1}^N \left(V_{0jk}(\mathbf{k}) \frac{\partial u_\ell}{\partial y_k} + V_{0\ell k}(\mathbf{k}) \frac{\partial u_j}{\partial y_k} \right), \tag{2.9}$$

where δ_{ij} is the Kronecker delta symbol such that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Putting (2.7) and (2.8) together, we observe that

$$\begin{aligned} \sum_{j=1}^N (\delta_{\ell j} + V_{0\ell j}) \frac{\partial q}{\partial y_j} = &\left(\rho \frac{\partial u_\ell}{\partial t} + \sum_{j,k=1}^N \left(u_j - \rho \frac{\partial \Psi_j}{\partial t} \right) (\delta_{jk} + V_{0jk}(\mathbf{k})) \frac{\partial u_\ell}{\partial y_k} \right) \\ &- \mu \sum_{j,k=1}^N (\delta_{jk} + V_{0jk}) \frac{\partial}{\partial y_k} \Big(\mathbf{D}(\mathbf{u})_{\ell j} + (\mathbf{V}_1(\mathbf{k})\nabla \mathbf{u})_{\ell j} \Big) \\ &- \mathbf{a}_\ell(x,t) - \delta_{\ell N} \alpha g \theta. \end{aligned}$$

Since $(\mathbf{I} + \nabla \Psi_h)(\mathbf{I} + \mathbf{V}_0) = (\frac{\partial x}{\partial y})(\frac{\partial y}{\partial x}) = \mathbf{I}$, we get

$$\begin{aligned} \frac{\partial q}{\partial y_i} = &\sum_{\ell=1}^N (\delta_{i\ell} + \partial_i \Psi_\ell) \left(\rho \frac{\partial u_\ell}{\partial t} + \sum_{j,k=1}^N \left(u_j - \rho \frac{\partial \Psi_j}{\partial t} \right) (\delta_{jk} + V_{0jk}(\mathbf{k})) \frac{\partial u_\ell}{\partial y_k} \right) \\ &- \mu \sum_{\ell,j,k=1}^N (\delta_{i\ell} + \partial_i \Psi_\ell)(\delta_{jk} + V_{0jk}) \frac{\partial}{\partial y_k} \Big(\mathbf{D}(\mathbf{u})_{\ell j} + (\mathbf{V}_1(\mathbf{k})\nabla \mathbf{u})_{\ell j} \Big) \\ &- \sum_{\ell=1}^N (\delta_{i\ell} + \partial_i \Psi_\ell)(\mathbf{a}_\ell + \delta_{\ell N} \alpha g \theta). \end{aligned}$$

Thus, we define an N-vector of functions $\mathbf{N}_1(\mathbf{u}, \mathbf{a}, \theta, \Psi_h)$ by letting

$$\begin{aligned} &\mathbf{N}_1(\mathbf{u}, \mathbf{a}, \theta, \Psi_h)|_i \\ &= - \sum_{j,k=1}^N \left(u_j - \rho \frac{\partial \Psi_j}{\partial t} \right) (\delta_{jk} + V_{0jk}(\mathbf{k})) \frac{\partial u_i}{\partial y_k} \\ &\quad - \sum_{\ell=1}^N \partial_i \Psi_\ell \left(\rho \frac{\partial u_\ell}{\partial t} + \sum_{j,k=1}^N \left(u_j - \rho \frac{\partial \Psi_j}{\partial t} \right) (\delta_{jk} + V_{0jk}(\mathbf{k})) \frac{\partial u_\ell}{\partial y_k} \right) \end{aligned}$$

$$+\mu\left(\sum_{j=1}^{N}\frac{\partial}{\partial y_j}(\mathbf{V}_1(\mathbf{k})\nabla\mathbf{u})_{ij}+\sum_{j,k=1}^{N}V_{0jk}(\mathbf{k})\frac{\partial}{\partial y_k}\left(\mathbf{D}(\mathbf{u})_{ij}+(\mathbf{V}_1(\mathbf{k})\nabla\mathbf{u})_{ij}\right)\right.$$
$$\left.+\sum_{j,k,\ell=1}^{N}\partial_i\Psi_\ell\left(\delta_{jk}+V_{0jk}(\mathbf{k})\right)\frac{\partial}{\partial y_k}\left(\mathbf{D}(\mathbf{u})_{\ell j}+(\mathbf{V}_1(\mathbf{k})\nabla\mathbf{u})_{\ell j}\right)\right)$$
$$-\sum_{\ell=1}^{N}\partial_i\Psi_\ell\mathbf{a}_\ell-\sum_{\ell=1}^{N}\partial_i\Psi_\ell\delta_{\ell N}\alpha g\theta. \tag{2.10}$$

Next, we consider the divergence free condition $\operatorname{div}\mathbf{v}=0$. By (2.5), we have

$$\operatorname{div}_x\mathbf{v}=\sum_{i=1}^{N}\frac{\partial v_i}{\partial x_i}=\sum_{i,j=1}^{N}\left(\delta_{ij}+V_{0ij}(\mathbf{k})\right)\frac{\partial u_i}{\partial y_j}=\operatorname{div}_y\mathbf{u}+\sum_{i,j=1}^{N}V_{0ij}(\mathbf{k})\frac{\partial u_i}{\partial y_j}.$$

Let J be the Jacobian of the transformation: $x=y+\Psi_h(y,t)$. For any test function $\varphi\in C_0^\infty(\Omega_t)$, we obtain

$$(\operatorname{div}_x\mathbf{v},\varphi)_{\dot\Omega_t}=-(\mathbf{v},\nabla_x\varphi)_{\dot\Omega_t}=-\sum_{i=1}^{N}\left(Ju_i,\sum_{j=1}^{N}\left(\delta_{ij}+V_{0ij}(\mathbf{k})\right)\frac{\partial\varphi}{\partial y_j}\right)_{\dot\Omega}$$
$$=\sum_{i,j=1}^{N}\left(\frac{\partial}{\partial y_j}\left(J\left(\delta_{ij}+V_{0ij}(\mathbf{k})\right)u_i\right),\varphi\right)_{\dot\Omega}.$$

Choosing $\lambda>0$ small enough, we may assume that $J=J(\mathbf{k})=1+J_0(\mathbf{k})$, where $J_0(\mathbf{k})$ is a C^∞ function defined for $|\mathbf{k}|<\lambda$ such that $J_0(0)=0$. We define $N_2(\mathbf{u},\Psi_h)$ and $\mathbf{N}_3(\mathbf{u},\Psi_h)$ by

$$N_2\left(\mathbf{u},\Psi_h\right)=-J_0\operatorname{div}\mathbf{u}+(1+J_0)\left(\sum_{i,j=1}^{N}V_{0ij}\left(\mathbf{k}\right)\frac{\partial u_i}{\partial y_j}\right),$$
$$\mathbf{N}_3\left(\mathbf{u},\Psi_h\right)|_k=\sum_{i=1}^{N}V_{0ik}\left(\mathbf{k}\right)u_i+J_0\left(\mathbf{k}\right)\sum_{i=1}^{N}\left(\delta_{ik}+V_{0ik}\left(\mathbf{k}\right)\right)u_i. \tag{2.11}$$

Since $V_{0ij}(0)=J_0(0)=0$, we may symbolically write as

$$N_2\left(\mathbf{u},\Psi_h\right)=\mathbf{V}_2\left(\nabla\Psi_h\right)\nabla\Psi_h\otimes\nabla\mathbf{u},\quad \mathbf{N}_3\left(\mathbf{u},\Psi_h\right)=\mathbf{V}_3\left(\nabla\Psi_h\right)\nabla\Psi_h\otimes\mathbf{u}, \tag{2.12}$$

with some matrix of C^1 functions $\mathbf{V}_2\left(\nabla\Psi_h\right)$ and $\mathbf{V}_3\left(\nabla\Psi_h\right)$ defined on $|\nabla\Psi_h|\leqslant\lambda$.

Finally, we consider the second equation in (1.1), by (2.6)

$$\Delta=\nabla\cdot\nabla=\sum_{i=1}^{N}\left(\sum_{k=1}^{N}\left(\delta_{ik}+V_{0ik}\left(\mathbf{k}\right)\right)\frac{\partial}{\partial y_k}\right)\left(\sum_{j=1}^{N}\left(\delta_{ij}+V_{0ij}\left(\mathbf{k}\right)\right)\frac{\partial}{\partial y_j}\right)$$
$$=\sum_{i=1}^{N}\left\{\frac{\partial^2}{\partial y_i^2}+\sum_{j=1}^{N}\frac{\partial}{\partial y_i}\left(V_{0ij}\left(\mathbf{k}\right)\frac{\partial}{\partial y_j}\right)+\sum_{j,k=1}^{N}V_{0ik}\left(\mathbf{k}\right)\right.$$
$$\left.\times\frac{\partial}{\partial y_k}\left(\left(\delta_{ij}+V_{0ij}\left(\mathbf{k}\right)\right)\frac{\partial}{\partial y_j}\right)\right\}. \tag{2.13}$$

Thus, in view of (2.7) and (2.13), we have

$$\partial_t\theta-\kappa\Delta\theta=N_4\left(\mathbf{u},\theta,\Psi_h\right)\qquad\text{in }\dot\Omega\times(0,T) \tag{2.14}$$

with

$$
\begin{aligned}
N_4\left(\mathbf{u}, \theta, \Psi_h\right) = & -\sum_{i,j=1}^{N}\left(u_i-\frac{\partial \Psi_i}{\partial t}\right)\left(\delta_{ij}+V_{0ij}(\mathbf{k})\right)\frac{\partial \theta}{\partial y_j}+\kappa\sum_{i,j=1}^{N}\frac{\partial V_{0ij}\left(\mathbf{k}\right)}{\partial y_i}\frac{\partial \theta}{\partial y_j} \\
& +2\kappa\sum_{i,j=1}^{N}\frac{\partial^2\theta}{\partial y_i\partial y_j}V_{0jk}\left(\mathbf{k}\right)+\kappa\sum_{i,j,k=1}^{N}V_{0ik}(\mathbf{k})V_{0ij}(\mathbf{k})\frac{\partial^2\theta}{\partial y_j\partial y_k} \\
& +\kappa\sum_{i,j,k=1}^{N}V_{0ik}(\mathbf{k})\frac{\partial \theta}{\partial y_j}\frac{\partial V_{0ij}(\mathbf{k})}{\partial y_k}.
\end{aligned} \tag{2.15}
$$

2.3. **Reformulation of boundary conditions.** Since Γ is a hypersurface of class C^3 in $\mathbb{R}^N$, there exists a bijective mapping: $\mathbb{R}^N \ni x \mapsto \Phi_\ell(x) \in \mathbb{R}^N$ for $\ell \in \mathbb{N}$ and $\Phi_\ell \in C^3(\mathbb{R}^N)$. To represent Γ locally, we use the local coordinates near $x_\ell \in \Gamma$ such that

$$
\begin{aligned}
&\Omega_\pm \cap B_{d'}\left(x_\ell\right) = \left\{y=\Phi_\ell(z) \mid z\in\mathbb{R}^N_\pm\right\}\cap B_{d'}\left(x_\ell\right), \\
&\Gamma\cap B_{d'}\left(x_\ell\right) = \left\{y=\Phi_\ell\left(z',0\right) \mid z'=\left(z_1,\ldots,z_{N-1}\right)\in\mathbb{R}^{N-1}\right\}\cap B_{d'}\left(x_\ell\right),
\end{aligned} \tag{2.16}
$$

where $B_{d'}(x_\ell)=\{x\in\mathbb{R}^N \mid |x-x_\ell|< d'\}$. Let $\{\zeta_\ell\}_{\ell\in\mathbb{N}}$ be a partition of unity such that $\operatorname{supp}\zeta_\ell \subset B_{d'}(x_\ell)$. In the following we use the formula

$$
f=\sum_{\ell=1}^{\infty}\zeta_\ell f \quad \text{ in } \Gamma,
$$

for any function f defined on Γ. Recall that $\Psi_h = h$ on Γ, so the derivatives of H_h coincide with the derivatives of h on Γ. We set $h = h\left(y\left(z',0\right),t\right)$ in the following, by the chain rule, we have

$$
\frac{\partial h}{\partial z_i}=\frac{\partial}{\partial z_i}\Psi_h\left(\Phi_\ell\left(z',0\right),t\right)=\sum_{m=1}^{N}\frac{\partial \Psi_h}{\partial y_m}\frac{\partial \Phi_{\ell,m}}{\partial z_i}\bigg|_{z_N=0}, \tag{2.17}
$$

where we have set $\Phi_\ell=\left(\Phi_{\ell,1},\ldots,\Phi_{\ell,N}\right)^\top$; thus, $\frac{\partial h}{\partial z_i}$ is defined in $B_{d'}\left(x_\ell\right)$ by

$$
\frac{\partial h}{\partial z_i}=\sum_{m=1}^{N}\frac{\partial \Psi_h}{\partial y_m}\circ\Phi_\ell\frac{\partial \Phi_{\ell,m}}{\partial z_i}. \tag{2.18}
$$

Next, we give a representation formula of $\mathbf{n}_t$. Since Γ_t is represented by $x = \Phi_\ell(z',0)+h(\Phi_\ell(z',0),t)\mathbf{n}(\Phi_\ell(z',0),t)$, we set

$$
\mathbf{n}_t = a\left(\mathbf{n}+\sum_{i=1}^{N-1}b_i\tau_i\right) \quad \text{ with } \tau_i=\frac{\partial}{\partial z_i}y=\frac{\partial}{\partial z_i}\Phi_\ell\left(z',0\right),
$$

where a and b_i are unknown functions, and the vectors $\tau_i (i=1,\ldots,N-1)$ form a basis of the tangent space of Γ at $y=y\left(z_1,\ldots,z_{N-1}\right)$. Since $\frac{\partial x}{\partial z_i}\cdot\mathbf{n}_t=0$, we have

$$
0=a\left(\mathbf{n}+\sum_{j=1}^{N-1}b_j\tau_j\right)\cdot\left(\frac{\partial y}{\partial z_i}+\frac{\partial h}{\partial z_i}\mathbf{n}+h\frac{\partial \mathbf{n}}{\partial z_i}\right).
$$

Moreover, $\mathbf{n}\cdot\frac{\partial y}{\partial z_i}=\mathbf{n}\cdot\tau_i=0$, $\frac{\partial \mathbf{n}}{\partial z_i}\cdot\mathbf{n}=0$ (because of $|\mathbf{n}|^2=1$) and $\frac{\partial y}{\partial z_i}\cdot\frac{\partial y}{\partial z_j}=\tau_i\cdot\tau_j=g_{ij}$, we have

$$\frac{\partial h}{\partial z_i}+\sum_{i=1}^{N-1}\left(g_{ij}+h\frac{\partial \mathbf{n}}{\partial z_i}\cdot\tau_j\right)b_j=0. \tag{2.19}$$

Since $\mathbf{n}$ is defined in $\mathbb{R}^N$ as an N-vector of C^2 functions with $\|\mathbf{n}\|_{H^2_\infty(\mathbb{R}^N)}<\infty$, we can write

$$M_{ij}=\frac{\partial \mathbf{n}}{\partial z_i}\cdot\tau_j=\sum_{m=1}^{N}\frac{\partial \mathbf{n}}{\partial y_m}\circ\Phi_\ell\frac{\partial \Phi_{\ell,m}}{\partial z_i}\cdot\frac{\partial \Phi_\ell}{\partial z_j},$$

where M is defined in $\mathbb{R}^N$ and $\|M\|_{H^2_\infty(\mathbb{R}^N)}\leqslant C$ with some constant C independent of $\ell\in\mathbb{N}$. By (2.19), we have

$$\nabla' h=-\left(G+Mh\right)\mathbf{b}, \tag{2.20}$$

where G is an $(N-1)\times(N-1)$ matrix with the $(i,j)^{\text{th}}$ component g_{ij}, $\mathbf{b}=(b_1,\dots,b_{N-1})^\top$ and $\nabla' h=(\frac{\partial h}{\partial z_1},\dots,\frac{\partial h}{\partial z_{N-1}})^\top$. Since $|\mathbf{n}_t|^2=1$, it follows that

$$1=a^2\left(1+\sum_{i,j=1}^{N-1}g_{ij}b_ib_j\right). \tag{2.21}$$

Putting (2.19) and (2.21) together gives

$$\mathbf{b}=-(G+hM)^{-1}\nabla' h=-G^{-1}\left(\mathbf{I}+hMG^{-1}\right)^{-1}\nabla' h,$$

and

$$a=\left\{\left(1+\left(\mathbf{I}+hMG^{-1}\right)^{-1}\nabla' h,G^{-1}\left(\mathbf{I}+hMG^{-1}\right)^{-1}\nabla' h\right)\right\}^{-1/2}.$$

Then, by (2.18), $\nabla' h$ is extended to $\mathbb{R}^N$ by letting $\nabla' h=\nabla\Psi_h\circ\Phi_\ell\left(\nabla\Phi_\ell\right)=:L_{h,\ell}$, then we get

$$\mathbf{n}_t=\mathbf{n}-\sum_{i,j=1}^{N-1}g^{ij}\tau_i\frac{\partial h}{\partial z_j}+\bar{\mathbf{V}}_4\left(h,\nabla' h\right), \tag{2.22}$$

where

$$\begin{aligned}
&\bar{\mathbf{V}}_4\left(h,\nabla' h\right)\\
&=-\left(G^{-1}\left(\left(\mathbf{I}+\Psi_hMG^{-1}\right)^{-1}-\mathbf{I}\right)L_{h,\ell},\tau\right)\\
&\quad+\left\{\left(1+\left(\left(\mathbf{I}+\Psi_hMG^{-1}\right)^{-1}L_{h,\ell},G^{-1}\left(\mathbf{I}+\Psi_hMG^{-1}\right)^{-1}L_{h,\ell}\right)\right)^{-1/2}-1\right\}\\
&\quad\times\left\{\mathbf{n}-\left(G^{-1}\left(\mathbf{I}+\Psi_hMG^{-1}\right)^{-1}L_{H,\ell},\tau\right)\right\}.
\end{aligned}$$

Thus, we may write $\bar{\mathbf{V}}_4\left(h,\nabla' h\right)=\bar{\mathbf{V}}_{4,\ell}(\bar{\mathbf{k}})\bar{\nabla}\Psi_h\otimes\bar{\nabla}\Psi_h$ on $B_{d'}\left(x_\ell\right)$ with a matrix of functions $\bar{\mathbf{V}}_{4,\ell}(\bar{\mathbf{k}})=\bar{\mathbf{V}}_{4,\ell}(y,\bar{\mathbf{k}})$ defined on $B_{d'}(x_\ell)\times\{\bar{\mathbf{k}}||\bar{\mathbf{k}}\,|\leqslant\lambda\}$ with $\bar{\mathbf{V}}_{4,\ell}(0)=0$ possessing the estimate

$$\left\|\left(\bar{\mathbf{V}}_{4,\ell},\partial_{\bar{\mathbf{k}}}\bar{\mathbf{V}}_{4,\ell}\right)(\cdot,\bar{\mathbf{k}})\right\|_{H^1_\infty(B_{d'}(x_\ell))}\leqslant C, \tag{2.23}$$

for some constant C independent of $\ell\in\mathbb{N}$, here and in the following $\bar{\mathbf{k}}$ is the variable corresponding to $\bar{\nabla}\Psi_h=(\Psi_h,\nabla\Psi_h)$ and $\partial_{\bar{\mathbf{k}}}$ denotes the partial derivative with respect to variable $\bar{\mathbf{k}}$. Let $\mathbf{V}_4(\bar{\mathbf{k}})=\sum_{\ell=1}^\infty\zeta_\ell\bar{\mathbf{V}}_{4,\ell}(\bar{\mathbf{k}})$, then

$$\left\|\left(\mathbf{V}_4,\partial_{\mathbf{k}}\mathbf{V}_4\right)(\cdot,\bar{\mathbf{k}})\right\|_{H^1_\infty(\Omega)}\leqslant C. \tag{2.24}$$

In view of (2.18) and (2.22), the unit outer normal $\mathbf{n}_t$ is also represented by

$$\mathbf{n}_t = \mathbf{n} - \sum_{i,j=1}^{N-1} \sum_{m=1}^{N} g^{ij} \tau_i \frac{\partial \Psi_h}{\partial z_m} \circ \Phi_\ell \frac{\partial \Phi_{\ell,m}}{\partial z_j} + \bar{\mathbf{V}}_{4,\ell}(\bar{\mathbf{k}}) \bar{\nabla} \Psi_h \otimes \bar{\nabla} \Psi_h$$

on $B_{d'}(x_\ell)$, and we can also write

$$\mathbf{n}_t = \mathbf{n} + \tilde{\mathbf{V}}_{4,\ell} \left(\bar{\nabla} \Psi_h \right) \bar{\nabla} \Psi_h, \tag{2.25}$$

for some functions $\tilde{\mathbf{V}}_{4,\ell}(\bar{\mathbf{k}}) = \tilde{\mathbf{V}}_{4,\ell}(y, \bar{\mathbf{k}})$ defined on $B_{d'}(x_\ell) \times \{\bar{\mathbf{k}} \mid |\bar{\mathbf{k}}| \leqslant \lambda\}$ possessing the estimate:

$$\left\| (\tilde{\mathbf{V}}_{4,\ell}, \partial_{\bar{\mathbf{k}}} \tilde{\mathbf{V}}_{4,\ell})(\cdot, \bar{\mathbf{k}}) \right\|_{H^1_\infty(B_{d'}(x_\ell))} \leqslant C. \tag{2.26}$$

Let $\tilde{\mathbf{V}}_4(\bar{\mathbf{k}}) = \sum_{\ell=1}^\infty \zeta_\ell \tilde{\mathbf{V}}_{4,\ell}(\bar{\mathbf{k}})$, then

$$\left\| (\tilde{\mathbf{V}}_4, \partial_{\mathbf{k}} \tilde{\mathbf{V}}_4)(\cdot, \bar{\mathbf{k}}) \right\|_{H^1_\infty(\Omega)} \leqslant C. \tag{2.27}$$

Therefore, we obtain

$$\mathbf{n}_t = \mathbf{n} + \tilde{\mathbf{V}}_4 \left(\bar{\nabla} \Psi_h \right) \bar{\nabla} \Psi_h. \tag{2.28}$$

We now consider the kinematic equation $V_n = \mathbf{v} \cdot \mathbf{n}_t$. Note that $x = y + h(y,t)\mathbf{n}$ on Γ_t, by (2.22), we have

$$\begin{aligned} V_n &= \frac{\partial x}{\partial t} \cdot \mathbf{n}_t = \frac{\partial h}{\partial t} \mathbf{n} \cdot \mathbf{n}_t \\ &= \sum_{\ell=1}^\infty \zeta_\ell \left\{ \partial_t h + \partial_t h \left(\mathbf{n}, - \sum_{i,j=1}^{N-1} g^{ij} \tau_i \frac{\partial h}{\partial z_j} + \bar{\mathbf{V}}_{4,\ell}(\bar{\mathbf{k}}) \bar{\nabla} \Psi_h \otimes \bar{\nabla} \Psi_h \right) \right\} \\ &= \partial_t h + \partial_t h \left(\mathbf{n}, \mathbf{V}_4(\bar{\mathbf{k}}) \bar{\nabla} \Psi_h \otimes \bar{\nabla} \Psi_h \right). \end{aligned}$$

On the other hand, we get

$$\mathbf{v} \cdot \mathbf{n}_t = \mathbf{n} \cdot \mathbf{u} - \langle \mathbf{u} \perp \nabla' h \rangle + \mathbf{u} \cdot \mathbf{V}_4(\bar{\mathbf{k}}) \bar{\nabla} \Psi_h \otimes \bar{\nabla} \Psi_h,$$

due to $[\![\mathbf{v}]\!] = 0$, where

$$\langle \mathbf{u} \perp \nabla' h \rangle = \sum_{i,j=1}^{N-1} g^{ij} (\tau_i, \mathbf{u}) \frac{\partial h}{\partial z_j}.$$

Thus, the kinematic equation is transformed to

$$\partial_t h - \mathbf{n} \cdot \mathbf{u} + \langle \mathbf{u} \perp \nabla' h \rangle = d(\mathbf{u}, \Psi_h) \quad \text{ on } \Gamma \times (0, T), \tag{2.29}$$

with

$$d(\mathbf{u}, \Psi_h) := \mathbf{u} \cdot \mathbf{V}_4(\bar{\mathbf{k}}) \bar{\nabla} \Psi_h \otimes \bar{\nabla} \Psi_h - \partial_t h \left(\mathbf{n}, \mathbf{V}_4(\bar{\mathbf{k}}) \bar{\nabla} \Psi_h \otimes \bar{\nabla} \Psi_h \right). \tag{2.30}$$

In fact, if we attempt to use the standard fixed point argument, we have to move the term $\langle \mathbf{u} \perp \nabla' h \rangle$ to the right hand side, then we need to assume that both initial data $\mathbf{u}_0$ and h_0 are small enough. However, we are unwilling to make any restriction on the size of the initial velocity $\mathbf{u}_0$. For this purpose, we approximate $\mathbf{u}$ as follows. Let $\mathbf{u}_0$ be an initial data belonging to $B^{2/p'}_{q,p}(\Omega)$ with $1/p + 1/p' = 1$ and set $\mathbf{u}_{0+} = \mathbf{u}_0|_{\Omega_+}$. Notice that $[\![\mathbf{u}_0]\!] = 0$ on Γ, which is one of the compatibility conditions for the initial data. Let $\tilde{\mathbf{u}}_{0+}$ be an extension of $\mathbf{u}_{0+}$ to $\mathbb{R}^N$ such that $\tilde{\mathbf{u}}_{0+} = \mathbf{u}_{0+}$ in Ω_+. We approximate $\mathbf{u}_0$ by $\mathbf{u}_\iota$ defined by

$$\mathbf{u}_\iota = \frac{1}{\iota} \int_0^\iota T(s) \tilde{\mathbf{u}}_0 ds, \tag{2.31}$$

where $T(s)$ is the heat semigroup on $\mathbb{R}^N$ given by

$$T(s)f = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}e^{-|\xi|^2 s}\mathcal{F}[f](\xi)d\xi, \quad \mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}f(x)dx.$$

Since $T(s)$ is a bounded analytic semigroup, we have

$$\begin{aligned}\|T(\cdot)\tilde{\mathbf{u}}_0\|_{L^\infty\left((0,\infty),B^{2/p'}_{q,p}(\mathbb{R}^N)\right)} + \|T(\cdot)\tilde{\mathbf{u}}_0\|_{L^p\left((0,\infty),H^2_q(\mathbb{R}^N)\right)}\\ + \|T(\cdot)\tilde{\mathbf{u}}_0\|_{H^1_p((0,\infty),L^q(\mathbb{R}^N))} \leqslant C\,\|\mathbf{u}_0\|_{B^{2/p'}_{q,p}(\Omega_+)},\end{aligned} \tag{2.32}$$

which yields

$$\begin{aligned}\|\mathbf{u}_\iota\|_{B^{2/p'}_{q,p}(\mathbb{R}^N)} &\leqslant C\,\|\mathbf{u}_0\|_{B^{2/p'}_{q,p}(\Omega_+)},\\ \|\mathbf{u}_\iota\|_{H^2_q(\mathbb{R}^N)} &\leqslant C\iota^{-1/p}\,\|\mathbf{u}_0\|_{B^{2/p'}_{q,p}(\Omega_+)},\end{aligned} \tag{2.33}$$

for $1 < p, q < \infty$. Thus, as a kinematic condition, we use the following equation:

$$\partial_t h - \mathbf{n}\cdot\mathbf{u} + \langle \mathbf{u}_\iota \perp \nabla' h\rangle = N_5(\mathbf{u},\Psi_h), \tag{2.34}$$

with

$$N_5(\mathbf{u},\Psi_h) = d(\mathbf{u},\Psi_h) + \langle \mathbf{u}_\iota - \mathbf{u} \perp \nabla' h\rangle. \tag{2.35}$$

We next consider the boundary condition:

$$[\![(\mu\mathbf{D}(\mathbf{v}) - p\mathbf{I})\mathbf{n}_t]\!] = \sigma H(\Gamma_t)\,\mathbf{n}_t \quad \text{ on } \Gamma_t,$$

for $0 < t < T$ in equations (1.1). It is convenient to divide the formula into the tangential part and normal part on Γ_t as follows:

$$\begin{aligned}[\![\Pi_t\mu\mathbf{D}(\mathbf{v})\mathbf{n}_t]\!] &= 0,\\ [\![(\mu\mathbf{D}(\mathbf{v})\mathbf{n}_t,\mathbf{n}_t) - p]\!] &= \sigma(H(\Gamma_t)\,\mathbf{n}_t,\mathbf{n}_t),\end{aligned}$$

where Π_t is defined by $\Pi_t\mathbf{d} = \mathbf{d} - (\mathbf{d},\mathbf{n}_t)\mathbf{n}_t$ for any N-vector $\mathbf{d}$. A similar result for the Stokes equations with free boundary conditions is given in [27]. We omit the details and get the following conclusions directly. We have

$$\begin{aligned}[\![\mu\mathbf{D}(\mathbf{u})\mathbf{n}]\!]_\tau &= [\![\mathbf{N}_6(\mathbf{u},\Psi_h)]\!] \quad \text{ on } \Gamma\times[0,T],\\ [\![(\mu\mathbf{D}(\mathbf{u})\mathbf{n},\mathbf{n}) - q]\!] - \sigma(\Delta_\Gamma h + \mathcal{F}h) &= [\![N_7(\mathbf{u},\Psi_h)]\!] \quad \text{ on } \Gamma\times[0,T],\end{aligned} \tag{2.36}$$

where $\mathbf{d}_\tau = \mathbf{d} - (\mathbf{d},\mathbf{n})\mathbf{n}$ for any N-vector $\mathbf{d}$, Δ_Γ is the Laplace-Beltrami operator on Γ, $\mathcal{F}$ is bounded C^1 function, and by (2.25), we get

$$\begin{aligned}\mathbf{N}_6(\mathbf{u},\Psi_h) = &-\mu\Big\{\Pi_0\mathbf{V}_1(\mathbf{k})\nabla\mathbf{u} + \Big(\tilde{\mathbf{V}}_4(\bar{\mathbf{k}})\bar{\mathbf{k}},\mathbf{D}(\mathbf{u}) + \mathbf{V}_1(\mathbf{k})\nabla\mathbf{u}\Big)\,\mathbf{n}\\ &+ (\mathbf{n},\mathbf{D}(\mathbf{u}) + \mathbf{V}_1(\mathbf{k})\nabla\mathbf{u})\,\tilde{\mathbf{V}}_4(\bar{\mathbf{k}})\bar{\mathbf{k}}\\ &+ \Big(\tilde{\mathbf{V}}_4(\bar{\mathbf{k}})\bar{\mathbf{k}},\mathbf{D}(\mathbf{u}) + \mathbf{V}_1(\mathbf{k})\nabla\mathbf{u}\Big)\,\tilde{\mathbf{V}}_{4,\ell}(\bar{\mathbf{k}})\bar{\mathbf{k}}\Big\},\end{aligned} \tag{2.37}$$

$$\begin{aligned}N_7(\mathbf{u},\Psi_h) = &-\big(\mathbf{n},\mu\mathbf{D}(\mathbf{u})\mathbf{V}_3(\bar{\mathbf{k}})\bar{\mathbf{k}}\big) - \big(\mathbf{n},\mu\left(\mathbf{V}_1(\mathbf{k})\nabla\mathbf{u}\right)\big(\mathbf{n} + \mathbf{V}_3(\bar{\mathbf{k}})\bar{\mathbf{k}}\big)\big)\\ &+ \sigma\mathbf{V}_5(\bar{\mathbf{k}})\bar{\mathbf{k}}\otimes\bar{\bar{\mathbf{k}}},\end{aligned} \tag{2.38}$$

for $\bar{\mathbf{k}} = (\Psi_h,\nabla\Psi_h)$ and $\bar{\bar{\mathbf{k}}} = \big(\Psi_h,\nabla\Psi_h,\nabla^2\Psi_h\big)$. In view of (2.37) and (2.38), we may symbolically write

$$\mathbf{N}_6(\mathbf{u},\Psi_h) = \mathbf{V}_6\left(\bar{\mathbf{k}}\right)\bar{\mathbf{k}}\otimes\nabla\mathbf{u}, \tag{2.39}$$

and

$$N_7(\mathbf{u},\Psi_h) = \mathbf{V}_7\left(\bar{\mathbf{k}}\right)\bar{\mathbf{k}}\otimes\nabla\mathbf{u} + \sigma\mathbf{V}_5(\bar{\mathbf{k}})\bar{\mathbf{k}}\otimes\bar{\bar{\mathbf{k}}}, \tag{2.40}$$

where for $i=5,6,7$, $\mathbf{V}_i(\bar{\mathbf{k}})=\mathbf{V}_i(y,\bar{\mathbf{k}})$ are some matrices of C^1 functions consisting of products of elements of $\mathbf{n}$ defined on $\dot{\Omega}\times\{\bar{\mathbf{k}}\mid|\bar{\mathbf{k}}|<\lambda\}$ possessing the estimate

$$\sup_{|\bar{\mathbf{k}}|\leqslant\lambda}\left\|\left(\mathbf{V}_i(\cdot,\bar{\mathbf{k}}),\partial_{\bar{\mathbf{k}}}\mathbf{V}_i(\cdot,\bar{\mathbf{k}})\right)\right\|_{H^1_\infty(\dot{\Omega})}\leqslant C.$$

Finally, we see the interface condition $[\![\kappa\nabla\theta'\cdot\mathbf{n}_t]\!]=0$. In view of (2.6) and (2.22), we can write

$$[\![\kappa\nabla\theta\cdot\mathbf{n}]\!]=N_8(\theta,\Psi_h),$$

with

$$N_8\left(\theta,\Psi_h\right)=[\![\kappa\nabla\theta\cdot(\mathbf{n}-\mathbf{n}_t)]\!]-[\![\kappa\mathbf{V}_0(\mathbf{k})\nabla\theta\cdot\mathbf{n}_t]\!]=\mathbf{V}_8(\bar{\mathbf{k}})\bar{\mathbf{k}}\otimes[\![\kappa\nabla\theta]\!]. \tag{2.41}$$

Here, $\mathbf{V}_8(\bar{\mathbf{k}})$ is a matrix and a set of matrices of functions consisting of products of elements of $\mathbf{n}$ and smooth functions defined for $|\mathbf{k}|<\lambda$, possessing the estimate

$$\sup_{\bar{\mathbf{k}}|\leqslant\lambda}\left\|\left(\mathbf{V}_8(\cdot,\bar{\mathbf{k}}),\partial_{\bar{\mathbf{k}}}\mathbf{V}_8(\cdot,\bar{\mathbf{k}})\right)\right\|_{H^1_\infty(\Omega)}\leqslant C. \tag{2.42}$$

2.4. **Main theorem.** Let h and Ψ_h be the same functions as in subsections 2.1 and 2.2, combining (2.10) with (2.11), (2.14), (2.34), (2.39), (2.40) and (2.41), for $t\in(0,T)$, we have seen that (1.1) is reformulated as follows:

$$\begin{cases}\rho\partial_t\mathbf{u}-\mathrm{Div}(\mu\mathbf{D}(\mathbf{u})-q\mathbf{I})=\mathbf{N}_1(\mathbf{u},\mathbf{a},\theta,\Psi_h)+\mathbf{a}(x(y,t),t)+\alpha\mathbf{g}\theta & \text{in } \dot{\Omega},\\ \operatorname{div}\mathbf{u}=N_2(\mathbf{u},\Psi_h)=\operatorname{div}\mathbf{N}_3(\mathbf{u},\Psi_h) & \text{in } \dot{\Omega},\\ \partial_t\theta-\kappa\Delta\theta=N_4(\mathbf{u},\theta,\Psi_h) & \text{in } \dot{\Omega},\\ \partial_t h-\mathbf{n}\cdot\mathbf{u}+\langle\mathbf{u}_\iota\perp\nabla'h\rangle=N_5(\mathbf{u},\Psi_h) & \text{on } \Gamma,\\ [\![\mu\mathbf{D}(\mathbf{u})\mathbf{n}]\!]_\tau=[\![\mathbf{N}_6\left(\mathbf{u},\Psi_h\right)]\!],\quad [\![\mathbf{u}]\!]=0 & \text{on } \Gamma,\\ [\![(\mu\mathbf{D}(\mathbf{u})\mathbf{n},\mathbf{n})-q]\!]-\sigma\left(\Delta_\Gamma h+\mathcal{F}h\right)=[\![N_7\left(\mathbf{u},\Psi_h\right)]\!] & \text{on } \Gamma,\\ [\![\kappa\nabla\theta\cdot\mathbf{n}]\!]=N_8(\theta,\Psi_h),\quad [\![\theta]\!]=0 & \text{on } \Gamma,\\ \mathbf{u}=0,\quad \nabla\theta_-\cdot\mathbf{n}_-+\beta\theta_-=b(y,t) & \text{on } \Gamma_-,\\ \mathbf{u}|_{t=0}=\mathbf{u}_0\ \text{ in } \dot{\Omega},\quad \theta|_{t=0}=\theta_0\ \text{ in } \dot{\Omega},\quad h|_{t=0}=h_0\ \text{ on } \Gamma.\end{cases} \tag{2.43}$$

The main theorem about local well-posedness is stated as follows.

Theorem 2.1. *Let $2<p<\infty$, $N<q<\infty$, $2/p+N/q<1$, $B>1$ and let Γ be a compact hypersurface of class C^3. Let $\mathbf{a}(x(y,t),t)\in L^p((0,T),L^q(\dot{\Omega}))$ and $b(y,t)\in L^p\left((0,T),H^1_q(\Omega_-)\right)\cap H^{1/2}_p\left((0,T),L^q(\Omega_-)\right)$. Let $(\mathbf{u}_0,\theta_0)\in B^{2(1-1/p)}_{q,p}(\dot{\Omega})$ and $h_0\in B^{3-1/p-1/q}_{q,p}(\Gamma)$ be initial data satisfying $\|(\mathbf{u}_0,\theta_0)\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})}\leqslant B$ and $\|h_0\|_{B^{3-1/p-1/q}_{q,p}}\leqslant\epsilon$ for some small number $\epsilon>0$ and the compatibility conditions*

$$\begin{aligned}&\mathbf{u}_0-\mathbf{N}_3\left(\mathbf{u}_0,\Psi_h|_{t=0}\right)\in\mathcal{D}(\Omega),\quad \operatorname{div}\mathbf{u}_0=N_2\left(\mathbf{u}_0,\Psi_h|_{t=0}\right) && \text{in } \dot{\Omega},\\ &[\![(\mu\mathbf{D}\left(\mathbf{u}_0\right)\mathbf{n})_\tau]\!]=[\![(\mathbf{N}_6\left(\mathbf{u}_0,\Psi_h|_{t=0}\right))_\tau]\!],\quad [\![\mathbf{u}_0]\!]=0 && \text{on } \Gamma,\\ &[\![\kappa\nabla\theta_0\cdot\mathbf{n}]\!]=N_8\left(\theta_0,\Psi_h|_{t=0}\right),\quad [\![\theta_0]\!]=0 && \text{on } \Gamma,\\ &\mathbf{u}_0=0,\quad \nabla\theta_{0-}\cdot\mathbf{n}_-+\beta\theta_{0-}=b|_{t=0} && \text{on } \Gamma_-,\end{aligned} \tag{2.44}$$

where $\mathcal{D}(\Omega)$ can be found in the appendix, then there exists a small time $T>0$ such that problem (2.43) admits a unique solution $(\mathbf{u},\theta,h)$ with

$$\mathbf{u}\in H^1_p\left((0,T),L^q(\dot{\Omega})\right)\cap L^p\left((0,T),H^2_q(\dot{\Omega})\right),$$

$$\theta \in H_p^1\left((0,T), L^q(\dot\Omega)\right) \cap L^p\left((0,T), H_q^2(\dot\Omega)\right),$$
$$h \in H_p^1\left((0,T), W_q^{2-1/q}(\Gamma)\right) \cap L^p\left((0,T), W_q^{3-1/q}(\Gamma)\right)$$

possessing the estimate (2.3) and the estimate

$$\begin{aligned} &E_{p,q,T}(\mathbf{u},\theta,h) \\ &\lesssim B + \|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)} + \|b\|_{H_p^{1/2}((0,T),L^q(\Omega_-))} + \|b\|_{L^p\left((0,T),H_q^1(\Omega_-)\right)}, \end{aligned} \tag{2.45}$$

where the symbol "$\lesssim$" denotes "$\leqslant C$" for some constant $C>0$, and

$$\begin{aligned} E_{p,q,T}(\mathbf{u},\theta,h) :=&\|(\mathbf{u},\theta)\|_{L^p\left((0,T),H_q^2(\dot\Omega)\right)} + \|\partial_t(\mathbf{u},\theta)\|_{L^p\left((0,T),L^q(\dot\Omega)\right)} \\ &+ \|h\|_{L^p\left((0,T),W_q^{3-1/q}(\Gamma)\right)} + \|\partial_t h\|_{L^p\left((0,T),W_q^{2-1/q}(\Gamma)\right)} \\ &+ \|\partial_t h\|_{L^\infty\left((0,T),W_q^{1-1/q}(\Gamma)\right)}. \end{aligned} \tag{2.46}$$

3. **Linearization Theory.** We consider the L^p-L^q maximal regularity to two linearized equations. One is the Stokes equations with transmission conditions on Γ and non-slip conditions on Γ_-, the other is the heat equations with interface and boundary conditions on Γ and Γ_-, respectively.

3.1. **Maximal regularity for the two-phase problem for the Stokes equations with interface conditions.** This subsection is devoted to presenting the L^p-L^q maximal regularity for the two-phase problem of the Stokes equations with free boundary conditions given as follows:

$$\begin{cases} \rho\partial_t\mathbf{u} - \mathrm{Div}(\mu\mathbf{D}(\mathbf{u}) - q\mathbf{I}) = \mathbf{f} & \text{in } \dot\Omega\times(0,T), \\ \mathrm{div}\,\mathbf{u} = g = \mathrm{div}\,\mathbf{g} & \text{in } \dot\Omega\times(0,T), \\ \partial_t h - \mathbf{n}\cdot\mathbf{u} + \langle A_\iota \perp \nabla' h\rangle = d & \text{on } \Gamma\times(0,T), \\ [\![(\mu\mathbf{D}(\mathbf{u}) - q\mathbf{I})\mathbf{n}]\!] - ((\mathcal{F}+\sigma\Delta_\Gamma)h)\,\mathbf{n} = [\![\mathbf{h}]\!], \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma\times(0,T), \\ \mathbf{u} = 0 & \text{on } \Gamma_-\times(0,T), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \dot\Omega, \quad h|_{t=0} = h_0 \quad \text{on } \Gamma. \end{cases} \tag{3.1}$$

Here, A_ι, for $\iota\in[0,1)$, is an $(N-1)$-vector of functions defined on Γ possessing the following properties: $A_0 = 0$ and for any $\iota\in(0,1)$

$$\begin{aligned} &|A_\iota(x)| \leqslant m_1, \quad |A_\iota(x) - A_\iota(y)| \leqslant m_1|x-y|^b \text{ for any } x,y\in\Gamma, \\ &\|A_\iota\|_{W_r^{2-1/r}(\Gamma)} \leqslant m_2\iota^{-c}, \end{aligned} \tag{3.2}$$

for $N<r<\infty$ and some positive constants m_1, m_2, b and c independent of $\iota\in(0,1)$. Moreover, we assume that

$$\|\mathcal{F}h\|_{W_q^{1-1/q}(\Gamma)} \leqslant C\|h\|_{W_q^{2-1/q}(\Gamma)}. \tag{3.3}$$

Theorem 3.1 (cf. [27])**.** *Let $1<p,q<\infty$ with $2/p+1/q\notin\{1,2\}$ and $T>0$. Assume that Γ is a compact hypersurface of class C^3. There exists a γ_0 such that the following assertion holds: Let $\mathbf{u}_0\in B_{q,p}^{2(1-1/p)}(\dot\Omega)$ and $h_0\in B_{q,p}^{3-1/p-1/q}(\Gamma)$ be initial data for equations (3.1) and let $\mathbf{f},g,\mathbf{g},d$ and $\mathbf{h}$ be functions appearing in the right side of equations (3.1) and satisfying the conditions:*

$$\mathbf{f}\in L^p\left((0,T), L^q(\dot\Omega)\right), \quad d\in L^p\left((0,T), W_q^{2-1/q}(\Gamma)\right),$$

$$e^{-\gamma t} g \in L^p\left(\mathbb{R}, H_q^1(\dot\Omega)\right) \cap H_p^{1/2}\left(\mathbb{R}, L^q(\dot\Omega)\right), \quad e^{-\gamma t}\mathbf{g} \in H_p^1\left(\mathbb{R}, L^q(\dot\Omega)\right),$$
$$e^{-\gamma t}\mathbf{h} \in L^p\left(\mathbb{R}, H_q^1(\dot\Omega)\right) \cap H_p^{1/2}\left(\mathbb{R}, L^q(\dot\Omega)\right),$$

for any $\gamma \geqslant \gamma_0$. Assume that $\mathbf{u}_0$, g and $\mathbf{h}$ satisfy the following compatibility conditions

$$\mathbf{u}_0 - \mathbf{g}|_{t=0} \in \mathcal{D}(\Omega) \text{ and } \operatorname{div}\mathbf{u}_0 = g|_{t=0} \text{ in } \dot\Omega.$$

In addition, we assume that

$$\begin{aligned} &[\![(\mu\mathbf{D}(\mathbf{u}_0)\,\mathbf{n})_\tau]\!] = [\![\mathbf{h}_\tau]\!]|_{t=0} && \text{on } \Gamma, && \text{for } 2/p+1/q<1,\\ &[\![\mathbf{u}_0]\!] = 0 \text{ on } \Gamma, \quad \mathbf{u}_0 = 0 && \text{on } \Gamma_-, && \text{for } 2/p+1/q<2.\end{aligned}$$

Then, (3.1) admits a unique solution $(\mathbf{u}, q, h)$ with

$$\mathbf{u} \in L^p\left((0,T), H_q^2(\dot\Omega)\right) \cap H_p^1\left((0,T), L^q(\dot\Omega)\right), \quad q \in L^p\left((0,T), H_q^1(\dot\Omega) + \hat H_q^1(\Omega)\right),$$
$$h \in L^p\left((0,T), W_q^{3-1/q}(\Gamma)\right) \cap H_p^1\left((0,T), W_q^{2-1/q}(\Gamma)\right),$$

where $\hat H_q^1$ is defined in the appendix, possessing the estimates

$$\begin{aligned} &\|\mathbf{u}\|_{L^p\left((0,T),H_q^2(\dot\Omega)\right)} + \|\partial_t\mathbf{u}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\\ &\quad + \|h\|_{L^p\left((0,T),W_q^{3-1/q}(\Gamma)\right)} + \|\partial_t h\|_{L^p\left((0,T),W_q^{2-1/q}(\Gamma)\right)}\\ &\lesssim e^{\gamma\iota^{-c}T}\Big\{\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \iota^{-c}\|h_0\|_{B_{q,p}^{3-1/p-1/q}(\Gamma)} + \|\mathbf{f}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\\ &\quad + \|d\|_{L^p\left((0,T),W_q^{2-1/q}(\Gamma)\right)} + \left\|e^{-\gamma t}\partial_t\mathbf{g}\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)} + \left\|e^{-\gamma t}(g,\mathbf{h})\right\|_{L^p\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\ &\quad + \left(1+\gamma^{1/2}\right)\left(\left\|e^{-\gamma t}g\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot\Omega)\right)} + \left\|e^{-\gamma t}\mathbf{h}\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot\Omega)\right)}\right)\Big\}\end{aligned}$$

for any $\gamma \geqslant \gamma_0$ with some constant $C>0$ independent of γ.

3.2. **The maximal regularity for the heat equations with interface conditions.** This subsection is devoted to presenting the maximal L^p-L^q regularity for the heat equations with interface conditions,

$$\begin{cases} \partial_t\theta - \kappa\Delta\theta = f & \text{in } \dot\Omega\times(0,T),\\ [\![\kappa\nabla\theta\cdot\mathbf{n}]\!] = g, \quad [\![\theta]\!] = 0 & \text{on } \Gamma\times(0,T),\\ \nabla\theta_-\cdot\mathbf{n}_- + \beta\theta - = h & \text{on } \Gamma_-\times(0,T),\\ \theta|_{t=0} = \theta_0 & \text{in } \dot\Omega.\end{cases} \tag{3.4}$$

Theorem 3.2. *Let $1<p,q<\infty$ with $2/p+1/q \notin \{1,2\}$ and $t>0$. Assume that Γ is a compact hypersurface of class C^3. Let $\theta_0 \in B_{q,p}^{2(1-1/p)}(\dot\Omega)$ be initial data for equation (3.4). Let f, g and h be functions appearing on the right side of equation (3.4) which satisfy the following conditions:*

$$e^{-\gamma t}f \in L^p\left((0,T), L^q(\dot\Omega)\right), \quad e^{-\gamma t}g \in L^p\left(\mathbb{R}, H_q^1(\dot\Omega)\right) \cap H_p^{1/2}\left(\mathbb{R}, L^q(\dot\Omega)\right),$$
$$e^{-\gamma t}h \in L^p\left(\mathbb{R}, H_q^1(\Omega_-)\right) \cap H_p^{1/2}\left(\mathbb{R}, L^q(\Omega_-)\right),$$

for any $\gamma \geqslant \gamma_0$ with some γ_0. Assume that the compatibility conditions hold:

$$\begin{aligned} &\begin{cases} [\![\kappa\nabla\theta_0\cdot\mathbf{n}]\!] = g|_{t=0} & \text{on } \Gamma,\\ \nabla\theta_{0-}\cdot\mathbf{n}_- + \beta\theta_{0-} = h|_{t=0} & \text{on } \Gamma_-,\end{cases} && \text{if } 2/p+1/q<1;\\ &\quad [\![\theta_0]\!] = 0 \quad \text{on } \Gamma, && \text{if } 2/p+1/q<2.\end{aligned} \tag{3.5}$$

Then, problem (3.4) admits a unique solution θ with

$$\theta \in L^p\left((0,\infty), H_q^2(\dot{\Omega})\right) \cap H_p^1\left((0,\infty), L^q(\dot{\Omega})\right)$$

possessing the estimate

$$\begin{aligned}
&\|\partial_t\theta\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)} + \|\theta\|_{L^p\left((0,T),H_q^2(\dot{\Omega})\right)} \\
&\lesssim e^{\gamma T}\left\{\|\theta_0\|_{B_{q,p}^{2(1-1/p)}(\dot{\Omega})} + \|f\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)}\right. \\
&\quad + \left\|e^{-\gamma t}g\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)} + \left\|e^{-\gamma t}h\right\|_{L^p\left(\mathbb{R},H_q^1(\Omega_-)\right)} \\
&\quad \left. + \left(1+\gamma^{1/2}\right)\left(\left\|e^{-\gamma t}g\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)} + \left\|e^{-\gamma t}h\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\Omega_-)\right)}\right)\right\}. \qquad (3.6)
\end{aligned}$$

Remark 3.3. (1) Theorem 3.2 has been proved by Hao and Zhang [10] in the case where $e^{-\gamma t}f \in L^p(\mathbb{R}, L^q(\dot{\Omega}))$. But we can set $\tilde{f}$ to be the zero extension of f outside $(0,T)$, that is, $\tilde{f}(t) = f(t)$ for $t \in (0,T)$. Then we can replace f with $\tilde{f}$ in Theorem 3.2 and get

$$\left\|e^{-\gamma t}\tilde{f}\right\|_{L^p\left((0,\infty),L^q(\dot{\Omega})\right)} \leqslant \|f\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)}.$$

Thus, (3.6) is satisfied.

(2) Theorem 3.2 was proved in [10] under the assumption that Ω is a uniform $W_r^{2-1/r}$ domain for $1 < r < \infty$. In fact, if Ω is a uniform C^3 domain, we can also get the same result as in [26]. In this paper, we assume Γ is a compact hypersurface of class C^3, obviously, Ω is a uniform C^3 domain.

4. Estimates of nonlinear terms.

We give an iteration scheme to prove Theorem 2.1 by the Banach fixed point theorem. We define an underlying space $\mathbf{U}_T$ by

$$\begin{aligned}
\mathbf{U}_T = \Big\{ &(\mathbf{v},\Theta,\varrho) \mid (\mathbf{v},\Theta) \in H_p^1\left((0,T), L^q(\dot{\Omega})\right) \cap L^p\left((0,T), H_q^2(\dot{\Omega})\right), \\
&\varrho \in H_p^1\left((0,T), W_q^{2-1/q}(\Gamma)\right) \cap L^p\left((0,T), W_q^{3-1/q}(\Gamma)\right), \\
&(\mathbf{v},\Theta)|_{t=0} = (\mathbf{u}_0,\theta_0) \quad \text{in } \dot{\Omega}, \quad \varrho|_{t=0} = h_0 \quad \text{on } \Gamma, \\
&E_{p,q,T}(\mathbf{v},\Theta,\varrho) \leqslant L, \quad \sup_{t\in(0,T)} \|\varrho(\cdot,t)\|_{W_\infty^1(\Gamma)} \leqslant \lambda \Big\},
\end{aligned}$$

where L is a sufficiently large number determined later. For $t \in (0,T)$, let $(\mathbf{v},\Theta,\varrho) \in \mathbf{U}_T$ and let $\mathbf{u}$, q, θ and h be solutions of the linear equations:

$$\begin{cases}
\rho\partial_t\mathbf{u} - \operatorname{Div}(\mu\mathbf{D}(\mathbf{u}) - q\mathbf{I}) = \mathbf{N}_1(\mathbf{v},\mathbf{a},\Theta,\Psi_\varrho) + \mathbf{a}(x(y,t),t) + \alpha\mathbf{g}\theta & \text{in } \dot{\Omega}, \\
\operatorname{div}\mathbf{u} = N_2(\mathbf{v},\Psi_\varrho) = \operatorname{div}\mathbf{N}_3(\mathbf{v},\Psi_\varrho) & \text{in } \dot{\Omega}, \\
\partial_t\theta - \kappa\Delta\theta = N_4(\mathbf{v},\Theta,\Psi_\varrho) & \text{in } \dot{\Omega}, \\
\partial_t h - \mathbf{n}\cdot\mathbf{u} + \langle \mathbf{u}_\iota \perp \nabla' h\rangle = N_5(\mathbf{v},\Psi_\varrho) & \text{on } \Gamma, \\
[\![(\mu\mathbf{D}(\mathbf{u})\mathbf{n})_\tau]\!] = [\![\mathbf{N}_6(\mathbf{v},\Psi_\varrho)]\!], \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma, \\
[\![(\mu\mathbf{D}(\mathbf{u})\mathbf{n},\mathbf{n}) - q]\!] - \sigma(\Delta_\Gamma h + \mathcal{F}h) = [\![N_7(\mathbf{v},\Psi_\varrho)]\!] & \text{on } \Gamma, \\
[\![\kappa\nabla\theta\cdot\mathbf{n}]\!] = N_8(\mathbf{v},\Theta,\Psi_\varrho), \quad [\![\theta]\!] = 0 & \text{on } \Gamma, \\
\mathbf{u} = 0, \quad \nabla\theta_-\cdot\mathbf{n}_- + \beta\theta_- = b(y,t) & \text{on } \Gamma_-, \\
\mathbf{u}|_{t=0} = \mathbf{u}_0 \text{ in } \dot{\Omega}, \quad \theta|_{t=0} = \theta_0 \text{ in } \dot{\Omega}, \quad h|_{t=0} = h_0 \text{ on } \Gamma,
\end{cases} \qquad (4.1)$$

where $\Psi_\varrho = \omega(y)H_\varrho(y,t)\mathbf{n}(y)$. In view of (2.3), we may assume

$$\sup_{t\in(0,T)} \|\Psi_\varrho(\cdot,t)\|_{H^1_\infty(\Omega)} \leqslant \lambda, \tag{4.2}$$

because we have assumed $\sup_{t\in(0,T)} \|\varrho(\cdot,t)\|_{W^1_\infty(\Gamma)} \leqslant \lambda$. Since $E_{p,q,T}(\mathbf{v},\Theta,\varrho) \leqslant L$, we have

$$\begin{aligned}&\|\partial_t\mathbf{v}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)} + \|\mathbf{v}\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)} + \|\partial_t\Theta\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\\ &\quad + \|\Theta\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)} + \|\partial_t\varrho\|_{L^\infty\left((0,T),W^{1-1/q}_q(\Gamma)\right)} + \|\partial_t\varrho\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)}\\ &\quad + \|\varrho\|_{L^p\left((0,T),W^{3-1/q}_q(\Gamma)\right)} \leqslant L. \end{aligned}\tag{4.3}$$

Moreover, for initial data $\mathbf{u}_0$, θ_0 and h_0 we assume

$$\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} \leqslant B, \quad \|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} \leqslant B, \quad \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)} \leqslant \epsilon. \tag{4.4}$$

where $\epsilon > 0$ is a small constant and B is a given positive number. Since we mainly consider the case where $\mathbf{u}_0$ and θ_0 are large, we may assume that $B > 1$ hereafter. And we may assume that $0 < \epsilon < 1 < B < L$.

Remark 4.1. On the right side of the first equation in (4.1), we also need to consider θ and $\mathbf{a}$. From Theorem 3.1, in fact we just estimate $\|(\mathbf{a},\theta)\|_{L^p((0,T),L^q(\dot\Omega))}$ with a given function $\mathbf{a}$ and

$$\|\theta\|_{L^p((0,T),L^q(\dot\Omega))} \lesssim \|\theta\|_{L^p((0,T),H^2_q(\dot\Omega))},$$

then, we can get estimates of $\|\theta\|_{L^p((0,T),L^q(\dot\Omega))}$ by (3.6). Thus, we only need to consider the nonlinear term $\mathbf{N}_1(\mathbf{v},\mathbf{a},\Theta,\Psi_\varrho)$.

Since $N < q < \infty$, by Sobolev's inequality we have the following estimates:

$$\begin{aligned}&\|f\|_{L^\infty(\dot\Omega)} \lesssim \|f\|_{H^1_q(\dot\Omega)}, \quad \|fg\|_{H^1_q(\dot\Omega)} \lesssim \|f\|_{H^1_q(\dot\Omega)}\|g\|_{H^1_q(\dot\Omega)},\\ &\|fg\|_{H^2_q(\dot\Omega)} \lesssim \|f\|_{H^2_q(\dot\Omega)}\|g\|_{H^1_q(\dot\Omega)} + \|f\|_{H^1_q(\dot\Omega)}\|g\|_{H^2_q(\dot\Omega)},\\ &\|fg\|_{W^{1-1/q}_q(\Gamma)} \lesssim \|f\|_{W^{1-1/q}_q(\Gamma)}\|g\|_{W^{1-1/q}_q(\Gamma)},\\ &\|fg\|_{W^{2-1/q}_q(\Gamma)} \lesssim \|f\|_{W^{2-1/q}_q(\Gamma)}\|g\|_{W^{1-1/q}_q(\Gamma)} + \|f\|_{W^{1-1/q}_q(\Gamma)}\|g\|_{W^{2-1/q}_q(\Gamma)}.\end{aligned}\tag{4.5}$$

Moreover, since $2/p + N/q < 1$, we have

$$\|f\|_{H^1_\infty(\dot\Omega)} \lesssim \|f\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}, \quad \|f\|_{H^2_\infty(\Gamma)} \lesssim \|f\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)}. \tag{4.6}$$

In order to obtain the estimates of $\mathbf{u}$, θ and h, we shall use Theorems 3.1 and 3.2. For this purpose, we shall estimate the nonlinear functions appearing on the right-hand side of equations (4.1). The definition of $\mathbf{N}_1(\mathbf{v},\mathbf{a},\Theta,\Psi_\varrho)$ is given by replacing $\mathbf{u}$, θ and h with $\mathbf{v}$, Θ and ϱ in (2.10), respectively. Since $|\mathbf{V}_0(\mathbf{k})| \lesssim |\mathbf{k}|$ when $|\mathbf{k}| \leqslant \lambda$, by (4.2), we have

$$\begin{aligned}\|\mathbf{N}_1(\mathbf{v},\mathbf{a},\Theta,\Psi_\varrho)\|_{L^q(\dot\Omega)} \lesssim &\|\mathbf{v}\|_{L^\infty(\dot\Omega)}\|\nabla\mathbf{v}\|_{L^q(\dot\Omega)} + \|\partial_t\Psi_\varrho\|_{L^\infty(\dot\Omega)}\|\nabla\mathbf{v}\|_{L^q(\dot\Omega)}\\ &+ \|\nabla\Psi_\varrho\|_{L^\infty(\dot\Omega)}\|\partial_t\mathbf{v}\|_{L^q(\dot\Omega)} + \|\nabla\Psi_\varrho\|_{L^\infty(\dot\Omega)}\left\|\nabla^2\mathbf{v}\right\|_{L^q(\dot\Omega)}\\ &+ \left\|\nabla^2\Psi_\varrho\right\|_{L^q(\dot\Omega)}\|\nabla\mathbf{v}\|_{L^\infty(\dot\Omega)} + \|\nabla\Psi_\varrho\|_{L^\infty(\dot\Omega)}\|\mathbf{a}\|_{L^q(\dot\Omega)}\\ &+ \|\nabla\Psi_\varrho\|_{L^\infty(\dot\Omega)}\|\Theta\|_{L^q(\dot\Omega)}.\end{aligned}$$

Thus, by (2.1), (4.5) and the fact that for any function f, g

$$\|fg\|_{L^p((0,T))} \leqslant \|f\|_{L^p((0,T))}\|g\|_{L^\infty((0,T))}, \quad \|g\|_{L^p((0,T))} \leqslant T^{1/p}\|g\|_{L^\infty((0,T))},$$

we get

$$\begin{aligned}
&\|N_1(\mathbf{v},\Theta,\mathbf{a},\Psi_\varrho)\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\\
&\lesssim T^{1/p}\|\mathbf{v}\|^2_{L^\infty\left((0,T),H^1_q(\dot\Omega)\right)}+T^{1/p}\|\mathbf{v}\|_{L^\infty\left((0,T),H^1_q(\dot\Omega)\right)}\|\partial_t\varrho\|_{L^\infty\left((0,T),W^{1-1/q}_q(\Gamma)\right)}\\
&\quad+\|\varrho\|_{L^\infty\left((0,T),W^{2-1/q}_q(\Gamma)\right)}\Big(\|\partial_t\mathbf{v}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}+\|\mathbf{v}\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)}\\
&\quad+\|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}+\|\Theta\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)}\Big). \qquad (4.7)
\end{aligned}$$

In what follows, we shall use the following inequalities:

$$\begin{aligned}
&\|\mathbf{w}\|_{L^\infty\left((0,T),B^{2/p'}_{q,p}(\dot\Omega)\right)}\\
&\lesssim \|\mathbf{w}_0\|_{B^{2/p'}_{q,p}(\dot\Omega)}+\|\mathbf{w}\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)}+\|\partial_t\mathbf{w}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}, \qquad (4.8)
\end{aligned}$$

for $\mathbf{w}\in\{\mathbf{v},\Theta\}$,

$$\begin{aligned}
&\|\varrho\|_{L^\infty\left((0,T),B^{3-1/p-1/q}_{q,p}(\Gamma)\right)}\\
&\lesssim \|h_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)}+\|\varrho\|_{L^p\left((0,T),W^{3-1/q}_q(\Gamma)\right)}+\|\partial_t\varrho\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)}, \qquad (4.9)
\end{aligned}$$

the inequalities: (4.8) and (4.9) will be proved later. Obviously, we also have

$$\begin{aligned}
\|\varrho\|_{L^\infty\left((0,T),W^{2-1/q}_q(\Gamma)\right)}&\leqslant\|h_0\|_{W^{2-1/q}_0(\Gamma)}+\int_0^T\|\partial_s\varrho(\cdot,s)\|_{W^{2-1/q}_q(\Gamma)}\,ds\\
&\leqslant\|h_0\|_{W^{2-1/q}_q(\Gamma)}+T^{1/p'}\|\partial_t\varrho\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)}\\
&\leqslant T^{1/p'}L. \qquad (4.10)
\end{aligned}$$

In what follows, we assume $0<\epsilon=T=\iota<1$ such that $T^{1/p'}L\leqslant 1$ and $1<B<L$. Combining (4.7) with (4.3), (4.4), (4.8) and (4.10), we have

$$\begin{aligned}
&\|\mathbf{N}_1(\mathbf{v},\mathbf{a},\Theta,\Psi_\varrho)\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\\
&\lesssim T^{1/p}(L+B)^2+T^{1/p'}L^2+T^{1/p'}L\,\|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\\
&\lesssim T^{1/p}(B+L)^2+\|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}, \qquad (4.11)
\end{aligned}$$

due to $1/p'>1/p$ from $1<p'=p/(p-1)<2<p<\infty$.

Let $N_4(\mathbf{v},\Theta,\Psi_\varrho)$ be a nonlinear term given in (2.15), by (4.2), we have

$$\begin{aligned}
\|\mathbf{N}_4(\mathbf{v},\Theta,\Psi_\varrho)\|_{L^q(\dot\Omega)}\lesssim&\|\mathbf{v}\|_{L^\infty(\dot\Omega)}\|\nabla\Theta\|_{L^q(\dot\Omega)}+\|\partial_t\Psi_\varrho\|_{L^\infty(\dot\Omega)}\|\nabla\Theta\|_{L^q(\dot\Omega)}\\
&+\|\nabla\Psi_\varrho\|_{L^\infty(\dot\Omega)}\left\|\nabla^2\Theta\right\|_{L^q(\dot\Omega)}+\left\|\nabla^2\Psi_\varrho\right\|_{L^q(\dot\Omega)}\|\nabla\Theta\|_{L^\infty(\dot\Omega)}.
\end{aligned}$$

Employing a similar argument as that in proving (4.11), we get

$$\|N_4(\mathbf{v},\Theta,\Psi_\varrho)\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\lesssim T^{1/p}(L+B)^2. \qquad (4.12)$$

We next consider $N_5(\mathbf{v},\Psi_\varrho)$ given in (2.35), while the detailed proof was given in [27]. Thus, we have

$$\begin{aligned}
&\|N_5(\mathbf{v},\Psi_\varrho)\|_{L^\infty\left((0,T),W^{1-1/q}_q(\Gamma)\right)}\lesssim(L+B)T^{1/p'}L,\\
&\|N_5(\mathbf{v},\Psi_\varrho)\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)}\leqslant C_sL^2(B+L)T^{(1-1/s)/p'}, \qquad (4.13)
\end{aligned}$$

where $s \in (1, 2/p')$ is a constant and ι is small enough. According to Theorems 3.1 and 3.2, we have to extend other nonlinear terms to the whole time interval $\mathbb{R}$. Before turning to the extension of these functions, we make a few definitions. We first consider $\mathbf{v}$, let $\tilde{\mathbf{u}}_0 \in B^{2/p'}_{q,p}\left(\mathbb{R}^N\right)$ be an extension of $\mathbf{u}_0 \in B^{2/p'}_{q,p}(\Omega_\pm)$ to $\mathbb{R}^N$ such that

$$\mathbf{u}_0 = \tilde{\mathbf{u}}_0 \quad \text{in } \Omega_\pm, \quad \|\tilde{\mathbf{u}}_0\|_{B^{2/p'}_{q,p}(\mathbb{R}^N)} \lesssim \|\mathbf{u}_0\|_{B^{2/p'}_{q,p}(\Omega_\pm)}.$$

Let

$$T_{\mathbf{v}_\pm}(t)\mathbf{u}_0 = e^{-(2-\Delta)t}\tilde{\mathbf{u}}_{0\pm} = \mathcal{F}^{-1}_\xi\left[e^{-\left(|\xi|^2+2\right)t}\mathcal{F}\left[\tilde{\mathbf{u}}_{0\pm}\right](\xi)\right], \tag{4.14}$$

where

$$\mathcal{F}^{-1}_\xi[g(\xi)](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)d\xi,$$

and $T_{\mathbf{v}}(t)\mathbf{u}_0 = T_{\mathbf{v}\pm}(t)\mathbf{u}_0|_{\Omega_\pm}$for $x \in \Omega_\pm$, then $T_{\mathbf{v}}(0)\mathbf{u}_0 = \mathbf{u}_0$ in $\dot{\Omega}$. By the analytic semigroup theory and a standard real-interpolation method, we directly have

$$\begin{aligned}
&\left\|e^t T_{\mathbf{v}}(\cdot)\mathbf{u}_0\right\|_{X(\dot{\Omega})} \lesssim \|\mathbf{u}_0\|_{X(\dot{\Omega})} \qquad \text{for } X \in \left\{H^k_q, B^{2/p'}_{q,p}(\dot{\Omega})\right\},\\
&\left\|e^t T_{\mathbf{v}}(\cdot)\mathbf{u}_0\right\|_{H^1_p\left((0,\infty),L^q(\dot{\Omega})\right)} + \left\|e^t T_{\mathbf{v}}(\cdot)\mathbf{u}_0\right\|_{L^p\left((0,\infty),H^2_q(\dot{\Omega})\right)} \lesssim \|\mathbf{u}_0\|_{B^{2/p'}_{q,p}(\dot{\Omega})}.
\end{aligned} \tag{4.15}$$

We can also define similar operators for $T_\Theta(t)\theta_0$ and $T_\varrho(t)h_0$, then get the same inequalities as (4.15).

Let $\mathbf{W}$, P and Ξ be solutions of the equations:

$$\begin{cases}
\rho\partial_t\mathbf{W} + \lambda_0\mathbf{W} - m^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{W}) - P\mathbf{I}) = 0, \quad \operatorname{div}\mathbf{W} = 0 & \text{in } \dot{\Omega}\times(0,\infty),\\
\partial_t\Xi + \lambda_0\Xi - \mathbf{W}\cdot\mathbf{n} = 0 & \text{on } \Gamma\times(0,\infty),\\
[\![\mu\mathbf{D}(\mathbf{W}) - P\mathbf{I})\mathbf{n}]\!] - \sigma\left(\Delta_\Gamma\Xi\right)\ \mathbf{n} = 0 & \text{on } \Gamma\times(0,\infty),\\
[\![\mathbf{W}]\!] = 0 & \text{on } \Gamma\times(0,\infty),\\
\mathbf{W} = 0 & \text{on } \Gamma_-\times(0,\infty),\\
(\mathbf{W},\Xi)|_{t=0} = (0, h_0) & \text{in } \dot{\Omega}\times\Gamma,
\end{cases}$$

where $\lambda_0 > 0$ is large enough. Then by Theorem 3.1 and real interpolation theory, we obtain the uniqueness and existence of $\mathbf{W}$, P and Ξ and

$$\begin{aligned}
&\left\|e^t\mathbf{W}\right\|_{L^\infty\left((0,\infty),B^{2/p'}_{q,p}(\dot{\Omega})\right)} + \left\|e^t\mathbf{W}\right\|_{L^p\left((0,\infty),H^2_q(\dot{\Omega})\right)} + \left\|e^t\partial_t\mathbf{W}\right\|_{L^p\left((0,\infty),L^q(\dot{\Omega})\right)}\\
&\quad + \left\|e^t\Xi\right\|_{L^\infty\left((0,\infty),B^{3-1/p-1/q}_{q,p}(\Gamma)\right)} + \left\|e^t\Xi\right\|_{L^p\left((0,\infty),W^{3-1/q}_q(\Gamma)\right)}\\
&\quad + \left\|e^t\partial_t\Xi\right\|_{L^p\left((0,\infty),W^{1-1/q}_q(\Gamma)\right)} \leqslant C\|h_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)} \lesssim \epsilon.
\end{aligned}$$

Setting $T_\varrho(t)h_0 = \Psi_\Xi$, we have $T_\varrho(0)h_0 = \Psi_{h_0}$ in $\dot{\Omega}$ and $T_\varrho(0)h_0 = h_0$ in Γ, then by (2.1)

$$\begin{aligned}
&\left\|e^t T_\varrho(\cdot)h_0\right\|_{\left(L^\infty(0,\infty),B^{3-1/p}_{q,p}(\Omega)\right)} + \left\|e^t\partial_t T_\varrho(\cdot)h_0\right\|_{L^\infty\left((0,\infty),H^1_q(\Omega)\right)}\\
&\quad + \left\|e^t T_\varrho(\cdot)h_0\right\|_{L^p\left(\left((0,\infty),H^3_q(\Omega)\right)\right)} + \left\|e^t\partial_t T_\varrho(\cdot)h_0\right\|_{L^p\left((0,\infty),H^2_q(\Omega)\right)}\\
&\leqslant C\|h_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)} \lesssim \epsilon.
\end{aligned} \tag{4.16}$$

Given a function $f(t)$ on $(0,T)$, the extension $E_T[f]$ of f is defined by

$$E_T[f](t)=\begin{cases}0 & \text{for } t<0,\\ f(t) & \text{for } 0<t<T,\\ f(2T-t) & \text{for } T<t<2T,\\ 0 & \text{for } t>2T.\end{cases} \tag{4.17}$$

Obviously, $E_T[f](t)=f(t)$ for $t\in(0,T)$ and $E_T[f](t)=0$ for $t\notin(0,2T)$. Moreover, if $f|_{t=0}=0$, then

$$\partial_t E_T[f](t)=\begin{cases}0 & \text{for } t<0,\\ (\partial_t f)(t) & \text{for } 0<t<T,\\ -(\partial_t f)(2T-t) & \text{for } T<t<2T,\\ 0 & \text{for } t>2T.\end{cases} \tag{4.18}$$

Let $\psi(t)\in C^\infty(\mathbb{R})$ equal one for $t>-1$ and zero for $t<-2$. Under these preparations, we now define the extensions $\mathcal{E}_1[\mathbf{v}]$, $\mathcal{E}_2[\Theta]$ and $\mathcal{E}_3[\Psi_\varrho]$ of $\mathbf{v}$, Θ and Ψ_ϱ to $\mathbb{R}$, respectively, by

$$\begin{aligned}\mathcal{E}_1[\mathbf{v}]&=E_T[\mathbf{v}-T_{\mathbf{v}}(t)\mathbf{u}_0]+\psi(t)T_{\mathbf{v}}(|t|)\mathbf{u}_0,\\ \mathcal{E}_2[\Theta]&=E_T[\Theta-T_\Theta(t)\theta_0]+\psi(t)T_\Theta(|t|)\theta_0,\\ \mathcal{E}_3[\Psi_\varrho]&=E_T[\Psi_\varrho-T_\varrho(t)h_0]+\psi(t)T_\varrho(|t|)h_0.\end{aligned} \tag{4.19}$$

Since $\mathbf{v}|_{t=0}=T_{\mathbf{v}}(0)\mathbf{u}_0=\mathbf{u}_0$, $\Theta|_{t=0}=T_\Theta(0)\theta_0=\theta_0$ and $\Psi_\varrho|_{t=0}=T_\varrho(0)h_0$, we can differentiate $\mathcal{E}_1[\mathbf{v}]$, $\mathcal{E}_2[\Theta]$ and $\mathcal{E}_3[\Psi_\varrho]$ once with respect to t by using the formula (4.17). Obviously, we have

$$\mathcal{E}_1[\mathbf{v}]=\mathbf{v},\quad \mathcal{E}_2[\Theta]=\Theta,\quad \mathcal{E}_3[\Psi_\varrho]=\Psi_\varrho\quad \text{in } \dot\Omega\times(0,T), \tag{4.20}$$

and then, applying the Hanzawa transformation: $x=y+\mathcal{E}_3[\Psi_\varrho]$ instead of $x=y+\Psi_\varrho$, by (2.1), (4.10) and (4.16), we get

$$\begin{aligned}\sup_{t\in\mathbb{R}}\|\mathcal{E}_3[\Psi_\varrho]\|_{H^1_\infty(\dot\Omega)}&\lesssim \sup_{t\in(0,T)}\|\Psi_\varrho\|_{H^2_q(\dot\Omega)}+\|T_\varrho(\cdot)h_0\|_{L^\infty((0,\infty),H^2_q(\dot\Omega))}\\ &\lesssim \|h_0\|_{W_q^{3-1/p-1/q}(\Gamma)}+T^{1/p'}L.\end{aligned}$$

Thus, we choose T and $\|h_0\|_{W_{q,p}^{3-1/p-1/q}(\Gamma)}$ so small that

$$\sup_{t\in\mathbb{R}}\|\mathcal{E}_3[\Psi_\varrho]\|_{H^1_\infty(\dot\Omega)}\leqslant\lambda. \tag{4.21}$$

Now, we consider $N_8(\theta,\Psi_h)$ given in (2.41). We extend $N_8(\Theta,\Psi_\varrho)$ to the whole time interval $\mathbb{R}$, let

$$\tilde N_8(\Theta,\Psi_\varrho)=\mathbf{V}_8\left(\bar\nabla\mathcal{E}_3[\Psi_\varrho]\right)\bar\nabla\mathcal{E}_3[\Psi_\varrho]\otimes[\![\kappa\nabla\mathcal{E}_2[\Theta]]\!], \tag{4.22}$$

Obviously, we have

$$\tilde N_8(\Theta,\Psi_\varrho)=N_8(\Theta,\Psi_\varrho)\quad \text{in } \dot\Omega\times(0,T).$$

Let $\mathcal{E}_\mp$ be the extension map acting on $\Theta_\pm\in H^2_q(\Omega_\pm)$ satisfying $\mathcal{E}_\mp(\Theta_\pm)\in H^2_q(\Omega)$, $\mathcal{E}_-(\Theta_+)=\Theta_-$ in Ω_-, $\mathcal{E}_+(\Theta_-)=\Theta_+$ in Ω_+, then we have

$$(\partial_x^\alpha\mathcal{E}_\mp(\Theta_\pm))(x_0)=\lim_{\substack{x\to x_0\\ x\in\Omega_\pm}}\partial_x^\alpha\Theta_\pm(x),\quad [\![\partial_x^\alpha\Theta]\!]=\partial_x^\alpha\mathcal{E}_-(\Theta_+)|_\Gamma-\partial_x^\alpha\mathcal{E}_+(\Theta_-)|_\Gamma, \tag{4.23}$$

for $x_0\in\Gamma$ and $|\alpha|\leqslant 1$, and

$$\|\mathcal{E}_\mp(\Theta_\pm)\|_{H^i_q(\Omega)}\leqslant C_{i,q}\left(\|\Theta_+\|_{H^i_q(\Omega)}+\|\Theta_-\|_{H^i_q(\Omega)}\right)=C_{i,q}\|\Theta\|_{H^i_q(\dot\Omega)}, \tag{4.24}$$

for $i=0,1,2$. Thus, we obtain

$$\tilde{N}_8\left(\Theta,\Psi_\varrho\right)=\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\otimes\nabla\mathcal{E}_2\left[\kappa_-\mathcal{E}_-\Theta_+|_\Gamma-\kappa_+\mathcal{E}_+\Theta_-|_\Gamma\right]. \tag{4.25}$$

Since

$$\begin{aligned}\partial_t\left(\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)=&\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\partial_t\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\\&+\mathbf{V}_8'\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\partial_t\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right],\end{aligned} \tag{4.26}$$

where $\mathbf{V}_8'$ denotes the derivative of $\mathbf{V}_8(\bar{\mathbf{k}})$ with respect to $\bar{\mathbf{k}}$, by (2.1), (2.42), (4.3), (4.5), (4.10), (4.15), (4.16), (4.19), (4.21) and (4.26), we have the following estimates:

$$\begin{aligned}&\left\|\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\\&\leqslant C\left\|\mathcal{E}_3\left[\Psi_\varrho\right]\right\|_{L^\infty\left(\mathbb{R},H_q^2(\dot{\Omega})\right)}\leqslant C\left\|T_\varrho(|t|)h_0\right\|_{L^\infty\left((-2,\infty),H_q^2(\dot{\Omega})\right)}+\left\|\Psi_\varrho\right\|_{L^\infty\left((0,T),H_q^2(\dot{\Omega})\right)}\\&\leqslant C\left(\epsilon+T^{1/p'}L\right),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\left\|\partial_t\left(\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\right\|_{L^\infty\left(\mathbb{R},L^q(\dot{\Omega})\right)}&\leqslant C\left\|\partial_t\nabla\mathcal{E}_3\left[\Psi_h\right]\right\|_{L^\infty\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\&\leqslant C(\epsilon+L),\end{aligned} \tag{4.28}$$

$$\begin{aligned}\left\|\partial_t\left(\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}&\leqslant C\left\|\partial_t\nabla\mathcal{E}_3\left[\Psi_h\right]\right\|_{L^p\left(\left(\mathbb{R},H_q^1(\dot{\Omega})\right)\right)}\\&\leqslant C(\epsilon+L).\end{aligned} \tag{4.29}$$

For some $\gamma>0$ given in Theorem 3.2, we also have

$$\begin{aligned}&\left\|e^{-\gamma t}\left(\nabla\mathcal{E}_2\left[\Theta\right]\right)\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\\&\leqslant C\left(\|\Theta\|_{L^p\left((0,T),H_q^2(\dot{\Omega})\right)}+e^{2\gamma}\left\|T_\Theta(|t|)\theta_0\right\|_{L^p\left((-2,\infty),H_q^2(\dot{\Omega})\right)}\right)\\&\leqslant C\left(L+e^{2\gamma}B\right),\end{aligned} \tag{4.30}$$

$$\begin{aligned}&\left\|e^{-\gamma t}\mathcal{E}_2[\Theta]\right\|_{H_p^1\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\&\leqslant C\left(\|\partial_t\Theta\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)}+e^{2\gamma}\left\|\partial_tT_\Theta(|t|)\theta_0\right\|_{L^p\left((-2,\infty),L^q(\dot{\Omega})\right)}\right)\\&\leqslant C\left(L+e^{2\gamma}B\right).\end{aligned} \tag{4.31}$$

Thus, from the Theorem 3.2, by (A.1), (A.3), (4.25), (4.27)–(4.31) and the Hölder inequality, it follows

$$\begin{aligned}&\left\|e^{-\gamma t}\tilde{N}_8\left(\Theta,\Psi_\varrho\right)\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}+\left\|e^{-\gamma t}\tilde{N}_8\left(\Theta,\Psi_\varrho\right)\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\&\leqslant C\Big\{T^{(q-N)/(pq)}\left\|\partial_t\left(\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\right\|_{L^\infty\left(\mathbb{R},L^q(\dot{\Omega})\right)}^{1-N/(2q)}\\&\quad\times\left\|\partial_t\left(\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}^{N/(2q)}\\&\quad+\left\|\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_\varrho\right]\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\Big\}\\&\quad\times\left(\left\|e^{-\gamma t}\nabla\mathcal{E}_2[\Theta]\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}+\left\|e^{-\gamma t}\nabla\mathcal{E}_2[\Theta]\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\right)\\&\lesssim L\left(T^{1/p'}+T^{(q-N)/(pq)}\right)\left(L+e^{2\gamma}B\right).\end{aligned} \tag{4.32}$$

Moreover, we extend $N_2\left(\mathbf{v},\Psi_\varrho\right)$, $\mathbf{N}_3\left(\mathbf{v},\Psi_\varrho\right)$, $\mathbf{N}_6\left(\mathbf{v},\Psi_\varrho\right)$ and $N_7\left(\mathbf{v},\Psi_\varrho\right)$ to the whole time interval $\mathbb{R}$, by (2.12), (2.39) and (2.40), let

$$\tilde{N}_2\left(\mathbf{v},\Psi_\varrho\right)=\mathbf{V}_2\left(\nabla\mathcal{E}_3\left[\Psi_\varrho\right]\right)\nabla\mathcal{E}_3\left[\Psi_\varrho\right]\otimes\nabla\mathcal{E}_1[\mathbf{v}],$$

$$\begin{aligned}\tilde{\mathbf{N}}_3(\mathbf{v},\Psi_\varrho) =&\mathbf{V}_3\left(\nabla\mathcal{E}_3[\Psi_\varrho]\right)\nabla\mathcal{E}_3[\Psi_\varrho]\otimes\mathcal{E}_1[\mathbf{v}],\\ \tilde{\mathbf{N}}_6(\mathbf{v},\Psi_\varrho) =&\mathbf{V}_6\left(\bar{\nabla}\mathcal{E}_3[\Psi_\varrho]\right)\bar{\nabla}\mathcal{E}_3[\Psi_\varrho]\otimes\nabla\mathcal{E}_1[\mathbf{v}],\\ \tilde{N}_7(\mathbf{v},\Psi_\varrho) =&\mathbf{V}_7\left(\bar{\nabla}\mathcal{E}_3[\Psi_\varrho]\right)\bar{\nabla}\mathcal{E}_3[\Psi_\varrho]\otimes\nabla\mathcal{E}_1[\mathbf{v}]\\ &+\sigma\mathbf{V}_5\left(\bar{\nabla}\mathcal{E}_3[\Psi_\varrho]\right)\bar{\nabla}\mathcal{E}_3[\Psi_\varrho]\otimes\bar{\nabla}^2\mathcal{E}_3[\Psi_\varrho].\end{aligned}\tag{4.33}$$

By the fact that $H_p^1\left(\mathbb{R},L^q(\dot{\Omega})\right)\subset H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)$ and (4.16), we have

$$\begin{aligned}&\left\|e^{-\gamma t}\bar{\nabla}^2\mathcal{E}_3[\Psi_\varrho]\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}+\left\|e^{-\gamma t}\bar{\nabla}^2\mathcal{E}_3[\Psi_\varrho]\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\ &\lesssim\left\|e^{-\gamma t}\mathcal{E}_3[\Psi_\varrho]\right\|_{H_p^1\left(\mathbb{R},H_q^2(\dot{\Omega})\right)}+\left\|e^{-\gamma t}\mathcal{E}_3[\Psi_\varrho]\right\|_{L^p\left(\mathbb{R},H_q^3(\dot{\Omega})\right)}\leqslant CL.\end{aligned}$$

Then, by (4.23) and (4.24), $[\![\tilde{\mathbf{N}}_6]\!]$ and $[\![\tilde{N}_7]\!]$ can be replaced by $\tilde{\mathbf{N}}_6$ and $\tilde{N}_7$, then employing the same argument as in proving (4.32), we have

$$\begin{aligned}&\left\|e^{-\gamma t}\left(\tilde{N}_2(\mathbf{v},\Psi_\varrho),\tilde{\mathbf{N}}_6(\mathbf{v},\Psi_\varrho),\tilde{N}_7(\mathbf{v},\Psi_\varrho)\right)\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\ &\quad+\left\|e^{-\gamma t}\left(\tilde{N}_2(\mathbf{v},\Psi_\varrho),\tilde{\mathbf{N}}_6(\mathbf{v},\Psi_\varrho),\tilde{N}_7(\mathbf{v},\Psi_\varrho)\right)\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\\ &\quad+\left\|e^{-\gamma t}\left(\tilde{\mathbf{N}}_3(\mathbf{v},\Psi_\varrho)\right)\right\|_{H_p^1\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\ &\lesssim\left(\epsilon+L\left(T^{1/p'}+T^{1/p}+T^{(q-N)/(pq)}\right)\right)\left(L+e^{2\gamma}B\right).\end{aligned}\tag{4.34}$$

Finally, we consider the interface condition on Γ_-. Noting that $x=y$ near $\Gamma\times(0,T)$, we consider the following equation

$$\nabla\mathcal{E}_2[\theta_-]\cdot\mathbf{n}_-+\beta\mathcal{E}_2[\theta_-]=\tilde{b}(y,t)$$

with

$$\tilde{b}=\begin{cases}0 & \text{for } t<-2,\\ \psi(t)\nabla T_{\theta_-}(|t|)\theta_0\cdot\mathbf{n}_-+\beta\psi(t)T_{\theta_-}(|t|)\theta_0 & \text{for } -2<t<-1,\\ \nabla T_{\theta_-}(|t|)\theta_0\cdot\mathbf{n}_-+\beta T_{\theta_-}(|t|)\theta_0 & \text{for } -1<t<0,\\ b(y,t) & \text{for } 0<t<T,\\ b(y,2T-t) & \text{for } T<t<2T,\\ \nabla T_{\theta_-}(t)\theta_0\cdot\mathbf{n}_-+\beta T_{\theta_-}(t)\theta_0 & \text{for } t>2T.\end{cases}$$

Obviously, we have

$$\nabla\mathcal{E}_2[\theta_-]\cdot\mathbf{n}_-+\beta\mathcal{E}_2[\theta_-]=\nabla\theta_-\cdot\mathbf{n}_-+\beta\theta_-=b(y,t)\quad\text{on }\Gamma_-\times(0,T).$$

By (A.2), (A.3), (4.4), (4.15) and $\|\mathbf{n}_-\|_{H^1_\infty(\mathbb{R}^N)}\leqslant C$, we have

$$\begin{aligned}&\left\|e^{-\gamma t}\tilde{b}\right\|_{L^p\left(\mathbb{R},H_q^1(\Omega_-)\right)}+\left\|e^{-\gamma t}\tilde{b}\right\|_{H_p^{1/2}(\mathbb{R},L^q(\Omega_-))}\\ &\leqslant C\Big\{e^{2\gamma}\left\|T_{\theta_-}(|t|)\theta_0\right\|_{L^p\left((-2,\infty),H_q^2(\Omega_-)\right)}+e^{2\gamma}\left\|T_{\theta_-}(|t|)\theta_0\right\|_{H_p^1((-2,\infty),L^q(\Omega_-))}\\ &\quad+\|b\|_{L^p\left((0,T),H_q^1(\Omega_-)\right)}+\|b\|_{H_p^{1/2}((0,T),L^q(\Omega_-))}\Big\}\\ &\lesssim e^{2\gamma}\|\theta_0\|_{B_{q,p}^{2/p'}(\dot{\Omega})}+\|b\|_{L^p\left((0,T),H_q^1(\Omega_-)\right)}+\|b\|_{H_p^{1/2}((0,T),L^q(\Omega_-))}.\end{aligned}\tag{4.35}$$

Let

$$\tilde{E}_{p,q,T}(\mathbf{u},\theta,h)=\|(\mathbf{u},\theta)\|_{L^p\left((0,T),H_q^2(\dot{\Omega})\right)}+\|\partial_t(\mathbf{u},\theta)\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)}$$

$$+ \|h\|_{L^p\left((0,T),W_q^{3-1/q}(\Gamma)\right)} + \|\partial_t h\|_{L^p\left((0,T),W_q^{2-1/q}(\Gamma)\right)},$$

then

$$E_{p,q,T}(\mathbf{u},\theta,h) = \tilde{E}_{p,q,T}(\mathbf{u},\theta,h) + \|\partial_t h\|_{L^\infty\left((0,T),W_q^{1-1/q}(\dot{\Omega})\right)}.$$

Applying Theorems 3.1 and 3.2, (4.11)–(4.13), (4.32), (4.34) and (4.35), we have

$$\begin{aligned}\tilde{E}_{p,q,T}(\mathbf{u},\theta,h) \leqslant & Ce^{\gamma\iota^{-c}T}(1+\gamma^{1/2})\left(\mathcal{M} + \|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)}\right.\\ &\left. + \|b\|_{H_p^{1/2}((0,T),L^q(\Omega_-))} + \|b\|_{H_p^{1/2}\left((0,T),H_q^1(\Omega_-)\right)}\right),\end{aligned} \tag{4.36}$$

with

$$\begin{aligned}\mathcal{M} =& B + e^{2\gamma}B + T^{1/p}(L+B)^2 + L^2(B+L)T^{(1-1/s)/p'}\\ &+ L\left(T^{1/p'} + T^{1/p} + T^{(q-N)/(pq)}\right)\left(L+e^{2\gamma}B\right),\end{aligned}$$

for some positive constants γ and C independent of B and T. Combining (4.8) with (4.4) yields that

$$\|(\mathbf{u},\theta)\|_{L^\infty\left((0,T),B_{q,p}^{2(1-1/p)}(\Omega)\right)} \leqslant C\left(B + \tilde{E}_{p,q,T}(\mathbf{u},\theta,h)\right). \tag{4.37}$$

Noting that $3-1/p-1/q > 2-1/q$, then by (2.33), (4.1), (4.10), (4.13), (4.37) and the trace theorem, we have

$$\begin{aligned}&\|\partial_t h(\cdot,t)\|_{L^\infty\left((0,T),W_q^{1-1/q}(\Gamma)\right)}\\ &\leqslant \|\mathbf{u}_\iota\|_{H_q^1(\Omega_+)}\|h\|_{L^\infty\left((0,T),W_q^{2-1/q}(\Gamma)\right)} + C\|\mathbf{u}(\cdot,t)\|_{L^\infty\left((0,T),H_q^1(\Omega)\right)}\\ &\quad + \|N_5(\cdot,t)\|_{L^\infty\left((0,T),W_q^{1-1/q}(\Gamma)\right)}\\ &\leqslant CBT^{1/p'}\tilde{E}_{p,q,T} + C\left(B+\tilde{E}_{p,q,T}\right) + C(L+B)T^{1/p'}L.\end{aligned} \tag{4.38}$$

Since we may assume that $0<\iota=\epsilon=T<1$ is small enough, $L>B>1$, we have

$$\begin{aligned}E_{p,q,T}(\mathbf{u},\theta,h) \leqslant & B + (L+B)T^{1/p'}L + (1+BT^{1/p'})(1+\gamma^{1/2})e^{\gamma\iota^{-c}T}\\ &\times\left(\mathcal{M} + \|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)} + \|b\|_{H_p^{1/2}((0,T),L^q(\Omega_-))}\right.\\ &\left. + \|b\|_{H_p^{1/2}\left((0,T),H_q^1(\Omega_-)\right)}\right).\end{aligned} \tag{4.39}$$

Let

$$L = C\left\{B + \|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)} + \|b\|_{H_p^{1/2}((0,T),L^q(\Omega_-))} + \|b\|_{H_p^{1/2}\left((0,T),H_q^1(\Omega_-)\right)}\right\},$$

for some constant C depend on γ and independent of B,ϵ,ι and T. But γ is fixed in such a way that the estimates given in Theorems 3.1 and 3.2 hold, so we do not mention the dependence on γ. Choosing ι and T such that

$$\mathcal{M} \leqslant 2e^{2\gamma}B, \quad \gamma\iota^{-c}T \leqslant 1, \quad (L+B)T^{1/p'}L \leqslant B, \quad BT^{1/p'} \leqslant 1, \tag{4.40}$$

we obtain

$$E_{p,q,T}(\mathbf{u},\theta,h) \leqslant L. \tag{4.41}$$

If we define a mapping Φ by $\Phi(\mathbf{v},\Theta,\varrho) = (\mathbf{u},\theta,h)$, then, by (4.41), Φ maps $\mathbf{U}_T$ into itself. Thus, we can use the contraction mapping principle in next section to complete the proof of Theorem 2.1.

Finally, we prove (4.8) and (4.9). Let $\mathcal{E}_1[\mathbf{v}]$ be the function given in (4.19), we just consider $\mathbf{v}$ to get

$$\begin{aligned}\|\mathbf{v}\|_{L^\infty\left((0,T),B^{2/p'}_{q,p}(\dot\Omega)\right)} &\leqslant \|\mathcal{E}_1[\mathbf{v}]\|_{L^\infty\left((0,T),B^{2/p'}_{q,p}(\dot\Omega)\right)}\\ &\lesssim \|\mathcal{E}_1[\mathbf{v}]\|_{L^p\left((0,\infty),H^2_q(\dot\Omega)\right)} + \|\partial_t\mathcal{E}_1[\mathbf{v}]\|_{L^p\left((0,\infty),L^q(\dot\Omega)\right)},\end{aligned}$$

which, combined with (4.15), leads to the inequality (4.8). Analogously, using $\mathcal{E}_3\left[\Psi_\varrho\right]$ given in (4.19) and the real interpolation theory, we have

$$\|\Psi_\varrho\|_{L^\infty\left((0,T),B^{3-1/p-1/q}_{q,p}(\Gamma)\right)} \lesssim \|\Psi_\varrho\|_{L^p\left((0,T),H^3_q(\dot\Omega)\right)} + \|\partial_t\Psi_\varrho\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)},$$

which, combined with (4.16) and (2.1), yields the inequality in (4.9).

5. **Completion of the proof of Theorem 2.1.** If we prove that Φ is a contraction mapping, then by the contraction mapping principle, we will complete the proof of Theorem 2.1.

Let $(\mathbf{v}_i,\Theta_i,\varrho_i)\in\mathbf{U}_T(i=1,2)$. We shall mainly estimate $E_{p,q,T}(\mathbf{u}_1-\mathbf{u}_2,\theta_1-\theta_2,h_1-h_2)$ with $(\mathbf{u}_i,\theta_i,h_i)=\Phi(\mathbf{v}_i,\Theta_i,\varrho_i)$, then we shall prove that Φ is a contraction map on $\mathbf{U}_T$. We set

$$\begin{aligned}&\tilde{\mathbf{u}}=\mathbf{u}_1-\mathbf{u}_2,\quad \tilde\theta=\theta_1-\theta_2,\quad \tilde h=h_1-h_2,\\ &\mathcal{N}_1=\mathbf{N}_1\left(\mathbf{v}_1,\mathbf{a},\Theta_1,\Psi_{\varrho 1}\right)-\mathbf{N}_1\left(\mathbf{v}_2,\mathbf{a},\Theta_2,\Psi_{\varrho 2}\right),\\ &\mathcal{N}_i=N_i\left(\mathbf{v}_1,\Psi_{\varrho 1}\right)-N_i\left(\mathbf{v}_2,\Psi_{\varrho 2}\right),\quad \mathcal{N}_j=\mathbf{N}_j\left(\mathbf{v}_1,\Psi_{\varrho 1}\right)-\mathbf{N}_j\left(\mathbf{v}_2,\Psi_{\varrho 2}\right),\\ &\mathcal{N}_4=N_4\left(\mathbf{v}_1,\Theta_1,\Psi_{\varrho 1}\right)-N_4\left(\mathbf{v}_2,\Theta_2,\Psi_{\varrho 2}\right),\quad \mathcal{N}_8=N_8\left(\Theta_1,\Psi_{\varrho 1}\right)-N_8\left(\Theta_2,\Psi_{\varrho 2}\right),\end{aligned}$$

for $i=2,5,7$ and $j=3,6$. Because $\mathbf{a}(x,t)$ and $b(y,t)$ are given functions, then by (4.1), $\tilde{\mathbf{u}}$ and $\tilde h$ satisfy the following equations with some pressure term Q:

$$\begin{cases}\rho\partial_t\tilde{\mathbf{u}}-\operatorname{Div}(\mu\mathbf{D}(\tilde{\mathbf{u}})-Q\mathbf{I})=\mathcal{N}_1+\alpha\mathbf{g}\tilde\theta, & \text{in } \dot\Omega\times(0,T),\\ \operatorname{div}\tilde{\mathbf{u}}=\mathcal{N}_2=\operatorname{div}\mathcal{N}_3 & \text{in } \dot\Omega\times(0,T),\\ \partial_t\tilde\theta-\kappa\Delta\tilde\theta=\mathcal{N}_4 & \text{in } \dot\Omega\times(0,T),\\ \partial_t\tilde h-\mathbf{n}\cdot\tilde{\mathbf{u}}+\langle\mathbf{u}_\iota\perp\nabla'\tilde h\rangle=\mathcal{N}_5 & \text{on } \Gamma\times(0,T),\\ [\![\tilde{\mathbf{u}}]\!]=0,\quad [\![(\mu\mathbf{D}(\tilde{\mathbf{u}})_\tau)]\!]=[\![\mathcal{N}_6]\!] & \text{on } \Gamma\times(0,T),\\ [\![(\mu\mathbf{D}(\tilde{\mathbf{u}})\mathbf{n},\mathbf{n})-Q]\!]-\sigma\left(\Delta_\Gamma\tilde h+\mathcal{F}\tilde h\right)=[\![\mathcal{N}_7]\!] & \text{on } \Gamma\times(0,T),\\ [\![\tilde\theta]\!]=0,\quad [\![\kappa\nabla\tilde\theta\cdot\mathbf{n}]\!]=\mathcal{N}_8 & \text{on } \Gamma\times(0,T),\\ \tilde{\mathbf{u}}=0,\quad \nabla\tilde\theta_-\cdot\mathbf{n}_-+\beta\tilde\theta_-=0 & \text{on } \Gamma_-\times(0,T),\\ (\tilde{\mathbf{u}},\tilde\theta)\Big|_{t=0}=(0,0) \text{ in } \dot\Omega\quad \tilde h\Big|_{t=0}=0\quad \text{on } \Gamma,\end{cases}\tag{5.1}$$

where $\tilde\theta_-=\theta_{1-}-\theta_{2-}$. We have to estimate the nonlinear terms appearing on the right side of equations (5.1). By Remark 4.1, we only need to estimate $\mathcal{N}_1$ in the first equation in (5.1), as was written in (2.10), we write

$$N_1\left(\mathbf{v},\mathbf{a},\Theta,\Psi_\varrho\right)=\mathbf{V}'(\bar\nabla\Psi_\varrho)\mathbf{f}_1\left(\mathbf{v},\mathbf{a},\Theta,\Psi_\varrho\right)$$

where $\mathbf{V}'(\bar\nabla\Psi_\varrho)$ is a matrix of bounded functions defined on $|\bar\nabla\Psi_\varrho|\leqslant\lambda$ satisfying the estimate:

$$\sup_{|\bar{\mathbf{k}}|\leqslant\lambda}\left\|\left(\mathbf{V}'(\cdot,\bar{\mathbf{k}}),\partial_{\bar{\mathbf{k}}}\mathbf{V}'(\cdot,\bar{\mathbf{k}})\right)\right\|_{H^1_\infty(\Omega)}\leqslant C,\tag{5.2}$$

with $\bar{\mathbf{k}}$ corresponding to $\bar{\nabla}\Psi_{\varrho}$. Let

$$\begin{aligned}\mathbf{f}_1\left(\mathbf{v},\Theta,\mathbf{a},\Psi_{\varrho}\right)=&\nabla\Psi_{\varrho}\otimes\left(\partial_t\mathbf{v},\nabla^2\mathbf{v}\right)+\partial_t\Psi_{\varrho}\otimes\nabla\mathbf{v}+\mathbf{v}\otimes\nabla\mathbf{v}+\nabla^2\Psi_{\varrho}\otimes\nabla\mathbf{v}\\&+\nabla\Psi_{\varrho}\otimes\mathbf{a}+\nabla\Psi_{\varrho}\otimes\alpha\mathbf{g}\Theta,\end{aligned}$$

then we can write $\mathcal{N}_1$ as

$$\begin{aligned}\mathcal{N}_1=&\left(\mathbf{V}'\left(\bar{\nabla}\Psi_{\varrho_1}\right)-\mathbf{V}'\left(\bar{\nabla}\Psi_{\varrho_2}\right)\right)\mathbf{f}_1\left(\mathbf{v}_1,\Theta_1,\mathbf{a},\Psi_{\varrho_1}\right)\\&+\mathbf{V}'\left(\bar{\nabla}\Psi_{\varrho_2}\right)\left(\mathbf{f}_1\left(\mathbf{v}_1,\Theta_1,\mathbf{a},\Psi_{\varrho_1}\right)-\mathbf{f}_1\left(\mathbf{v}_2,\Theta_2,\mathbf{a},\Psi_{\varrho_2}\right)\right),\end{aligned}$$

with

$$\begin{aligned}&\mathbf{f}_1\left(\mathbf{v}_1,\Theta_1,\mathbf{a},\Psi_{\varrho_1}\right)-\mathbf{f}_1\left(\mathbf{v}_2,\Theta_2,\mathbf{a},\Psi_{\varrho_2}\right)\\&=\nabla\left(\Psi_{\varrho_1}-\Psi_{\varrho_2}\right)\otimes\partial_t\mathbf{v}_1+\nabla\Psi_{\varrho_2}\otimes\partial_t\left(\mathbf{v}_1-\mathbf{v}_2\right)+\nabla\left(\Psi_{\varrho_1}-\Psi_{\varrho_2}\right)\otimes\nabla^2\mathbf{v}_1\\&\quad+\nabla\Psi_{\varrho_2}\otimes\nabla^2\left(\mathbf{v}_1-\mathbf{v}_2\right)+\partial_t\left(\Psi_{\varrho_1}-\Psi_{\varrho_2}\right)\otimes\nabla\mathbf{v}_1+\partial_t\Psi_{\varrho_2}\otimes\nabla\left(\mathbf{v}_1-\mathbf{v}_2\right)\\&\quad+\left(\mathbf{v}_1-\mathbf{v}_2\right)\otimes\nabla\mathbf{v}_1+\mathbf{v}_2\otimes\nabla\left(\mathbf{v}_1-\mathbf{v}_2\right)+\nabla^2\left(\Psi_{\varrho_1}-\Psi_{\varrho_2}\right)\otimes\nabla\mathbf{v}_1\\&\quad+\nabla^2\Psi_{\varrho_2}\otimes\nabla\left(\mathbf{v}_1-\mathbf{v}_2\right)+\left(\nabla\Psi_{\varrho_1}-\nabla\Psi_{\varrho_2}\right)\otimes\mathbf{a}+\left(\nabla\Psi_{\varrho_1}-\nabla\Psi_{\varrho_2}\right)\otimes\alpha\mathbf{g}\Theta_1\\&\quad+\nabla\Psi_{\varrho_2}\otimes\left(\alpha\mathbf{g}\Theta_1-\alpha\mathbf{g}\Theta_2\right).\end{aligned}$$

Let $f\in H^1_p(\mathbb{R},H^1_q(\dot{\Omega}))$ and $f|_{t=0}=0$, then $f(t)=\int_0^t\partial_s f(\cdot,s)ds$. Applying Hölder's inequality, we have

$$\|f\|_{L^\infty\left((0,T),H^1_q(\dot{\Omega})\right)}\leqslant T^{1/p'}\|\partial_t f\|_{L^p\left((0,T),H^1_q(\dot{\Omega})\right)}.\tag{5.3}$$

Notice that $\Psi_{\varrho_1}-\Psi_{\varrho_2}=0$ for $t=0$, in view of the integral mean value theorem, (2.1), (4.2), (5.2) and (5.3), we can get

$$\begin{aligned}&\left\|\mathbf{V}'(\bar{\nabla}\Psi_{\varrho_1})-\mathbf{V}'(\bar{\nabla}\Psi_{\varrho_2})\right\|_{L^\infty\left((0,T),H^1_q(\dot{\Omega})\right)}\\&\leqslant C\left\|\bar{\nabla}(\Psi_{\varrho_1}-\Psi_{\varrho_2})\right\|_{L^\infty\left((0,T),H^1_q(\dot{\Omega})\right)}\\&\leqslant CT^{1/p'}\left\|\partial_t\left(\Psi_{\varrho_1}-\Psi_{\varrho_2}\right)\right\|_{L^p\left(\mathbb{R},H^2_q(\dot{\Omega})\right)}\\&\leqslant CT^{1/p'}\left\|\partial_t\left(\varrho_1-\varrho_2\right)\right\|_{L^p\left(\mathbb{R},W^{2-1/q}_q(\Gamma)\right)}\\&\leqslant CT^{1/p'}E_{p,q,T}\left(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2\right).\end{aligned}\tag{5.4}$$

In view of (4.11) and (5.2), we have

$$\|\mathbf{f}_1(\mathbf{v}_1,\Theta_1,\mathbf{a},\varrho_1)\|_{L^p((0,T),L^q(\dot{\Omega}))}\leqslant CT^{1/p}(B+L)^2+\|\mathbf{a}\|_{L^p((0,T),L^q(\dot{\Omega}))}.\tag{5.5}$$

By (2.1) and (4.5), we obtain that

$$\begin{aligned}&\left\|\mathbf{f}_1\left(\mathbf{v}_1,\Theta_1,\mathbf{a},\Psi_{\varrho_1}\right)-\mathbf{f}_1\left(\mathbf{v}_2,\Theta_2,\mathbf{a},\Psi_{\varrho_2}\right)\right\|_{L^q(\dot{\Omega})}\\&\lesssim\|\varrho_1-\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\|\partial_t\mathbf{v}_1\|_{L^q(\dot{\Omega})}+\|\varrho_1-\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\left\|\nabla^2\mathbf{v}_1\right\|_{L^q(\dot{\Omega})}\\&\quad+\|\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\|\partial_t\left(\mathbf{v}_1-\mathbf{v}_2\right)\|_{L^q(\dot{\Omega})}+\|\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\left\|\nabla^2\left(\mathbf{v}_1-\mathbf{v}_2\right)\right\|_{L^q(\dot{\Omega})}\\&\quad+\|\partial_t\left(\varrho_1-\varrho_2\right)\|_{W^{1-1/q}_q(\Gamma)}\|\nabla\mathbf{v}_1\|_{L^q(\dot{\Omega})}+\|\partial_t\varrho_2\|_{W^{1-1/q}_q(\Gamma)}\|\nabla\left(\mathbf{v}_1-\mathbf{v}_2\right)\|_{L^q(\dot{\Omega})}\\&\quad+\|(\mathbf{v}_1,\mathbf{v}_2)\|_{H^1_q(\dot{\Omega})}\|\mathbf{v}_1-\mathbf{v}_2\|_{H^1_q(\dot{\Omega})}+\|\varrho_1-\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\|\mathbf{v}_1\|_{H^2_q(\dot{\Omega})}\\&\quad+\|\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\|\mathbf{v}_1-\mathbf{v}_2\|_{H^2_q(\dot{\Omega})}+\|\varrho_1-\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\|\mathbf{a}\|_{L^q(\dot{\Omega})}\\&\quad+\|\varrho_1-\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\|\Theta_1\|_{L^q(\dot{\Omega})}+\|\varrho_2\|_{W^{2-1/q}_q(\Gamma)}\|\Theta_1-\Theta_2\|_{L^q(\dot{\Omega})}.\end{aligned}$$

Since $\varrho_1|_{t=0} - \varrho_2|_{t=0} = 0$, it follows from (5.3) that

$$\|\varrho_1 - \varrho_2\|_{L^\infty\left((0,T),W_q^{2-1/q}(\Gamma)\right)} \leqslant T^{1/p'} \|\partial_t (\varrho_1 - \varrho_2)\|_{L^p\left((0,T),W_q^{2-1/q}(\Gamma)\right)}, \tag{5.6}$$

then, noticing that $\mathbf{v}_1|_{t=0} - \mathbf{v}_2|_{t=0} = 0$ and $\Theta_1|_{t=0} - \Theta_2|_{t=0} = 0$, and in view of (2.1), (4.5), (4.8), (4.10) and (5.6), we have

$$\begin{aligned}
&\|\mathbf{f}_1(\mathbf{v}_1, \Theta_1, \mathbf{a}, \Psi_{\varrho_1}) - \mathbf{f}_1(\mathbf{v}_2, \Theta_2, \mathbf{a}, \Psi_{\varrho_2})\|_{L^p\left((0,T),L^q(\dot\Omega)\right)} \\
&\leqslant \left(T^{1/p'}\left(L + \|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\right) + T^{1/p}(B+L)\right) \\
&\quad \times E_{p,q,T}(\mathbf{v}_1 - \mathbf{v}_2, \Theta_1 - \Theta_2, \varrho_1 - \varrho_2).
\end{aligned} \tag{5.7}$$

Then, we can get the following estimate by combining with (5.2), (5.4), (5.5) and (5.7),

$$\begin{aligned}
&\|\mathcal{N}_1\|_{L^p\left((0,T),L^q(\dot\Omega)\right)} \\
&\leqslant C\left(T^{1/p}T^{1/p'}(B+L)^2 + T^{1/p'}\|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)} + T^{1/p'}L + T^{1/p}(B+L)\right) \\
&\quad \times E_{p,q,T}(\mathbf{v}_1 - \mathbf{v}_2, \Theta_1 - \Theta_2, \varrho_1 - \varrho_2) \\
&\leqslant CT^{1/p}\left(L + B + \|\mathbf{a}\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}\right) E_{p,q,T}(\mathbf{v}_1 - \mathbf{v}_2, \Theta_1 - \Theta_2, \varrho_1 - \varrho_2).
\end{aligned} \tag{5.8}$$

In fact, we have used the estimates: $1/p < 1/p'$ for $2 \leqslant p$ and $T^{1/p'}T^{1/p}(L+B)^2 \leqslant 2(T^{1/p'}L)T^{1/p}(L+B) \leqslant 2T^{1/p}(L+B)$ due to $1 < B < L$ and T being small enough.

Next we consider $\mathcal{N}_4$. By (2.13), we may write

$$N_4(\mathbf{v}, \Theta, \varrho) = \mathbf{V}''(\bar\nabla\Psi_\varrho)\mathbf{f}_2(\mathbf{v}, \Theta, \varrho),$$

where $\mathbf{V}''(\bar\nabla\Psi_\varrho)$ is a matrix of bounded functions defined on $\left|\bar\nabla\Psi_\varrho\right| \leqslant \lambda$ and satisfies the estimate:

$$\sup_{|\bar{\mathbf{k}}| \leqslant \lambda} \left\|\left(\mathbf{V}''(\cdot, \bar{\mathbf{k}}), \partial_{\bar{\mathbf{k}}}\mathbf{V}''(\cdot, \bar{\mathbf{k}})\right)\right\|_{H^1_\infty(\Omega)} \leqslant C, \tag{5.9}$$

with $\bar{\mathbf{k}}$ corresponding to $\bar\nabla\Psi_\varrho$. Let

$$\mathbf{f}_2(\mathbf{v}, \Theta, \Psi_\varrho) = \mathbf{v} \otimes \nabla\Theta + \partial_t\Psi_\varrho \otimes \nabla\Theta + \kappa\nabla^2\Psi_\varrho \otimes \nabla\Theta + \kappa\nabla^2\Theta \otimes \nabla\Psi_\varrho.$$

Employing the same argument as that in proving (5.8), we can get

$$\|\mathcal{N}_4\|_{L^p\left((0,T),L^q(\dot\Omega)\right)} \leqslant CT^{1/p}(L+B)E_{p,q,T}(\mathbf{v}_1 - \mathbf{v}_2, \Theta_1 - \Theta_2, \varrho_1 - \varrho_2). \tag{5.10}$$

We now consider $\mathcal{N}_5$, in view of (2.35), we have

$$\mathcal{N}_5 = d(\mathbf{v}_1, \Psi_{\varrho_1}) - d(\mathbf{v}_2, \Psi_{\varrho_2}) + \langle \mathbf{v}_1 - \mathbf{v}_2 \perp \nabla'\varrho_2\rangle + \langle \mathbf{v}_1 - \mathbf{u}_\iota \perp \nabla'(\varrho_1 - \varrho_2)\rangle,$$

where $\mathbf{u}_\iota = \frac{1}{\iota}\int_0^\iota T(s)\tilde{\mathbf{u}}_0 ds$ and by (2.30)

$$\begin{aligned}
&d(\mathbf{v}_1, \Psi_{\varrho_1}) - d(\mathbf{v}_2, \Psi_{\varrho_2}) \\
&= \left(\mathbf{v}_1 - \mathbf{v}_2 - \partial_t(\varrho_1 - \varrho_2)\mathbf{n}, \mathbf{V}_4\left(\cdot, \bar\nabla\Psi_{\varrho_1}\right)\bar\nabla\Psi_{\varrho_1} \otimes \bar\nabla\Psi_{\varrho_1}\right) \\
&\quad + \left(\mathbf{v}_2 - \partial_t\varrho_2\mathbf{n}, \left(\mathbf{V}_4\left(\cdot, \bar\nabla\Psi_{\varrho_1}\right)\bar\nabla\Psi_{\varrho_1} - \mathbf{V}_4\left(\cdot, \bar\nabla\Psi_{\varrho_2}\right)\bar\nabla\Psi_{\varrho_2}\right) \otimes \bar\nabla\Psi_{\varrho_1}\right) \\
&\quad + \left(\mathbf{v}_2 - \partial_t\varrho_2\mathbf{n}, \mathbf{V}_4\left(\cdot, \bar\nabla\Psi_{\varrho_2}\right)\bar\nabla\Psi_{\varrho_2} \otimes \bar\nabla\left(\Psi_{\varrho_1} - \Psi_{\varrho_2}\right)\right).
\end{aligned}$$

By the definition of $\mathbf{u}_\iota$, we have

$$\mathbf{u}_\iota - \mathbf{u}_0 = \frac{1}{\iota}\int_0^\iota \left(T(s)\tilde{\mathbf{u}}_0 - \mathbf{u}_0\right) ds,$$

and so, by the interpolation theory, (2.32) and (2.33) we have

$$\|\mathbf{u}_\iota - \mathbf{u}_0\|_{L^q(\dot{\Omega})} \leqslant \frac{1}{\iota}\int_0^\iota \left(\int_0^s \|\partial_r T(r)\tilde{\mathbf{u}}_0\|_{L^q(\dot{\Omega})}\, dr\right) ds \leqslant C\iota^{1/p'} \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})}, \tag{5.11}$$

$$\|\mathbf{u}_\iota - \mathbf{u}_0\|_{H^1_q(\dot{\Omega})} \leqslant C\|\mathbf{u}_\iota - \mathbf{u}_0\|^{1-1/s}_{L^q(\dot{\Omega})} \|\mathbf{u}_\iota - \mathbf{u}_0\|^{1/s}_{B^{2-(1-1/p)}_{q,p}(\dot{\Omega})} \leqslant C\iota^{(1-1/s)/p'} \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})}, \tag{5.12}$$

where s is a positive number such that $1 < s < 2/p'$. Putting (5.11) and (5.12) together gives that

$$\sup_{t\in(0,T)} \|\mathbf{v}(\cdot,t) - \mathbf{u}_\iota\|_{H^1_q(\dot{\Omega})} = \sup_{t\in(0,T)} \|\mathbf{v}(\cdot,t) - \mathbf{u}_0 + \mathbf{u}_0 - \mathbf{u}_\iota\|_{H^1_q(\dot{\Omega})} \leqslant C\left(\iota^{(1-1/s)/p'} + T^{(1-1/s)/p'}\right)(L+B), \tag{5.13}$$

where we have used $\mathbf{v}_1(\cdot,t)-\mathbf{u}_0=\int_0^t \partial_r \mathbf{v}_1(\cdot,r)dr$ and $\|\mathbf{v}_1(\cdot,t)-\mathbf{u}_{0+}\|_{L^q(\dot{\Omega})} \leqslant LT^{1/p'}$.

Finally, we set $1 \leqslant B < L$, $0 < \iota = T < 1$ and choose T small enough, then in view of (2.33), (4.5), (5.6) and (5.13), we can get

$$\begin{aligned}
&\|\langle \mathbf{u}_\iota - \mathbf{v}_1 \perp \nabla'(\varrho_1 - \varrho_2)\rangle\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)} \\
&\lesssim \left(\|\mathbf{u}_\iota\|_{H^2_q(\dot{\Omega})} T^{1/p} + \|\mathbf{v}_1\|_{L^p\left((0,T),H^2_q(\dot{\Omega})\right)}\right)\|\varrho_1 - \varrho_2\|_{L^\infty\left((0,T),W^{2-1/q}_q(\Gamma)\right)} \\
&\qquad + \|\mathbf{u}_\iota - \mathbf{v}_1\|_{L^\infty\left((0,T),H^1_q(\dot{\Omega})\right)} \|\varrho_1 - \varrho_2\|_{L^p\left((0,T),W^{3-1/q}_q(\Gamma)\right)} \\
&\leqslant CL(L+B)T^{(1-1/s)/p'} E_{p,q,T}(\mathbf{v}_1 - \mathbf{v}_2, \Theta_1 - \Theta_2, \varrho_1 - \varrho_2).
\end{aligned} \tag{5.14}$$

By interpolation theory, (4.8) and (5.3), we have

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_{H^1_q(\dot{\Omega})} \leqslant C\|\mathbf{v}_1 - \mathbf{v}_2\|^{1-1/s}_{L^q(\dot{\Omega})} \|\mathbf{v}_1 - \mathbf{v}_2\|^{1/s}_{H^s_q(\dot{\Omega})} \leqslant CT^{(1-1/s)/p'} E_{p,q,T}(\mathbf{v}_1 - \mathbf{v}_2, \Theta_1 - \Theta_2, \varrho_1 - \varrho_2), \tag{5.15}$$

then in view of (4.10), we get

$$\|\varrho_2\|_{L^\infty\left((0,T),W^{2-1/q}_q(\Gamma)\right)} \leqslant CT^{1/p'}L. \tag{5.16}$$

By (4.5) and combining with (5.15) and (5.16), we have

$$\begin{aligned}
&\|\langle \mathbf{v}_1 - \mathbf{v}_2 \perp \nabla'\varrho_2\rangle\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)} \\
&\lesssim \|\varrho_2\|_{L^p\left((0,T),W^{3-1/q}_q(\Gamma)\right)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^\infty\left((0,T),H^1_q(\dot{\Omega})\right)} \\
&\quad + \|\varrho_2\|_{L^\infty\left((0,T),W^{2-1/q}_q(\Gamma)\right)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^p\left((0,T),H^2_q(\dot{\Omega})\right)} \\
&\leqslant C\left(T^{(1-1/s)/p'}L + T^{1/p'}L\right) E_{p,q,T}(\mathbf{v}_1 - \mathbf{v}_2, \Theta_1 - \Theta_2, \varrho_1 - \varrho_2).
\end{aligned} \tag{5.17}$$

Then, by (4.5), (5.15) and $\|\mathbf{n}\|_{H^2_\infty(\dot{\Omega})} \leqslant C$, we have

$$\begin{aligned}
&\left\|\left(\mathbf{v}_1 - \mathbf{v}_2 - \partial_t(\varrho_1 - \varrho_2)\mathbf{n}, \mathbf{V}_4\left(\cdot, \bar{\nabla}\Psi_{\varrho_1}\right)\bar{\nabla}\Psi_{\varrho_1} \otimes \bar{\nabla}\Psi_{\varrho_1}\right)\right\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)} \\
&\leqslant C\left(\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)} + \|\partial_t(\varrho_1 - \varrho_2)\|_{L^p\left((0,T),W^{2-1/q}_q(\Gamma)\right)}\right)
\end{aligned}$$

$$\begin{aligned}
&\times \left\|\mathbf{V}_4\left(\cdot, \bar{\nabla}\Psi_{\varrho_1}\right)\bar{\nabla}\Psi_{\varrho_1}\otimes\bar{\nabla}\Psi_{\varrho_1}\right\|_{L^\infty\left((0,T),W_q^{1-1/q}(\Gamma)\right)}\\
&+C\left\|\mathbf{v}_1-\mathbf{v}_2\right\|_{L^\infty\left((0,T),W_q^{1-1/q}(\Gamma)\right)}+\left\|\partial_t\left(\varrho_1-\varrho_2\right)\right\|_{L^\infty\left((0,T),W_q^{1-1/q}(\Gamma)\right)}\\
&\times \left\|\mathbf{V}_4\left(\cdot, \bar{\nabla}\Psi_{\varrho_1}\right)\bar{\nabla}\Psi_{\varrho_1}\otimes\bar{\nabla}\Psi_{\varrho_1}\right\|_{L^p\left((0,T),W_q^{1-1/q}(\Gamma)\right)}\\
\leqslant\,& CL(L+B)T^{(1-1/s)/p'}E_{p,q,T}\left(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2\right), \qquad (5.18)
\end{aligned}$$

then for the others in $d\left(\mathbf{v}_1,\Psi_{\varrho_1}\right)-d\left(\mathbf{v}_2,\Psi_{\varrho_2}\right)$, we can get similar conclusions and omit details. Thus, we obtain

$$\begin{aligned}
&\left\|d\left(\mathbf{v}_1,\Psi_{\varrho_1}\right)-d\left(\mathbf{v}_2,\Psi_{\varrho_2}\right)\right\|_{L^p\left((0,T),W_q^{2-1/q}(\Gamma)\right)}\\
\leqslant\,& CL(L+B)T^{(1-1/s)/p'}E_{p,q,T}\left(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2\right). \qquad (5.19)
\end{aligned}$$

By (5.14), (5.17) and (5.19), we have

$$\begin{aligned}
&\left\|\mathcal{N}_5\right\|_{L^p\left((0,T),W_q^{2-1/q}(\Gamma)\right)}\\
\leqslant\,& CL(L+B)T^{(1-1/s)/p'}E_{p,q,T}\left(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2\right). \qquad (5.20)
\end{aligned}$$

Next, by (4.33), we get

$$\begin{aligned}
\tilde{\mathcal{N}}_2 =&\tilde{N}_2\left(\mathbf{v}_1,\Psi_{\varrho_1}\right)-\tilde{N}_2\left(\mathbf{v}_2,\Psi_{\varrho_2}\right)\\
=&(\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}])\otimes\nabla\mathcal{E}_1[\mathbf{v}_1]\\
&-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\otimes(\nabla\mathcal{E}_1[\mathbf{v}_1]-\nabla\mathcal{E}_1[\mathbf{v}_2]), \qquad (5.21)\\
\tilde{\mathcal{N}}_3 =&\tilde{N}_3\left(\mathbf{v}_1,\Psi_{\varrho_1}\right)-\tilde{N}_3\left(\mathbf{v}_2,\Psi_{\varrho_2}\right)\\
=&(\mathbf{V}_3\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\mathbf{V}_3\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}])\otimes\mathcal{E}_1[\mathbf{v}_1]\\
&-\mathbf{V}_3\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\otimes(\mathcal{E}_1[\mathbf{v}_1]-\mathcal{E}_1[\mathbf{v}_2]). \qquad (5.22)
\end{aligned}$$

Owing to $(\mathbf{v}_i,\Theta_i,\varrho_i)\in\mathbf{U}_T(i=1,2)$, we have

$$\mathbf{v}_1=\mathbf{v}_2=\mathbf{u}_0,\quad \Theta_1=\Theta_2=\theta_0,\quad \varrho_1=\varrho_2=h_0\quad \text{for } t=0,$$

thus, we can get $\mathcal{E}_1[\mathbf{v}_1]-\mathcal{E}_1[\mathbf{v}_2]=E_t[\mathbf{v}_1-\mathbf{v}_2]$. By (A.1), (A.3), (2.1), (4.8), (4.16), (4.27), (4.28) and (4.29), we have

$$\begin{aligned}
&\left\|e^{-\gamma t}\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\otimes(\nabla\mathcal{E}_1[\mathbf{v}_1]-\nabla\mathcal{E}_1[\mathbf{v}_2])\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot\Omega)\right)}\\
&\quad+\left\|e^{-\gamma t}\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\otimes(\nabla\mathcal{E}_1[\mathbf{v}_1]-\nabla\mathcal{E}_1[\mathbf{v}_2])\right\|_{L^p\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\
\leqslant\,& C\left(T^{1/p'}+T^{(q-N)/pq}\right)L\left(\left\|e^{-\gamma t}E_t[\mathbf{v}_1-\mathbf{v}_2]\right\|_{H_p^1\left(\mathbb{R},L^q(\dot\Omega)\right)}\right. \qquad (5.23)\\
&\quad\left.+\left\|e^{-\gamma t}E_t[\mathbf{v}_1-\mathbf{v}_2]\right\|_{L^p\left(\mathbb{R},H_q^2(\dot\Omega)\right)}\right)\\
\leqslant\,& C\left(T^{1/p'}+T^{(q-N)/pq}\right)LE_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2). \qquad (5.24)
\end{aligned}$$

By (2.1), (4.5), (4.10), (4.15), (4.21), (5.6) and $\mathcal{E}_3[\Psi_{\varrho_1}]-\mathcal{E}_3[\Psi_{\varrho_2}]=E_t[\Psi_{\varrho_1}-\Psi_{\varrho_2}]$, we can also get

$$\begin{aligned}
&\left\|\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\
\leqslant\,&\left\|(\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right))\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\
&\quad+\left\|\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\
\leqslant\,& CT^{1/p'}\left\|\partial_t(\varrho_1-\varrho_2)\right\|_{L^p((0,T),W_q^{2-1/q}(\Gamma))}\left\|\mathcal{E}_3[\Psi_{\varrho_2}]\right\|_{L^\infty(\mathbb{R},H_q^2(\dot\Omega))}
\end{aligned}$$

$$\leqslant CLT^{1/p'}E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2). \tag{5.25}$$

Then by (2.1), (4.5), (4.16) (4.21), (4.26) and (5.6), we have

$$\begin{aligned}
&\|\partial_t(\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}])\|_{L^\infty\left(\mathbb{R},L^q(\dot\Omega)\right)}\\
\lesssim&\|\partial_t\nabla\left(\mathcal{E}_3\left[\Psi_{\varrho_1}\right],\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\|_{L^\infty\left(\mathbb{R},L^q(\dot\Omega)\right)}\|\varrho_1-\varrho_2\|_{L^\infty\left((0,T),W_q^{2-1/q}(\Gamma)\right)}\\
&+\|\nabla\left(\mathcal{E}_3\left[\Psi_{\varrho_1}\right],\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\|\partial_t\left(\varrho_1-\varrho_2\right)\|_{L^\infty\left((0,T),W_q^{1-1/q}(\Gamma)\right)}\\
\leqslant&CLT^{1/p'}E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2).
\end{aligned}\tag{5.26}$$

Thus, using (A.2), (A.3), (5.25) and (5.26), we have

$$\begin{aligned}
&\left\|e^{-\gamma t}(\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}])\nabla\mathcal{E}_1[\mathbf{v}_1]\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot\Omega)\right)}\\
&\quad+\left\|e^{-\gamma t}(\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}])\nabla\mathcal{E}_1[\mathbf{v}_1]\right\|_{L^p\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\
&\leqslant\Big(\|\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\
&\quad+\|\partial_t(\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_1}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_1}]-\mathbf{V}_2\left(\nabla\mathcal{E}_3[\Psi_{\varrho_2}]\right)\nabla\mathcal{E}_3[\Psi_{\varrho_2}])\|_{L^\infty\left(\mathbb{R},L^q(\dot\Omega)\right)}\Big)\\
&\quad\times\Big(\left\|e^{-\gamma t}\mathcal{E}_1[\mathbf{v}_1]\right\|_{H_p^1\left(\mathbb{R},L^q(\dot\Omega)\right)}+\left\|e^{-\gamma t}\mathcal{E}_1[\mathbf{v}_1]\right\|_{L^p\left(\mathbb{R},H_q^2(\dot\Omega)\right)}\Big)\\
&\leqslant CT^{1/p'}L(L+e^{2\gamma}B)E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2).
\end{aligned}\tag{5.27}$$

Combining with (5.23) and (5.27), we have

$$\begin{aligned}
&\left\|e^{-\gamma t}\tilde{\mathcal{N}}_2\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot\Omega)\right)}+\left\|e^{-\gamma t}\tilde{\mathcal{N}}_2\right\|_{L^p\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\
&\leqslant CL(T^{1/p'}+T^{q-N/(pq)})(L+e^{2\gamma}B)E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2).
\end{aligned}\tag{5.28}$$

Next we consider $\tilde{\mathcal{N}}_3$, from (2.1), (4.3), (4.8), (4.15) and (5.6), it follows

$$\begin{aligned}
&\big\|e^{-\gamma t}\partial_t\left(\left(\mathbf{V}_3\left(\nabla\mathcal{E}_3\left[\Psi_{\varrho_1}\right]\right)\nabla\mathcal{E}_3\left[\Psi_{\varrho_1}\right]\right.\right.\\
&\qquad\left.\left.-\mathbf{V}_3\left(\nabla\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\nabla\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\otimes\mathcal{E}_1\left[\mathbf{v}_1\right]\right)\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}\\
&\leqslant C\left\|\nabla\left(\mathcal{E}_3\left[\Psi_{\varrho_1}\right]-\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\left\|e^{-\gamma t}\partial_t\mathcal{E}_1\left[\mathbf{v}_1\right]\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}\\
&\quad+\left\|\nabla\partial_t\left(\mathcal{E}_3\left[\Psi_{\varrho_1}\right]-\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}\left\|e^{-\gamma t}\mathcal{E}_1\left[\mathbf{v}_1\right]\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\\
&\quad+\Big(\left\|\nabla\partial_t\mathcal{E}_3\left[\Psi_{\varrho_1}\right]\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}+\left\|\nabla\partial_t\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}\Big)\\
&\quad\times\Big(\left\|\nabla(\mathcal{E}_3\left[\Psi_{\varrho_1}\right]-\mathcal{E}_3\left[\Psi_{\varrho_2}\right])\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\left\|e^{-\gamma t}\mathcal{E}_1\left[\mathbf{v}_1\right]\right\|_{L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)}\Big)\\
&\leqslant CT^{1/p}(e^{2\gamma}B+L)E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2),
\end{aligned}\tag{5.29}$$

where we have used $T^{1/p'}L\leqslant 1$ and $1/p<1/p'$. By (2.1), (4.3), (4.8), (4.15), (5.3) and (5.15), we have

$$\begin{aligned}
&\left\|e^{-\gamma t}\partial_t\left(\mathbf{V}_3\left(\nabla\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\nabla\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\otimes\left(\mathcal{E}_1\left[\mathbf{v}_1\right]-\mathcal{E}_1\left[\mathbf{v}_2\right]\right)\right)\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}\\
&\leqslant CLT^{1/p'}\left\|\partial_t\left(\mathbf{v}_1-\mathbf{v}_2\right)\right\|_{L^p\left((0,T),L^q(\dot\Omega)\right)}+L\left\|\mathbf{v}_1-\mathbf{v}_2\right\|_{L^\infty\left((0,T),H_q^1(\dot\Omega)\right)}\\
&\leqslant CL(T^{(1-1/s)/p'}+T^{1/p'})E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2).
\end{aligned}\tag{5.30}$$

Combining (5.29) and (5.30) yields

$$\left\|e^{-\gamma t}\partial_t\tilde{\mathcal{N}}_3\right\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}$$

$$\leqslant C(L+e^{2\gamma}B)(T^{1/p}+T^{(1-1/s)/p'})E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2). \tag{5.31}$$

We finally consider $\tilde{\mathcal{N}}_6$, $\tilde{\mathcal{N}}_7$ and $\tilde{\mathcal{N}}_8$. By (4.25), we have

$$\begin{aligned}\tilde{\mathcal{N}}_8 &= \tilde{N}_8\left(\Theta_1,\Psi_{\varrho_1}\right)-\tilde{N}_8\left(\Theta_2,\Psi_{\varrho_2}\right)\\ &= \mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_1}\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_1}\right]\otimes\nabla\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_1]\right]\\ &\qquad -\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\otimes\nabla\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_2]\right]\\ &= \left(\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_1}\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_1}\right]-\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\otimes\nabla\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_1]\right]\\ &\quad +\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\otimes\nabla(\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_1]\right]-\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_2]\right]),\end{aligned}$$

where $\mathrm{tr}\,[\kappa\Theta_i]=\kappa_-\mathcal{E}_-\Theta_{i+}|_\Gamma-\kappa_+\mathcal{E}_+\Theta_{i-}|_\Gamma$ for $i=1,2$. By (4.3), (4.15) and (A.3), we have

$$\left\|e^{-\gamma t}\nabla\mathcal{E}_2\left[\mathrm{tr}\left[\Theta_1\right]\right]\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}+\left\|e^{-\gamma t}\nabla\mathcal{E}_2\left[\mathrm{tr}\left[\Theta_1\right]\right]\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\leqslant C(e^{2\gamma}B+L). \tag{5.32}$$

Since $\mathcal{E}_2\left[\mathrm{tr}\left[\Theta_1\right]\right]-\mathcal{E}_2\left[\mathrm{tr}\left[\Theta_2\right]\right]=e_T\left\{\left[\mathrm{tr}\left[\Theta_1\right]-\mathrm{tr}\left[\Theta_2\right]\right]\right\}$, by (A.3), we have

$$\begin{aligned}&\left\|e^{-\gamma t}\nabla\left(\mathcal{E}_2\left[\mathrm{tr}\left[\Theta_1\right]\right]-\mathcal{E}_2\left[\mathrm{tr}\left[\Theta_2\right]\right]\right)\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\ &\qquad+\left\|e^{-\gamma t}\nabla\left(\mathcal{E}_2\left[\mathrm{tr}\left[\Theta_1\right]\right]-\mathcal{E}_2\left[\mathrm{tr}\left[\Theta_2\right]\right]\right)\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\\ &\leqslant C\left(\left\|\Theta_1-\Theta_2\right\|_{L^p\left((0,T),H_q^2(\dot{\Omega})\right)}+\left\|\partial_t\left(\Theta_1-\Theta_2\right)\right\|_{L^p\left((0,T),L^q(\dot{\Omega})\right)}\right)\\ &\leqslant CE_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2).\end{aligned} \tag{5.33}$$

By (A.2), combined with (5.32) and (5.33), we have

$$\begin{aligned}&\left\|e^{-\gamma t}\tilde{\mathcal{N}}_8\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}+\left\|e^{-\gamma t}\tilde{\mathcal{N}}_8\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\\ &=\left\|e^{-\gamma t}\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\otimes\nabla(\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_1]\right]-\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_2]\right])\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\ &\quad+\left\|e^{-\gamma t}\mathbf{V}_8\left(\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\bar{\nabla}\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\otimes\nabla(\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_1]\right]-\mathcal{E}_2\left[\mathrm{tr}[\kappa\Theta_2]\right])\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\\ &\leqslant CL(e^{2\gamma}B+L)\left(T^{1/p'}+T^{(q-N)/(pq)}\right)E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2).\end{aligned} \tag{5.34}$$

Then, by (2.1), (4.16) and (A.3), we get

$$\begin{aligned}&\left\|\bar{\nabla}^2\mathcal{E}_3\left[\Psi_{\varrho_1}\right]\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}+\left\|\bar{\nabla}^2\mathcal{E}_3\left[\Psi_{\varrho_1}\right]\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\\ &\leqslant C\Bigg(\left\|\varrho_1\right\|_{H_p^1\left((0,T),W_q^{2-1/q}(\Gamma)\right)}+\left\|\varrho_1\right\|_{L^p\left((0,T),W_q^{3-1/q}(\Gamma)\right)}\\ &\qquad+\left\|T_{\varrho_1}(\cdot)h_0\right\|_{H_p^1\left((0,\infty),H_q^2(\dot{\Omega})\right)}+\left\|T_{\varrho_1}(\cdot)h_0\right\|_{L^p\left((0,\infty),H_q^3(\dot{\Omega})\right)}\Bigg)\\ &\leqslant CL,\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}&\left\|e^{-\gamma t}\bar{\nabla}^2\left(\mathcal{E}_3\left[\Psi_{\varrho_1}\right]-\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\right\|_{H_p^{1/2}\left(\mathbb{R},L^q(\dot{\Omega})\right)}\\ &\quad+\left\|e^{-\gamma t}\bar{\nabla}^2\left(\mathcal{E}_3\left[\Psi_{\varrho_1}\right]-\mathcal{E}_3\left[\Psi_{\varrho_2}\right]\right)\right\|_{L^p\left(\mathbb{R},H_q^1(\dot{\Omega})\right)}\\ &\leqslant C\left(\left\|\varrho_1-\varrho_2\right\|_{H_p^1\left((0,T),W_q^{2-1/q}(\Gamma)\right)}+\left\|\varrho_1-\varrho_2\right\|_{L^p\left((0,T),W_q^{3-1/q}(\Gamma)\right)}\right)\\ &\leqslant CE_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2,\Theta_1-\Theta_2,\varrho_1-\varrho_2).\end{aligned} \tag{5.36}$$

Since $\tilde{\mathcal{N}}_6$ and $\tilde{\mathcal{N}}_7$ are similar to $\tilde{\mathcal{N}}_8$, employing a similar argument in proving (5.34) and combined with (5.35) and (5.36), we have

$$\begin{aligned}&\left\|e^{-\gamma t}(\tilde{\mathcal{N}}_6, \tilde{\mathcal{N}}_7)\right\|_{H_p^{1/2}\left(\mathbb{R}, L^q(\dot{\Omega})\right)} + \left\|e^{-\gamma t}(\tilde{\mathcal{N}}_6, \tilde{\mathcal{N}}_7)\right\|_{L^p\left(\mathbb{R}, H_q^1(\dot{\Omega})\right)} \\ &\leqslant CL(L+e^{2\gamma}B)(T^{1/p'}+T^{(q-N)/pq})E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2, \Theta_1-\Theta_2, \varrho_1-\varrho_2). \end{aligned} \tag{5.37}$$

Moreover, by the fourth equation of (5.1), (5.6), (2.33) and (5.20), we have

$$\begin{aligned}&\|\partial_t(h_1-h_2)\|_{L^\infty\left((0,T), W_q^{1-1/q}(\Gamma)\right)} \\ &\lesssim BT^{1/p'}\|\partial_t(h_1-h_2)\|_{L^p\left((0,T), W_q^{2-1/q}(\Gamma)\right)} + \|\mathbf{u}_1-\mathbf{u}_2\|_{L^\infty\left((0,T), H_q^1(\dot{\Omega})\right)} \\ &\quad + T^{1/p'}(L+B)E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2, \Theta_1-\Theta_2, \varrho_1-\varrho_2). \end{aligned} \tag{5.38}$$

Applying Theorems 3.1 and 3.2, (5.8), (5.10), (5.20), (5.28), (5.31), (5.34), (5.37) and (5.38) gives

$$E_{p,q,T}(\mathbf{u}_1-\mathbf{u}_2, \theta_1-\theta_2, h_1-h_2) \leqslant \mathbf{f}_T(L,B)E_{p,q,T}(\mathbf{v}_1-\mathbf{v}_2, \Theta_1-\Theta_2, \varrho_1-\varrho_2), \tag{5.39}$$

with

$$\begin{aligned}\mathbf{f}_T(L,B) =& Ce^{\gamma c^{-\iota}T}(1+\gamma^2)L(T^{1/p}+T^{(1-1/s)/p'}+T^{(q-N)/pq}) \\ &\times\left(L+e^{2\gamma}B+\|\mathbf{a}\|_{L^p((0,T),L^q(\dot{\Omega}))}\right).\end{aligned}$$

Thus, choosing $\iota = T$ so small that $\mathbf{f}_T(L,B) \leqslant 1/2$, we see that Φ is a contraction mapping from U_T into itself, and so there is a unique fixed point $(\mathbf{u}, \theta, h) \in U_T$ of the mapping Φ. Then $(\mathbf{u}, \theta, h)$ solves equations (2.43) uniquely and possesses the properties mentioned in Theorem 2.1. This completes the proof of Theorem 2.1.

Appendix A. Notations and useful results.

A.1. Further notations. For any scalar function $f = f(x)$ and N-vector function $\mathbf{g} = (g_1(x), \ldots, g_N(x))$, we write

$$\begin{aligned}&\nabla f = (\partial_1 f(x), \ldots, \partial_N f(x)), \quad \nabla\mathbf{g} = (\nabla g_1(x), \ldots, \nabla g_N(x)), \\ &\operatorname{div}\mathbf{g} = \sum_{j=1}^N \partial_j g_j(x), \quad \nabla^2 f = (\partial_i\partial_j f)_{i,j=1}^N, \quad \nabla^2\mathbf{g} = (\nabla^2 g_1, \ldots, \nabla^2 g_N).\end{aligned}$$

For any m-vector $\mathbf{V} = (v_1, \ldots, v_m)$ and n-vector $\mathbf{W} = (w_1, \ldots, w_n)$, $\mathbf{V}\otimes\mathbf{W}$ denotes an (m,n) matrix whose $(i,j)^{\text{th}}$ component is $\mathbf{V}_i\mathbf{W}_j$. For any (mn, N) matrix $\mathbf{A} = (A_{ij,k})$ for $i = 1, \ldots, m$, $j = 1, \ldots, n, k = 1, \ldots, N$, $\mathbf{A}\mathbf{V}\otimes\mathbf{W}$ denotes an N column vector whose i^{th} component is the quantity: $\sum_{j=1}^m\sum_{k=1}^n A_{jk,i}v_jw_k$.

For an open set Ω of $\mathbb{R}^N$, $p, q \in [1,\infty]$ and $s \in \mathbb{R}$, let $L^q(\Omega)$, $H_q^s(\Omega)$ and $B_{q,p}^s(\Omega)$ denote Lebesgue spaces, Sobolev spaces and Besov spaces on Ω, with norms $\|\cdot\|_{L^q(\Omega)}$, $\|\cdot\|_{H_q^s(\Omega)}$ and $\|\cdot\|_{B_{q,p}^s(\Omega)}$, respectively. Let

$$X(\dot{\Omega}) = \left\{f : f|_{\Omega_\pm} \in X(\Omega_\pm)\right\}, \quad \|f\|_{X(\dot{\Omega})} = \left\|f|_{\Omega_+}\right\|_{X(\Omega_+)} + \left\|f|_{\Omega_-}\right\|_{X(\Omega_-)},$$

for $X \in \left\{L^q, H_q^s, B_{q,p}^s\right\}$. For simplicity, we write $\|\mathbf{g}\|_{X(\dot{\Omega})^N} = \|\mathbf{g}\|_{X(\dot{\Omega})}$. In this paper, for boundary Γ, we write $W_q^s(\Gamma) = B_{q,q}^s(\Gamma)$, while its norm is written by $\|\cdot\|_{W_q^s(\Gamma)}$. For any N-vectors $\mathbf{a}$ and $\mathbf{b}$, we set $\mathbf{a}\cdot\mathbf{b} = (\mathbf{a}, \mathbf{b}) = \sum_{j=1}^N a_ib_i$, and the tangential component $\mathbf{a}_\tau$ of $\mathbf{a}$ with respect to the normal $\mathbf{n}$ is defined by $\mathbf{a}_\tau = \mathbf{a} - (\mathbf{a}, \mathbf{n})\,\mathbf{n}$. For complex-valued functions f and g, we set $(f,g)_\Omega = \int_\Omega f(x)\overline{g(x)}dx$

where $\overline{g(x)}$ is the complex conjugate of $g(x)$, and for any two N-vector functions $\mathbf{f}$ and $\mathbf{g}$, denote $(\mathbf{f},\mathbf{g})_\Omega=\sum_{j=1}^N (f_j,g_j)_\Omega$. Let $1<q<\infty$, $\frac{1}{q}+\frac{1}{q'}=1$, we introduce the following spaces

$$\hat{H}_q^1(\Omega):=\left\{u\in L^{q,\mathrm{loc}}(\Omega)\mid \nabla u\in L^q(\Omega)\right\},$$
$$\mathcal{D}(\Omega):=\left\{\mathbf{f}\in L^q(\Omega)\mid (\mathbf{f},\nabla\varphi)_\Omega=0\quad\text{for any}\quad \varphi\in\hat{H}_{q'}^1(\Omega)\right\}.$$

Since $C_0^\infty(\Omega)\subset\hat{H}_q^1(\Omega)$, we see that $\operatorname{div}\mathbf{f}=0$ in Ω provided $\mathbf{f}\in\mathcal{D}(\Omega)$. But, the opposite direction does not hold in general.

Definition A.1. Let $1<r<\infty$, and Ω be a domain in $\mathbb{R}^N$ with boundaries Γ and Γ_-. We say that Ω is a *uniform C^3 domain*, if there exist some positive constants α, β, γ and K such that

(1) for any $x_0=(x_{01},x_{02},\dots,x_{0N})\in\Gamma$, there exist a coordinate number j and a C^3 function $h(x')$ (where $x'=(x_1,\dots,\hat{x}_j,\dots,x_N)=(x_1,\dots,x_{j-1},x_{j+1},\dots,x_N)$) for $x'\in B'_\alpha(x'_0)$ with $x'_0=(x_{01},\dots\hat{x}_{0j},\dots,x_{0N})$ and $\|h\|_{H^3_\infty(B'_\alpha(x'_0))}\leqslant K$ such that

$$\Omega\cap B_\beta(x_0)=\left\{x\in\mathbb{R}^N:-\gamma+h\left(x'_j\right)<x_j<h\left(x'_j\right)+\gamma\right\}\cap B_\beta(x_0),$$
$$\Gamma\cap B_\beta(x_0)=\left\{x\in\mathbb{R}^N:x_j=h\left(x'_j\right)\right\}\cap B_\beta(x_0).$$

(2) For any $x_0\in\Gamma_-$, there exist a coordinate number j and a C^3 function $h(x')$ for $x'\in B'_\alpha(x'_0)$ with $\|h\|_{H^3_\infty(B'_\alpha(x'_0))}\leqslant K$ such that

$$\Omega\cap B_\beta(x_0)=\left\{x\in\mathbb{R}^N:x_j>h(x')\right\}\cap B_\beta(x_0),$$
$$\Gamma_-\cap B_\beta(x_0)=\left\{x\in\mathbb{R}^N:x_j=h(x')\right\}\cap B_\beta(x_0).$$

Here, $B'_\alpha(x'_0)=\{x'\in\mathbb{R}^{N-1}:|x'-x'_0|<\alpha\}$, $B_\beta(x_0)=\{x\in\mathbb{R}^N:|x-x_0|<\beta\}$.

Lemma A.2 (cf. [28])**.** *Let $1<p<\infty$, $N<q<\infty$ and $0<T\leqslant 1$. Let $f\in H_p^1\left(\mathbb{R},H_q^1(\dot\Omega)\right)$ and $g\in H_p^{1/q}\left(\mathbb{R},L^q(\dot\Omega)\right)\cap L^p\left(\mathbb{R},H_q^1(\dot\Omega)\right)$. If f vanishes for $t\notin(0,2T)$, then we have*

$$\begin{aligned}&\|fg\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}+\|fg\|_{L^p(\mathbb{R},H_q^1(\dot\Omega))}\\&\leqslant C\left\{\|f\|_{L^\infty(\mathbb{R},H_q^1(\dot\Omega))}+T^{(q-N)/(pq)}\|\partial_t f\|^{1-N/(2q))}_{L^\infty(\mathbb{R},L^q(\dot\Omega))}\|\partial_t f\|^{N/(2q)}_{L^p(\mathbb{R},H_q^1(\dot\Omega))}\right\}\\&\quad\times\left(\|g\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}+\|g\|_{L^p(\mathbb{R},H_q^1(\dot\Omega))}\right).\end{aligned}\tag{A.1}$$

Lemma A.3 (cf. [28])**.** *Let $1<p<\infty$ and $N<q<\infty$. Let*

$$f\in L^\infty\left(\mathbb{R},H_q^1(\dot\Omega)\right)\cap H_\infty^1\left(\mathbb{R},L^q(\dot\Omega)\right),$$
$$g\in H_p^{1/2}\left(\mathbb{R},L^q(\dot\Omega)\right)\cap L^p\left(\mathbb{R},H_q^1(\dot\Omega)\right).$$

Then, we have

$$\begin{aligned}&\|fg\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}+\|fg\|_{L^p(\mathbb{R},H_q^1(\dot\Omega))}\\&\leqslant C\left(\|f\|_{L^\infty(\mathbb{R},H_q^1(\dot\Omega))}+\|f\|_{H^1_\infty(\mathbb{R},L^q(\dot\Omega))}\right)\times\left(\|g\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}+\|g\|_{L^p(\mathbb{R},H_q^1(\dot\Omega))}\right).\end{aligned}\tag{A.2}$$

Lemma A.4 (cf. [28])**.** *Let $1<p,q<\infty$. Then,*

$$H_p^1\left(\mathbb{R},L^q(\dot\Omega)\right)\cap L^p\left(\mathbb{R},H_q^2(\dot\Omega)\right)\subset H_p^{1/2}\left(\mathbb{R},H_q^1(\dot\Omega)\right)$$

and

$$\|u\|_{H^{1/2}_p\left(\mathbb{R},H^1_q(\dot\Omega)\right)} \leqslant C\left\{\|u\|_{L^p\left(\mathbb{R},H^2_q(\dot\Omega)\right)} + \|\partial_t u\|_{L^p\left(\mathbb{R},L^q(\dot\Omega)\right)}\right\}. \tag{A.3}$$

A.2. **Laplace-Beltrami Operator.** In this subsection, we introduce the Laplace-Beltrami operator and some important formulas from differential geometry, which is necessary to know to introduce the surface tension. Let Γ be a hypersurface of class C^3 in $\mathbb{R}^N$. Locally at $p \in \Gamma$ is parametrized as $p = \phi(\theta) = (\phi_1(\theta), \ldots, \phi_N(\theta))^\top$, where $\theta = (\theta_1, \ldots, \theta_{N-1})$ runs through a domain $\Theta \subset \mathbb{R}^{N-1}$. Let

$$\tau_i = \tau_i(p) = \frac{\partial}{\partial\theta_i}\phi(\theta) = \partial_i\phi \quad (i = 1, \ldots, N-1)$$

which forms a basis of the tangent space $T_p\Gamma$ of Γ at p. Let $\mathbf{n} = \mathbf{n}(p)$ denote the outer unit normal of Γ at p. Notice that

$$\langle \tau_i, \mathbf{n}\rangle = 0.$$

Here and in the following, $\langle\cdot,\cdot\rangle$ denotes a standard inner product in $\mathbb{R}^N$. Let

$$g_{ij} = g_{ij}(p) = \langle \tau_i, \tau_j\rangle \quad (i, j = 1, \ldots, N-1),$$

and let G be an $(N-1)\times(N-1)$ matrix whose $(i,j)^{\text{th}}$ components are g_{ij}. The matrix G is called the first fundamental form of Γ. G is a positive symmetric matrix, and therefore G^{-1} exists. Let g^{ij} be the $(i,j)^{\text{th}}$ component of G^{-1} and let $\tau^i = g^{ij}\tau_j$. We next consider $\tau_{ij} = \partial_i\partial_j\phi = \partial_j\tau_i$. Notice that $\tau_{ij} = \tau_{ji}$. Let

$$\Lambda^k_{ij} = \langle \tau_{ij}, \tau^k\rangle, \quad \ell_{ij} = \langle \tau_{ij}, \mathbf{n}\rangle,$$

and then, we have

$$\tau_{ij} = \Lambda^k_{ij}\tau_k + \ell_{ij}\mathbf{n}.$$

Let L be an $N-1\times N-1$ matrix whose $(i,j)^{\text{th}}$ component is ℓ_{ij} which is called the second fundamental form of Γ. Let

$$\mathcal{H}(\Gamma) = \frac{1}{N-1}\operatorname{tr}\left(G^{-1}L\right) = \frac{1}{N-1}g^{ij}\ell_{ij}. \tag{A.4}$$

Then $\mathcal{H}(\Gamma)$ is called the mean curvature of Γ. Let $g = \det G$. We can get the following formulas easily

$$\partial_i\left(\sqrt{g}g^{ij}\tau_j\right) = \sqrt{g}g^{ij}\ell_{ij}\mathbf{n}, \quad \partial_i\left(\sqrt{g}g^{ij}\right) = -\sqrt{g}g^{ik}\Lambda^j_{ik}. \tag{A.5}$$

We now introduce the Laplace-Beltrami operator Δ_Γ on Γ, which is defined by

$$\Delta_\Gamma f = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j f\right).$$

By (A.5), we have

$$\Delta_\Gamma f = g^{ij}\partial_i\partial_j f - g^{ik}\Lambda^j_{ik}\partial_j f.$$

By (A.3) and (A.5), we have

$$\Delta_\Gamma\phi = (N-1)\mathcal{H}(\Gamma)\mathbf{n}.$$

Usually, we put $H(\Gamma) = (N-1)\mathcal{H}(\Gamma)$, and thus, we have

$$\Delta_\Gamma x = H(\Gamma)\mathbf{n} \quad \text{for } x \in \Gamma.$$

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