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A priori estimates for free boundary problem of 3D incompressible inviscid rotating Boussinesq equations

Chengchun Hao and Wei Zhang

Abstract. In this paper, we consider the three-dimensional rotating Boussinesq equations (the "primitive" equations of geophysical fluid flows). Inspired by Christodoulou and Lindblad (Pure Appl Math 53:1536–1602, 2000), we establish a priori estimates of Sobolev norms for free boundary problem of inviscid rotating Boussinesq equations under the Taylor-type sign condition on the initial free boundary. Using the same method, we can also obtain a priori estimates for the incompressible inviscid rotating MHD system with damping.

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1. Introduction

The Boussinesq equations are of relevance to study a number of models coming from atmospheric or oceanographic turbulence where the rotation and stratification play an important role. Referring to [19], we consider the following inviscid rotating Boussinesq equations without heat diffusion in $\mathcal{D}$:

$$\begin{cases}\partial_t v + v\cdot\nabla v + fe_3\times v + \nabla p = \Upsilon h e_3,\\ \partial_t h + v\cdot\nabla h = -\Gamma v_3,\\ \nabla\cdot v = 0,\end{cases}\tag{1.1}$$

where $v=(v_1,v_2,v_3)$, p and h denote the velocity, the fluid pressure and the deviation of the temperature function from the basic temperature profile, respectively. The Coriolis parameter $f=2A\sin\phi$ is assumed to be a nonzero real constant in which A is the angular frequency of rotation and ϕ is the latitude; $e_3=(0,0,1)$ is the vertical unit vector; the Coriolis force $fe_3\times v$ gives rise to a vertical rigidity in the fluid. The number $\Upsilon>0$ is gravity and $\Gamma>0$ is the stratification parameter which represents the Brunt-Väisälä frequency (also buoyancy frequency). The stratification induces the term Γv_3 in the equations, which gives rise to a horizontal rigidity in the fluid. $\mathcal{D}\subset\cup_{0\leqslant t\leqslant T}\{t\}\times\mathbb{R}^3$ is an unknown time-space domain for some constant $T>0$.

We want to find a set $\mathcal{D}$ and (v,h) solving (1.1) and satisfying the initial conditions:

$$\{x:(0,x)\in\mathcal{D}\}=\mathcal{D}_0,\quad (v,h)|_{t=0}=(v_0(x),h_0(x))\ \text{for}\ x\in\mathcal{D}_0.\tag{1.2}$$

Let $\mathcal{D}_t=\{x\in\mathbb{R}^n:(t,x)\in\mathcal{D}\}$, then the conditions on the free boundary read

$$\begin{cases}v_N=\kappa, & \text{on }\partial\mathcal{D}_t,\\ p=0, & \text{on }\partial\mathcal{D}_t,\end{cases}\tag{1.3}$$

for each $t\in[0,T]$, where N is the exterior unit normal to $\partial\mathcal{D}_t$, $v_N=N^i v_i$ in the sense of Einstein's summation convention, κ is the normal velocity of $\partial\mathcal{D}_t$.

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We will prove a priori bounds for (1.1)–(1.3) in some Sobolev spaces under the assumption:

$$\nabla_N p \leqslant -\varepsilon < 0 \text{ on } \partial\mathcal{D}_t, \tag{1.4}$$

where ε is a constant. In fact, we can assume that the condition (1.4) holds initially and it will hold true within some time. In other words, (1.4) is natural physical condition, the pressure is larger in the interior than on the boundary. Moreover, (1.4) is called the Taylor sign condition for the Euler equations.

The free boundary problems of incompressible Euler equations have been studied by many people in recent decades. In two and three dimensions, Wu [24,25] obtained the well-posedness for the incompressible irrotational water wave problem. Christodoulou and Lindblad [6] proved the a priori energy estimates under Taylor's sign condition without surface tension for the incompressible Euler equations; Lindblad proved the local well-posedness for the motion of an incompressible liquid with free surface boundary in [15,16]; Coutand and Shkoller [7] obtained the local well-posedness of the problem with or without surface tension. More important progresses have been made for flows with some general data, see [18] for example.

When $f = \Gamma = 0$, there have been some results for the Boussinesq equations in $\mathbb{R}^n$. Chae and Nam [3] proved the local existence and blow-up criterion for the Boussinesq equations. In [1], Abidi and Hmidi got the global well-posedness for Boussinesq system. Danchin and Paicu proved the existence and uniqueness results for the Boussinesq system with data in Lorentz spaces in [10]. Sulaiman [20] obtained the global existence for the axisymmetric Euler-Boussinesq system in critical Besov spaces. Xu had done a lot of work involving the Boussinesq equations in [21–23].

When $f \neq 0$, in [5], Charve proved the global well-posedness for the primitive equations with some less regular initial data. Charve also studied asymptotics and lower bound for the lifespan of solutions to the primitive equations in [4]. Babin, Mahalov and Nicolaenko had considered regularity of three-dimensional rotating Euler–Boussinesq equations in [2]. Iwabuchi, Mahalov and Takada proved global solutions for the incompressible rotating stably stratified fluids in [14].

However, there have been only few results on the free boundary problems for the Boussinesq equations. In the Hölder spaces, the local and global existence theorem for the problem in the Oberbeck–Boussinesq approximation was established by Denisova and Solonnikov in [8,9]. Hao and Zhang proved the maximal L^p-L^q regularity for the linearized equations in [12] and the local well-posedness in [13] for the two-phase fluid motion in the Oberbeck–Boussinesq approximation.

In this paper, we adopt a geometrical point of view used in [6], and estimate quantities such as the second fundamental form. The energy contains interior and boundary parts involving projected spatial derivatives which is crucial due to the loss of regularity for the estimates of pressure on the boundary. We denote the material derivative by $D_t = \partial_t + v^k \partial_k$, then the system (1.1) can be rewritten as:

$$\begin{aligned} D_t v_j + \tilde{v}_j + \partial_j p &= \delta_{j3} \Upsilon h, && \text{in } \mathcal{D}, \\ D_t h &= -\Gamma v_3, && \text{in } \mathcal{D}, \\ \partial_j v^j &= 0, && \text{in } \mathcal{D}, \\ v_N &= \kappa, && \text{on } [0,T] \times \partial\mathcal{D}_t, \\ p &= 0, && \text{on } [0,T] \times \partial\mathcal{D}_t, \\ \nabla_N p &< -\varepsilon, && \text{on } \{t=0\} \times \partial\mathcal{D}_0, \end{aligned} \tag{1.5}$$

where $\tilde{v} := (-fv_2, fv_1, 0)$ and δ_{ij} is the Kronecker delta symbol such that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Remark 1.1. Just consider fixed boundary problem, we need to add additional conditions to (1.5). When $\Gamma = \Upsilon$ and $v_N = 0$ on $\partial\mathcal{D}_0$, the energy

$$E_0(t) = \frac{1}{2} \int\limits_{\mathcal{D}_0} \left(|v(t,x)|^2 + |h(t,x)|^2 \right) dx \tag{1.6}$$

is conserved. In fact, the rotation and stratification do not cause the above energy loss.

Remark 1.2. Similar to the fixed boundary problem, in this paper, we do not need to assume the condition of temperature on the boundary. Different from Euler equations, the energy of the system is not conserved, but it can be controlled by the initial data and time T. In the following proof, we can find that the higher-order energy of temperature is actually controlled by velocity and the initial energies, and is not affected by the boundary condition.

In order to define higher-order energies, we introduce the second fundamental form of the free surface and tensor products given in [6]. We want to project the system to the tangent space of the boundary. The orthogonal projection Π to the tangent space of the boundary of a $(0,r)$ tensor α is defined to be the projection of each component along the normal:

$$(\Pi\alpha)_{i_1\cdots i_r} = \Pi_{i_1}^{j_1}\cdots\Pi_{i_r}^{j_r}\alpha_{j_1\cdots j_r}, \quad \text{where} \quad \Pi_i^j = \delta_i^j - N_iN^j.$$

Let $\bar{\partial}_i = \Pi_i^j\partial_j$ be a tangential derivative. If $p=0$ on $\partial\mathcal{D}_t$, it follows that $\bar{\partial}_i p = 0$ and

$$\left(\Pi\partial^2 p\right)_{ij} = \theta_{ij}\nabla_N p, \tag{1.7}$$

where $\theta_{ij} = \bar{\partial}_i N_j$ is the second fundamental form of $\partial\mathcal{D}_t$. Then we define the quadratic form Q of the form:

$$Q(\alpha,\beta) = \langle\Pi\alpha,\Pi\beta\rangle = q^{i_1j_1}\cdots q^{i_rj_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r},$$

where

$$q^{ij} = \delta^{ij} - \eta^2(d)N^iN^j, \quad d(x) = \text{dist}\,(x,\partial\mathcal{D}_t)\,, \quad N^i = -\delta^{ij}\partial_j d.$$

Here η is a smooth cut-off function satisfying $0\leqslant\eta(d)\leqslant 1$, $\eta(d)=1$ when $d<d_0/4$, and $\eta(d)=0$ when $d>d_0/2$. d_0 is a fixed number that is smaller than the injectivity radius ς_0 of the normal exponential map, defined to be the largest number ς_0 such that the map

$$\partial\mathcal{D}_t\times(-\varsigma_0,\varsigma_0)\to\{x\in\mathbb{R}^n : \text{dist}\,(x,\partial\mathcal{D}_t)<\varsigma_0\}\,,$$

given by

$$(\bar{x},\varsigma)\to x = \bar{x}+\varsigma N(\bar{x}),$$

is an injection. Then we define the higher energies for $r\geqslant 1$ as

$$\begin{aligned} E_r(t) = &\int_{\mathcal{D}_t}\delta^{ij}Q\left(\partial^r v_i,\partial^r v_j\right) + \int_{\mathcal{D}_t}\left|\partial^{r-1}\text{curl}v\right|^2 dx \\ &+\int_{\mathcal{D}_t}|\partial^r h|^2\,dx + \text{sgn}(r-1)\int_{\partial\mathcal{D}_t}Q\left(\partial^r p,\partial^r p\right)\vartheta dS, \end{aligned}\tag{1.8}$$

where sgn denotes the sign function and

$$\vartheta = \left(-\nabla_N p\right)^{-1}.$$

In the present paper, we prove the following main result.

Theorem 1.1. *Let*

$$\begin{aligned} &\mathcal{K}(0) = \max\left(\|\theta(0,\cdot)\|_{L^\infty(\partial\mathcal{D}_0)}, 1/\varsigma_0(0)\right), \\ &\mathcal{E}(0) = \|1/\left(\nabla_N p(0,\cdot)\right)\|_{L^\infty(\partial\mathcal{D}_0)} = 1/\varepsilon(0) > 0. \end{aligned}\tag{1.9}$$

There exists a continuous function $\mathcal{T}>0$ such that if

$$T\leqslant\mathcal{T}\left(|f|,\Upsilon,\Gamma,\mathcal{K}(0),\mathcal{E}(0),E_0(0),\cdots,E_4(0),\mathit{Vol}\mathcal{D}_0\right), \tag{1.10}$$

then any smooth solution of the free boundary problem for inviscid rotating Boussinesq Eq. (1.5) without heat diffusion satisfies

$$\sum_{s=0}^{4} E_s(t) \leqslant 2\sum_{s=0}^{4} E_s(0), \quad 0 \leqslant t \leqslant T. \tag{1.11}$$

Let us now outline the proof of Theorem 1.1. Firstly, for the rotating Boussinesq Eq. (1.5), we transform the free boundary problem to a fixed boundary problem in the Lagrangian coordinates in Sect. 2. In Sect. 3, we prove the zero-order and the first-order energy estimates. Section 4 is devoted to the higher-order energy estimates by using the identities derived in Sect. 2, then, we justify the a priori assumptions in Sect. 5. Finally, for the rotating MHD equations with damping, we can get a similar conclusion in Sect. 6.

2. Reformulation in Lagrangian coordinates

We introduce the Lagrangian coordinates to transform the free boundary problem to a fixed boundary problem. Let Ω be a bounded domain in $\mathbb{R}^3$, and $f_0 : \Omega \to \mathcal{D}_0$ where f_0 is a diffeomorphism. The connection between the Eulerian coordinates x and the Lagrangian coordinates y is given by $x = x(t, y) = f_t(y)$ and

$$\frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega. \tag{2.1}$$

The Euclidean metric δ_{ij} in $\mathcal{D}_t$, then in Ω for each fixed t, induces a metric

$$g_{ab}(t, y) = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}, \tag{2.2}$$

and its inverse

$$g^{cd}(t, y) = \delta^{kl}\frac{\partial y^c}{\partial x^k}\frac{\partial y^d}{\partial x^l}. \tag{2.3}$$

Furthermore, expressed in the y-coordinates, we have

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}. \tag{2.4}$$

Let us introduce the notation for the material derivative

$$D_t = \frac{\partial}{\partial t}\Big|_{y=\text{ const}} = \frac{\partial}{\partial t}\Big|_{x=\text{ const}} + v^k\frac{\partial}{\partial x^k}.$$

If $k(t, x)$ is the (0,r) tensor expressed in the x-coordinates, we have

$$D_t w_{a_1\cdots a_r} = \frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}\left(D_t k_{i_1\cdots i_r} + \frac{\partial v^\ell}{\partial x^{i_1}}k_{\ell\cdots i_r} + \cdots + \frac{\partial v^\ell}{\partial x^{i_r}}w_{i_1\cdots \ell}\right)$$

where $w_{a_1\cdots a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}k_{i_1\cdots i_r}(t, x)$.

Let $u(t, y)$, $\Theta(t, y)$, $P(t, y)$ represent the velocity, deviation of the temperature function, pressure in the Lagrangian coordinates, respectively. Then from [17, Lemma 2.1] and (1.5), we have

$$\begin{aligned} D_t u_a =& \frac{\partial x^j}{\partial y^a}\left(-\tilde{v}_j - \partial_j p + \Upsilon\delta_{3j}h\right) + v_j\frac{\partial x^k}{\partial y^a}\frac{\partial v^j}{\partial x^k} \\ =& -\tilde{u}_a - \nabla_a P + \Upsilon\delta_{3a}\Theta + u^c\nabla_a u_c, \end{aligned} \tag{2.5}$$

where $\tilde{u}_a = \frac{\partial x^j}{\partial y^a}\tilde{v}_j$. Similarly, since the deviation of the temperature function Θ is scalar, we directly get

$$D_t\Theta = -\Gamma u_3. \tag{2.6}$$

Thus, the system (1.5) can be rewritten in the Lagrangian coordinates as

$$\begin{aligned} D_t u_a + \tilde{u}_a + \nabla_a P &= \Upsilon \delta_{3a} \Theta + u^c \nabla_a u_c, &&\text{in } [0,T] \times \Omega, \\ D_t \Theta &= -\Gamma u_3, &&\text{in } [0,T] \times \Omega, \\ \nabla_a u^a &= 0, &&\text{in } [0,T] \times \Omega, \\ P &= 0, &&\text{on } [0,T] \times \partial\Omega. \end{aligned} \tag{2.7}$$

3. The zero-order and the first-order energy estimates

In this section, we define the zero-order energy as

$$E_0(t) = \frac{1}{2} \int_\Omega \left(|u(t,y)|^2 + |\Theta(t,y)|^2 \right) dy. \tag{3.1}$$

By [17, Lemma 2.1] and (2.7), Gauss' formula, it yields

$$\begin{aligned} \frac{d}{dt} E_0(t) =& \frac{1}{2} \int_\Omega D_t \left(g^{ab} u_a u_b + |\Theta|^2 \right) d\mu_g \\ =& \int_\Omega \left(u^a D_t u_a + \Theta D_t \Theta \right) d\mu_g + \int_\Omega \frac{1}{2} \left(D_t g^{ab} \right) (u_a u_b) \, d\mu_g \\ =& \int_\Omega \left[-u^a \tilde{u}_a - u^a \nabla_a P + \Upsilon u^a \delta_{3a} \Theta + u^a u^c \nabla_a u_c - \Gamma \Theta u_3 \right] d\mu_g - \int_\Omega h^{ab} (u_a u_b) \, d\mu_g \\ =& -\int_{\partial\Omega} N_a u^a P d\mu_\gamma + \int_\Omega (\Upsilon - \Gamma) u_3 \Theta d\mu_g + \int_\Omega u^a u^c \nabla_a u_c d\mu_g \\ & - \frac{1}{2} \int_\Omega g^{ac} \left(\nabla_c u_d + \nabla_d u_c \right) g^{db} u_a u_b d\mu_g \\ =& \int_\Omega (\Upsilon - \Gamma) u_3 \Theta d\mu_g, \end{aligned}$$

where $d\mu_g = \sqrt{\det g} dy$ is the Riemannian volume element on Ω in the metric g. In fact, we can easily obtain $D_t d\mu_g = 0$ and $u^a \tilde{u}_a = 0$ by using $\mathrm{div} u = 0$. Obviously, when $\Upsilon = \Gamma$, the energy of the system is conserved. Using the Hölder inequality

$$\frac{d}{dt} E_0(t) \leqslant C(\Upsilon, \Gamma) \|u\|_{L^2(\Omega)} \|\Theta\|_{L^2(\Omega)} \leqslant C(\Upsilon, \Gamma) E_0(t). \tag{3.2}$$

From the Gronwall inequality, for $t \in [0,T]$ with a constant $T > 0$, it follows that

$$E_0(t) \leqslant C(T, \Upsilon, \Gamma) E_0(0). \tag{3.3}$$

Due to the initial energy is given, we can get the zero-order energy estimate. Before dealing with the first-order energy estimates, we need the following Identities. From [11, Lemma 2.3], (2.5) and (2.7), we have

$$\begin{aligned} & D_t (\nabla_b u_a) + \nabla_b \nabla_a P \\ &= [D_t, \nabla_b] u_a + \nabla_b D_t u_a + \nabla_b \nabla_a P \\ &= -\left(\nabla_a \nabla_b u^d \right) u_d + \nabla_b \tilde{u}_a + \Upsilon \delta_{3a} \nabla_b \Theta + \nabla_b u^c \nabla_a u_c + u^c \nabla_b \nabla_a u_c \\ &= \nabla_b u^c \nabla_a u_c + \nabla_b \tilde{u}_a + \delta_{3a} \Upsilon \nabla_b \Theta. \end{aligned} \tag{3.4}$$

Note that by (2.6) and [11, Lemma 2.3], we directly find that

$$D_t\left(\nabla\Theta\right)=\left[D_t,\nabla\right]\Theta+\nabla D_t\Theta=-\Gamma\nabla u_3. \tag{3.5}$$

Now we calculate the first-order energy estimates. From (3.4), [17, Lemma 2.1], and [11, (A.13)], we derive the material derivative of $g^{bd}\gamma^{ae}\nabla_a u_b\nabla_e u_d$,

$$\begin{aligned}
&D_t\left(g^{bd}\gamma^{ae}\nabla_a u_b\nabla_e u_d\right)\\
&=\left(D_t g^{bd}\right)\gamma^{ae}\nabla_a u_b\nabla_e u_d+g^{bd}\left(D_t\gamma^{ae}\right)\nabla_a u_b\nabla_e u_d+2g^{bd}\gamma^{ae}\left(D_t\nabla_a u_b\right)\nabla_e u_d\\
&=-4\gamma^{ae}\gamma^{fc}\nabla_e u_f\nabla_a u^d\nabla_c u_d+2\Upsilon\gamma^{ae}\nabla_e u^b\delta_{3b}\nabla_a\Theta-\gamma^{ae}\nabla_e u^b\nabla_a\tilde{u}_b-2\nabla_b\left(\gamma^{ae}\nabla_e u^b\nabla_a P\right)\\
&\quad+2\left(\nabla_b\gamma^{ae}\right)\left(\nabla_e u^b\nabla_a P\right).
\end{aligned} \tag{3.6}$$

In fact, $\gamma^{ae}\nabla_e u^b\nabla_a\tilde{u}_b=0$ by using symmetry and the definition of $\tilde{u}$. Similarly, by (3.5), we obviously have

$$\begin{aligned}
D_t\left(|\nabla\Theta|^2\right)&=D_t(g^{ab}\nabla_a\Theta\nabla_b\Theta)=(D_t g^{ab})\nabla_a\Theta\nabla_b\Theta+2g^{ab}D_t\nabla_a\Theta\nabla_b\Theta\\
&=4g^{ac}g^{bd}\nabla_c u_d\nabla_a\Theta\nabla_b\Theta-2\Gamma g^{ab}\nabla_a u_3\nabla_b\Theta.
\end{aligned} \tag{3.7}$$

Next, we shall calculate the material derivative of $|\mathrm{curl}u|^2$. Indeed, we can get

$$\begin{aligned}
D_t|\mathrm{curl}u|^2=&D_t\left(g^{ac}g^{bd}(\mathrm{curl}u)_{ab}(\mathrm{curl}u)_{cd}\right)\\
=&2\left(D_t g^{ac}\right)g^{bd}(\mathrm{curl}u)_{ab}(\mathrm{curl}u)_{cd}+4g^{ac}g^{bd}\left(D_t\nabla_a u_b\right)(\mathrm{curl}u)_{cd}\\
=&-2g^{ae}g^{fc}g^{bd}\left(\nabla_e u_f+\nabla_f u_e\right)(\mathrm{curl}u)_{ab}(\mathrm{curl}u)_{cd}\\
&+4g^{ac}g^{bd}(\mathrm{curl}u)_{cd}\nabla_a u^e\nabla_b u_e-4g^{ac}g^{bd}(\mathrm{curl}u)_{cd}\nabla_a\tilde{u}_b\\
&-4g^{ac}g^{bd}(\mathrm{curl}u)_{cd}\nabla_a\nabla_b P+4\Upsilon g^{ac}g^{bd}(\mathrm{curl}u)_{cd}\delta_{3b}\nabla_a\Theta\\
=&-4g^{ae}g^{bd}\nabla_e u^c(\mathrm{curl}u)_{ab}(\mathrm{curl}u)_{cd}-4g^{ac}g^{bd}(\mathrm{curl}u)_{cd}\nabla_a\tilde{u}_b\\
&+4\Upsilon g^{ac}g^{bd}(\mathrm{curl}u)_{cd}\delta_{3b}\nabla_a\Theta.
\end{aligned} \tag{3.8}$$

Define the first-order energy as

$$E_1(t)=\int\limits_\Omega g^{bd}\gamma^{ae}\nabla_a u_b\nabla_e u_d d\mu_g+\int\limits_\Omega|\mathrm{curl}u|^2 d\mu_g+\int\limits_\Omega|\nabla\Theta|^2 d\mu_g. \tag{3.9}$$

Then we get the following estimates.

Theorem 3.1. *For any smooth solution of system (2.7) satisfying*

$$\begin{aligned}
&|\nabla P|\leqslant M,\quad|\nabla u|\leqslant M, && \text{in } [0,T]\times\Omega,\\
&|\theta|+|\nabla u|+\frac{1}{\varsigma_0}\leqslant K, && \text{on } [0,T]\times\partial\Omega,
\end{aligned}$$

we have for $t\in[0,T]$

$$E_1(t)\leqslant 2e^{CMt}E_1(0)+CK^2\left(e^{CMt}-1\right), \tag{3.10}$$

where C depends only on $\Gamma,\Upsilon,|f|$ and VolΩ.

Proof. By (3.6), (3.7), (3.8), [17, Lemma 2.1] and Gauss' formula, it holds

$$\begin{aligned}\frac{d}{dt}E_1(t)=&\int_\Omega \left(-4\gamma^{ae}\gamma^{fc}\nabla_e u_f\nabla_a u^d\nabla_c u_d+2\Upsilon\gamma^{ae}\nabla_e u^b\delta_{3b}\nabla_a\Theta\right)d\mu_g\\&+2\int_\Omega(\nabla_b\gamma^{ae})\left(\nabla_e u^b\nabla_a P\right)d\mu_g-4\int_\Omega g^{ae}g^{bd}\nabla_e u^c(\text{curl}u)_{ab}(\text{curl}u)_{cd}d\mu_g\\&-4\int_\Omega g^{ac}g^{bd}(\text{curl}u)_{cd}\nabla_a\tilde{u}_b+4\Upsilon\int_\Omega g^{ac}g^{bd}(\text{curl}u)_{cd}\delta_{3b}\nabla_a\Theta d\mu_g\\&+4\int_\Omega g^{ac}g^{bd}\nabla_c u_d\nabla_a\Theta\nabla_b\Theta-2\Gamma\int_\Omega g^{ab}\nabla_a u_3\nabla_b\Theta-2\int_{\partial\Omega}N_b\left(\gamma^{ae}\nabla_e u^b\nabla_a P\right)d\mu_\gamma,\end{aligned}$$

where $d\mu_\gamma$ is the Riemannian volume element on $\partial\Omega$. Since $P=0$ on $\partial\Omega$, it follows that $\gamma^{ae}\nabla_a P=0$. Thus, the integral on the boundary is zero.

Next, from [11, (A.3) and (A.5)], we get

$$\nabla_b\gamma^{ae}=-\theta_b^a N^e-\theta_b^e N^a.$$

By the Hölder inequality and [11, (A.5)], we directly get that

$$\begin{aligned}\frac{d}{dt}E_1(t)\leqslant&CKM(\text{Vol}\Omega)^{1/2}E_1^{1/2}(t)+C(\Upsilon+\Gamma+|f|)E_1(t)\\&+C\|\nabla u\|_{L^\infty(\Omega)}\left(\|\nabla u\|^2_{L^2(\Omega)}+\|\nabla\Theta\|^2_{L^2(\Omega)}+\|\text{curl}u\|^2_{L^2(\Omega)}\right)\\\leqslant&CKM(\text{Vol}\Omega)^{1/2}E_1^{1/2}(t)+C(M,\Upsilon,\Gamma,|f|)E_1(t).\end{aligned}$$

From the Gronwall inequality, it yields the desired estimate. □

Remark 3.1. Whether in the lower order or the higher-order energy estimates later in this paper, we can find that the Coriolis force $\tilde{u}$ does not affect energy of tangential velocity, but it will affect Θ and the energy of curl$u$. In fact, the integral involving P is zero, so we do not need to estimate the boundary integral in E_1. But for the higher-order estimates, we have to introduce boundary integrals for P.

4. The general r-th order energy estimates

In this section, we establish the higher-order energy estimates. Applying [11, Lemma 2.2] and (1.5), we get

$$\begin{aligned}D_t\nabla^r u_a=&D_t\nabla_{a_1}\cdots\nabla_{a_r}u_a\\=&D_t\left(\frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}\frac{\partial x^i}{\partial y^a}\partial_{i_1}\cdots\partial_{i_r}v_i\right)\\=&-\nabla^r\tilde{u}_a-\nabla^r\nabla_a P-\sum_{s=1}^{r-1}\binom{r}{s+1}\left(\nabla^{1+s}u\right)\cdot\nabla^{r-s}u_a+\Upsilon\delta_{3a}\nabla^r\Theta\\&+\nabla_a u^c\nabla^r u_c,\end{aligned}$$

and so, we get for $r\geqslant 2$,

$$\begin{aligned}D_t\nabla^r u_a+\nabla^r\nabla_a P=&-\nabla^r\tilde{u}_a+(\text{curl}u)_{ac}\nabla^r u^c+\Upsilon\delta_{3a}\nabla^r\Theta\\&+\text{sgn}(2-r)\sum_{s=1}^{r-2}\binom{r}{s+1}\left(\nabla^{1+s}u\right)\cdot\nabla^{r-s}u_a.\end{aligned}\tag{4.1}$$

Similarly, by (2.7) and [11, Lemma 2.3], we have

$$D_t\nabla^r\Theta = [D_t, \nabla^r]\,\Theta + \nabla^r D_t\Theta = -\sum_{s=1}^{r-1}\begin{pmatrix} r \\ s+1 \end{pmatrix}\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}\Theta - \Gamma\nabla^r u_3. \tag{4.2}$$

Define the r-th order energy for $r \geqslant 2$ as

$$\begin{aligned} E_r(t) = &\int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d d\mu_g + \int_\Omega \left|\nabla^{r-1}\text{curlu}\right|^2 d\mu_g \\ &\int_\Omega |\nabla^r\Theta|^2 d\mu_g + \int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\vartheta d\mu_\gamma, \end{aligned} \tag{4.3}$$

where $\vartheta = 1/\left(-\nabla_N P\right)$ as before, then we have the following theorem.

Theorem 4.1. *For the integer $r \in [2,4]$, there exists a constant $T > 0$ such that, for any smooth solution to system (2.7) for $0 \leqslant t \leqslant T$ satisfying*

$$\begin{aligned} |\nabla P| \leqslant M, \quad |\nabla u| \leqslant M, \quad |\nabla\Theta| \leqslant M, &\quad \text{in } [0,T]\times\Omega, \\ |\theta| + 1/\varsigma_0 \leqslant K, &\quad \text{on } [0,T]\times\partial\Omega, \\ -\nabla_N P \geqslant \varepsilon > 0, &\quad \text{on } [0,T]\times\partial\Omega, \\ \left|\nabla^2 P\right| + |\nabla_N D_t P| \leqslant L, &\quad \text{on } [0,T]\times\partial\Omega, \end{aligned} \tag{4.4}$$

we have, for $t \in [0,T]$,

$$E_r(t) \leqslant e^{C_1 t}E_r(0) + C_2\left(e^{C_1 t} - 1\right), \tag{4.5}$$

where the constants C_1 and C_2 depend on $\Upsilon, \Gamma, |f|, K, M, L, 1/\varepsilon$, Vol$\Omega, E_0(0), E_1(0), \cdots$ and $E_{r-1}(0)$.

Proof. By (4.3), the derivative of E_r with respect to t is

$$\begin{aligned} \frac{d}{dt}E_r(t) = &\int_\Omega D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\right)d\mu_g + \int_\Omega D_t\left(|\nabla^r\Theta|^2\right)d\mu_g \\ &+ \int_\Omega D_t\left|\nabla^{r-1}\text{curlu}\right|^2 d\mu_g + \int_{\partial\Omega} D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\right)\vartheta d\mu_\gamma \\ &+ \int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\left(\frac{\vartheta_t}{\vartheta} - h_{NN}\right)\vartheta d\mu_\gamma, \end{aligned} \tag{4.6}$$

where $h_{NN} = h_{ab}N^a N^b$ and $h_{ab} = D_t g_{ab}/2$. By using [17, Lemma 2.1], (4.1) and (4.2), we can directly get

$$\begin{aligned} &D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\right) \\ &\quad= \left(D_t g^{bd}\right)\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d + r g^{bd}\left(D_t\gamma^{af}\right)\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d \\ &\qquad+ 2g^{bd}\gamma^{af}\gamma^{AF}D_t\left(\nabla_A^{r-1}\nabla_a u_b\right)\nabla_F^{r-1}\nabla_f u_d \\ &= -2\nabla_c u_e\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^c\nabla_F^{r-1}\nabla_f u^e - 4r\nabla_c u_e\gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a u^d\nabla_F^{r-1}\nabla_f u_d \\ &\qquad- 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a\tilde{u}_b - 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a\nabla_b P \\ &\quad+ 2\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b(\text{curlu})_{bc}\nabla_A^{r-1}\nabla_a u^c + 2\Upsilon\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u^b\delta_{3b}\nabla_A^{r-1}\nabla_a\Theta \\ &\qquad+ 2\text{sgn}(2-r)\gamma^{af}\gamma^{AF}\nabla_F^{r-1}\nabla_f u_d\sum_{s=1}^{r-2}\begin{pmatrix} r \\ s+1 \end{pmatrix}\left(\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}u^d\right)_{Aa}. \end{aligned}$$

Similarly,

$$\begin{aligned} D_t\left(|\nabla^r\Theta|^2\right) =& D_t(g^{af}g^{AF}\nabla_A^{r-1}\nabla_a\Theta\nabla_F^{r-1}\nabla_f\Theta)\\ =& r(D_t g^{af})g^{AF}\nabla_A^{r-1}\nabla_a\Theta\nabla_F^{r-1}\nabla_f\Theta + 2g^{af}g^{AF}D_t(\nabla_A^{r-1}\nabla_a\Theta)\nabla_F^{r-1}\nabla_f\Theta\\ =& -2rg^{ae}\nabla_e u^f g^{AF}\nabla_A^{r-1}\nabla_a\Theta\nabla_F^{r-1}\nabla_f\Theta - 2\Gamma g^{af}g^{AF}\nabla_F^{r-1}\nabla_f\Theta\nabla_A^{r-1}\nabla_a u_3\\ & -2g^{af}g^{AF}\nabla_F^{r-1}\nabla_f\Theta\left(\sum_{s=1}^{r-1}\binom{r}{s+1}\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}\Theta\right)_{Aa}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} & D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\right)\\ & = -2r\nabla_c u_e\gamma^{ac}\gamma^{ef}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P + 2\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P D_t\left(\nabla_F^{r-1}\nabla_f P\right). \end{aligned}$$

In fact, the difficulty is how to deal with the integration of the higher-order derivatives of P on the boundary.

We can apply (4.7), the Hölder inequality and

$$\sum_{s=1}^{r-1}\tbinom{r}{s+1}\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}\Theta = \nabla^r u\cdot\nabla\Theta + \mathrm{sgn}(r-2)\sum_{s=1}^{r-2}\tbinom{r}{s+1}\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}\Theta,$$

to get the estimates for $|\nabla^r\Theta|^2$, it follows that

$$\begin{aligned} \left|\int_\Omega D_t\left(|\nabla^r\Theta|^2\right)d\mu_g\right| \leqslant & C\|\nabla u\|_{L^\infty(\Omega)}E_r(t) + C\Gamma E_r(t) + C\|\nabla\Theta\|_{L^\infty(\Omega)}E_r(t)\\ & + C\mathrm{sgn}(r-2)\int_\Omega g^{af}g^{AF}\nabla_F^{r-1}\nabla_f\Theta\left(\sum_{s=1}^{r-2}\tbinom{r}{s+1}\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}\Theta\right)_{Aa}d\mu_g\\ \leqslant & C(\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla\Theta\|_{L^\infty(\Omega)} + \Gamma)E_r(t)\\ & + C\mathrm{sgn}(r-2)E_r^{1/2}(t)\sum_{s=1}^{r-2}\left\|\nabla^{s+1}u\right\|_{L^4(\Omega)}\left\|\nabla^{r-s}\Theta\right\|_{L^4(\Omega)}. \end{aligned} \tag{4.8}$$

Similarly, by the Hölder inequality, we finally obtain that

$$\begin{aligned} & \int_\Omega D_t\left(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b\nabla_F^{r-1}\nabla_f u_d\right)d\mu_g\\ & \quad + \int_{\partial\Omega} D_t\left(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P\nabla_F^{r-1}\nabla_f P\right)\vartheta d\mu_\gamma\\ & \leqslant C\left\|\nabla u\right\|_{L^\infty(\Omega)}E_r(t) + C\Upsilon E_r(t)\\ & \quad + C\mathrm{sgn}(r-2)E_r^{1/2}(t)\sum_{s=1}^{r-2}\left\|\nabla^{s+1}u\right\|_{L^4(\Omega)}\left\|\nabla^{r-s}u\right\|_{L^4(u)}\\ & + 2\int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_{Aa}^r P\left(D_t\nabla_{Ff}^r P - \frac{1}{\vartheta}N_b\nabla_{Ff}^r u^b\right)\vartheta d\mu_\gamma\\ & \quad + 2\int_\Omega \nabla_b\left(\gamma^{af}\gamma^{AF}\right)\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a P d\mu_g. \end{aligned} \tag{4.9}$$

Hence, by the Hölder inequality, we get

$$2\int_{\Omega} \nabla_b \left(\gamma^{af}\gamma^{AF}\right) \nabla_F^{r-1}\nabla_f u^b \nabla_A^{r-1}\nabla_a P d\mu_g \leqslant CKE_r^{1/2}(t) \left\|\nabla^r P\right\|_{L^2(\Omega)}. \tag{4.10}$$

Now we need to estimate $\|\nabla^r P\|_{L^2(\Omega)}$. The first step is to find an equation for P, taking divergence on the first equation of (2.7), by [11, Lemma 2.2], we obtain

$$\Delta P = -\nabla_a \tilde{u}^a - \nabla_a u^b \nabla_b u^a + \Upsilon \partial_3 \Theta. \tag{4.11}$$

We get for $r \geqslant 2$

$$\nabla^{r-2}\Delta P = -\nabla^{r-2}(\nabla_a \tilde{u}^a) - \sum_{s=0}^{r-2} \binom{r-2}{s} \nabla^s \nabla_a u^b \nabla^{r-2-s}\nabla_b u^a + \Upsilon \nabla^{r-2}\partial_3\Theta.$$

Definition 4.1. Let $0 < \varepsilon_1 < 2$ be a fixed number, and let $\iota_1 = \iota_1(\varepsilon_1)$ the largest number such that

$$|N(\bar{x}_1) - N(\bar{x}_2)| \leq \varepsilon_1 \quad \textit{whenever}\ |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t.$$

Suppose $1/\varsigma_1 \leqslant K_1$, then from [11, (A.28)], we see that, for $\varsigma_1 \geqslant 1/K_1$,

$$\|u\|_{L^\infty(\Omega)} \leqslant C \sum_{s=0}^{2} K_1^{n/2-s} \|\nabla^s u\|_{L^2(\Omega)} \leqslant C(K_1) \sum_{s=0}^{2} E_s^{1/2}(t). \tag{4.12}$$

In view of (4.12), for $s \geqslant 0$, one has

$$\|\nabla^s u\|_{L^\infty(\Omega)} \leqslant C \sum_{\ell=0}^{2} K_1^{n/2-\ell} \left\|\nabla^{\ell+s} u\right\|_{L^2(\Omega)} \leqslant C(K_1) \sum_{\ell=0}^{2} E_{s+\ell}^{1/2}(t), \tag{4.13}$$

and similarly,

$$\|\nabla^s \Theta\|_{L^\infty(\Omega)} \leqslant C \sum_{\ell=0}^{2} K_1^{n/2-\ell} \left\|\nabla^{\ell+s} \Theta\right\|_{L^2(\Omega)} \leqslant C(K_1) \sum_{\ell=0}^{2} E_{s+\ell}^{1/2}(t).$$

From the Hölder inequality, (4.12) and (4.13), we get for $r \in \{3, 4\}$,

$$\begin{aligned}
\left\|\nabla^{r-2}\Delta P\right\|_{L^2(\Omega)} \leqslant & C|f| \|\nabla^{r-1} u\|_{L^2(\Omega)} + C\sum_{s=0}^{r-2} \left\|\nabla^s \nabla_a u^b \nabla^{r-2-s}\nabla_b u^a\right\|_{L^2(\Omega)} \\
& + C\Upsilon \left\|\nabla^{r-1}\Theta\right\|_{L^2(\Omega)} \\
\leqslant & C\|\nabla u\|_{L^\infty(\Omega)} \left\|\nabla^{r-1} u\right\|_{L^2(\Omega)} + (r-3)C\left\|\nabla^2 u\right\|_{L^\infty(\Omega)} \left\|\nabla^2 u\right\|_{L^2(\Omega)} \\
& + C(|f|, \Upsilon)(\left\|\nabla^{r-1} u\right\|_{L^2(\Omega)} + \left\|\nabla^{r-1}\Theta\right\|_{L^2(\Omega)}) \\
\leqslant & C(K_1) \sum_{\ell=1}^{r-1} E_\ell(t) + C(K_1) E_2^{1/2}(t) E_r^{1/2}(t) + C(\Upsilon, |f|) E_{r-1}^{1/2}(t) \\
\leqslant & C(K_1, \Upsilon, |f|) \sum_{\ell=0}^{r-1} E_\ell(t) + C(K_1) E_2^{1/2}(t) E_r^{1/2}(t).
\end{aligned} \tag{4.14}$$

The last inequality is attributed to the zero-order energy estimate and the inequality $E_{r-1}^{1/2}(t) \leqslant CE_0(t) + E_{r-1}(t)$. For $r = 2$, we have the following estimate from the assumption of (4.4) and the Hölder inequality, i.e.,

$$\|\Delta P\|_{L^2(\Omega)} \leqslant C(M, |f|, \Upsilon) E_1^{1/2}(t), \tag{4.15}$$

which is a lower-order energy term. Then, by [6, (A.17)], (4.14) and (4.15), we obtain for any $\delta_r > 0$

$$\begin{aligned} \|\nabla^r P\|_{L^2(\Omega)} \leqslant &\delta_r \|\Pi\nabla^r P\|_{L^2(\partial\Omega)} + C(1/\delta_r, K, \mathrm{Vol}\Omega) \sum_{s\leqslant r-2} \|\nabla^s \Delta P\|_{L^2(\Omega)} \\ \leqslant &\delta_r \|\Pi\nabla^r P\|_{L^2(\partial\Omega)} + C(1/\delta_r, \Upsilon, |f|, K, K_1, M, \mathrm{Vol}\Omega) \sum_{\ell=0}^{r-1} E_\ell(t) \\ &+ (r-2)C(1/\delta_r, K, K_1, M, \mathrm{Vol}\Omega)(E_2^{1/2}(t)E_r^{1/2}(t)). \end{aligned} \tag{4.16}$$

Since $P = 0$ on $\partial\Omega$, due to [11, (A.18)], we obtain for $r \geqslant 1$,

$$\|\Pi\nabla^r P\|_{L^2(\partial\Omega)} \leqslant C(K, K_1)\left(\|\theta\|_{L^\infty(\partial\Omega)} + (r-2)\sum_{k\leqslant r-3}\left\|\bar\nabla^k\theta\right\|_{L^2(\partial\Omega)}\right)\sum_{k\leqslant r-1}\left\|\nabla^k P\right\|_{L^2(\partial\Omega)}. \tag{4.17}$$

From [11, (A.7)], we get the fact that $\Pi\nabla^2 P = \theta\nabla_N P$, and then by (4.4), we have

$$\|\theta\|_{L^2(\partial\Omega)} = \left\|\frac{\Pi\nabla^2 P}{\nabla_N P}\right\|_{L^2(\partial\Omega)} \leqslant \frac{1}{\varepsilon}\left\|\Pi\nabla^2 P\right\|_{L^2(\partial\Omega)}. \tag{4.18}$$

Next, we will estimate $\|\Pi\nabla^r P\|_{L^2(\partial\Omega)}$ and $\|\nabla^r P\|_{L^2(\Omega)}$ for $r \in \{2, 3, 4\}$.

For $r = 2$, by using the trace theorem, (4.16) and (4.17), we get

$$\begin{aligned} \left\|\Pi\nabla^2 P\right\|_{L^2(\partial\Omega)} &\leqslant \|\theta\|_{L^\infty(\partial\Omega)}\|\nabla P\|_{L^2(\partial\Omega)} \\ &\leqslant C(K, \mathrm{Vol}\Omega)\left(\left\|\nabla^2 P\right\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)}\right) \\ &\leqslant C(K, \mathrm{Vol}\Omega)\delta_2\left\|\Pi\nabla^2 P\right\|_{L^2(\partial\Omega)} + C(K, \mathrm{Vol}\Omega, M, \Upsilon, |f|)(\mathrm{Vol}\Omega)^{1/2}E_1(t) \\ &+ C(1/\delta_2, \Upsilon, |f|, K, K_1, M, \mathrm{Vol}\Omega, E_0(0))E_1(t). \end{aligned}$$

We can take δ_2 so small that the first term can be absorbed by the left-hand side. Thus,

$$\left\|\Pi\nabla^2 P\right\|_{L^2(\partial\Omega)}, \left\|\nabla^2 P\right\|_{L^2(\Omega)} \leqslant C(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, E_0(0))(1 + E_1(t)), \tag{4.19}$$

$$\|\theta\|_{L^2(\partial\Omega)} \leqslant C(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_0(0))(1 + E_1(t)). \tag{4.20}$$

By Theorem 3.1, there exists a constant $T > 0$ such that $E_1(t) \leqslant CE_1(0)$ for $t \in [0, T]$.

For $r = 3$, from (4.4), (4.17), (4.19) and (4.20), we get

$$\begin{aligned} \left\|\Pi\nabla^3 P\right\|_{L^2(\partial\Omega)} \leqslant &C(K, K_1)\left(K + \|\theta\|_{L^2(\partial\Omega)}\right)\sum_{k\leqslant 2}\left\|\nabla^k P\right\|_{L^2(\partial\Omega)} \\ \leqslant &C(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_0(0)E_1(0))\left\|\nabla^3 P\right\|_{L^2(\Omega)} \\ &+ C(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_0(0), E_1(0)), \end{aligned} \tag{4.21}$$

and it follows from (4.16) that

$$\begin{aligned} \left\|\nabla^3 P\right\|_{L^2(\Omega)} \leqslant &\delta_3 C(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_1(0))\left\|\nabla^3 P\right\|_{L^2(\Omega)} \\ &+ \delta_3 C(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_1(0)) \\ &+ C(1/\delta_3, K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega)(E_0(t) + E_1(t) + E_2(t)) \\ &+ C(1/\delta_3, K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega)(E_2^{1/2}(t)E_3^{1/2}(t)). \end{aligned} \tag{4.22}$$

Hence, we can choose a sufficiently small $\delta_3 > 0$, and by using (4.21) and (4.22), it implies

$$\begin{aligned}\left\|\nabla^3 P\right\|_{L^2(\Omega)}, \left\|\Pi^3 P\right\|_{L^2(\partial\Omega)} \leqslant & C\left(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_0(0), E_1(0)\right) \\ & + C\left(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega\right)\left(\sum_{\ell=0}^{2} E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t)\right).\end{aligned} \tag{4.23}$$

For $r = 4$, since

$$\begin{aligned}\bar{\nabla}_b \nabla_N P = \gamma_b^d \nabla_d\left(N^a \nabla_a P\right) &= \left(\delta_b^d - N_b N^d\right)\left(\left(\nabla_d N^a\right) \nabla_a P + N^a \nabla_d \nabla_a P\right) \\ &= \theta_b^a \nabla_a P + N^a \nabla_b \nabla_a P - N_b N^d\left(\theta_d^a \nabla_a P + N^a \nabla_d \nabla_a P\right),\end{aligned}$$

then from [11, (A.31) and (A.8)], (4.15), (4.19), (4.20) and (4.23), it follows that

$$\begin{aligned}\left\|\bar{\nabla} \nabla_N P\right\|_{L^2(\partial\Omega)} \leqslant & C\|\theta\|_{L^\infty(\partial\Omega)}\|\nabla P\|_{L^2(\partial\Omega)} + C\left\|\nabla^2 P\right\|_{L^2(\partial\Omega)} \\ \leqslant & C(K, \mathrm{Vol}\Omega)\left(\left\|\nabla^3 P\right\|_{L^2(\Omega)} + \left\|\nabla^2 P\right\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)}\right) \\ \leqslant & C\left(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_0(0), E_1(0)\right) \\ & \cdot\left(1 + \sum_{\ell=0}^{2} E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t)\right).\end{aligned}$$

Thus, by [11, (A.8)], it follows that $(\bar{\nabla}\theta)\nabla_N P = \Pi\nabla^3 P - 3\theta\tilde{\otimes}\bar{\nabla}\nabla_N P$, and we have

$$\begin{aligned}\|\bar{\nabla}\theta\|_{L^2(\partial\Omega)} \leqslant & \frac{1}{\varepsilon}\left(\left\|\Pi\nabla^3 P\right\|_{L^2(\partial\Omega)} + C\|\theta\|_{L^\infty(\partial\Omega)}\left\|\bar{\nabla}\nabla_N p\right\|_{L^2(\partial\Omega)}\right) \\ \leqslant & C\left(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_0(0), E_1(0)\right) \\ & \cdot\left(1 + \sum_{\ell=0}^{2} E_\ell(t) + E_2^{1/2}(t) E_3^{1/2}(t)\right).\end{aligned}$$

Hence, by using (4.17), it yields

$$\left\|\Pi\nabla^4 P\right\|_{L^2(\partial\Omega)} \leqslant C\left(K, K_1\right)\left(K + \|\theta\|_{L^2(\partial\Omega)} + \|\bar{\nabla}\theta\|_{L^2(\partial\Omega)}\right) \sum_{k\leqslant 4}\left\|\nabla^k P\right\|_{L^2(\Omega)}.$$

Consequently, from (4.16), we choose a sufficiently small $\delta_4 > 0$ which can absorb the highest-order term in the right-hand side, and get

$$\begin{aligned}&\left\|\nabla^4 P\right\|_{L^2(\Omega)}, \left\|\Pi\nabla^4 P\right\|_{L^2(\partial\Omega)} \\ &\leqslant C\left(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_0(0), E_1(0)\right)\left(1 + \sum_{\ell=0}^{3} E_\ell(t) + E_2^{1/2}(t) E_4^{1/2}(t)\right).\end{aligned} \tag{4.24}$$

Therefore, thanks to (4.19), (4.23) and (4.24), we can get for $r \geqslant 2$

$$\begin{aligned}\|\nabla^r P\|_{L^2(\Omega)} \leqslant & C\left(K, K_1, \Upsilon, |f|, M, \mathrm{Vol}\Omega, 1/\varepsilon, E_0(0), E_1(0)\right) \\ & \cdot\left(1 + \sum_{\ell=0}^{r-1} E_\ell(t) + (r-2)(E_2^{1/2}(t) E_r^{1/2}(t))\right),\end{aligned}$$

from which and (4.10), we obtain

$$
\begin{aligned}
&2\int_\Omega \nabla_b\left(\gamma^{af}\gamma^{AF}\right)\nabla_F^{r-1}\nabla_f u^b\nabla_A^{r-1}\nabla_a P d\mu_g\\
&\quad\leqslant C\left(K,K_1,\Upsilon,|f|,M,\mathrm{Vol}\Omega,1/\varepsilon,E_0(0),E_1(0)\right)E_r^{1/2}(t)\\
&\quad\quad\cdot\left(1+\sum_{\ell=0}^{r-1}E_\ell(t)+(r-2)E_2^{1/2}(t)E_r^{1/2}(t)\right).
\end{aligned}
$$

Now, we recall that $P=0$ on $\partial\Omega$, so $\gamma_b^a\nabla_a P=0$ on $\partial\Omega$. Then we also obtain

$$
-\vartheta^{-1}N_b=\nabla_N P N_b=N^a\nabla_a P N_b=\delta_b^a\nabla_a P-\gamma_b^a\nabla_a P=\nabla_b P. \tag{4.25}
$$

Next, by the Hölder inequality and (4.25), we have

$$
\begin{aligned}
&\int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_{Aa}^r P\left(D_t\nabla_{Ff}^r P-\frac{1}{\vartheta}N_b\nabla_{Ff}^r u^b\right)\vartheta d\mu_\gamma\\
&\leqslant C\|\vartheta\|_{L^\infty(\partial\Omega)}^{1/2}E_r^{1/2}(t)\left\|\Pi\left(D_t\left(\nabla^r P\right)-\vartheta^{-1}N_b\nabla^r u^b\right)\right\|_{L^2(\partial\Omega)}\\
&=C\|\vartheta\|_{L^\infty(\partial\Omega)}^{1/2}E_r^{1/2}(t)\left\|\Pi\left(D_t\left(\nabla^r P\right)+\nabla^r u\cdot\nabla P\right)\right\|_{L^2(\partial\Omega)},
\end{aligned}\tag{4.26}
$$

then we need to estimate $\Pi D_t\left(\nabla^r P\right)$ and $\nabla^r u\cdot\nabla P$, by [11, Lemma 2.3], it follows that

$$
\begin{aligned}
D_t\nabla^r P+\nabla^r u\cdot\nabla P&=\left[D_t,\nabla^r\right]P+\nabla^r D_t P+\nabla^r u\cdot\nabla P\\
&=\mathrm{sgn}(2-r)\sum_{s=1}^{r-2}\binom{r}{s+1}\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}P+\nabla^r D_t P.
\end{aligned}\tag{4.27}
$$

By [11, (A.18), (A.31) and (A.17)], we have, for $2\leqslant r\leqslant 4$

$$
\begin{aligned}
\left\|\Pi\nabla^r D_t P\right\|_{L^2(\partial\Omega)}\leqslant{}& C\left(K,K_1,\mathrm{Vol}\Omega\right)\left(\|\theta\|_{L^\infty(\partial\Omega)}+(r-2)\sum_{k\leqslant r-3}\left\|\bar\nabla^k\theta\right\|_{L^2(\partial\Omega)}\right)\\
&\cdot\sum_{k\leqslant r}\left\|\nabla^k D_t P\right\|_{L^2(\Omega)},
\end{aligned}\tag{4.28}
$$

and

$$
\left\|\nabla^r D_t P\right\|_{L^2(\Omega)}\leqslant\delta\left\|\Pi\nabla^r D_t P\right\|_{L^2(\partial\Omega)}+C(1/\delta,K,\mathrm{Vol}\Omega)\sum_{s\leqslant r-2}\left\|\nabla^s\Delta D_t P\right\|_{L^2(\Omega)}. \tag{4.29}
$$

Now, from [11, Lemmas 2.1 and 2.3], (2.7), (3.4), (4.11) and

$$
\begin{aligned}
D_t(\nabla_a\tilde u^a)=&D_t f(-\nabla_1 u_2+\nabla_2 u_1)\\
=&f(-[D_t,\nabla_1]u_2-\nabla_1 D_t u_2+[D_t,\nabla_2]u_1+\nabla_2 D_t u_1)\\
=&f(\nabla_b u^c\nabla_a u_c+\nabla_b\tilde u_a-\nabla_b\nabla_a P-\nabla_a u^c\nabla_b u_c-\nabla_a\tilde u_b+\nabla_a\nabla_b P)\\
=&f(-f\partial_1 u_1-f\partial_2 u_2)\\
=&f^2\partial_3 u_3,
\end{aligned}
$$

it follows that

$$\begin{aligned}\Delta D_t P =& -2h^{ab}\nabla_a\nabla_b P + (\Delta u^e)\nabla_e P - D_t\left(g^{bd}g^{ac}\nabla_a u_d\nabla_b u_c + \nabla_a\tilde{u}^a + \Upsilon\partial_3\Theta\right)\\ =& -2h^{ab}\nabla_a\nabla_b P + (\Delta u^e)\nabla_e P - 2D_t\left(g^{bd}\right)\nabla_a u_d\nabla_b u^a - 2g^{bd}D_t\left(\nabla_a u_d\right)\nabla_b u^a\\ &+ D_t(\nabla_a\tilde{u}^a) + \Upsilon D_t(\partial_3\Theta)\\ =& -2h^{ab}\nabla_a\nabla_b P + (\Delta u^e)\nabla_e P + 4h^{bd}\nabla_a u_d\nabla_b u^a + 2g^{bd}\nabla_b u^a\nabla_a\nabla_d P + \Upsilon D_t(\partial_3\Theta)\\ &+ 2g^{bd}\nabla_b u^a\nabla_a\tilde{u}_d - 2g^{bd}\nabla_b u^a\nabla_a u^c\nabla_d u_c + f^2\partial_3 u_3 - 2\Upsilon g^{bd}\nabla_b u^a\delta_{3d}\nabla_a\Theta\\ =& 4g^{ac}\nabla_c u^b\nabla_a\nabla_b P + (\Delta u^e)\nabla_e P + 2\nabla_e u^b\nabla_b u^a\nabla_a u^e + 2g^{bd}\nabla_b u^a\nabla_a\tilde{u}_d\\ &- 2\Upsilon g^{bd}\nabla_b u^a\delta_{nb}\nabla_a\Theta + (f^2 - \Gamma\Upsilon)\partial_3 u_3.\end{aligned}$$

By (4.13), (4.16) and [11, Lemma A.12], it implies that, for $s=2$ (similarly for $s=0,1$)

$$\begin{aligned}&\left\|\nabla^2\Delta D_t P\right\|_{L^2(\Omega)}\\ &\leqslant C\left\|\nabla^3 u\nabla^2 P + \nabla^2 u\nabla^3 P + \nabla u\nabla^4 P + \nabla P\nabla^4 u\right\|_{L^2(\Omega)}\\ &\quad + C\left\|\nabla^3 u\nabla u\nabla u + \nabla^2 u\nabla^2 u\nabla u\right\|_{L^2(\Omega)} + C|f|\left\|\nabla^2 u\nabla^2 u + \nabla^3 u\nabla u\right\|_{L^2(\Omega)}\\ &\quad + C\Upsilon\left\|\nabla^3 u\nabla\Theta + \nabla^2 u\nabla^2\Theta + \nabla u\nabla^3\Theta\right\|_{L^2(\Omega)}\\ &+ C(\Gamma\Upsilon + |f|^2)\|\nabla^3 u\|_{L^2(\Omega)}\\ &\leqslant C\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^4 P\right\|_{L^2(\Omega)} + C\left\|\nabla^3 u\right\|_{L^2(\Omega)}\left\|\nabla^2 P\right\|_{L^\infty(\Omega)}\\ &\quad + C\left\|\nabla^2 u\right\|_{L^4(\Omega)}\left\|\nabla^3 P\right\|_{L^4(\Omega)} + C\left\|\nabla^4 u\right\|_{L^2(\Omega)}\|\nabla P\|_{L^\infty(\Omega)}\\ &\quad + C\|\nabla u\|_{L^\infty(\Omega)}\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^3 u\right\|_{L^2(\Omega)} + C\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^2 u\right\|_{L^4(\Omega)}\left\|\nabla^2 u\right\|_{L^4(\Omega)}\\ &\quad + C\Upsilon\|\nabla u\|_{L^\infty(\Omega)}\left\|\nabla^3\Theta\right\|_{L^2(\Omega)} + C\left\|\nabla^3 u\right\|_{L^2(\Omega)}(\Upsilon\|\nabla\Theta\|_{L^\infty(\Omega)} + |f|\|\nabla u\|_{L^\infty(\Omega)})\\ &\quad + C\left\|\nabla^2 u\right\|_{L^4(\Omega)}(\Upsilon\left\|\nabla^2\Theta\right\|_{L^4(\Omega)} + |f|\left\|\nabla^2 u\right\|_{L^4(\Omega)}) + C(\Gamma\Upsilon + |f|^2)\|\nabla^3 u\|_{L^2(\Omega)}.\end{aligned}$$

From [11, (A.11)] and (4.13), we can get

$$\begin{aligned}\left\|\nabla^{s+1}u\right\|_{L^4(\Omega)} &\leqslant C\left\|\nabla^s u\right\|_{L^\infty(\Omega)}^{1/2}\left(\sum_{\ell=0}^{2}\left\|\nabla^{s+\ell}u\right\|_{L^2(\Omega)}K_1^{2-\ell}\right)^{1/2}\\ &\leqslant C(K_1)\sum_{\ell=0}^{2}E_{s+\ell}^{1/2}(t).\end{aligned}\tag{4.30}$$

Similarly, it follows that

$$\left\|\nabla^{s+1}\Theta\right\|_{L^4(\Omega)} \leqslant C(K_1)\sum_{\ell=0}^{2}E_{s+\ell}^{1/2}(t).\tag{4.31}$$

By (4.30) and (4.31), we can estimate all terms with $L^4(\Omega)$ norms and the similar estimate of P by the assumptions. Thus, we obtain the bound which is linear about the highest-order energy $E_r^{1/2}(t)$, i.e.,

$$\begin{aligned}\|\nabla^s\Delta D_t P\|_{L^2(\Omega)} \leqslant& C\left(K, K_1, \Gamma, \Upsilon, |f|, M, L, 1/\varepsilon, \mathrm{Vol}\Omega, E_0(0), E_1(0)\right)\\ &\cdot\left(1 + \sum_{\ell=0}^{r-1}E_\ell(t)\right)\left(1 + E_r^{1/2}(t)\right).\end{aligned}\tag{4.32}$$

Therefore, by (4.28), (4.29), (4.32), for small δ independent of $E_r(t)$, we obtain, by induction argument for r, that

$$\begin{aligned}\|\Pi\nabla^r D_t P\|_{L^2(\partial\Omega)} \leqslant & C\left(K, K_1, \Gamma, \Upsilon, |f|, M, L, 1/\varepsilon, \mathrm{Vol}\Omega, E_0(0), E_1(0)\right) \\ & \cdot\left(1+\sum_{\ell=0}^{r-1} E_\ell(t)\right)\left(1+E_r^{1/2}(t)\right).\end{aligned} \tag{4.33}$$

Then, we estimate the remaining term $\Pi\left(\left(\nabla^{s+1}u\right)\cdot\nabla^{r-s}P\right)$, for the case $r=3,4$ and $s=r-2$, indeed, we have, by (A.6), [11, Lemma A.14] and (4.4),

$$\begin{aligned}&\left\|\Pi\left(\left(\nabla^{r-1}u\right)\cdot\nabla^2 P\right)\right\|_{L^2(\partial\Omega)}\\ \leqslant&\left\|\nabla^{r-1}u\right\|_{L^2(\partial\Omega)}\left\|\nabla^2 P\right\|_{L^\infty(\partial\Omega)} \leqslant CL\left\|\nabla^2 u\right\|_{L^{2(n-1)/(n-2)}(\partial\Omega)}\\ \leqslant& C(K,\mathrm{Vol}\Omega)L\left(\|\nabla^r u\|_{L^2(\Omega)}+\left\|\nabla^{r-1}u\right\|_{L^2(\Omega)}\right)\\ \leqslant& C(K,L,\mathrm{Vol}\Omega)\left(E_{r-1}^{1/2}(t)+E_r^{1/2}(t)\right).\end{aligned}$$

For $r=4$ and $s=1$, we get similarly

$$\begin{aligned}&\left\|\Pi\left(\left(\nabla^2 u\right)\cdot\nabla^3 P\right)\right\|_{L^2(\partial\Omega)}\\ &=\left\|\Pi\nabla^2 u\cdot\Pi\nabla^3 P+\Pi\left(\nabla^2 u\cdot N\right)\tilde{\otimes}\Pi\left(N\cdot\nabla^3 P\right)\right\|_{L^2(\partial\Omega)}\\ &\leqslant C\left\|\Pi\nabla^2 u\right\|_{L^4(\partial\Omega)}\left\|\Pi\nabla^3 P\right\|_{L^4(\partial\Omega)}+C\left\|\Pi\left(N^a\nabla^2 u_a\right)\right\|_{L^4(\partial\Omega)}\left\|\Pi\left(\nabla_N\nabla^2 P\right)\right\|_{L^4(\partial\Omega)}\\ &\leqslant C\left\|\nabla^2 u\right\|_{L^4(\partial\Omega)}\left\|\nabla^3 P\right\|_{L^4(\partial\Omega)}\\ &\leqslant C(K,\mathrm{Vol}\Omega)\left(\left\|\nabla^3 u\right\|_{L^2(\Omega)}+\left\|\nabla^2 u\right\|_{L^2(\Omega)}\right)\left(\left\|\nabla^4 P\right\|_{L^2(\Omega)}+\left\|\nabla^3 P\right\|_{L^2(\Omega)}\right)\\ &\leqslant C\left(K,K_1,\Upsilon,|f|,M,\mathrm{Vol}\Omega\right)\left(E_3^{1/2}(t)+E_2^{1/2}(t)\right)\left(\sum_{s=0}^{3}E_s(t)+\left(\sum_{\ell=0}^{2}E_\ell^{1/2}(t)\right)E_4^{1/2}(t)\right)\\ &\leqslant C\left(K,K_1,\Upsilon,|f|,M,\mathrm{Vol}\Omega\right)\sum_{s=0}^{3}E_s(t)\sum_{\ell=0}^{4}E_\ell^{1/2}(t).\end{aligned}$$

Hence, we have

$$|(4.26)|\leqslant C\left(K,K_1,\Gamma,\Upsilon,|f|,M,L,1/\varepsilon,\mathrm{Vol}\Omega,E_0(0),E_1(0)\right)\left(1+\sum_{s=0}^{r-1}E_s(t)\right)\left(1+E_r(t)\right).$$

By combining (4.30) with (4.31), we can get

$$\begin{aligned}|(4.8)|+|(4.9)|\leqslant & C\left(K,K_1,\Gamma,\Upsilon,|f|,M,L,1/\varepsilon,\mathrm{Vol}\Omega,E_0(0),E_1(0)\right)\\ &\cdot\left(1+\sum_{s=0}^{r-1}E_s(t)\right)\left(1+E_r(t)\right).\end{aligned}$$

Now we calculate the material derivatives of $\left|\nabla^{r-1}\mathrm{curl}u\right|^2$. By [17, Lemma 2.1] and (4.1), we have

$$\begin{aligned}
&D_t\left(\left|\nabla^{r-1}\mathrm{curl}u\right|^2\right)\\
&\quad= D_t\left(g^{ac}g^{bd}g^{AF}\nabla_A^{r-1}(\mathrm{curl}u)_{ab}\nabla_F^{r-1}(\mathrm{curl}u)_{cd}\right)\\
&\quad= (r+1)D_t\left(g^{ac}\right)g^{bd}g^{AF}\nabla_A^{r-1}(\mathrm{curl}u)_{ab}\nabla_F^{r-1}(\mathrm{curl}u)_{cd}\\
&\qquad+4g^{ac}g^{bd}g^{AF}D_t\left(\nabla_A^{r-1}\nabla_a u_b\right)\nabla_F^{r-1}(\mathrm{curl}u)_{cd}\\
&\quad= -2(r+1)g^{ae}\nabla_e u^c g^{bd}g^{AF}\nabla_A^{r-1}(\mathrm{curl}u)_{ab}\nabla_F^{r-1}(\mathrm{curl}u)_{cd}\\
&\qquad-4g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\mathrm{curl}u)_{cd}\nabla_{Aa}^r\nabla_b P+4g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\mathrm{curl}u)_{cd}\nabla_{Aa}^r\tilde{u}_b\\
&\quad-4\Upsilon g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\mathrm{curl}u)_{cd}\delta_{b3}\nabla_{Aa}^r\Theta+4g^{ac}g^{bd}g^{AF}\nabla_F^{r-1}(\mathrm{curl}u)_{cd}(\mathrm{curl}u)_{be}\nabla_{Aa}^r u^e\\
&\qquad+4\mathrm{sgn}(2-r)g^{ac}g^{AF}\nabla_F^{r-1}(\mathrm{curl}u)_{cd}\sum_{s=1}^{r-2}\binom{r}{s+1}\left(\left(\nabla^{1+s}u\right)\cdot\nabla^{r-s}u^d\right)_{Aa}.
\end{aligned}$$

The higher-order term involving pressure P will vanish by symmetry. For other terms, we can apply the Hölder inequality and the Gauss formula to get that

$$\begin{aligned}
&\int_\Omega D_t\left(\left|\nabla^{r-1}\mathrm{curl}u\right|^2\right)d\mu_g\\
&\qquad\leqslant (K,\Upsilon,|f|,M,L,1/\varepsilon,\mathrm{Vol}\Omega,E_0(0))\cdot\left(1+\sum_{s=0}^{r-1}E_s(t)\right)(1+E_r(t)).
\end{aligned}$$

Finally, we only need to estimate the last term in (4.6). By [11, (A.12)], we have

$$\frac{\vartheta_t}{\vartheta}=\frac{2h_d^a N^d\nabla_a P}{\nabla_N P}-h_{NN}-\frac{\nabla_N D_t P}{\nabla_N P}.$$

Thus, the integrals can be controlled by $C(K,\Upsilon,\Gamma,|f|,M,L,1/\varepsilon)E_r(t)$.

In summary, we obtain

$$\frac{d}{dt}E_r(t)\leqslant C\left(K,K_1,\Upsilon,\Gamma,|f|,M,L,1/\varepsilon,\mathrm{Vol}\Omega,E_0(0)\right)\left(1+\sum_{s=0}^{r-1}E_s(t)\right)(1+E_r(t)),$$

which implies the desired result by Gronwall's inequality and the induction argument for $r\in\{2,3,4\}$.

□

5. Justification of a priori assumptions

In this section, we justify the a priori assumptions in Sect. 4. At time t, denote

$$\begin{aligned}
&\mathcal{K}(t)=\max\left(\|\theta(t,\cdot)\|_{L^\infty(\partial\Omega)},1/\varsigma_0(t)\right),\\
&\mathcal{E}(t)=\|1/\left(\nabla_N P(t,\cdot)\right)\|_{L^\infty(\partial\Omega)},\quad \varepsilon(t)=\frac{1}{\mathcal{E}(t)}.
\end{aligned}\tag{5.1}$$

In fact, our judgment is very similar to those in [6,11], so we only state the results and omit their proofs as follows.

Lemma 5.1. *Let $K_1 \geqslant 1/\varsigma_1(t)$, then there are continuous functions F_j, $j=1,2,3,4$, such that*

$$\begin{aligned}
\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla \Theta\|_{L^\infty(\Omega)} &\leqslant F_1\left(K_1, E_0, \cdots, E_4\right), \\
\|\nabla P\|_{L^\infty(\Omega)} + \left\|\nabla^2 P\right\|_{L^\infty(\Omega)} &\leqslant F_2\left(K_1, \mathcal{E}, E_0, \cdots, E_4, \, Vol\Omega\right), \\
\|\nabla D_t P\|_{L^\infty(\partial\Omega)} &\leqslant F_4\left(K_1, \mathcal{E}, E_0, \cdots, E_4, \, Vol\Omega\right), \\
\|\theta\|_{L^\infty(\partial\Omega)} &\leqslant F_3\left(K_1, \mathcal{E}, E_0, \cdots, E_4, \, Vol\Omega\right), \\
\left|\frac{d}{dt}\mathcal{E}\right| &\leqslant C_r\left(K_1, \mathcal{E}, E_0, \cdots, E_4, \, Vol\Omega\right), \\
\left|\frac{d}{dt}E_r\right| &\leqslant C_r\left(K_1, \mathcal{E}, E_0, \cdots, E_4, \, Vol\Omega\right)\sum_{s=0}^{r} E_s.
\end{aligned}$$

Lemma 5.2. *There exists a continuous function $\mathcal{T} > 0$ depending on K_1, $|f|, \Upsilon, \Gamma$, $E_0(0)$, $E_1(0)$, $\cdots$, $E_4(0)$ and $Vol\Omega$ such that for*

$$0 \leqslant t \leqslant \mathcal{T}\left(K_1, |f|, \Upsilon, \Gamma, \mathcal{E}(0), E_0(0), \cdots, E_4(0), \, Vol\Omega\right)$$

the following statements hold

$$E_s(t) \leqslant 2E_s(0), \quad 0 \leqslant s \leqslant 4, \quad \mathcal{E}(t) \leqslant 2\mathcal{E}(0).$$

Furthermore,

$$\frac{1}{2} g_{ab}(0,y)Y^aY^b \leqslant g_{ab}(t,y)Y^aY^b \leqslant 2g_{ab}(0,y)Y^aY^b,$$

and with ε_1 as in Definition 4.1,

$$\begin{aligned}
|N(x(t,\bar{y})) - N(x(0,\bar{y}))| &\leqslant \frac{\varepsilon_1}{16}, \quad \bar{y} \in \partial\Omega, \\
|x(t,y) - x(t,y)| &\leqslant \frac{\varsigma_1}{16}, \quad y \in \Omega, \\
\left|\frac{\partial x(t,\bar{y})}{\partial y} - \frac{\partial(0,\bar{y})}{\partial y}\right| &\leqslant \frac{\varepsilon_1}{16}, \quad \bar{y} \in \partial\Omega.
\end{aligned}$$

Lemma 5.3. *Let $\mathcal{T}$ be as in Lemma 5.2. There exists some $\varepsilon_1 > 0$ such that, if*

$$\left|N\left(x\left(0, y_1\right)\right) - N\left(x\left(0, y_2\right)\right)\right| \leqslant \frac{\varepsilon_1}{2},$$

then for $t \leqslant \mathcal{T}$, it holds

$$\left|N\left(x\left(t, y_1\right)\right) - N\left(x\left(t, y_2\right)\right)\right| \leqslant \varepsilon_1.$$

Proof.

$$\begin{aligned}
&\left|N\left(x\left(t, y_1\right)\right) - N\left(x\left(t, y_2\right)\right)\right| \\
&\quad \leqslant \left|N\left(x\left(t, y_1\right)\right) - N\left(x\left(0, y_1\right)\right)\right| + \left|N\left(x\left(0, y_1\right)\right) - N\left(x\left(0, y_2\right)\right)\right| \\
&\qquad + \left|N\left(x\left(0, y_2\right)\right) - N\left(x\left(t, y_2\right)\right)\right|,
\end{aligned}$$

and follows from Lemma 5.2. □

Consequently, Lemmas 5.2 and 5.3 yield immediately Theorem 1.1.

6. A priori estimates for rotating magnetohydrodynamics with damping

As everyone knows, the rotating MHD has wide application including planetary flows, stellar flows and accretion discs. An incompressible inviscid MHD system with damping under solid body rotation and in the presence of a uniform background magnetic field will be considered. The equations in the rotating frame of reference are:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \alpha e_3 \times b + \nabla p = b\cdot\nabla b, \\ \partial_t b + v\cdot\nabla b = b\cdot\nabla v - \eta b, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \end{cases} \tag{6.1}$$

in $\mathcal{D}$ (the same as before), where v, b, p denote the velocity, the magnetic field, the total pressure, respectively; α is the rotation rate and $\eta > 0$ is a damping coefficient. Obviously, when $\alpha = \eta = 0$, (6.1) are the incompressible inviscid MHD equations considered in [11]. The system (6.1) with the conditions $p = 0$, $b\cdot N = 0$ and $v_N = \kappa$ (the normal velocity of free surface) on the free boundary $\partial\mathcal{D}_t$ can be written in the Lagrangian coordinates as

$$\begin{aligned} D_t u_a + \nabla_a P &= -\tilde{\beta}_a + \beta u^c \nabla_a u_c + \beta^d \nabla_d \beta_a, &&\text{in } [0,T]\times\Omega, \\ D_t \beta_a &= \beta^d \nabla_d u_a + \beta^c \nabla_a u_c - \eta\beta_a, &&\text{in } [0,T]\times\Omega, \\ \nabla_a u^a = 0, &\quad \nabla_a \beta^a = 0, &&\text{in } [0,T]\times\Omega, \\ \beta_a N^a = 0, &\quad P = 0, &&\text{on } [0,T]\times\partial\Omega, \end{aligned} \tag{6.2}$$

where $\tilde{\beta} = (-\alpha\beta_2, \alpha\beta_1, 0)$, $\Omega = \mathcal{D}_0$; u, β, P denote the velocity, the magnetic field, the total pressure in the new coordinates. Thus, in view of (6.2) and [17, Lemma 2.1], we also have the zero-order energy

$$E_0(t) = \frac{1}{2}\int_\Omega \left(|u(t,x)|^2 + |\beta(t,x)|^2\right) d\mu_g + \eta \int_0^t \int_\Omega |\beta|^2 d\mu_g d\tau.$$

A direct computation yields that the energy of the system is conserved.

Similarly, we can define the first-order energy as

$$\begin{aligned} E_1(t) = &\int_\Omega \left(g^{bd}\gamma^{ae}\nabla_a u_b \nabla_e u_d + g^{bd}\gamma^{ae}\nabla_a \beta_b \nabla_e \beta_d\right) d\mu_g \\ &+ \int_\Omega \left(|\mathrm{curl} u|^2 + |\mathrm{curl}\beta|^2\right) d\mu_g + \eta \int_0^t \int_\Omega |\nabla\beta|^2 d\mu_g d\tau. \end{aligned}$$

Theorem 6.1. *For any smooth solution of system (6.2) for $0 \leqslant t \leqslant T$ satisfying*

$$|\nabla P| \leqslant M, \quad |\nabla u| \leqslant M, \quad \text{in } [0,T]\times\Omega, \tag{6.3}$$

$$|\theta| + |\nabla u| + \frac{1}{\varsigma_0} \leqslant K, \quad \text{on } [0,T]\times\partial\Omega, \tag{6.4}$$

we have for $t \in [0,T]$

$$E_1(t) \leqslant 2e^{CMt}E_1(0) + CK^2\left(Vol\Omega + E_0(0)\right)\left(e^{CMt} - 1\right), \tag{6.5}$$

where C depend on α, η.

Define the r-th order energy for $r \geqslant 2$ as

$$\begin{aligned} E_r(t) = & \int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a u_b \nabla_F^{r-1}\nabla_f u_d d\mu_g + \int_\Omega \left|\nabla^{r-1}\mathrm{curl} u\right|^2 d\mu_g \\ & + \int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \beta_b \nabla_F^{r-1}\nabla_f \beta_d d\mu_g + \int_\Omega \left|\nabla^{r-1}\mathrm{curl}\beta\right|^2 d\mu_g \\ & + \eta \int_0^t \int_\Omega |\nabla^r \beta|^2 d\mu_g d\tau + \int_{\partial\Omega} \gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a P \nabla_F^{r-1}\nabla_f P \vartheta d\mu_\gamma. \end{aligned} \tag{6.6}$$

Theorem 6.2. *For the integer $2 \leqslant r \leqslant 4$, there exists a constant $T > 0$ such that, for any smooth solution to system (6.2) for $0 \leqslant t \leqslant T$ satisfying*

$$\begin{aligned} |\beta| \leqslant M_1, \quad for \quad r = 2, \quad & in\ [0,T]\times\Omega, \\ |\nabla P| \leqslant M, \quad |\nabla u| \leqslant M, \quad |\nabla\beta| \leqslant M, \quad & in\ [0,T]\times\Omega, \\ |\theta| + 1/\varsigma_0 \leqslant K, \quad & on\ [0,T]\times\partial\Omega, \\ -\nabla_N P \geqslant \varepsilon > 0, \quad & on\ [0,T]\times\partial\Omega, \\ \left|\nabla^2 P\right| + |\nabla_N D_t P| \leqslant L, \quad & on\ [0,T]\times\partial\Omega, \end{aligned} \tag{6.7}$$

we have for $t \in [0,T]$,

$$E_r(t) \leqslant e^{C_1 t} E_r(0) + C_2 \left(e^{C_1 t} - 1\right), \tag{6.8}$$

where the constants C_1 and C_2 depend on $K, \alpha, \eta, M, M_1, L, 1/\varepsilon$, $Vol\Omega, E_0(0), E_1(0), \cdots$ and $E_{r-1}(0)$.

Remark 6.1. It is meaningfully different from the Boussinesq equations that in such case the Taylor sign condition does involve the total pressure rather than just the pure hydrostatic pressure.

Remark 6.2. Because the nonlinear term involves β, then we have to estimate the L^2 norm of $\nabla_b\beta^a\beta^e\nabla_e \nabla_a u^a$ when we estimate $\|\nabla^s \Delta D_t P\|_{L^2(\Omega)}$ for $s \leqslant 2$. Obviously, we have to assume $|\beta| \leqslant M_1$ when $r = 2$. It is different from the rotating Boussinesq equations.

Similarly, we can obtain the following a priori estimates.

Theorem 6.3. *Let*

$$\begin{aligned} \mathcal{K}(0) &= \max\left(\|\theta(0,\cdot)\|_{L^\infty(\partial\Omega)}, 1/\varsigma_0(0)\right), \\ \mathcal{E}(0) &= \|1/\left(\nabla_N P(0,\cdot)\right)\|_{L^\infty(\partial\Omega)} = 1/\varepsilon(0) > 0. \end{aligned}$$

There exists a continuous function $\mathcal{T} > 0$ such that if

$$T \leqslant \mathcal{T}\left(\alpha, \eta, \mathcal{K}(0), \mathcal{E}(0), E_0(0), \cdots, E_4(0), Vol\Omega\right),$$

then any smooth solution of the free boundary problem for incompressible inviscid rotating MHD system (6.1) with damping satisfies

$$\sum_{s=0}^4 E_s(t) \leqslant 2\sum_{s=0}^4 E_s(0), \quad 0 \leqslant t \leqslant T.$$

Theorems 6.1, 6.2 and 6.3 can be proved similarly as those of rotating Boussinesq equations and the non-rotating MHD case in [11]. We omit the details of the proof.

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Chengchun Hao
Academy of Mathematics and Systems Science, Chinese Academy of Sciences
Beijing 100190
China
e-mail: hcc@amss.ac.cn

Chengchun Hao
School of Mathematical Sciences
University of Chinese Academy of Sciences
Beijing 100049
China

Wei Zhang
School of Mathematical Sciences
Capital Normal University
Beijing 100048
China
e-mail: zhangwei16@mails.ucas.edu.cn

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