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Splash singularity for the free boundary incompressible viscous MHD

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Abstract

In this paper, we prove the existence of smooth initial data for the two-dimensional free boundary incompressible viscous magnetohydrodynamics (MHD) equations, for which the interface remains regular but collapses into a splash singularity (self-intersects in at least one point) in finite time. The existence of the splash singularities is guaranteed by a local existence theorem, in which we need suitable spaces for the modified magnetic field together with the modification of the velocity and pressure such that the modified initial velocity is zero, and a stability result which allows us to construct a class of initial velocities and domains for an arbitrary initial magnetic field. It turns out that the presence of the magnetic field does not prevent the viscous fluid from forming splash singularities for certain smooth initial data.

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Keywords: Finite-time singularity; Free boundary problem; Incompressible viscous magnetohydrodynamics; Interface singularity; Splash singularity

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1. Introduction

In the present paper, we are concerned with the formation of splash singularities for two-dimensional incompressible viscous magnetohydrodynamics (MHD) equations without magnetic diffusivity. It consists of solving a bounded variable domain $\Omega(t) \subset \mathbb{R}^2$ filled with the incompressible viscous electrically conducting homogeneous plasma, the density of which is a positive constant, together with the vector field of velocity $u(t,x) = (u^1, u^2)^\top$ and magnetic field $H(t,x) = (H^1, H^2)^\top$, and the scalar pressure $q(t,x)$ satisfying the system of MHD. The boundary $\partial\Omega(t)$ of $\Omega(t)$ is the free-surface of the plasma.

The problem can be represented in the following form (cf. [19,20]). In the plasma region, the incompressible viscous MHD equations read

$$\begin{cases} \partial_t u + u\cdot\nabla u - \operatorname{div} T(u,q) = \operatorname{div} T_M(H), & \text{in } \Omega(t), \\ \partial_t H + u\cdot\nabla H = H\cdot\nabla u, & \text{in } \Omega(t), \\ \operatorname{div} u = 0, \quad \operatorname{div} H = 0, & \text{in } \Omega(t), \end{cases}$$

where the time $t > 0$; $u\cdot\nabla$ and $H\cdot\nabla$ are directional derivatives; $T(u,q) = -q\mathcal{I} + \nu S(u)$ is the viscous stress tensor; $\mathcal{I}$ is the 2×2 identity matrix; ν is the kinematic viscosity; $S(u) = \nabla u + (\nabla u)^\top$ is the doubled rate-of-strain tensor; $(\nabla u)_{ij} = \partial_j u^i$; $T_M(H) = \mu(H\otimes H - \frac{1}{2}|H|^2\mathcal{I})$ is the magnetic stress tensor; $(H\otimes H)_{ij} = H^i H^j$; μ is the magnetic permeability. We assume $\nu = 1$ and $\mu = 1$ for simplicity.

The plasma surface is free to move and the presence of the kinematic viscosity leads to the following condition that must be satisfied on the surface, i.e.,

$$(T(u,q) + T_M(H))n = 0, \text{ on } \partial\Omega(t),$$

where $n = (n^1, n^2)^\top$ is the unit outer normal to $\partial\Omega(t)$.

For convenience, we denote the total pressure

$$p = q + \frac{1}{2}|H|^2.$$

The system can be rewritten as

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = H\cdot\nabla H, & \text{in } \Omega(t), \\ \partial_t H + u\cdot\nabla H = H\cdot\nabla u, & \text{in } \Omega(t), \\ \operatorname{div} u = 0, \quad \operatorname{div} H = 0, & \text{in } \Omega(t), \\ (-p\mathcal{I} + \nabla u + (\nabla u)^\top + H\otimes H)n = 0, & \text{on } \partial\Omega(t), \\ u(0,\cdot) = u_0, \quad H(0,\cdot) = H_0, & \text{in } \Omega_0, \end{cases} \tag{1.1}$$

where Ω_0, u_0 and H_0 are prescribed initial data which satisfy the compatibility conditions

$$\begin{cases} \operatorname{div} u_0 = 0, \quad \operatorname{div} H_0 = 0, & \text{in } \Omega_0, \\ n_0^\perp(\nabla u_0 + (\nabla u_0)^\top + H_0\otimes H_0)n_0 = 0, & \text{on } \partial\Omega_0, \end{cases} \tag{1.2}$$

where $n_0 = (n_0^1, n_0^2)^\top$ is the unit outer normal to $\partial\Omega_0$ and $n_0^\perp = (-n_0^2, n_0^1)$.

1.1. Background and the main result

The splash singularity (cf. [3,4] for the rigorous definition) for the two-dimensional inviscid water wave equations was studied by Castro *et al.* in [5]. They established some smooth initial data for which the smoothness of the interface breaks down in finite time, i.e., collapses in a splash singularity, by proving a local existence and a structural stability result in a transformed domain. Moreover, their stability result shows that a sufficiently small perturbation of initial data which leads to a splash singularity still generates a splash singularity. The idea of their proof is to adapt a conformal map to transform the equation from $\Omega(t)$ to a new domain $\tilde{\Omega}(t)$ and the transformed water wave equations behave similarly to the original equations. The existence of solutions in the transformed domain for times in $[t^* - \varepsilon, t^* + \varepsilon]$, with a given splash domain $\Omega(t^*)$ was proved by adapting the energy estimates in [1] and [6]. Then, they selected a suitable initial velocity to complete the proofs.

Apart from the splash singularity, the splat singularity defined in [3] is another singularity that can arise in the water wave equations, i.e., the solutions collapse along an arc in finite time but remain smooth otherwise. It turned out that there exist some smooth initial data for which the smoothness of the interface breaks down in finite time, resulting in a splash singularity or a splat singularity in [3].

In the presence of viscosity, the strategy for the inviscid case could not work because the equations failed to be solved backward in time, i.e., proving the existence of solutions in the transformed domain for times $\left[t^* - \varepsilon, t^*\right)$ is rather difficult. For this problem, Castro *et al.* utilized the transformation to $\tilde{\Omega}(t)$ in a new way and proved that there exist solutions to the viscous water wave equations that remain smooth for short time but form a splash singularity in finite time in [4]. For the three-dimensional case, Coutand and Shkoller have proved that the free-surface incompressible Euler equations with regular initial geometries and velocity fields satisfying the Taylor sign condition have solutions that form a finite-time splash or splat singularity in [7]. Their approach is based on approximating the self-intersecting domain Ω_s with a sequence of regular domains Ω_ε and applying the local well-posedness, which gives the time of existence (T_ε) independent of ε. Since the proof depends primarily on the local well-posedness, their method can be adapted to other time-reversible PDEs that have a local well-posedness theorem. For the two-dimensional or three-dimensional viscous water wave equations, given a sufficiently smooth initial boundary which is close to self-intersection, they designed a divergence-free initial velocity field to push the boundary towards self-intersection and proved that the free-surface will indeed self-intersect in finite time in [8].

The two-dimensional free boundary viscoelastic fluid model of the Oldroyd-B type at a high Weissenberg number was studied in [10] by applying the classical conformal mapping method and the existence of splash singularities was proved. It turned out that the action of the viscoelastic deformation does not prevent the existence of splash singularities.

In many important physical situations, magnetic fields are essential (cf. [9,21,22]), e.g., solar flares in astrophysics. For the incompressible inviscid MHD in bounded domains, Hao and Luo established a priori estimates for the free boundary problem in [13] under the Taylor-type sign condition, and the ill-posedness for the two-dimensional case in [14] when the Taylor-type sign condition is violated. Luo and Zhang obtained a priori estimates for the low regularity solutions in the case when the domain has small volume in [18]. A local existence was established in [12], for which the detailed proof is given in an initial flat domain of the form $\mathbb{T}^2 \times (0, 1)$, for a two-dimensional period box $\mathbb{T}^2$ in x_1 and x_2. With the same set-up of the initial domain, the local well-posedness is obtained in [11] by Gu, Luo and Zhang with surface tension. For the case where the magnetic field is zero on the free boundary and in vacuum, in the three-dimensional space with infinite and finite depth settings, Lee proved the local existence and uniqueness of the free boundary problem of incompressible viscous-diffusive MHD flow in [15], he also proved in [16] a local unique solution to the free boundary MHD without kinetic viscosity and magnetic diffusivity via zero kinetic viscosity-magnetic diffusivity limit.

To our knowledge, we are not aware of any previous mathematical research on the splash singularity for the MHD equations with free-surface. To prove the existence of splash singularity for the two-dimensional incompressible viscous MHD model, we reduce the original system to a system in conformal Lagrangian coordinates to form a fixed boundary that is not self-intersecting.

Our main result is stated in the following theorem, see Theorem 6.2 for details.

Theorem 1.1. *There exists a bounded domain Ω_0 such that for any divergence-free $H_0 \in H^k(\Omega_0)$ with the integer k large enough, there exists a solution (u, p, H) to viscous MHD equations (1.1) in $\big[0, t^*)$ for some $t^* > 0$ and the surface $\partial\Omega(t)$ self-intersects at time t^* in at least one point which creates a splash singularity.*

To prove the local existence by applying Lemma 4.1, we must separate the velocity and pressure from the flux and magnetic field. As a result, the coupling effect between the velocity and magnetic field might be hidden but cannot be avoided. In fact, the effect of coupling can be observed in the proof of the local existence as well as the stability estimates. For example, to apply Lemmas 4.2, 4.3 and some lemmas in the appendix, some estimates involving the magnetic fields necessitate particularly complicated splitting, even into more than twenty terms.

Unlike the viscoelastic fluid model in [10], where the authors have chosen $\mathcal{A}^{s+1,\gamma} \times \mathcal{K}^{s+1}_{(0)} \times \mathcal{K}^{s}_{pr(0)} \times \mathcal{A}^{s,\gamma-1}$ for $2 < s < \frac{5}{2}$ and $1 < \gamma < s-1$ to prove the existence of splash singularity, we choose $\mathcal{A}^{s+1,\gamma+1} \times \mathcal{K}^{s+1}_{(0)} \times \mathcal{K}^{s}_{pr(0)} \times \mathcal{A}^{s,\gamma}$ for the MHD equations. Certainly, s and γ have the same range as before. The main difference is that, compared to the viscoelastic fluid model, MHD equations cannot be estimated due to the low spatial regularity. Indeed, the estimates of the magnetic field involve some product of the iterative solutions, but the insufficient regularity of the functional space for spatial variables makes it impossible to use the bilinear inequalities as in the viscoelastic case. For example, in our case, $\|F_1F_2F_3F_4\|_{H^1_{(0)}H^{\gamma-1}}$ could not be controlled by

$$C\|F_1\|_{H^1_{(0)}H^{\gamma}}\|F_2\|_{H^1_{(0)}H^{\gamma-1}}\|F_3\|_{H^1_{(0)}H^{\gamma-1}}\|F_4\|_{H^1_{(0)}H^{\gamma-1}}$$

since $\gamma - 1 < \frac{1}{2}$. Because of this, we need to choose the indices of the functional spaces carefully. Let us explain why we chose these indices. On the one hand, we could not simply raise the regularities for all spatial variables since the spaces to which the velocity and pressure belong are space-time coupled Beale spaces. For example,

$$\mathcal{K}^{s}_{(0)}([0,T];\Omega) = L^2([0,T];H^s(\Omega)) \cap H^{\frac{s}{2}}_{(0)}([0,T];L^2(\Omega)),$$

and we note that s is related to both the time and space, which means the spaces

$$L^2([0,T];H^{s_1}(\Omega)) \cap H^{s_2}_{(0)}([0,T];L^2(\Omega)),$$

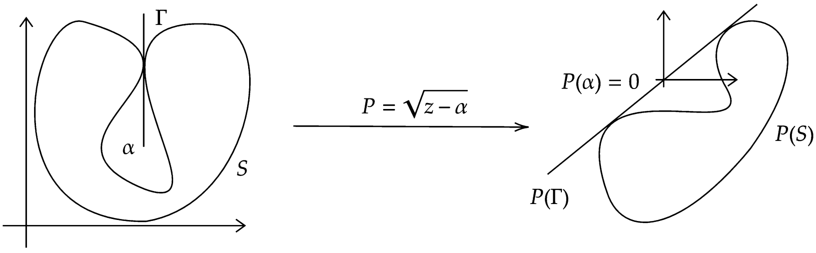

Fig. 1. Conformal mapping P.

for arbitrary s_1, s_2 are not appropriate. On the other hand, if we raise the regularities for all variables, the initial second-order derivatives of the solutions must be zero due to the definition of $H^3_{(0)}([0,T])$. Consequently, more intricate modifications are required and some key lemmas fail. Additionally, we attempted to establish the higher indices version of Lemma 4.1, but we found it difficult due to the failures to apply embedding theorems. Next, we observe that the index γ in the space

$$\mathcal{A}^{s,\gamma} = L^\infty_{\frac{1}{4}}([0,T]; H^s(\Omega)) \cap H^2_{(0)}([0,T]; H^\gamma(\Omega))$$

can be chosen suitably. In fact, we choose suitable spaces for the modified magnetic field and flux without changing the spaces for the velocity and pressure to keep some lemmas valid.

Furthermore, in order to acquire the necessary estimates, we must re-estimate in the current spaces. We prove the estimates of the flux (cf. Lemma 4.2) and the magnetic field (cf. Lemma 4.3) to control the iterative sequence and show that the sequence is Cauchy.

Moreover, to apply Lemma 4.1, we still need a modification of the velocity and pressure such that the modified initial velocity is zero. Due to the presence of the magnetic field, we extend the analysis for the choice of the approximated solution made in [4]. Finally, we show that the presence of the magnetic field does not prevent the fluid to form splash singularities either.

1.2. *Overview of the proof and the structure of the paper*

To prove the existence of splash singularities for two-dimensional viscous MHD, let us first recall the classical method of conformal mapping (see, for example [6]). This method has recently been utilized to solve this kind of problem in [4].

We introduce the map defined as an analytic branch of $\sqrt{z-\alpha}$:

$$P(z) = \sqrt{z-\alpha}, \qquad \text{for } z \in \mathbb{C} \setminus \Gamma,$$

where Γ is a branch cut, passed through the splash point (see Fig. 1). Also, we note that $P^{-1}(w) = w^2 + \alpha$ is an entire function.

The idea to prove our main theorem is to reduce system (1.1), in Eulerian coordinates, to a system in Lagrangian coordinates to form a fixed boundary, as in [2]. The second key observation regards the behavior of the magnetic field at the Lagrangian boundary. As a result, it shows that the way the viscokinematic deformation acts on the boundary does not prevent the natural tendency of the fluid to form splash singularities.

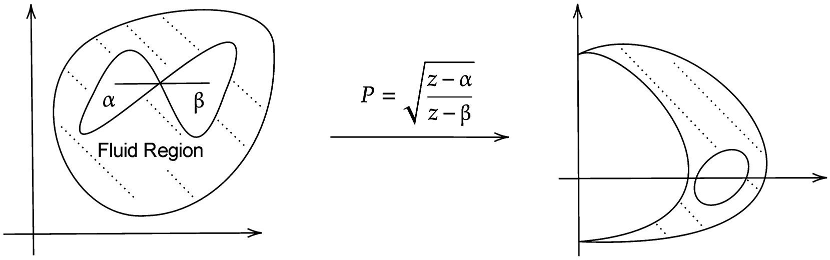

Fig. 2. Another type of splash scenario.

Remark 1.2. Another type of splash scenario is illustrated in Fig. 2. The proof can be easily adapted by replacing $P(z) = \sqrt{z-\alpha}$ by a branch of $\sqrt{\frac{z-\alpha}{z-\beta}}$ with suitable α, β and the branch cut.

From now on, we assume that $\alpha = 0$ for simplicity, then $P(z) = \sqrt{z}$ for $z \in \mathbb{C} \setminus \Gamma$.

Step 1. Changing to conformal Lagrangian coordinates.

Let the initial domain Ω_0 be a non-regular domain as in Fig. 3. Applying the conformal map P, we change coordinates from $\Omega(t)$ to $\tilde{\Omega}(t) := P(\Omega(t))$. In particular, the initial domain Ω_0 becomes $\tilde{\Omega}_0 := P(\Omega_0)$, which is a non-splash type domain. After that, we use the Lagrangian coordinates to transform the free boundary problem into a fixed boundary problem, i.e.,

$$\begin{cases} \partial_t \tilde{v} - Q^2(\tilde{X})\nabla(\nabla\tilde{v}\tilde{\zeta})\tilde{\zeta} + J(\tilde{X})^\top \tilde{\zeta}^\top \nabla \tilde{q} = \nabla \tilde{G}\tilde{\zeta} J(\tilde{X})\tilde{G}, & \text{in } \tilde{\Omega}_0, \\ \partial_t \tilde{G} = \nabla \tilde{v}\tilde{\zeta} J(\tilde{X})\tilde{G}, & \text{in } \tilde{\Omega}_0, \\ \mathrm{Tr}(\nabla \tilde{v}\tilde{\zeta} J(\tilde{X})) = 0, \quad \mathrm{Tr}(\nabla \tilde{G}\tilde{\zeta} J(\tilde{X})) = 0, & \text{in } \tilde{\Omega}_0, \\ (-\tilde{q}\mathcal{I} + (\nabla\tilde{v}\tilde{\zeta} J(\tilde{X}) + (\nabla\tilde{v}\tilde{\zeta} J(\tilde{X}))^\top) + \tilde{G}\tilde{G}^\top)(J(\tilde{X}))^{-1}\nabla_\Lambda \tilde{X}\tilde{n}_0 = 0, & \text{on } \partial\tilde{\Omega}_0, \\ \tilde{v}(0,\cdot) = \tilde{v}_0, \quad \tilde{G}(0,\cdot) = \tilde{G}_0, & \text{in } \tilde{\Omega}_0, \end{cases}$$

where J and ∇_Λ are defined in Section 3.

Then, we take a one-parameter family of initial data $\{(\tilde{\Omega}_\varepsilon(0), \tilde{v}'_{\varepsilon,0}, \tilde{G}'_{\varepsilon,0}) : \varepsilon > 0\}$ with $\tilde{\Omega}_\varepsilon(0) = \tilde{\Omega}_0 + \varepsilon b$ and $|b| = 1$, such that $P^{-1}(\partial\tilde{\Omega}_\varepsilon(0))$ is regular as in Fig. 6. Similarly, we change them to the conformal Lagrangian coordinates.

Step 2. Local existence of smooth solutions in conformal Lagrangian coordinates.

To this end, we define the iterative sequence $\{(\tilde{X}^{(n)}, \tilde{v}^{(n)}, \tilde{q}^{(n)}, \tilde{G}^{(n)})\}$ and show that the modified sequence $\{(\tilde{X}^{(n)} - \hat{X}, \tilde{v}^{(n)} - \phi, \tilde{q}^{(n)} - \tilde{q}_\phi, \tilde{G}^{(n)} - \hat{G})\}$ is Cauchy in $\mathcal{A}^{s+1,\gamma+1} \times \mathcal{K}^{s+1}_{(0)} \times \mathcal{K}^{s}_{pr(0)} \times \mathcal{A}^{s,\gamma}$, provided $T > 0$ small enough, where $\hat{X}, \phi, \tilde{q}_\phi, \hat{G}$ are defined in Section 4.

For this, we solve the linear system for the velocity and pressure by applying Lemma 4.1. This existence result is an adaption of [2, Theorem 4.3] and it is essential to establish the existence in conformal Lagrangian coordinates. Because of this, it is challenging to estimate the velocity and magnetic field simultaneously to verify the existence in conformal Lagrangian coordinates. As a consequence, we separate the velocity and pressure iterations from the flux and magnetic field iterations.

To apply Lemma 4.1, the velocity and pressure must be adjusted so that the modified terms belong to X_0 and the modified initial velocity vanishes. Accordingly, we extend the analysis for the selection of the approximated solution in [4, Section 5]. The approximated solution turns out to depend on the initial velocity and magnetic field.

Also, due to the coupling between the magnetic field and velocity, the estimates for some terms are exceedingly difficult. The coupling effect is reflected in the following aspects:

- The initial velocity and magnetic field determine the technical modifications of the velocity field and pressure by changing coordinates via the inverse of the conformal mapping.
- The product estimates, unlike the viscoelastic fluid model in [10], fail as described in Section 1.1. Therefore, we must re-estimate the flux and magnetic field (cf. Lemmas 4.2 and 4.3) in the functional spaces with new indices since the viscoelastic treatment is no longer applicable.
- The estimates of some magnetic-field-related terms are extremely complicated. These items require particularly fine expansion and tedious derivation. For example, in the proof of Proposition 4.8, **part 2**, we expand into 20 terms to estimate a portion of $\tilde{G}^{(n+1)} - \tilde{G}^{(n)}$ in $H^2_{(0)}H^\gamma$. Moreover, in the proof of Proposition 4.9, **part 2**, we expand $\tilde{h}^{(n)}_G - \tilde{h}^{(n-1)}_G$ into 32 terms via a fairly complex expansion to control $\|\tilde{h}^{(n)}_G - \tilde{h}^{(n-1)}_G\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}L^2}$.

Step 3. Stability estimates.

Recall that we take a one-parameter family of initial data $\{(\tilde{\Omega}_\varepsilon(0), \tilde{v}'_{\varepsilon,0}, \tilde{G}'_{\varepsilon,0}) : \varepsilon > 0\}$ by shifting the initial domain such that $P^{-1}(\tilde{\Omega}_\varepsilon(0))$ is regular. Then, we estimate the difference of the solutions and obtain

$$\mathrm{dist}(\partial\tilde{\Omega}(t), \partial\tilde{\Omega}_\varepsilon(t)) \leq C\varepsilon$$

for $t > 0$ small enough. The coupling effect between the velocity and magnetic field still exists. For example, in the proof of Proposition 5.3, we expand into 25 terms to estimate $\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{L^\infty_{\frac{1}{4}}H^s}$ and 24 terms to estimate $\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{H^2_{(0)}H^\gamma}$. These expansions also necessitate particularly precise analysis.

Step 4. Construction of the initial data and the existence of splash singularity.

We extend the analysis for the initial velocity choice already performed in [4, Section 7] since we are looking for some initial data such that the compatibility conditions (1.2) hold and the inner product of the velocity and the outer normal is positive as in Fig. 4. It follows that the evolution of the splash curve is demonstrated in Fig. 5.

As a result, combining the above considerations together, we demonstrate the existence of splash singularity as follows.

1) Let the initial domain Ω_0 be a non-regular domain as in Fig. 3. Applying the conformal map P, we change coordinates from $\Omega(t)$ to $\tilde{\Omega}(t) := P(\Omega(t))$. In particular, the initial domain Ω_0 becomes $\tilde{\Omega}_0 := P(\Omega_0)$, which is a non-splash type domain. Then, we use the Lagrangian coordinates to transform the free boundary problem into a fixed boundary problem.
2) For smooth initial data $(\tilde{\Omega}_0, \tilde{v}_0, \tilde{G}_0)$, from (local existence) Theorem 4.4 (Section 4), we obtain a solution $(\tilde{\Omega}(t), \tilde{v}(t,\cdot), \tilde{q}(t,\cdot), \tilde{G}(t,\cdot))$ for $t \in [0, T]$ and $T > 0$.
3) From the construction of the initial velocity (Section 6) as in Fig. 4 such that $\tilde{v}_0(\tilde{\omega}_j)\cdot\tilde{n}_0(\tilde{\omega}_j) > 0$ for $j = 1, 2$, it follows that $P^{-1}(\partial\tilde{\Omega}(\bar{t}))$ is self-intersecting for some $\bar{t} \in (0, T)$ small enough as in Fig. 5. Because this solution exists exclusively in the complex plane, it cannot be reversed by P^{-1} into a solution in $\Omega(t)$. Consequently, it is insufficient to prove the existence of a splash singularity.

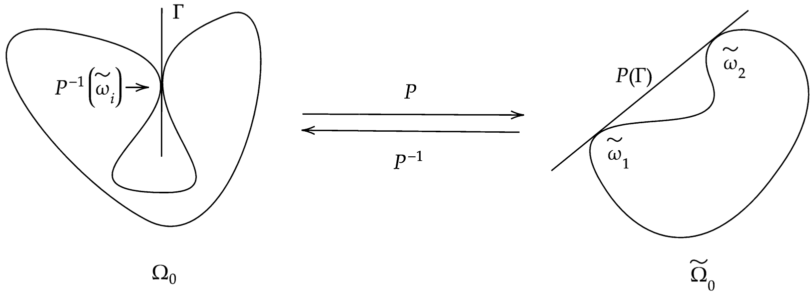

Fig. 3. Initial domains Ω_0 and $\tilde{\Omega}_0$.

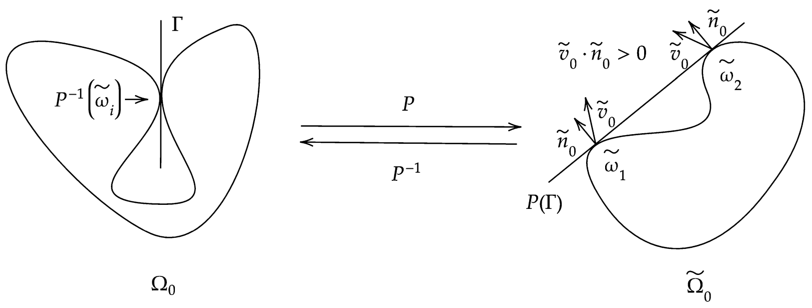

Fig. 4. A suitable choice of the initial velocity field.

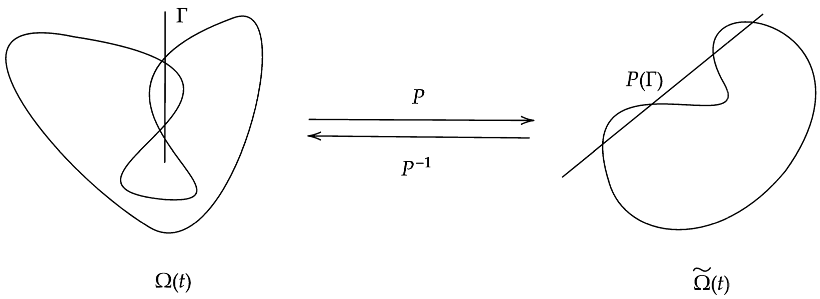

Fig. 5. Solutions in the complex domain.

4) To solve this problem, we take a one-parameter family of initial data $\{(\tilde{\Omega}_\varepsilon(0), \tilde{v}'_{\varepsilon,0}, \tilde{G}'_{\varepsilon,0}) : \varepsilon > 0\}$ with $\tilde{\Omega}_\varepsilon(0) = \tilde{\Omega}_0 + \varepsilon b$ and $|b| = 1$, such that $P^{-1}(\partial\tilde{\Omega}_\varepsilon(0))$ is regular as in Fig. 6. Similarly, we change to the conformal Lagrangian coordinates. Then, there exists a local-in-time smooth solution $(\tilde{\Omega}_\varepsilon(t), \tilde{v}'_\varepsilon(t,\cdot), \tilde{q}'_\varepsilon(t,\cdot), \tilde{G}'_\varepsilon(t,\cdot))$ in the complex plane. Therefore, by the inverse mapping, a local-in-time smooth solution $(\Omega_\varepsilon(t), u'_\varepsilon(t,\cdot), p'_\varepsilon(t,\cdot), H'_\varepsilon(t,\cdot))$ exists in the original domain.
5) For sufficiently small $\varepsilon > 0$, the stability result (Section 5) measures the difference between $(\tilde{\Omega}_\varepsilon(t), \tilde{v}'_\varepsilon(t,\cdot), \tilde{q}'_\varepsilon(t,\cdot), \tilde{G}'_\varepsilon(t,\cdot))$ and $(\tilde{\Omega}(t), \tilde{v}(t,\cdot), \tilde{q}(t,\cdot), \tilde{G}(t,\cdot))$ by shifting the solution $(\tilde{\Omega}_\varepsilon(t), \tilde{v}'_\varepsilon(t,\cdot), \tilde{q}'_\varepsilon(t,\cdot), \tilde{G}'_\varepsilon(t,\cdot))$ to $(\tilde{\Omega}_\varepsilon(t), \tilde{v}_\varepsilon(t,\cdot), \tilde{q}_\varepsilon(t,\cdot), \tilde{G}_\varepsilon(t,\cdot))$ and changing to Lagrangian coordinates. As a result, we obtain $\mathrm{dist}(\partial\tilde{\Omega}_\varepsilon(\bar{t}), \partial\tilde{\Omega}(\bar{t})) \le C\varepsilon$. It follows that $\mathrm{dist}(P^{-1}(\partial\tilde{\Omega}_\varepsilon(\bar{t})), P^{-1}(\partial\tilde{\Omega}(\bar{t}))) \le C\varepsilon$ and $P^{-1}(\partial\tilde{\Omega}_\varepsilon(\bar{t}))$ self-intersects as in Fig. 7.

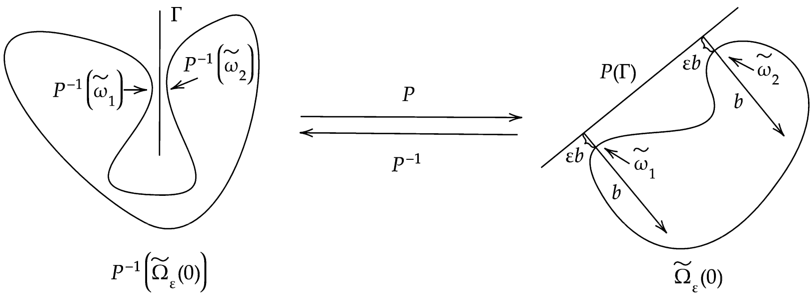

Fig. 6. The regular domain $P^{-1}(\partial\tilde{\Omega}_\varepsilon(0))$.

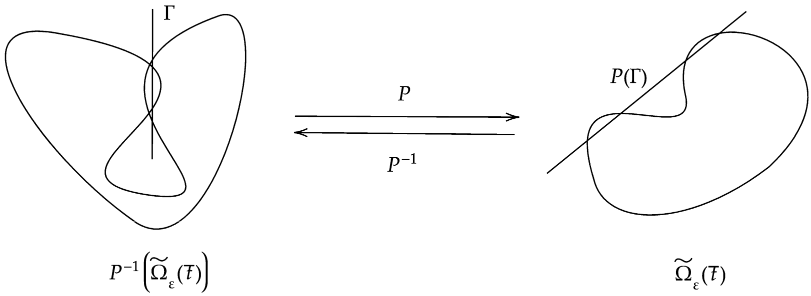

Fig. 7. Self-intersecting domain $P^{-1}(\tilde{\Omega}_\varepsilon(\bar{t}))$.

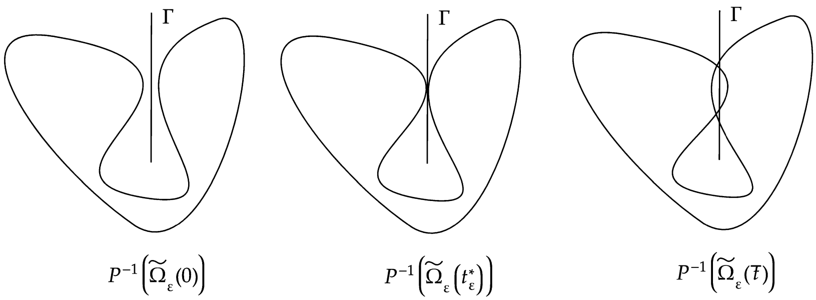

Fig. 8. The existence of a splash singularity.

6) From the regular domain $P^{-1}(\partial\tilde{\Omega}_\varepsilon(0))$ and self-intersecting domain $P^{-1}(\partial\tilde{\Omega}_\varepsilon(\bar{t}))$, there exists a time $t_\varepsilon^* \in (0, \bar{t})$ such that $P^{-1}(\partial\tilde{\Omega}_\varepsilon(t_\varepsilon^*))$ has a splash singularity as in Fig. 8.

The paper is organized as follows. In Section 2, we introduce the functional spaces for estimates. In Section 3, we define all the variables and deal with the transformations from $\Omega(t)$ into a fixed domain $\tilde{\Omega}_0$ by applying the conformal map and changing to the Lagrangian coordinates. In Section 4, we solve the viscous MHD model by constructing an iterative Cauchy sequence for $T > 0$ small enough. This local existence result in the functional spaces with new indices is one of the main ingredients for proving the existence of a splash singularity. In Section 5, we show the stability estimate which is another key ingredient. In Section 6, we construct a suitable initial velocity via the initial magnetic field and other initial data which will allow the splash singularity to occur. Finally, we obtain a finite-time splash singularity for the MHD model in the presence of the kinematic viscosity. In the appendix, we provide some estimates and key lemmas for readers to consult.

2. Beale spaces

For convenience, we introduce the Beale spaces and define the spaces that will be used.

As in the classical paper [2], the space $H^s_{(0)}([0,T])$ is defined by the interpolation between $L^2([0,T])$ and $H^1_{(0)}([0,T])$ for $0<s<1$ with the operator $S=1-\partial_t^2$ and domain $D(S)=\{u\in H^2([0,T]):u(0)=\partial_t u(T)=0\}$. The details can be found in [2].

The norm of $H^s_{(0)}([0,T])$ can be explicitly formulated. Note that the operator S has eigenvectors $\left\{\sin\left(\frac{(2n+1)\pi}{2T}t\right)\sqrt{\frac{2}{T}}\right\}_{n=0}^{\infty}$ with eigenvalues $\left\{1+\frac{(2n+1)^2\pi^2}{4T^2}\right\}_{n=0}^{\infty}$ which is an orthogonal basis of $L^2([0,T])$. Therefore, $H^s_{(0)}([0,T])$ consists of functions $u\in L^2([0,T])$ such that

$$\|u\|^2_{H^s_{(0)}}=\sum_{n=0}^{\infty}\left(u_n^{\sin}\right)^2\left(\frac{(2n+1)\pi}{2T}\right)^{2s}<\infty,$$

where

$$u_n^{\sin}:=\int_0^T u(t)\sin\left(\frac{(2n+1)\pi}{2T}t\right)\sqrt{\frac{2}{T}}dt.$$

For $s>\frac{1}{2}$, from $u\in H^s_{(0)}([0,T])$, we have $u(0)=0$.

For $m=1,2,3,\ldots$ and $0\le s<1$, the space $H^{m+s}_{(0)}([0,T])$ is regarded as the subspace of $\{u\in H^m([0,T]):(\partial_t^k u)(0)=0, k=0,\ldots,m-1\}$ with $\partial_t^m u\in H^s_{(0)}([0,T])$. We equip $H^{m+s}_{(0)}([0,T])$ with the norm for fractional derivatives in time

$$\|u\|^2_{H^{m+s}_{(0)}([0,T])}=\|u\|^2_{L^2([0,T])}+\|\partial_t u\|^2_{L^2([0,T])}+\ldots+\|\partial_t^m u\|^2_{H^s_{(0)}([0,T])}.$$

We also introduce the space $H^s([0,T])$ for $0<s<1$, which is defined as the interpolation of $L^2([0,T])$ and $H^1([0,T])$ with $S=1-\partial_t^2$ and domain $D(S)=\{u\in H^2([0,T]):(\partial_t u)(0)=(\partial_t u)(T)=0\}$. In this case, S has eigenvectors $\left\{\frac{1}{\sqrt{T}},\left\{\cos\left(\frac{n\pi}{T}t\right)\sqrt{\frac{2}{T}}\right\}_{n=1}^{\infty}\right\}$ with eigenvalues $\left\{1+\frac{n^2\pi^2}{T^2}\right\}_{n=0}^{\infty}$, which is also a basis of $L^2([0,T])$. Thus, we define

$$\|u\|^2_{H^s([0,T])}=\sum_{n=0}^{\infty}\left(1+\frac{n^2\pi^2}{T^2}\right)^s(u_n^{\cos})^2,$$

where

$$u_0^{\cos} := \int_0^T \frac{u(t)}{\sqrt{T}} dt, \quad u_n^{\cos} := \int_0^T u(t) \cos\left(\frac{n\pi}{T} t\right) \sqrt{\frac{2}{T}} dt, \quad n \ge 1.$$

For larger exponents, we regard $H^{m+s}([0,T]), m = 1, 2, 3, \ldots$, and $0 < s < 1$ as the subspace of $H^m([0,T])$ with $\partial_t^m u \in H^s([0,T])$. It happens that $H^{m+s}_{(0)}([0,T]) = \{u \in H^{m+s}([0,T]) : (\partial_t^k u)(0) = 0, k = 0, 1, \ldots, m\}$, for $s > \frac{1}{2}$ and $H^{m+s}_{(0)}([0,T]) = \{u \in H^{m+s}([0,T]) : (\partial_t^k u)(0) = 0, k = 0, 1, \ldots, m-1\}$, for $s < \frac{1}{2}$.

Now, we introduce the spaces that will be used to solve the free boundary MHD equations

$$\begin{aligned}
&\mathcal{K}^s_{(0)}([0,T];\Omega) := L^2([0,T]; H^s(\Omega)) \cap H^{\frac{s}{2}}_{(0)}([0,T]; L^2(\Omega)),\\
&\mathcal{K}^s_{pr(0)}([0,T];\Omega) := \big\{q \in L^\infty([0,T]; \dot{H}^1(\Omega)) : \nabla q \in \mathcal{K}^{s-1}_{(0)}([0,T];\Omega),\\
&\qquad\qquad q \in \mathcal{K}^{s-\frac{1}{2}}_{(0)}([0,T];\partial\Omega)\big\},\\
&\overline{\mathcal{K}}^s_{(0)}([0,T];\Omega) := L^2([0,T]; H^s(\Omega)) \cap H^{\frac{s+1}{2}}_{(0)}([0,T]; H^{-1}(\Omega)),\\
&\mathcal{A}^{s,\gamma}([0,T];\Omega) := L^\infty_{\frac{1}{4}}([0,T]; H^s(\Omega)) \cap H^2_{(0)}([0,T]; H^\gamma(\Omega)),
\end{aligned}$$

where $s - 1 - \varepsilon < \gamma < s - 1$ for some sufficiently small $\varepsilon > 0$ and

$$\|u\|_{L^\infty_{\frac{1}{4}} H^s} := \sup_{t \in [0,T]} t^{-\frac{1}{4}} \|u(t)\|_{H^s}.$$

These spaces are essential for applying embedding theorems and interpolation estimates to obtain the constants that are independent of time (see, for example [2,17]).

3. MHD system in conformal Lagrangian coordinates

In this section, we consider the conformal Lagrangian formulation of system (1.1).

For the two-dimensional free boundary incompressible viscous MHD model (1.1), Ω_0 is a domain in $\mathbb{R}^2$; $\Omega(t) = X(t, \Omega_0)$ moves according to the flux X associated to the velocity, which solves

$$\begin{cases} \dfrac{d}{dt} X(t,\omega) = u(t, X(t,\omega)), \\ X(0,\omega) = \omega, \quad \text{in } \Omega_0. \end{cases}$$

Remark 3.1. Let ω and $\tilde{\omega}$ be the typical points in Ω_0 and $\tilde{\Omega}_0$, respectively. Let α and β be the typical points in $\Omega(t)$ and $\tilde{\Omega}(t)$, respectively. $H = (H^1, H^2)^\top, u = (u^1, u^2)^\top, X = (X^1, X^2)^\top$ and $\nabla p = (\partial_1 p, \partial_2 p)^\top$ are column vectors. Moreover, we define

$$\nabla u = \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 \\ \partial_1 u^2 & \partial_2 u^2 \end{pmatrix},$$

and $\nabla u^i = (\partial_1 u^i, \partial_2 u^i)$ is a row vector.

First, we change the coordinates from $\Omega(t)$ to $\tilde{\Omega}(t) = P(\Omega(t))$ by applying the conformal map P. To this end, we define the transformed velocity field, pressure and magnetic field

$$\begin{cases} \tilde{u}(t,\beta) := u(t, P^{-1}(\beta)), \\ \tilde{p}(t,\beta) := p(t, P^{-1}(\beta)), \\ \tilde{H}(t,\beta) := H(t, P^{-1}(\beta)), \end{cases}$$

and hence

$$\begin{cases} \tilde{u}(t, P(\alpha)) = u(t,\alpha), \\ \tilde{p}(t, P(\alpha)) = p(t,\alpha), \\ \tilde{H}(t, P(\alpha)) = H(t,\alpha). \end{cases}$$

Also, we define

$$J_{kj}(\cdot) := (\partial_j P^k) \circ P^{-1}(\cdot).$$

Then, for the derivatives of the transformed velocity field, it follows that

$$\sum_{k=1}^{2} J_{kj}(\beta)(\partial_k \tilde{u}^i)(t,\beta) = (\partial_j u^i)(t, P^{-1}(\beta)),$$

i.e.,

$$\sum_{k=1}^{2} J_{kj}\partial_k \tilde{u}^i = (\partial_j u^i) \circ P^{-1}.$$

For the transformation in conformal coordinates, we have the following lemma:

Lemma 3.2. *Let $P = \sqrt{z}$ be the conformal map defined in Section 1.2, $J = (J_{ij})$ and $Q^2(\cdot) = \left|\frac{dP}{dz} \circ P^{-1}(\cdot)\right|^2$. Under this conformal transformation, system (1.1) becomes*

$$\begin{cases} \partial_t \tilde{u} + \nabla\tilde{u} J \tilde{u} - Q^2 \Delta\tilde{u} + J^\top \nabla\tilde{p} = \nabla\tilde{H} J \tilde{H}, & in\ \tilde{\Omega}(t), \\ \partial_t \tilde{H} + \nabla\tilde{H} J \tilde{u} = \nabla\tilde{u} J \tilde{H}, & in\ \tilde{\Omega}(t), \\ \mathrm{Tr}(\nabla\tilde{u} J) = 0, \quad \mathrm{Tr}(\nabla\tilde{H} J) = 0, & in\ \tilde{\Omega}(t), \\ (-\tilde{p}\mathcal{I} + (\nabla\tilde{u} J + (\nabla\tilde{u} J)^\top) + \tilde{H}\tilde{H}^\top) J^{-1}\tilde{n} = 0, & on\ \partial\tilde{\Omega}(t), \\ \tilde{u}(0,\cdot) = \tilde{u}_0, \quad \tilde{H}(0,\cdot) = \tilde{H}_0, & in\ \tilde{\Omega}_0, \end{cases} \tag{3.1}$$

or in components

$$\begin{cases} \partial_t \tilde{u}^i + \sum_{k,j=1}^{2} \partial_k \tilde{u}^i J_{kj} \tilde{u}^j - Q^2 \Delta \tilde{u}^i + \sum_{k=1}^{2} \partial_k \tilde{p} J_{ki} = \sum_{k,j=1}^{2} \partial_k \tilde{H}^i J_{kj} \tilde{H}^j, & in\ \tilde{\Omega}(t), \\ \partial_t \tilde{H}^i + \sum_{k,j=1}^{2} \partial_k \tilde{H}^i J_{kj} \tilde{u}^j = \sum_{k,j=1}^{2} \partial_k \tilde{u}^i J_{kj} \tilde{H}^j, & in\ \tilde{\Omega}(t), \\ \sum_{k,i=1}^{2} \partial_k \tilde{u}^i J_{ki} = 0, \quad \sum_{k,i=1}^{2} \partial_k \tilde{H}^i J_{ki} = 0, & in\ \tilde{\Omega}(t), \\ \left(-\tilde{p}\mathcal{I} + (\nabla \tilde{u} J + (\nabla \tilde{u} J)^\top) + \tilde{H}\tilde{H}^\top \right) J^{-1} \tilde{n} = 0, & on\ \partial\tilde{\Omega}(t), \\ \tilde{u}(0,\cdot) = \tilde{u}_0, \quad \tilde{H}(0,\cdot) = \tilde{H}_0, & in\ \tilde{\Omega}_0, \end{cases}$$

where $\tilde{n} = -\Lambda J|_{\partial\tilde{\Omega}(t)} \Lambda n$ *and* $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. Note that the conformal map $P = P_1 + iP_2$ satisfies the Cauchy-Riemann equations

$$\begin{cases} \partial_1 P_1 = \partial_2 P_2, \\ \partial_2 P_1 = -\partial_1 P_2. \end{cases}$$

Straightforward calculations give

$$u \cdot \nabla H \to \nabla \tilde{H} J \tilde{u}, \quad \nabla p \to J^\top \nabla \tilde{p}, \quad \Delta u \to Q^2 \Delta \tilde{u}.$$

The other terms can be calculated similarly. □

Next, to handle the free boundary, a traditional way is to consider it in the Lagrangian coordinates. For $\tilde{\omega} \in \tilde{\Omega}_0$, we define

$$\begin{cases} \frac{d}{dt}\tilde{X}(t,\tilde{\omega}) = J(\tilde{X}(t,\tilde{\omega}))\tilde{u}(t,\tilde{X}(t,\tilde{\omega})), & \text{in } \tilde{\Omega}(t), \\ \tilde{X}(0,\tilde{\omega}) = \tilde{\omega}, & \text{in } \tilde{\Omega}(0), \end{cases}$$

i.e.,

$$\begin{cases} \frac{d}{dt}\tilde{X}^k(t,\tilde{\omega}) = J_{kj}(\tilde{X}(t,\tilde{\omega}))\tilde{u}^j(t,\tilde{X}(t,\tilde{\omega})), & \text{in } \tilde{\Omega}(t), \\ \tilde{X}^k(0,\tilde{\omega}) = \tilde{\omega}^k, & \text{in } \tilde{\Omega}(0). \end{cases}$$

Then, we define the Lagrangian variables

$$\begin{cases} \tilde{v}(t,\tilde{\omega}) := \tilde{u}(t,\tilde{X}(t,\tilde{\omega})), \\ \tilde{q}(t,\tilde{\omega}) := \tilde{p}(t,\tilde{X}(t,\tilde{\omega})), \\ \tilde{G}(t,\tilde{\omega}) := \tilde{H}(t,\tilde{X}(t,\tilde{\omega})), \end{cases} \tag{3.2}$$

and

$$\begin{pmatrix} \tilde{\zeta}_{11} & \tilde{\zeta}_{12} \\ \tilde{\zeta}_{21} & \tilde{\zeta}_{22} \end{pmatrix}(t,\tilde{\omega}) := \begin{pmatrix} \partial_1 \tilde{X}^1 & \partial_2 \tilde{X}^1 \\ \partial_1 \tilde{X}^2 & \partial_2 \tilde{X}^2 \end{pmatrix}^{-1}(t,\tilde{\omega}). \tag{3.3}$$

Changing to Lagrangian coordinates, we have the following lemma:

Lemma 3.3. *In Lagrangian coordinates, system* (3.1) *becomes*

$$\begin{cases} \partial_t \tilde{v}^i - Q^2(\tilde{X}) \displaystyle\sum_{j,k,m=1}^{2} \partial_m(\partial_k \tilde{v}^i \tilde{\zeta}_{kj})\tilde{\zeta}_{mj} + \sum_{k,l=1}^{2} \partial_l \tilde{q} \tilde{\zeta}_{lk} J_{ki}(\tilde{X}) \\ \quad = \displaystyle\sum_{j,k,l=1}^{2} \partial_l \tilde{G}^i \tilde{\zeta}_{lk} J_{kj}(\tilde{X}) \tilde{G}^j, & \text{in } \tilde{\Omega}_0, \\ \partial_t \tilde{G}^i = \displaystyle\sum_{j,k,l=1}^{2} \partial_j \tilde{v}^i \tilde{\zeta}_{jk} J_{kl}(\tilde{X}) \tilde{G}^l, & \text{in } \tilde{\Omega}_0, \\ \mathrm{Tr}(\nabla \tilde{v} \tilde{\zeta} J(\tilde{X})) = 0, \quad \mathrm{Tr}(\nabla \tilde{G} \tilde{\zeta} J(\tilde{X})) = 0, & \text{in } \tilde{\Omega}_0, \\ (-\tilde{q}\mathcal{I} + (\nabla \tilde{v} \tilde{\zeta} J(\tilde{X}) + (\nabla \tilde{v} \tilde{\zeta} J(\tilde{X}))^\top) + \tilde{G}\tilde{G}^\top)(J(\tilde{X}))^{-1} \nabla_\Lambda \tilde{X} \tilde{n}_0 = 0, & \text{on } \partial\tilde{\Omega}_0, \\ \tilde{v}(0,\cdot) = \tilde{v}_0, \quad \tilde{G}(0,\cdot) = \tilde{G}_0, & \text{in } \tilde{\Omega}_0, \end{cases} \tag{3.4}$$

where $\nabla_\Lambda \tilde{X} = -\Lambda \nabla \tilde{X} \Lambda$.

Proof. From (3.2) and (3.3), we obtain system (3.4) in Lagrangian coordinates from standard calculations. □

With a slight abuse of notation, we define

$$(\nabla(\nabla \tilde{v} \tilde{\zeta})\tilde{\zeta})^i := \sum_{j,k,m=1}^{2} \partial_m(\partial_k \tilde{v}^i \tilde{\zeta}_{kj})\tilde{\zeta}_{mj}.$$

Note that the notation $(\nabla(\nabla \tilde{v} \tilde{\zeta})\tilde{\zeta})^i$ does not represent the matrix multiplication. It involves the first-order derivatives of $\tilde{\zeta}$, $\tilde{v}$ and the second-order derivatives of $\tilde{v}$, which is sufficient for us to calculate.

Finally, system (3.4) becomes

$$\begin{cases} \partial_t \tilde{v} - Q^2(\tilde{X}) \nabla(\nabla \tilde{v} \tilde{\zeta})\tilde{\zeta} + J(\tilde{X})^\top \tilde{\zeta}^\top \nabla \tilde{q} = \nabla \tilde{G} \tilde{\zeta} J(\tilde{X}) \tilde{G}, & \text{in } \tilde{\Omega}_0, \\ \partial_t \tilde{G} = \nabla \tilde{v} \tilde{\zeta} J(\tilde{X}) \tilde{G}, & \text{in } \tilde{\Omega}_0, \\ \mathrm{Tr}(\nabla \tilde{v} \tilde{\zeta} J(\tilde{X})) = 0, \quad \mathrm{Tr}(\nabla \tilde{G} \tilde{\zeta} J(\tilde{X})) = 0, & \text{in } \tilde{\Omega}_0, \\ (-\tilde{q}\mathcal{I} + (\nabla \tilde{v} \tilde{\zeta} J(\tilde{X}) + (\nabla \tilde{v} \tilde{\zeta} J(\tilde{X}))^\top) + \tilde{G}\tilde{G}^\top)(J(\tilde{X}))^{-1} \nabla_\Lambda \tilde{X} \tilde{n}_0 = 0, & \text{on } \partial\tilde{\Omega}_0, \\ \tilde{v}(0,\cdot) = \tilde{v}_0, \quad \tilde{G}(0,\cdot) = \tilde{G}_0, & \text{in } \tilde{\Omega}_0. \end{cases} \tag{3.5}$$

4. Local existence of smooth solutions to system (3.5)

In this section, we prove local existence of smooth solutions to system (3.5).

We will define the iterative sequence

$$\{(\tilde{X}^{(n)}, \tilde{v}^{(n)}, \tilde{q}^{(n)}, \tilde{G}^{(n)}) : n = 0, 1, \cdots\}$$

and show that

$$\{(\tilde{X}^{(n)} - \hat{X}, \tilde{v}^{(n)} - \phi, \tilde{q}^{(n)} - \tilde{q}_\phi, \tilde{G}^{(n)} - \hat{G}) : n = 0, 1, \cdots\}$$

is Cauchy in suitable spaces, provided $T > 0$ small enough. Here we have introduced

$$\hat{X} = \tilde{\omega} + tJ\tilde{v}_0, \quad \hat{G} = \tilde{G}_0 + t\nabla\tilde{v}_0 J\tilde{G}_0,$$

and the construction of $(\phi, \tilde{q}_\phi)$ will be clear in Section 4.1. Let $n \geq 0$. We separate the iteration for $(\tilde{v}, \tilde{q})$ from $\tilde{X}$ and $\tilde{G}$, i.e.,

$$\begin{cases} \partial_t \tilde{v}^{(n+1)} - Q^2\Delta\tilde{v}^{(n+1)} + J^\top\nabla\tilde{q}^{(n+1)} = \tilde{f}^{(n)}, & \text{in } (0,T)\times\tilde{\Omega}_0, \\ \mathrm{Tr}(\nabla\tilde{v}^{(n+1)}J) = \tilde{g}^{(n)}, & \text{in } (0,T)\times\tilde{\Omega}_0, \\ [-\tilde{q}^{(n+1)}\mathcal{I} + ((\nabla\tilde{v}^{(n+1)}J) + (\nabla\tilde{v}^{(n+1)}J)^\top)]J^{-1}\tilde{n}_0 = \tilde{h}^{(n)}, & \text{on } (0,T)\times\partial\tilde{\Omega}_0, \\ \tilde{v}(0,\tilde{\omega}) = \tilde{v}_0(\tilde{\omega}), & \text{in } \tilde{\Omega}_0, \end{cases} \tag{4.1}$$

where $\tilde{f}^{(n)}$, $\tilde{g}^{(n)}$ and $\tilde{h}^{(n)}$ collect all the nonlinear terms at the n-th step, i.e.,

$$\begin{aligned} \tilde{f}^{(n)} =& -Q^2\Delta\tilde{v}^{(n)} + J^\top\nabla\tilde{q}^{(n)} + Q^2(\tilde{X}^{(n)})\nabla(\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)})\tilde{\zeta}^{(n)} \\ & - J(\tilde{X}^{(n)})^\top\tilde{\zeta}^{(n)\top}\nabla\tilde{q}^{(n)} + \nabla\tilde{G}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}, \end{aligned} \tag{4.2}$$

$$\tilde{g}^{(n)} = \mathrm{Tr}(\nabla\tilde{v}^{(n)}J) - \mathrm{Tr}(\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})), \tag{4.3}$$

$$\begin{aligned} \tilde{h}^{(n)} =& -\tilde{q}^{(n)}J^{-1}\tilde{n}_0 + \tilde{q}^{(n)}(J(\tilde{X}^{(n)}))^{-1}\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0 \\ & + ((\nabla\tilde{v}^{(n)}J) + (\nabla\tilde{v}^{(n)}J)^\top)J^{-1}\tilde{n}_0 \\ & - ((\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})) + (\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)}))^\top)J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0 \\ & - \tilde{G}^{(n)}\tilde{G}^{(n)\top}J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0. \end{aligned} \tag{4.4}$$

The magnetic field $\tilde{G}^{(n+1)}$ solves

$$\begin{cases} \partial_t\tilde{G}^{(n+1)}(t,\tilde{\omega}) = \nabla\tilde{v}^{(n)}(t,\tilde{\omega})\tilde{\zeta}^{(n)}(\tilde{\omega})J(\tilde{X}^{(n)}(t,\tilde{\omega}))\tilde{G}^{(n)}(t,\tilde{\omega}), \\ \tilde{G}(0,\tilde{\omega}) = \tilde{G}_0(\tilde{\omega}), \quad \text{in } \tilde{\Omega}_0. \end{cases} \tag{4.5}$$

Finally, we define

$$\begin{cases} \dfrac{d}{dt}\tilde{X}^{(n+1)}(t,\tilde{\omega}) = J(\tilde{X}^{(n)}(t,\tilde{\omega}))\tilde{v}^{(n)}(t,\tilde{\omega}), \\ \tilde{X}(0,\tilde{\omega}) = \tilde{\omega}, \quad \text{in } \tilde{\Omega}_0. \end{cases} \tag{4.6}$$

We will study (4.1), (4.5) and (4.6) separately. For linear system (4.1), we consider the following system with initial data $\tilde{v}(0,\tilde{\omega}) = 0$:

$$\begin{cases} \partial_t \tilde{v} - Q^2 \Delta \tilde{v} + J^\top \nabla \tilde{q} = \tilde{f}, & \text{in } (0,T)\times\tilde{\Omega}_0, \\ \mathrm{Tr}(\nabla \tilde{v} J) = \tilde{g}, & \text{in } (0,T)\times\tilde{\Omega}_0, \\ [-\tilde{q}\mathcal{I} + ((\nabla \tilde{v} J) + (\nabla \tilde{v} J)^\top)] J^{-1}\tilde{n}_0 = \tilde{h}, & \text{on } (0,T)\times\partial\tilde{\Omega}_0, \\ \tilde{v}(0,\tilde{\omega}) = 0, & \text{in } \tilde{\Omega}_0, \end{cases} \tag{4.7}$$

with the compatibility conditions

$$\begin{cases} \mathrm{Tr}(\nabla \tilde{v}_0 J) = \tilde{g}(0), & \text{in } \tilde{\Omega}_0, \\ (J^{-1}\tilde{n}_0)^\perp [((\nabla \tilde{v} J) + (\nabla \tilde{v} J)^\top)] J^{-1}\tilde{n}_0 = \tilde{h}(0)(J^{-1}\tilde{n}_0)^\perp, & \text{on } \partial\tilde{\Omega}_0. \end{cases} \tag{4.8}$$

Define

$$X_0 := \{(\tilde{v},\tilde{q}) \in \mathcal{K}^{s+1}_{(0)}([0,T];\tilde{\Omega}_0) \times \mathcal{K}^{s}_{pr(0)}([0,T];\tilde{\Omega}_0)\},$$

$$Y_0 := \{(\tilde{f},\tilde{g},\tilde{h}) \in \mathcal{K}^{s-1}_{(0)}([0,T];\tilde{\Omega}_0) \times \bar{\mathcal{K}}^{s}_{(0)}([0,T];\tilde{\Omega}_0) \times \mathcal{K}^{s-\frac{1}{2}}_{(0)}([0,T];\partial\tilde{\Omega}_0) : \text{(4.8) are satisfied}\},$$

for the solutions $(\tilde{v},\tilde{q})$ and data $(\tilde{f},\tilde{g},\tilde{h})$, respectively. Also, we define a linear operator $L : X_0 \to Y_0$, related to system (4.7)

$$L(\tilde{v},\tilde{q}) = (\tilde{f},\tilde{g},\tilde{h}). \tag{4.9}$$

From [4, Theorem 4.1], we have the following lemma:

Lemma 4.1. *For system (4.7), the operator L defined in (4.9) is invertible for* $2 < s < \frac{5}{2}$. *Moreover,* $\|L^{-1}\|$ *is bounded uniformly if T is bounded above and the following estimate holds*

$$\|(\tilde{v},\tilde{q})\|_{X_0} \le C\|(\tilde{f},\tilde{g},\tilde{h})\|_{Y_0}.$$

4.1. Technical modifications of the velocity field and pressure

To apply Lemma 4.1, we modify the velocity field and pressure such that the modified terms belong to X_0 and the time derivative of the modified initial velocity field vanishes. For this purpose, we extend the construction of the approximated solution in [4, Section 5].

For our problem, we need to choose ϕ such that

$$\tilde{v}^{(n)}(0) = \tilde{v}_0 = \phi(0), \quad (\partial_t \tilde{v}^{(n)})(0) = (\partial_t \phi)(0),$$

for all $n \ge 0$. Note that, if $t = 0$, the first equation in (3.5) reads

$$(\partial_t \tilde{v})(0) - Q^2 \Delta \tilde{v}_0 + J^\top \nabla \tilde{q}(0) = \nabla \tilde{G}_0 J \tilde{G}_0, \quad \text{in } \tilde{\Omega}_0.$$

Therefore, ϕ must satisfy

$$(\partial_t \phi)(0) = Q^2 \Delta \tilde{v}_0 - J^\top \nabla \tilde{q}(0) + \nabla \tilde{G}_0 J \tilde{G}_0, \quad \phi(0) = \tilde{v}_0,$$

and a reasonable choice is

$$\phi(t) = \tilde{v}_0 + t(Q^2\Delta\tilde{v}_0 - J^\top\nabla\tilde{q}(0) + \nabla\tilde{G}_0 J\tilde{G}_0).$$

To specify $\nabla\tilde{q}(0)$, we study the iterative system for the velocity and pressure in $\Omega_0 = P^{-1}(\tilde{\Omega}_0)$ through the inverse of the conformal mapping P.

More precisely, we define

$$\begin{cases} v^{(n)}(t, P^{-1}(\tilde{\omega})) := \tilde{v}^{(n)}(t, \tilde{\omega}), \\ q^{(n)}(t, P^{-1}(\tilde{\omega})) := \tilde{q}^{(n)}(t, \tilde{\omega}), \\ f^{(n)}(t, P^{-1}(\tilde{\omega})) := \tilde{f}^{(n)}(t, \tilde{\omega}), \\ g^{(n)}(t, P^{-1}(\tilde{\omega})) := \tilde{g}^{(n)}(t, \tilde{\omega}), \\ h^{(n)}(t, P^{-1}(\tilde{\omega})) := \tilde{h}^{(n)}(t, \tilde{\omega}). \end{cases}$$

Then, system (4.1) becomes

$$\begin{cases} \partial_t v^{(n+1)} - \Delta v^{(n+1)} + \nabla q^{(n+1)} = f^{(n)}, & \text{in } (0,T)\times\Omega_0, \\ \operatorname{div} v^{(n+1)} = g^{(n)}, & \text{in } (0,T)\times\Omega_0, \\ [-q^{(n+1)}\mathcal{I} + (\nabla v^{(n+1)} + (\nabla v^{(n+1)})^\top)]n_0 = h^{(n)}, & \text{on } (0,T)\times\partial\Omega_0, \\ v(0,\omega) = v_0(\omega), & \text{in } \Omega_0. \end{cases}$$

A straightforward calculation gives

$$\partial_t g^{(n)} = \partial_t \operatorname{div} v^{(n+1)} = \Delta g^{(n)} - \Delta q^{(n+1)} + \operatorname{div} f^{(n)}.$$

Setting $t = 0$, it follows that

$$\begin{cases} -\Delta q^{(n+1)}(0) = (\partial_t g^{(n)})(0) - \Delta g^{(n)}(0) - \operatorname{div} f^{(n)}(0), & \text{in } \Omega_0, \\ q^{(n+1)}(0)n_0 = (\nabla v_0 + (\nabla v_0)^\top)n_0 - h^{(n)}n_0, & \text{on } \partial\Omega_0. \end{cases} \tag{4.10}$$

Hence, from $G_0 = H_0$, $\operatorname{div} H_0 = 0$, (4.2), (4.3) and (4.4), we conclude that

$$\begin{cases} f^{(n)}(0) = \nabla G_0 G_0, \\ g^{(n)}(0) = 0, \\ h^{(n)}(0) = -G_0 G_0^\top n_0, \\ \operatorname{div} f^{(n)}(0) = \operatorname{Tr}(\nabla G_0 \nabla G_0). \end{cases}$$

Observe from (3.3) that $\partial_t\tilde{\zeta}^{(n)} = -\tilde{\zeta}^{(n)}\nabla\partial_t\tilde{X}^{(n)}\tilde{\zeta}^{(n)}$. This, together with (4.3), implies that

$$(\partial_t\tilde{g}^{(n)})(0) = \operatorname{Tr}(\nabla\tilde{v}_0 J\nabla\tilde{v}_0 J).$$

As a result, we obtain

$$(\partial_t g^{(n)})(0) = \operatorname{Tr}(\nabla v_0 \nabla v_0),$$

and we conclude that $q^{(n+1)}(0)$ solves

$$\begin{cases} -\Delta q^{(n+1)}(0) = \mathrm{Tr}(\nabla v_0 \nabla v_0) - \mathrm{Tr}(\nabla G_0 \nabla G_0), & \text{in } \Omega_0, \\ q^{(n+1)}(0) n_0 = (\nabla v_0 + \nabla v_0^\top + G_0 G_0^\top) n_0, & \text{on } \partial\Omega_0, \end{cases}$$

and is independent of n.

Back to the domain $\tilde{\Omega}_0$, for all n, we have

$$\begin{cases} -Q^2 \Delta \tilde{q}^{(n+1)}(0) = \mathrm{Tr}(\nabla \tilde{v}_0 J \nabla \tilde{v}_0 J) - \mathrm{Tr}(\nabla \tilde{G}_0 J \nabla \tilde{G}_0 J), & \text{in } \tilde{\Omega}_0, \\ \tilde{q}^{(n+1)}(0) J^{-1} \tilde{n}_0 = (\nabla \tilde{v}_0 J + (\nabla \tilde{v}_0 J)^\top + \tilde{G}_0 \tilde{G}_0^\top) J^{-1} \tilde{n}_0, & \text{on } \partial\tilde{\Omega}_0. \end{cases}$$

Therefore,

$$\phi := \tilde{v}_0 + t\hat{\phi} := \tilde{v}_0 + t(Q^2 \Delta \tilde{v}_0 - J^\top \nabla \tilde{q}_\phi + \nabla \tilde{G}_0 J \tilde{G}_0),$$

and $\tilde{q}_\phi$ solving

$$\begin{cases} -Q^2 \Delta \tilde{q}_\phi = \mathrm{Tr}(\nabla \tilde{v}_0 J \nabla \tilde{v}_0 J) - \mathrm{Tr}(\nabla \tilde{G}_0 J \nabla \tilde{G}_0 J), & \text{in } \tilde{\Omega}_0, \\ \tilde{q}_\phi J^{-1} \tilde{n}_0 = (\nabla \tilde{v}_0 J + (\nabla \tilde{v}_0 J)^\top + \tilde{G}_0 \tilde{G}_0^\top) J^{-1} \tilde{n}_0, & \text{on } \partial\tilde{\Omega}_0, \end{cases}$$

are well-defined.

Once we obtain ϕ and $\tilde{q}_\phi$, we define the modified velocity field $\tilde{w}^{(n)}$ and the modified pressure $\tilde{q}_w^{(n)}$ by

$$\begin{cases} \tilde{w}^{(n)} := \tilde{v}^{(n)} - \phi, \\ \tilde{q}_w^{(n)} := \tilde{q}^{(n)} - \tilde{q}_\phi. \end{cases}$$

In terms of $\tilde{w}^{(n)}$ and $\tilde{q}_w^{(n)}$, system (4.1) can be rewritten as

$$\begin{cases} \partial_t \tilde{w}^{(n+1)} - Q^2 \Delta \tilde{w}^{(n+1)} + J^\top \nabla \tilde{q}_w^{(n+1)} \\ \quad = \tilde{f}^{(n)} - \partial_t \phi + Q^2 \Delta \phi - J^\top \nabla \tilde{q}_\phi, & \text{in } (0,T) \times \tilde{\Omega}_0, \\ \mathrm{Tr}(\nabla \tilde{w}^{(n+1)} J) = \tilde{g}^{(n)} - \mathrm{Tr}(\nabla \phi J), & \text{in } (0,T) \times \tilde{\Omega}_0, \\ [-\tilde{q}_w^{(n+1)} \mathcal{I} + ((\nabla \tilde{w}^{(n+1)} J) + (\nabla \tilde{w}^{(n+1)} J)^\top)] J^{-1} \tilde{n}_0 \\ \quad = \tilde{h}^{(n)} + \tilde{q}_\phi J^{-1} \tilde{n}_0 - ((\nabla \phi J) + (\nabla \phi J)^\top) J^{-1} \tilde{n}_0, & \text{on } (0,T) \times \partial\tilde{\Omega}_0, \\ \tilde{w}^{(n+1)}(0,\cdot) = 0, & \text{in } \tilde{\Omega}_0, \end{cases} \tag{4.11}$$

where

$$\begin{aligned} \tilde{f}^{(n)} = & -Q^2 \Delta (\tilde{w}^{(n)} + \phi) + J^\top \nabla (\tilde{q}_w^{(n)} + \tilde{q}_\phi) + Q^2 (\tilde{X}^{(n)}) \nabla (\nabla (\tilde{w}^{(n)} + \phi) \tilde{\zeta}^{(n)}) \tilde{\zeta}^{(n)} \\ & - J (\tilde{X}^{(n)})^\top \tilde{\zeta}^{(n)\top} \nabla (\tilde{q}_w^{(n)} + \tilde{q}_\phi) + \nabla \tilde{G}^{(n)} \tilde{\zeta}^{(n)} J (\tilde{X}^{(n)}) \tilde{G}^{(n)}, \\ \tilde{g}^{(n)} = & \ \mathrm{Tr}(\nabla (\tilde{w}^{(n)} + \phi) J) - \mathrm{Tr}(\nabla (\tilde{w}^{(n)} + \phi) \tilde{\zeta}^{(n)} J (\tilde{X}^{(n)})), \\ \tilde{h}^{(n)} = & -(\tilde{q}_w^{(n)} + \tilde{q}_\phi) J^{-1} \tilde{n}_0 + (\tilde{q}_w^{(n)} + \tilde{q}_\phi) (J(\tilde{X}^{(n)}))^{-1} \nabla_\Lambda \tilde{X}^{(n)} \tilde{n}_0 \end{aligned}$$

$$
\begin{aligned}
&+ ((\nabla(\tilde{w}^{(n)}+\phi)J) + (\nabla(\tilde{w}^{(n)}+\phi)J)^\top)J^{-1}\tilde{n}_0 \\
&- \left(\nabla(\tilde{w}^{(n)}+\phi)\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)}) + (\nabla(\tilde{w}^{(n)}+\phi)\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)}))^\top\right)J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&- \tilde{G}^{(n)}\tilde{G}^{(n)\top}J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0.
\end{aligned}
$$

Furthermore, we have

$$
\tilde{G}^{(n+1)}(t,\tilde{\omega}) = \tilde{G}_0 + \int_0^t \nabla(\tilde{w}^{(n)}(\tau,\tilde{\omega}) + \phi(\tau,\tilde{\omega}))\tilde{\zeta}^{(n)}(\tilde{\omega})J(\tilde{X}^{(n)}(\tau,\tilde{\omega}))\tilde{G}^{(n)}(\tau,\tilde{\omega})d\tau, \quad (4.12)
$$

and

$$
\tilde{X}^{(n+1)}(t,\tilde{\omega}) = \tilde{\omega} + \int_0^t J(\tilde{X}^{(n)}(\tau,\tilde{\omega}))(\tilde{w}^{(n)}(\tau,\tilde{\omega}) + \phi(\tau,\tilde{\omega}))d\tau \quad (4.13)
$$

for the magnetic field and flux, respectively.

4.2. Estimates to prove the local existence

The following estimates of the flux and magnetic field require being verified to prove the local existence theorem.

Without loss of generality, we assume $T < 1$ throughout the paper.

Lemma 4.2. *Let $2 < s < \frac{5}{2}$, $1 < \gamma < s-1$ and $\tilde{X} - \tilde{\omega} - \int_0^t J\nabla\phi d\tau \in \mathcal{A}^{s+1,\gamma+1}$. Let $\delta, \mu > 0$ be sufficiently small. Then, for $T > 0$ small enough, we have*

$$
\|\tilde{X}-\tilde{\omega}\|_{L^\infty H^{s+1}} \le T^{\frac{1}{4}} \left\| \tilde{X} - \tilde{\omega} - \int_0^t J\nabla\phi d\tau \right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0, \tilde{G}_0)T,
$$

$$
\|\tilde{X}-\tilde{\omega}\|_{L^\infty_{\frac{1}{4}} H^{s+1}} \le \left\| \tilde{X} - \tilde{\omega} - \int_0^t J\nabla\phi d\tau \right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0, \tilde{G}_0)T^{\frac{3}{4}},
$$

$$
\|\tilde{X}-\tilde{\omega}\|_{H^1_{(0)} H^{\gamma+1}} \le CT^{\delta} \left\| \tilde{X} - \tilde{\omega} - \int_0^t J\nabla\phi d\tau \right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0, \tilde{G}_0)T^{\frac{1}{2}},
$$

$$
\|\tilde{X}-\tilde{\omega}\|_{H^{\frac{s-1}{2}}_{(0)} H^{2+\mu}} \le CT^{\delta} \left\| \tilde{X} - \tilde{\omega} - \int_0^t J\nabla\phi d\tau \right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0, \tilde{G}_0)T^{\frac{1}{2}},
$$

$$
\|\tilde{X}-\tilde{\omega}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{2+\mu}} \le CT^{\delta} \left\| \tilde{X} - \tilde{\omega} - \int_0^t J\nabla\phi d\tau \right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0, \tilde{G}_0)T^{\frac{1}{8}},
$$

$$\|\tilde{X}-\hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}} \le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}},$$

$$\|\tilde{X}\|_{L^\infty H^{s+1}} \le T^{\frac{1}{4}}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0,\tilde{G}_0).$$

Proof. From the definition of the functional spaces, we have

$$\begin{aligned}
\|\tilde{X}-\tilde{\omega}\|_{L^\infty H^{s+1}} &\le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{L^\infty H^{s+1}} + \left\|\int_0^t J\nabla\phi d\tau\right\|_{L^\infty H^{s+1}}\\
&\le T^{\frac{1}{4}}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{L^\infty_{\frac{1}{4}} H^{s+1}} + \left\|\int_0^t J(\nabla\tilde{v}_0+\tau\nabla\hat{\phi})d\tau\right\|_{L^\infty H^{s+1}}\\
&\le T^{\frac{1}{4}}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0,\tilde{G}_0)T,
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{X}-\tilde{\omega}\|_{L^\infty_{\frac{1}{4}} H^{s+1}} &\le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{L^\infty_{\frac{1}{4}} H^{s+1}} + \left\|\int_0^t J\nabla\phi d\tau\right\|_{L^\infty_{\frac{1}{4}} H^{s+1}}\\
&\le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{L^\infty_{\frac{1}{4}} H^{s+1}} + \left\|\int_0^t J(\nabla\tilde{v}_0+\tau\nabla\hat{\phi})d\tau\right\|_{L^\infty_{\frac{1}{4}} H^{s+1}}\\
&\le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}} + C(\tilde{v}_0,\tilde{G}_0)T^{\frac{3}{4}}.
\end{aligned}$$

For some $\delta>0$ small enough and $\frac{1}{2}<\eta<1$, from Lemmas A.6 and A.10, we know that

$$\begin{aligned}
&\|\tilde{X}-\tilde{\omega}\|_{H^1_{(0)} H^{\gamma+1}}\\
&\le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{H^1_{(0)} H^{\gamma+1}} + \left\|\int_0^t J\nabla\phi d\tau\right\|_{H^1_{(0)} H^{\gamma+1}}\\
&\le \left\|\int_0^t \partial_\tau\Big(\tilde{X}-\tilde{\omega}-\int_0^\tau J\nabla\phi d\xi\Big)d\tau\right\|_{H^{1+\eta-\delta}_{(0)} H^{\gamma+1}} + C(\tilde{v}_0)\|t\|_{H^1_{(0)}} + C(\tilde{v}_0,\tilde{G}_0)\|t^2\|_{H^1_{(0)}}
\end{aligned}$$

$$\le CT^{\delta}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{H_{(0)}^{1+\eta}H^{\gamma+1}}+C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}$$

$$\le CT^{\delta}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}}+C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}},$$

and

$$\begin{aligned}\|\tilde{X}-\tilde{\omega}\|_{H_{(0)}^{\frac{s-1}{2}}H^{2+\mu}} &\le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{H_{(0)}^{\frac{s-1}{2}}H^{2+\mu}}+\left\|\int_0^t J\nabla\phi d\tau\right\|_{H_{(0)}^{\frac{s-1}{2}}H^{2+\mu}}\\ &\le \left\|\int_0^t \partial_\tau\Big(\tilde{X}-\tilde{\omega}-\int_0^\tau J\nabla\phi d\xi\Big)d\tau\right\|_{H_{(0)}^{\frac{s-1}{2}+\delta-\delta}H^{2+\mu}}\\ &\quad + C(\tilde{v}_0)\|t\|_{H_{(0)}^{\frac{s-1}{2}}}+C(\tilde{v}_0,\tilde{G}_0)\|t^2\|_{H_{(0)}^{\frac{s-1}{2}}}\\ &\le CT^{\delta}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{H_{(0)}^{\frac{s-1}{2}+\delta}H^{2+\mu}}+C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}\\ &\le CT^{\delta}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}}+C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}.\end{aligned}$$

Similarly,

$$\|\tilde{X}-\tilde{\omega}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{2+\mu}}\le CT^{\delta}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}}+C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{8}},$$

$$\begin{aligned}\|\tilde{X}-\hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}} &\le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}}+\left\|\int_0^t \tau J\nabla\hat{\phi} d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}}\\ &\le \left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}}+C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}},\\ \|\tilde{X}\|_{L^\infty H^{s+1}} &\le \|\tilde{X}-\tilde{\omega}\|_{L^\infty H^{s+1}}+\|\tilde{\omega}\|_{L^\infty H^{s+1}}\\ &\le T^{\frac{1}{4}}\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}}+C(\tilde{v}_0,\tilde{G}_0). \quad \square\end{aligned}$$

Lemma 4.3. *Let* $2 < s < \frac{5}{2}$, $1 < \gamma < s-1$ *and* $\tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \in \mathcal{A}^{s,\gamma}$. *Let* $\delta, \mu > 0$ *be sufficiently small. Then, for* $T > 0$ *small enough, we have*

$$\|\tilde{G} - \tilde{G}_0\|_{L^\infty H^s} \le T^{\frac{1}{4}} \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0, \tilde{G}_0) T,$$

$$\|\tilde{G} - \tilde{G}_0\|_{L^\infty_{\frac{1}{4}} H^s} \le \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0, \tilde{G}_0) T^{\frac{3}{4}},$$

$$\|\tilde{G} - \tilde{G}_0\|_{H^1_{(0)} H^\gamma} \le C T^\delta \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0, \tilde{G}_0) T^{\frac{1}{2}},$$

$$\|\tilde{G} - \tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\mu}} \le C T^\delta \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0, \tilde{G}_0) T^{\frac{1}{2}},$$

$$\|\tilde{G} - \tilde{G}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{1+\mu}} \le C T^\delta \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0, \tilde{G}_0) T^{\frac{1}{8}},$$

$$\|\tilde{G} - \hat{G}\|_{\mathcal{A}^{s,\gamma}} \le \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0, \tilde{G}_0) T^{\frac{1}{2}},$$

$$\|\tilde{G}\|_{L^\infty H^s} \le T^{\frac{1}{4}} \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0, \tilde{G}_0).$$

Proof. From the definition of the functional spaces, it follows that

$$\begin{aligned}
\|\tilde{G} - \tilde{G}_0\|_{L^\infty H^s} &= \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{L^\infty H^s} + \left\| \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{L^\infty H^s} \\
&\le T^{\frac{1}{4}} \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{L^\infty_{\frac{1}{4}} H^s} + \left\| \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{L^\infty H^s} \\
&\le T^{\frac{1}{4}} \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0, \tilde{G}_0) T.
\end{aligned}$$

Similarly,

$$\|\tilde G-\tilde G_0\|_{L^\infty_{\frac14}H^s}\le\left\|\tilde G-\tilde G_0-\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{\mathcal{A}^{s,\gamma}}+C(\tilde v_0,\tilde G_0)T^{\frac34}.$$

For some $\delta>0$ small enough and $\frac12<\eta<1$, from Lemmas A.6 and A.10, we have

$$\begin{aligned}\|\tilde G-\tilde G_0\|_{H^1_{(0)}H^\gamma}&=\left\|\tilde G-\tilde G_0-\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{H^1_{(0)}H^\gamma}+\left\|\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{H^1_{(0)}H^\gamma}\\&\le\left\|\int_0^t\partial_\tau(\tilde G-\tilde G_0-\int_0^\tau\nabla\phi J\tilde G_0d\xi)d\tau\right\|_{H^{1+\eta-\delta}_{(0)}H^\gamma}\\&\quad+C(\tilde v_0,\tilde G_0)(\|t\|_{H^1_{(0)}}+\|t^2\|_{H^1_{(0)}})\\&\le CT^\delta\left\|\tilde G-\tilde G_0-\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{H^{1+\eta}_{(0)}H^\gamma}+C(\tilde v_0,\tilde G_0)T^{\frac12}\\&\le CT^\delta\left\|\tilde G-\tilde G_0-\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{\mathcal{A}^{s,\gamma}}+C(\tilde v_0,\tilde G_0)T^{\frac12},\end{aligned}$$

and

$$\begin{aligned}\|\tilde G-\tilde G_0\|_{H^{\frac{s-1}{2}}_{(0)}H^{1+\mu}}&=\left\|\tilde G-\tilde G_0-\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{H^{\frac{s-1}{2}}_{(0)}H^{1+\mu}}+\left\|\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{H^{\frac{s-1}{2}}_{(0)}H^{1+\mu}}\\&\le\left\|\int_0^t\partial_\tau(\tilde G-\tilde G_0-\int_0^\tau\nabla\phi J\tilde G_0d\xi)d\tau\right\|_{H^{\frac{s-1}{2}+\delta-\delta}_{(0)}H^{1+\mu}}\\&\quad+C(\tilde v_0,\tilde G_0)(\|t\|_{H^1_{(0)}}+\|t^2\|_{H^1_{(0)}})\\&\le CT^\delta\left\|\tilde G-\tilde G_0-\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{H^{\frac{s-1}{2}+\delta}_{(0)}H^{1+\mu}}+C(\tilde v_0,\tilde G_0)T^{\frac12}\\&\le CT^\delta\left\|\tilde G-\tilde G_0-\int_0^t\nabla\phi J\tilde G_0d\tau\right\|_{\mathcal{A}^{s,\gamma}}+C(\tilde v_0,\tilde G_0)T^{\frac12}.\end{aligned}$$

Similarly,

$$\|\tilde{G}-\tilde{G}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{1+\mu}} \le CT^{\delta}\left\|\tilde{G}-\tilde{G}_0-\int_0^t \nabla\phi J\tilde{G}_0 d\tau\right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{8}},$$

$$\begin{aligned}\|\tilde{G}-\hat{G}\|_{\mathcal{A}^{s,\gamma}} &\le \left\|\tilde{G}-\tilde{G}_0-\int_0^t \nabla\phi J\tilde{G}_0 d\tau\right\|_{\mathcal{A}^{s,\gamma}} + \left\|\int_0^t \tau\nabla\hat{\phi}J\tilde{G}_0 d\tau\right\|_{\mathcal{A}^{s,\gamma}}\\ &\le \left\|\tilde{G}-\tilde{G}_0-\int_0^t \nabla\phi J\tilde{G}_0 d\tau\right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}},\\ \|\tilde{G}\|_{L^\infty H^s} &\le \|\tilde{G}-\tilde{G}_0\|_{L^\infty H^s} + \|\tilde{G}_0\|_{L^\infty H^s}\\ &\le T^{\frac{1}{4}}\left\|\tilde{G}-\tilde{G}_0-\int_0^t \nabla\phi J\tilde{G}_0 d\tau\right\|_{\mathcal{A}^{s,\gamma}} + C(\tilde{v}_0,\tilde{G}_0). \quad \square\end{aligned}$$

In the above, we assume that

$$\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau \in \mathcal{A}^{s+1,\gamma+1}, \quad \text{and } \tilde{G}-\tilde{G}_0-\int_0^t \nabla\phi J\tilde{G}_0 d\tau \in \mathcal{A}^{s,\gamma}.$$

We conclude that $\tilde{X}-\hat{X}$ and $\tilde{G}-\hat{G}$ belong to the functional spaces $\mathcal{A}^{s+1,\gamma+1}$ and $\mathcal{A}^{s,\gamma}$, respectively, since

$$\phi = \tilde{v}_0 + t\hat{\phi}, \quad \text{and } \partial_t\left(\int_0^t J\tau\nabla\hat{\phi}d\tau\right)|_{t=0} = 0.$$

However, $\tilde{X}-\tilde{\omega}$ does not belong to $\mathcal{A}^{s+1,\gamma+1}$ since the initial value of its time derivative does not vanish. Similarly, $\tilde{G}-\tilde{G}_0$ does not belong to $\mathcal{A}^{s,\gamma}$ either.

As a result, for the flux and magnetic field, we must adopt these technical modifications to ensure that the modified quantities belong to suitable spaces. In this manner, we are able to obtain the desired estimates and establish the local existence.

4.3. The local existence theorem

The main result of this section is the following theorem:

Theorem 4.4. *Let $2 < s < \frac{5}{2}$ and $1 < \gamma < s-1$. Let $\tilde{v}(0) = \tilde{v}_0 \in H^k(\tilde{\Omega}_0)$ and $\tilde{G}(0) = \tilde{G}_0 \in H^k(\tilde{\Omega}_0)$ for k large enough. There exists a sufficiently small $T > 0$ and a solution $(\tilde{X}-\hat{X}, \tilde{w}, \tilde{q}_w, \tilde{G}-\hat{G}) \in \mathcal{A}^{s+1,\gamma+1}\times\mathcal{K}^{s+1}_{(0)}\times\mathcal{K}^{s}_{pr(0)}\times\mathcal{A}^{s,\gamma}$ in $(0,T]\times\tilde{\Omega}_0$ such that*

$$\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}} \le N,$$

$$\|(\tilde{w},\tilde{q}_w) - L^{-1}(\tilde{f}^L_\phi + \tilde{f}_{G_0}, \bar{g}^L_\phi, \tilde{h}^L_\phi + \tilde{h}_{G_0}, 0)\|_{\mathcal{K}^{s+1}_{(0)}\times\mathcal{K}^s_{pr(0)}} \le N,$$

$$\left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} \le N,$$

where

$$N := \left\| \int_0^1 \tau J \nabla\hat{\phi} d\tau \right\|_{\mathcal{A}^{s+1,\gamma+1}([0,1];\tilde{\Omega}_0)} + \left\| \int_0^1 \tau \nabla\hat{\phi} J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}([0,1];\tilde{\Omega}_0)} + \|L^{-1}(\tilde{f}^L_\phi + \tilde{f}_{G_0}, \bar{g}^L_\phi, \tilde{h}^L_\phi + \tilde{h}_{G_0})\|_{\mathcal{K}^{s+1}_{(0)}([0,1];\tilde{\Omega}_0)\times\mathcal{K}^s_{pr(0)}([0,1];\tilde{\Omega}_0)}. \tag{4.14}$$

Remark 4.5. Recall that we have assumed that T is less than 1 since we only need to prove the existence of solutions for $T > 0$ small enough. Therefore, we take the integral from 0 to 1 in the above definition for simplicity.

Recall $\hat{X} = \tilde{\omega} + tJ\tilde{v}_0$ and $\hat{G} = \tilde{G}_0 + t\nabla\tilde{v}_0 J\tilde{G}_0$. It follows that

$$\tilde{X} - \hat{X} \in \mathcal{A}^{s+1,\gamma+1}, \quad \tilde{G} - \hat{G} \in \mathcal{A}^{s,\gamma},$$

since the initial values of the time derivative vanish. Also, we have introduced the terms $\tilde{f}_{G_0}, \tilde{h}_{G_0}, \tilde{f}^L_\phi, \bar{g}^{(n)}, \bar{g}^L_\phi$ and $\tilde{h}^L_\phi$ which will be defined later (cf. (4.15)) for technical modifications.

To prove Theorem 4.4, we need to verify the following propositions:

Proposition 4.6. *Let* $2 < s < \frac{5}{2}$ *and* $1 < \gamma < s - 1$. *Let* $((\tilde{w}^{(0)}, \tilde{q}^{(0)}_w), \tilde{X}^{(0)}, \tilde{G}^{(0)}) = ((0,0), \hat{X}, \hat{G})$ *and* N *be defined as in Theorem 4.4. For* $T > 0$ *small enough depending on* $N, \tilde{v}_0$ *and* $\tilde{G}_0$, *it follows that*

$$\tilde{X}^{(n)} - \hat{X} \in \mathcal{A}^{s+1,\gamma+1}, \quad (\tilde{w}^{(n)}, \tilde{q}^{(n)}_w) \in \mathcal{K}^{s+1}_{(0)} \times \mathcal{K}^s_{pr(0)}, \quad \tilde{G}^{(n)} - \hat{G} \in \mathcal{A}^{s,\gamma},$$

$$\left\| \tilde{X}^{(n)} - \tilde{\omega} - \int_0^t J\nabla\phi d\tau \right\|_{\mathcal{A}^{s+1,\gamma+1}} \le N,$$

$$\|(\tilde{w}^{(n)},\tilde{q}^{(n)}_w) - L^{-1}(\tilde{f}^L_\phi + \tilde{f}_{G_0}, \bar{g}^L_\phi, \tilde{h}^L_\phi + \tilde{h}_{G_0})\|_{\mathcal{K}^{s+1}_{(0)}\times\mathcal{K}^s_{pr(0)}} \le N,$$

$$\left\| \tilde{G}^{(n)} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{\mathcal{A}^{s,\gamma}} \le N,$$

where $n \ge 0$.

Moreover, we have

$$\begin{aligned}
&\|\tilde{w}^{(n+1)}-\tilde{w}^{(n)}\|_{\mathcal{K}^{s+1}_{(0)}}+\|\tilde{q}_w^{(n+1)}-\tilde{q}_w^{(n)}\|_{\mathcal{K}^s_{pr(0)}}+\|\tilde{X}^{(n+1)}-\tilde{X}^{(n)}\|_{\mathcal{A}^{s+1,\gamma+1}}\\
&+\|\tilde{G}^{(n+1)}-\tilde{G}^{(n)}\|_{\mathcal{A}^{s,\gamma}}\\
\leq{}& C(N,\tilde{v}_0,\tilde{G}_0)T^{\delta_1}\left(\|\tilde{w}^{(n)}-\tilde{w}^{(n-1)}\|_{\mathcal{K}^{s+1}_{(0)}}+\|\tilde{q}_w^{(n)}-\tilde{q}_w^{(n-1)}\|_{\mathcal{K}^s_{pr(0)}}\right)\\
&+C(N,\tilde{v}_0,\tilde{G}_0)T^{\delta_2}\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}}\\
&+C(N,\tilde{v}_0,\tilde{G}_0)T^{\delta_3}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}},
\end{aligned}$$

where $\delta_i>0$ *for* $i=1,2,3$ *and* $n\geq 1$. *In particular,* $((\tilde{w}^{(n)},\tilde{q}_w^{(n)}),\tilde{X}^{(n)}-\hat{X},\tilde{G}^{(n)}-\hat{G})$ *is Cauchy for* $T>0$ *small enough and the limit* $((\tilde{w},\tilde{q}_w),\tilde{X}-\hat{X},\tilde{G}-\hat{G})$ *satisfies*

$$\begin{aligned}
&\tilde{X}-\hat{X}\in\mathcal{A}^{s+1,\gamma+1},\quad (\tilde{w},\tilde{q}_w)\in\mathcal{K}^{s+1}_{(0)}\times\mathcal{K}^s_{pr(0)},\quad \tilde{G}-\hat{G}\in\mathcal{A}^{s,\gamma},\\
&\left\|\tilde{X}-\tilde{\omega}-\int_0^t J\nabla\phi\, d\tau\right\|_{\mathcal{A}^{s+1,\gamma+1}}\leq N,\\
&\|(\tilde{w},\tilde{q}_w)-L^{-1}(\tilde{f}^L_\phi+\tilde{f}_{G_0},\bar{g}^L_\phi,\tilde{h}^L_\phi+\tilde{h}_{G_0})\|_{\mathcal{K}^{s+1}_{(0)}\times\mathcal{K}^s_{pr(0)}}\leq N,\\
&\left\|\tilde{G}-\tilde{G}_0-\int_0^t \nabla\phi J\tilde{G}_0\, d\tau\right\|_{\mathcal{A}^{s,\gamma}}\leq N.
\end{aligned}$$

Therefore, we have solved system (3.5).

We will show that if the initial iterative sequence meets the conditions in the first part of the following lemmas and propositions, all subsequent quantities belong to the predetermined balls uniformly for $T>0$ small enough. Additionally, the iterative sequence is Cauchy and its limit solves system (3.5).

To start with, we write system (4.11) as follows:

$$\begin{aligned}
L(\tilde{w}^{(n+1)},\tilde{q}_w^{(n+1)})={}&\big(\tilde{f}^{(n)}-\partial_t\phi+Q^2\Delta\phi-J^\top\nabla\tilde{q}_\phi,\tilde{g}^{(n)}-\mathrm{Tr}(\nabla\phi J),\\
&\quad\tilde{h}^{(n)}+\tilde{q}_\phi J^{-1}\tilde{n}_0-((\nabla\phi J)+(\nabla\phi J)^\top)J^{-1}\tilde{n}_0\big)\\
={}&(\tilde{f}^{(n)}-\tilde{f}_{G_0},\tilde{g}^{(n)},\tilde{h}^{(n)}-\tilde{h}_{G_0})+(\tilde{f}^L_\phi+\tilde{f}_{G_0},\bar{g}^L_\phi,\tilde{h}^L_\phi+\tilde{h}_{G_0}),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{f}_{G_0} &:= \nabla \tilde{G}_0 J \tilde{G}_0, \\
\tilde{h}_{G_0} &:= -\tilde{G}_0 \tilde{G}_0^\top J^{-1} \tilde{n}_0, \\
\tilde{f}_\phi^L &:= -\partial_t \phi + Q^2 \Delta \phi - J^\top \nabla \tilde{q}_\phi, \\
\bar{g}^{(n)} &:= \tilde{g}^{(n)} + \mathrm{Tr}(\nabla \phi \tilde{\zeta}_\phi J_\phi) - \mathrm{Tr}(\nabla \phi J), \\
\bar{g}_\phi^L &:= -\mathrm{Tr}(\nabla \phi \tilde{\zeta}_\phi J_\phi), \\
\tilde{h}_\phi^L &:= \tilde{q}_\phi J^{-1} \tilde{n}_0 - (\nabla \phi J + (\nabla \phi J)^\top) J^{-1} \tilde{n}_0,
\end{aligned} \tag{4.15}$$

and

$$\tilde{\zeta}_\phi(t,\cdot) := \mathcal{I} + t\left(\partial_\tau \tilde{\zeta}^{(n)}(\tau,\cdot)|_{\tau=0}\right) = \mathcal{I} - t\nabla(J\tilde{v}_0)(\cdot),$$

$$(J_\phi)_{ij}(t,\cdot) := J_{ij}(\cdot) + t\left(\partial_\tau (J_{ij}(\tilde{X}^{(n)}(\tau,\cdot)))\big|_{\tau=0}\right) = J_{ij} + t\sum_{k,l=1}^{2} (\partial_k J_{ij} J_{kl} \tilde{v}_0^l)(\cdot).$$

In this fashion, we easily check that

$$\begin{aligned}
(\tilde{f}^{(n)} - \tilde{f}_{G_0}, \bar{g}^{(n)}, \tilde{h}^{(n)} - \tilde{h}_{G_0})(0,\cdot) &= 0, \\
(\tilde{f}_\phi^L + \tilde{f}_{G_0}, \bar{g}_\phi^L, \tilde{h}_\phi^L + \tilde{h}_{G_0})(0,\cdot) &= 0, \\
(\partial_t \bar{g}^{(n)})(0,\cdot) &= 0, \\
(\partial_t \bar{g}_\phi^L)(0,\cdot) &= 0, \\
(\partial_t (\tilde{g}^{(n)} - \mathrm{Tr}(\nabla \phi J)))(0,\cdot) &= 0.
\end{aligned}$$

These technical modifications are necessary as

$$(\tilde{f}^{(n)} - \tilde{f}_{G_0}, \bar{g}^{(n)}, \tilde{h}^{(n)} - \tilde{h}_{G_0}), \quad (\tilde{f}_\phi^L + \tilde{f}_{G_0}, \bar{g}_\phi^L, \tilde{h}_\phi^L + \tilde{h}_{G_0})$$

belong to Y_0. As a result, we are able to apply Lemma 4.1 to obtain some time-independent estimates.

To prove Proposition 4.6, we define the following notations:

$$B(X) := \left\{ \tilde{X} : \tilde{X} - \hat{X} \in \mathcal{A}^{s+1,\gamma+1}, \left\| \tilde{X} - \tilde{\omega} - \int_0^t J\nabla\phi \, d\tau \right\|_{\mathcal{A}^{s+1,\gamma+1}} \le N \right\},$$

$$\begin{aligned}
B(w,q) := \big\{ &(\tilde{w}, \tilde{q}_w) \in \mathcal{K}_{(0)}^{s+1} \times \mathcal{K}_{pr(0)}^{s} : \\
&\|(\tilde{w}, \tilde{q}_w) - L^{-1}(\tilde{f}_\phi^L + \tilde{f}_{G_0}, \bar{g}_\phi^L, \tilde{h}_\phi^L + \tilde{h}_{G_0})\|_{\mathcal{K}_{(0)}^{s+1} \times \mathcal{K}_{pr(0)}^{s}} \le N \big\},
\end{aligned}$$

$$B(G) := \left\{ \tilde{G} : \tilde{G} - \hat{G} \in \mathcal{A}^{s,\gamma}, \left\| \tilde{G} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 \, d\tau \right\|_{\mathcal{A}^{s,\gamma}} \le N \right\},$$

where N is given in Theorem 4.4.

We study (4.11)-(4.13) separately. From [4, Proposition 5.3], we have the following lemma for system (4.13):

Lemma 4.7. *Let $2 < s < \frac{5}{2}$, $1 < \gamma < s-1$. For $T > 0$ small enough depending on N and $\tilde{v}_0$, it follows that*

(1) For $n \geq 0$, if $\tilde{X}^{(n)} \in B(X)$ and $(\tilde{w}^{(n)}, \tilde{q}_w^{(n)}) \in B(w,q)$, we have $\tilde{X}^{(n+1)} \in B(X)$.
(2) For $n \geq 1$, if $\tilde{X}^{(n-1)}, \tilde{X}^{(n)}, \tilde{X}^{(n+1)} \in B(X)$ and $(\tilde{w}^{(n-1)}, \tilde{q}_w^{(n-1)}), (\tilde{w}^{(n)}, \tilde{q}_w^{(n)}), (\tilde{w}^{(n+1)}, \tilde{q}_w^{(n+1)}) \in B(w,q)$, the following estimate holds:

$$\|\tilde{X}^{(n+1)} - \tilde{X}^{(n)}\|_{\mathcal{A}^{s+1,\gamma+1}} \leq C(\tilde{v}_0, N)T^{\delta}(\|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\|_{\mathcal{K}^{s+1}_{(0)}} + \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}})$$

for some $\delta > 0$.

4.4. Estimates for the magnetic field

In this subsection, for the estimates of the magnetic field, we prove the following proposition:

Proposition 4.8. *Let $2 < s < \frac{5}{2}$, $1 < \gamma < s-1$. For $T > 0$ small enough depending on N, $\tilde{v}_0$ and $\tilde{G}_0$, we have*

(1) For $n \geq 0$, if $\tilde{X}^{(n)} \in B(X)$, $(\tilde{w}^{(n)}, \tilde{q}_w^{(n)}) \in B(w,q)$ and $\tilde{G}^{(n)} \in B(G)$, we have $\tilde{G}^{(n+1)} \in B(G)$.
(2) For $n \geq 1$, if $\tilde{X}^{(n-1)}, \tilde{X}^{(n)}, \tilde{X}^{(n+1)} \in B(X)$, $(\tilde{w}^{(n-1)}, \tilde{q}_w^{(n-1)}), (\tilde{w}^{(n)}, \tilde{q}_w^{(n)}), (\tilde{w}^{(n+1)}, \tilde{q}_w^{(n+1)}) \in B(w,q)$ and $\tilde{G}^{(n-1)}, \tilde{G}^{(n)}, \tilde{G}^{(n+1)} \in B(G)$, the following estimate holds:

$$\begin{aligned}\|\tilde{G}^{(n+1)} - \tilde{G}^{(n)}\|_{\mathcal{A}^{s,\gamma}} \leq C(N, \tilde{v}_0, \tilde{G}_0)T^{\delta}(&\|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}} \\ &+ \|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\|_{\mathcal{K}^{s+1}_{(0)}} + \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}}) \quad (4.16)\end{aligned}$$

for some $\delta > 0$.

Proof. Part 1. To show $\tilde{G}^{(n+1)} \in B(G)$.

We have

$$\begin{aligned}\left\|\tilde{G}^{(n+1)} - \tilde{G}_0 - \int_0^t \nabla\phi J\tilde{G}_0 d\tau\right\|_{\mathcal{A}^{s,\gamma}} &\leq \left\|\int_0^t \nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}d\tau\right\|_{\mathcal{A}^{s,\gamma}} \\ &+ \left\|\int_0^t \nabla\tilde{v}_0(\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0)d\tau\right\|_{\mathcal{A}^{s,\gamma}} \\ &+ \left\|\int_0^t \tau\nabla\hat{\phi}(\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0)d\tau\right\|_{\mathcal{A}^{s,\gamma}} \\ &=: \|I_1\|_{\mathcal{A}^{s,\gamma}} + \|I_2\|_{\mathcal{A}^{s,\gamma}} + \|I_3\|_{\mathcal{A}^{s,\gamma}}\,.\end{aligned}$$

Recalling $\mathcal{A}^{s,\gamma}([0,T];\Omega) = L^\infty_{\frac14}([0,T];H^s(\Omega)) \cap H^2_{(0)}([0,T];H^\gamma(\Omega))$, we start with the estimates $\|I_i\|_{L^\infty_{\frac14}H^s}$ for $i=1,2,3$. In order to apply Lemmas 4.2, 4.3 together with the estimates in Appendix A, these terms require being properly separated.

To control the first term, we use Minkowski's, Cauchy's inequalities and the definition of $L^\infty_{\frac14}H^s$ to obtain

$$\begin{aligned}
\|I_1\|_{L^\infty_{\frac14}H^s} &\le \sup_{t\in[0,T]} t^{-\frac14}\int_0^t \left\|\nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}\right\|_{H^s} d\tau \\
&\le \sup_{t\in[0,T]} t^{-\frac14}t^{\frac12}\|\nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}\|_{L^2([0,t];H^s)} \\
&\le T^{\frac14}\|\nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}\|_{L^2([0,T];H^s)} \\
&= T^{\frac14}\|\nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})(\tilde{G}^{(n)}-\tilde{G}_0)\|_{L^2H^s} + T^{\frac14}\|\nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}_0\|_{L^2H^s} \\
&=: I_{1,1} + I_{1,2}.
\end{aligned}$$

We will only focus on the estimate of $I_{1,1}$ since $\tilde{G}_0$ is the initial data. From the hypotheses

$$\tilde{X}^{(n)} \in B(X), \quad (\tilde{w}^{(n)}, \tilde{q}_w^{(n)}) \in B(w,q), \quad \tilde{G}^{(n)} \in B(G),$$

and applying Lemmas 4.3, A.1 and A.3, we have

$$\begin{aligned}
I_{1,1} &\le T^{\frac14}\|\nabla\tilde{w}^{(n)}\|_{L^2H^s}\|\tilde{\zeta}^{(n)}\|_{L^\infty H^s}\|J(\tilde{X}^{(n)})\|_{L^\infty H^s}\|\tilde{G}^{(n)}-\tilde{G}_0\|_{L^\infty H^s} \\
&\le T^{\frac14}C(\tilde{v}_0)\|\tilde{w}^{(n)}\|_{\mathcal{K}^{s+1}_{(0)}}\|\tilde{G}^{(n)}-\tilde{G}_0\|_{L^\infty H^s} \\
&\le T^{\frac12}C(N,\tilde{v}_0,\tilde{G}_0).
\end{aligned}$$

Similarly, $I_{1,2} \le T^{\frac14}C(N,\tilde{v}_0,\tilde{G}_0)$. Therefore, we have $\sum_{i=1}^2 I_{1,i} \le T^{\frac14}C(N,\tilde{v}_0,\tilde{G}_0)$.

For the second term, we have

$$\begin{aligned}
\|I_2\|_{L^\infty_{\frac14}H^s} &\le \sup_{t\in[0,T]} t^{-\frac14}\int_0^t \left\|\nabla\tilde{v}_0(\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0)\right\|_{H^s} d\tau \\
&\le \sup_{t\in[0,T]} t^{-\frac14}t^{\frac12}\|\nabla\tilde{v}_0(\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0)\|_{L^2([0,t];H^s)} \\
&\le T^{\frac14}\|\nabla\tilde{v}_0(\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0)\|_{L^2([0,T];H^s)} \\
&= T^{\frac14}\|\nabla\tilde{v}_0\tilde{\zeta}^{(n)}(J(\tilde{X}^{(n)})-J)\tilde{G}^{(n)}\|_{L^2H^s} + T^{\frac14}\|\nabla\tilde{v}_0(\tilde{\zeta}^{(n)}-\mathcal{I})J\tilde{G}^{(n)}\|_{L^2H^s} \\
&\quad + T^{\frac14}\|\nabla\tilde{v}_0J(\tilde{G}^{(n)}-\tilde{G}_0)\|_{L^2H^s} \\
&=: I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned}$$

We only concentrate on the estimate of $I_{2,1}$ since those of $I_{2,2}$ and $I_{2,3}$ are similar to that of $I_{2,1}$ by applying Lemmas 4.2, 4.3, together with

$$\begin{aligned}\|\tilde{\zeta}^{(n)}-\mathcal{I}\|_{L^\infty H^s} &= \|\tilde{\zeta}^{(n)}\nabla(\tilde{\omega}-\tilde{X}^{(n)}(t,\tilde{\omega}))\|_{L^\infty H^s}\\ &\le \|\tilde{\zeta}^{(n)}\|_{L^\infty H^s}\|\tilde{\omega}-\tilde{X}^{(n)}(t,\tilde{\omega})\|_{L^\infty H^{s+1}}\\ &\le T^{\frac{1}{4}}C(N,\tilde{v}_0,\tilde{G}_0).\end{aligned}$$

From Lemmas 4.3, A.1 and A.3, we have

$$\begin{aligned}I_{2,1} &\le T^{\frac{1}{4}}\|\nabla\tilde{v}_0\|_{L^2H^s}\|\tilde{\zeta}^{(n)}\|_{L^\infty H^s}\|J(\tilde{X}^{(n)})-J\|_{L^\infty H^s}\|\tilde{G}^{(n)}\|_{L^\infty H^s}\\ &\le T^{\frac{3}{4}}\|\nabla\tilde{v}_0\|_{L^\infty H^s}\|\tilde{\zeta}^{(n)}\|_{L^\infty H^s}\|J(\tilde{X}^{(n)})-J\|_{L^\infty H^s}\|\tilde{G}^{(n)}\|_{L^\infty H^s}\\ &\le T^{\frac{3}{4}}C(N,\tilde{v}_0,\tilde{G}_0).\end{aligned}$$

We conclude that $\sum_{i=1}^{3} I_{2,i}\le T^{\frac{3}{4}}C(N,\tilde{v}_0,\tilde{G}_0)$.

The estimate of I_3 can be achieved from a similar argument by replacing $\nabla\tilde{v}_0$ with $\tau\nabla\hat{\phi}$. In fact, from

$$\|\tau\nabla\hat{\phi}\|_{L^2H^s}\le\|\tau\|_{L^2}C(\tilde{v}_0,\tilde{G}_0)\le T^{\frac{3}{2}}C(\tilde{v}_0,\tilde{G}_0),$$

we obtain $\|I_3\|_{L^\infty_{\frac{1}{4}}H^s}\le T^{\frac{7}{4}}C(N,\tilde{v}_0,\tilde{G}_0)$.

Combining the above calculations, we have that

$$\left\|\tilde{G}^{(n+1)}-\tilde{G}_0-\int_0^t\nabla\phi J\tilde{G}_0 d\tau\right\|_{L^\infty_{\frac{1}{4}}H^s}\le\sum_{i=1}^{3}\|I_i\|_{L^\infty_{\frac{1}{4}}H^s}\le T^{\frac{1}{4}}C(N,\tilde{v}_0,\tilde{G}_0). \tag{4.17}$$

Next, we estimate $\|I_i\|_{H^2_{(0)}H^\gamma}$ for $i=1,2$ and 3. To control the first term, we apply Lemma A.6 to get

$$\|I_1\|_{H^2_{(0)}H^\gamma}\le\|\nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}\|_{H^1_{(0)}H^\gamma}.$$

In order to apply Lemmas 4.2, 4.3, together with the estimates in Appendix A, the above expression should be expanded into 8 terms as follows:

$$\nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}=\nabla\tilde{w}^{(n)}[(\tilde{\zeta}^{(n)}-\mathcal{I})+\mathcal{I}][(J(\tilde{X}^{(n)})-J)+J][(\tilde{G}^{(n)}-\tilde{G}_0)+\tilde{G}_0].$$

We concentrate on the most difficult term $\nabla\tilde{w}^{(n)}(\tilde{\zeta}^{(n)}-\mathcal{I})(J(\tilde{X}^{(n)})-J)(\tilde{G}^{(n)}-\tilde{G}_0)$ and the others are similar or easier owing to the initial data $\tilde{G}_0$.

From the hypotheses

$$\tilde{X}^{(n)}\in B(X),\quad(\tilde{w}^{(n)},\tilde{q}_w^{(n)})\in B(w,q),\quad\tilde{G}^{(n)}\in B(G),$$

combined with Lemmas 4.2, 4.3, A.1, A.3 and A.9 with $\gamma > 1$, it follows that

$$\begin{aligned}
&\|\nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J)(\tilde{G}^{(n)} - \tilde{G}_0)\|_{H^1_{(0)}H^\gamma} \\
&\le \|\nabla \tilde{w}^{(n)}\|_{H^1_{(0)}H^\gamma} \|\tilde{\zeta}^{(n)} - \mathcal{I}\|_{H^1_{(0)}H^\gamma} \|J(\tilde{X}^{(n)}) - J\|_{H^1_{(0)}H^\gamma} \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H^1_{(0)}H^\gamma} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0)\|\tilde{X}^{(n)} - \tilde{\omega}\|^2_{H^1_{(0)}H^{\gamma+1}} T^\delta \\
&\le T^{3\delta} C(N, \tilde{v}_0, \tilde{G}_0)
\end{aligned}$$

for some $\delta > 0$ small enough. We conclude that

$$\|I_1\|_{H^2_{(0)}H^\gamma} \le T^{\delta_1} C(N, \tilde{v}_0, \tilde{G}_0)$$

for some $\delta_1 > 0$.

For the second term, we have

$$\|I_2\|_{H^2_{(0)}H^\gamma} \le \|\nabla \tilde{v}_0(\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0)\|_{H^1_{(0)}H^\gamma}.$$

A slightly different way to write $\nabla \tilde{v}_0(\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0)$ is

$$\begin{aligned}
&\nabla \tilde{v}_0(\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0) \\
&= \nabla \tilde{v}_0[(\tilde{\zeta}^{(n)} - \mathcal{I}) + \mathcal{I}][(J(\tilde{X}^{(n)}) - J) + J][(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] - \nabla \tilde{v}_0 J \tilde{G}_0
\end{aligned}$$

where we split into 7 terms since $\nabla \tilde{v}_0 J\tilde{G}_0$ has been eliminated. From a similar argument we conclude that

$$\|I_2\|_{H^2_{(0)}H^\gamma} \le T^{\delta_2} C(N, \tilde{v}_0, \tilde{G}_0)$$

for some $\delta_2 > 0$ small enough.

Similarly,

$$\|I_3\|_{H^2_{(0)}H^\gamma} \le \|\tau \nabla \hat{\phi}(\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0)\|_{H^1_{(0)}H^\gamma},$$

and we rewrite it as

$$\begin{aligned}
&\tau \nabla \hat{\phi}(\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - J\tilde{G}_0) \\
&= \tau \nabla \hat{\phi}[(\tilde{\zeta}^{(n)} - \mathcal{I}) + \mathcal{I}][(J(\tilde{X}^{(n)}) - J) + J][(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] - \tau \nabla \hat{\phi} J \tilde{G}_0.
\end{aligned}$$

We point out that $\hat{\phi}$ depends only on the initial data and $\|\tau\|_{H^1_{(0)}} \le CT^{\frac{1}{2}}$, it follows that

$$\|I_3\|_{H^2_{(0)}H^\gamma} \le T^{\delta_3} C(N, \tilde{v}_0, \tilde{G}_0)$$

for some $\delta_3 > 0$ small enough.

Combining the above calculations, we conclude that

$$\left\| \tilde{G}^{(n+1)} - \tilde{G}_0 - \int_0^t \nabla\phi J \tilde{G}_0 d\tau \right\|_{H^2_{(0)}H^\gamma} \leq \sum_{i=1}^{3} \|I_i\|_{H^2_{(0)}H^\gamma} \leq T^\delta C(N, \tilde{v}_0, \tilde{G}_0)$$

where $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$.

This, together with (4.17), we complete the proof of **Part 1.**

Part 2. To prove the validity of (4.16).

We consider the difference

$$\tilde{G}^{(n+1)} - \tilde{G}^{(n)} = \int_0^t [\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - \nabla\tilde{v}^{(n-1)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)}]d\tau.$$

To control the estimate in $L^\infty_{\frac{1}{4}}H^s$, from Minkowski's and Cauchy's inequalities, we have

$$\begin{aligned}
&\|\tilde{G}^{(n+1)} - \tilde{G}^{(n)}\|_{L^\infty_{\frac{1}{4}}H^s} \\
&= \left\| \int_0^t [\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - \nabla\tilde{v}^{(n-1)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)}]d\tau \right\|_{L^\infty_{\frac{1}{4}}H^s} \\
&\leq \sup_{t\in[0,T]} t^{-\frac{1}{4}} \int_0^t \left\| \nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - \nabla\tilde{v}^{(n-1)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)} \right\|_{H^s} d\tau \\
&\leq T^{\frac{1}{4}} \|\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} - \nabla\tilde{v}^{(n-1)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)}\|_{L^2([0,T];H^s)} \\
&=: T^{\frac{1}{4}} \|J_1\|_{L^2H^s}.
\end{aligned}$$

We write J_1 as

$$\begin{aligned}
J_1 &= (\nabla\tilde{v}^{(n)} - \nabla\tilde{v}^{(n-1)})\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})\tilde{G}^{(n)} + \nabla\tilde{v}^{(n-1)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)}) J(\tilde{X}^{(n)})\tilde{G}^{(n)} \\
&\quad + \nabla\tilde{v}^{(n-1)}\tilde{\zeta}^{(n-1)}(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\tilde{G}^{(n)} \\
&\quad + \nabla\tilde{v}^{(n-1)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})(\tilde{G}^{(n)} - \tilde{G}^{(n-1)}) \\
&=: \sum_{i=1}^{4} J_{1,i}.
\end{aligned}$$

For $J_{1,1}$, from $\tilde{X}^{(n)} \in B(X)$, $(\tilde{w}^{(n-1)}, \tilde{q}_w^{(n-1)})$, $(\tilde{w}^{(n)}, \tilde{q}_w^{(n)}) \in B(w, q)$ and $\tilde{G}^{(n)} \in B(G)$, together with Lemmas 4.3, A.1 and A.3, we have

$$\begin{aligned}\|J_{1,1}\|_{L^2H^s} &\le \|\nabla\tilde{v}^{(n)}-\nabla\tilde{v}^{(n-1)}\|_{L^2H^s}\|\tilde{\zeta}^{(n)}\|_{L^\infty H^s}\|J(\tilde{X}^{(n)})\|_{L^\infty H^s}\|\tilde{G}^{(n)}\|_{L^\infty H^s}\\ &\le C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{w}^{(n)}-\tilde{w}^{(n-1)}\|_{L^2H^{s+1}}\\ &\le C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{w}^{(n)}-\tilde{w}^{(n-1)}\|_{\mathcal{K}^{s+1}_{(0)}}.\end{aligned}$$

Recalling $\tilde{v}^{(n)}=\tilde{w}^{(n)}+\tilde{v}_0+t\hat{\phi}$ and applying Lemmas 4.3, A.1 and A.4, we estimate $J_{1,2}$ as follows:

$$\begin{aligned}\|J_{1,2}\|_{L^2H^s} &\le \|\nabla\tilde{v}^{(n-1)}\|_{L^2H^s}\|\tilde{\zeta}^{(n)}-\tilde{\zeta}^{(n-1)}\|_{L^\infty H^s}\|J(\tilde{X}^{(n)})\|_{L^\infty H^s}\|\tilde{G}^{(n)}\|_{L^\infty H^s}\\ &\le (\|\tilde{w}^{(n)}\|_{L^2H^{s+1}}+\|\tilde{v}_0\|_{L^2H^{s+1}}+\|s\hat{\phi}\|_{L^2H^{s+1}})\\ &\quad\cdot\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{L^\infty H^{s+1}}C(N,\tilde{v}_0,\tilde{G}_0)\\ &\le C(N,\tilde{v}_0,\tilde{G}_0)T^{\frac14}\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{L^\infty_{\frac14}H^{s+1}}\\ &\le T^{\frac14}C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}}.\end{aligned}$$

The estimate of $J_{1,3}$ can be achieved from a similar argument thanks to Lemmas 4.3, A.2 and A.3. Thus, we have

$$\|J_{1,3}\|_{L^2H^s}\le T^{\frac14}C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s,\gamma+1}}.$$

For the last term, from Lemmas 4.3, A.1 and A.3, the following estimate holds:

$$\begin{aligned}\|J_{1,4}\|_{L^2H^s} &\le \|\nabla\tilde{v}^{(n-1)}\|_{L^2H^s}\|\tilde{\zeta}^{(n-1)}\|_{L^\infty H^s}\|J(\tilde{X}^{(n-1)})\|_{L^\infty H^s}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{L^\infty H^s}\\ &\le (\|\tilde{w}^{(n)}\|_{L^2H^s}+\|\tilde{v}_0\|_{L^2H^s}+\|t\hat{\phi}\|_{L^2H^s})\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{L^\infty H^s}C(N,\tilde{v}_0,\tilde{G}_0)\\ &\le T^{\frac14}C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{L^\infty H^s}\\ &\le T^{\frac14}C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}}.\end{aligned}$$

Next, we estimate $\|\tilde{G}^{(n+1)}-\tilde{G}^{(n)}\|_{H^2_{(0)}H^\gamma}$. From Lemma A.6, we have

$$\begin{aligned}\|\tilde{G}^{(n+1)}-\tilde{G}^{(n)}\|_{H^2_{(0)}H^\gamma} &\le \|\nabla\tilde{v}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}-\nabla\tilde{v}^{(n-1)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)}\|_{H^1_{(0)}H^\gamma}\\ &\le \|\nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}-\nabla\tilde{w}^{(n-1)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)}\|_{H^1_{(0)}H^\gamma}\\ &\quad+\|\nabla\tilde{v}_0\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}-\nabla\tilde{v}_0\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)}\|_{H^1_{(0)}H^\gamma}\\ &\quad+\|t\hat{\phi}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}-t\hat{\phi}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)}\|_{H^1_{(0)}H^\gamma}\\ &=:\|J_{2,1}\|_{H^1_{(0)}H^\gamma}+\|J_{2,2}\|_{H^1_{(0)}H^\gamma}+\|J_{2,3}\|_{H^1_{(0)}H^\gamma}.\end{aligned}$$

We expand $J_{2,1}$ as follows:

$$
\begin{aligned}
J_{2,1} &= \nabla \tilde{w}^{(n)}[(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)}) + \tilde{\zeta}^{(n-1)}][(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})) + J(\tilde{X}^{(n-1)})]\tilde{G}^{(n)} \\
&\quad - \nabla \tilde{w}^{(n-1)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)} \\
&= \nabla \tilde{w}^{(n)}[(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})(J(\tilde{X}^{(n-1)}) - J) + (\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J \\
&\quad + (\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})) \\
&\quad + (J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})) + \tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})]\tilde{G}^{(n)} \\
&\quad - \nabla \tilde{w}^{(n-1)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)} \\
&= \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})(J(\tilde{X}^{(n-1)}) - J)[(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J[(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))[(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla \tilde{w}^{(n)}(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))[(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla \tilde{w}^{(n)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})\tilde{G}^{(n)} \\
&\quad - \nabla \tilde{w}^{(n-1)}\tilde{\zeta}^{(n-1)} J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)} \\
&= \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})(J(\tilde{X}^{(n-1)}) - J)(\tilde{G}^{(n)} - \tilde{G}_0) \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})(J(\tilde{X}^{(n-1)}) - J)\tilde{G}_0 \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J(\tilde{G}^{(n)} - \tilde{G}_0) \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J\tilde{G}_0 \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))(\tilde{G}^{(n)} - \tilde{G}_0) \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\tilde{G}_0 \\
&\quad + \nabla \tilde{w}^{(n)}(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))(\tilde{G}^{(n)} - \tilde{G}_0) \\
&\quad + \nabla \tilde{w}^{(n)}(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\tilde{G}_0 \\
&\quad + (\nabla \tilde{w}^{(n)} - \nabla \tilde{w}^{(n-1)})(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n-1)}) - J)(\tilde{G}^{(n-1)} - \tilde{G}_0) \\
&\quad + (\nabla \tilde{w}^{(n)} - \nabla \tilde{w}^{(n-1)})(\tilde{\zeta}^{(n-1)} - \mathcal{I})J(\tilde{G}^{(n-1)} - \tilde{G}_0) \\
&\quad + (\nabla \tilde{w}^{(n)} - \nabla \tilde{w}^{(n-1)})\mathcal{I}(J(\tilde{X}^{(n-1)}) - J)(\tilde{G}^{(n-1)} - \tilde{G}_0) \\
&\quad + (\nabla \tilde{w}^{(n)} - \nabla \tilde{w}^{(n-1)})\mathcal{I}J(\tilde{G}^{(n-1)} - \tilde{G}_0) \\
&\quad + (\nabla \tilde{w}^{(n)} - \nabla \tilde{w}^{(n-1)})(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n-1)}) - J)\tilde{G}_0 \\
&\quad + (\nabla \tilde{w}^{(n)} - \nabla \tilde{w}^{(n-1)})(\tilde{\zeta}^{(n-1)} - \mathcal{I})J\tilde{G}_0 \\
&\quad + (\nabla \tilde{w}^{(n)} - \nabla \tilde{w}^{(n-1)})\mathcal{I}(J(\tilde{X}^{(n-1)}) - J)\tilde{G}_0 \\
&\quad + (\nabla \tilde{w}^{(n)} - \nabla \tilde{w}^{(n-1)})\mathcal{I}J\tilde{G}_0 \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n-1)}) - J)(\tilde{G}^{(n)} - \tilde{G}^{(n-1)}) \\
&\quad + \nabla \tilde{w}^{(n)}(\tilde{\zeta}^{(n-1)} - \mathcal{I})J(\tilde{G}^{(n)} - \tilde{G}^{(n-1)}) \\
&\quad + \nabla \tilde{w}^{(n)}\mathcal{I}(J(\tilde{X}^{(n-1)}) - J)(\tilde{G}^{(n)} - \tilde{G}^{(n-1)})
\end{aligned}
$$

$$+\nabla\tilde{w}^{(n)}\mathcal{I}J(\tilde{G}^{(n)}-\tilde{G}^{(n-1)})$$
$$=:\sum_{i=1}^{20}J_{2,1,i}.$$

We explain the procedure of separating the magnetic field in this process. To begin with, we avoid multiplying $(\tilde{\zeta}^{(n)}-\tilde{\zeta}^{(n-1)})$ and $(J(\tilde{X}^{(n)})-J(\tilde{X}^{(n-1)}))$. Then, in each product, we leave just one item as the difference between the iterative steps $n-1$ and n. To estimate the other terms, we subtract their initial values.

We will focus on the most complicated term $J_{2,1,1}$. From the hypotheses, Lemmas 4.2, 4.3, A.1, A.4 and A.9, we obtain

$$\begin{aligned}
&\|J_{2,1,1}\|_{H^1_{(0)}H^\gamma}\\
&=\|\nabla\tilde{w}^{(n)}(\tilde{\zeta}^{(n)}-\tilde{\zeta}^{(n-1)})(J(\tilde{X}^{(n-1)})-J)(\tilde{G}^{(n)}-\tilde{G}_0)\|_{H^1_{(0)}H^\gamma}\\
&\le\|\nabla\tilde{w}^{(n)}\|_{H^1_{(0)}H^\gamma}\|\tilde{\zeta}^{(n)}-\tilde{\zeta}^{(n-1)}\|_{H^1_{(0)}H^\gamma}\|J(\tilde{X}^{(n-1)})-J\|_{H^1_{(0)}H^\gamma}\|\tilde{G}^{(n)}-\tilde{G}_0\|_{H^1_{(0)}H^\gamma}\\
&\le T^\delta C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{H^1_{(0)}H^{\gamma+1}}\|\tilde{X}^{(n-1)}-\tilde{\omega}\|_{H^1_{(0)}H^\gamma}\\
&\le T^{2\delta}C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}},
\end{aligned}$$

for some $\delta>0$ small enough. The remaining estimates can be achieved from similar arguments. We conclude that

$$\begin{aligned}
&\sum_{i=2}^{8}\|J_{2,1,i}\|_{H^1_{(0)}H^\gamma}\le T^{\delta_{2,1,1}}C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}},\\
&\sum_{i=9}^{16}\|J_{2,1,i}\|_{H^1_{(0)}H^\gamma}\le T^{\delta_{2,1,2}}C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{w}^{(n)}-\tilde{w}^{(n-1)}\|_{\mathcal{K}^{s+1}_{(0)}},\\
&\sum_{i=17}^{20}\|J_{2,1,i}\|_{H^1_{(0)}H^\gamma}\le T^{\delta_{2,1,3}}C(N,\tilde{v}_0,\tilde{G}_0)\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}}
\end{aligned}$$

for some $\delta_{2,1,i}>0$.

For $J_{2,2}$ and $J_{2,3}$, the idea of their expansion is very similar to that explained before. These, combining with the fact that $\tilde{v}_0$ and $\hat{\phi}$ depend only on the initial data, we conclude that

$$\|J_{2,i}\|_{H^1_{(0)}H^\gamma}\le T^{\delta_{2,i}}C(N,\tilde{v}_0,\tilde{G}_0)(\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}}+\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}})$$

for $i=2,3$ and some $\delta_{2,i}>0$.

To complete the proof, we combine the estimates in $L^\infty_{\frac{1}{4}}H^s$ and $H^2_{(0)}H^\gamma$ to obtain

$$\begin{aligned}
\|\tilde{G}^{(n+1)}-\tilde{G}^{(n)}\|_{\mathcal{A}^{s,\gamma}}\le C(N,\tilde{v}_0,\tilde{G}_0)T^\delta(&\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}}+\|\tilde{w}^{(n)}-\tilde{w}^{(n-1)}\|_{\mathcal{K}^{s+1}_{(0)}}\\
&+\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}})
\end{aligned}$$

for $\delta = \min\{\frac{1}{4}, \frac{1}{2}, \delta_{2,1,1}, \delta_{2,1,2}, \delta_{2,1,3}, \delta_{2,2}, \delta_{2,3}\} > 0$. □

4.5. Estimates for the velocity field and pressure

In this subsection, for the velocity and pressure, the following estimates hold:

Proposition 4.9. *Let $2 < s < \frac{5}{2}$, $1 < \gamma < s-1$. For $T > 0$ small enough depending on N, $\tilde{v}_0$ and $\tilde{G}_0$, we have*

(1) For $n \geq 0$, if $\tilde{X}^{(n)} \in B(X)$, $(\tilde{w}^{(n)}, \tilde{q}_w^{(n)}) \in B(w,q)$ and $\tilde{G}^{(n)} \in B(G)$, we have $(\tilde{w}^{(n+1)}, \tilde{q}_w^{(n+1)}) \in B(w,q)$.
(2) For $n \geq 1$, if $\tilde{X}^{(n-1)}, \tilde{X}^{(n)}, \tilde{X}^{(n+1)} \in B(X)$, $(\tilde{w}^{(n-1)}, \tilde{q}_w^{(n-1)}), (\tilde{w}^{(n)}, \tilde{q}_w^{(n)}), (\tilde{w}^{(n+1)}, \tilde{q}_w^{(n+1)}) \in B(w,q)$ and $\tilde{G}^{(n-1)}, \tilde{G}^{(n)}, \tilde{G}^{(n+1)} \in B(G)$, the following estimate holds:

$$\begin{aligned}
&\|\tilde{w}^{(n+1)} - \tilde{w}^{(n)}\|_{\mathcal{K}^{s+1}_{(0)}} + \|\tilde{q}_w^{(n+1)} - \tilde{q}_w^{(n)}\|_{\mathcal{K}^{s}_{pr(0)}} \\
&\leq C(\tilde{v}_0, \tilde{G}_0) T^{\delta} (\|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}} + \|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\|_{\mathcal{K}^{s+1}_{(0)}} + \|\tilde{q}_w^{(n)} - \tilde{q}_w^{(n-1)}\|_{\mathcal{K}^{s}_{pr(0)}} \\
&\quad + \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}})
\end{aligned}$$

for some $\delta > 0$.

Proof. Part 1.

To begin with, from Lemma 4.1, we have

$$\begin{aligned}
&\|(\tilde{w}^{(n+1)}, \tilde{q}_w^{(n+1)}) - L^{-1}(\tilde{f}_\phi^L + \tilde{f}_{G_0}, \bar{g}_\phi^L, \tilde{h}_\phi^L + \tilde{h}_{G_0})\|_{X_0} \\
&= \|L^{-1}(\tilde{f}^{(n)} - \tilde{f}_{G_0}, \bar{g}^{(n)}, \tilde{h}^{(n)} - \tilde{h}_{G_0})\|_{X_0} \\
&\leq C(\|\tilde{f}^{(n)} - \tilde{f}_{G_0}\|_{\mathcal{K}^{s-1}_{(0)}} + \|\bar{g}^{(n)}\|_{\bar{\mathcal{K}}^{s}_{(0)}} + \|\tilde{h}^{(n)} - \tilde{h}_{G_0}\|_{\mathcal{K}^{s-\frac{1}{2}}_{(0)}}).
\end{aligned} \tag{4.18}$$

It is sufficient to estimate $\|\tilde{f}^{(n)} - \tilde{f}_{G_0}\|_{\mathcal{K}^{s-1}_{(0)}}$, $\|\bar{g}^{(n)}\|_{\bar{\mathcal{K}}^{s}_{(0)}}$ and $\|\tilde{h}^{(n)} - \tilde{h}_{G_0}\|_{\mathcal{K}^{s-\frac{1}{2}}_{(0)}}$.

Estimate for $\tilde{f}^{(n)}$.

We write $\tilde{f}^{(n)}$ as

$$\begin{aligned}
\tilde{f}^{(n)} &= -Q^2 \Delta(\tilde{w}^{(n)} + \phi) + J^{\top} \nabla(\tilde{q}^{(n)}) + Q^2(\tilde{X}^{(n)}) \nabla(\nabla(\tilde{w}^{(n)} + \phi)\tilde{\zeta}^{(n)})\tilde{\zeta}^{(n)} \\
&\quad - J(\tilde{X}^{(n)})^{\top} \tilde{\zeta}^{(n)\top} \nabla(\tilde{q}^{(n)}) \\
&= [-Q^2 \Delta \tilde{w}^{(n)} + Q^2(\tilde{X}^{(n)}) \nabla(\nabla \tilde{w}^{(n)} \tilde{\zeta}^{(n)}) \tilde{\zeta}^{(n)}] \\
&\quad + [-Q^2 \Delta\phi + Q^2(\tilde{X}^{(n)}) \nabla(\nabla\phi \tilde{\zeta}^{(n)}) \tilde{\zeta}^{(n)}] \\
&\quad + [J^{\top} \nabla \tilde{q}^{(n)} - J(\tilde{X}^{(n)})^{\top} \tilde{\zeta}^{(n)\top} \nabla \tilde{q}^{(n)}] \\
&\quad + [\nabla \tilde{G}^{(n)} \tilde{\zeta}^{(n)} J(\tilde{X}^{(n)}) \tilde{G}^{(n)}] \\
&=: \tilde{f}_w^{(n)} + \tilde{f}_\phi^{(n)} + \tilde{f}_q^{(n)} + \tilde{f}_G^{(n)}.
\end{aligned} \tag{4.19}$$

Note that the terms $\tilde{f}_w^{(n)}$, $\tilde{f}_\phi^{(n)}$ and $\tilde{f}_q^{(n)}$ have already been studied in [4, Proposition 5.4]. However, our definition of ϕ differs from [4] since we have introduced the initial magnetic field $\tilde{G}_0$. It turns out that this distinction has no significant effect on the estimates of $\tilde{f}_\phi^{(n)}$ in [4], but the constant will depend on $\tilde{G}_0$. As a result, it suffices to estimate $\|\tilde{f}_G^{(n)} - \tilde{f}_{G_0}\|_{L^2H^{s-1}}$ and $\|\tilde{f}_G^{(n)} - \tilde{f}_{G_0}\|_{H_{(0)}^{\frac{s-1}{2}}L^2}$.

To control the first term, we expand $\tilde{f}_G^{(n)} - \tilde{f}_{G_0}$ into

$$\begin{aligned}
\tilde{f}_G^{(n)} - \tilde{f}_{G_0} &= \nabla\tilde{G}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)} - \nabla\tilde{G}_0J\tilde{G}_0 \\
&= [(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0) + \nabla\tilde{G}_0]\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})[(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] - \nabla\tilde{G}_0J\tilde{G}_0 \\
&= (\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0)\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})(\tilde{G}^{(n)} - \tilde{G}_0) \\
&\quad + (\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0)\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}_0 + \nabla\tilde{G}_0\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})(\tilde{G}^{(n)} - \tilde{G}_0) \\
&\quad + \nabla\tilde{G}_0\tilde{\zeta}^{(n)}(J(\tilde{X}^{(n)}) - J)\tilde{G}_0 + \nabla\tilde{G}_0(\tilde{\zeta}^{(n)} - \mathcal{I})J\tilde{G}_0 \\
&=: \sum_{i=1}^{5} d_i^f.
\end{aligned}$$

We only focus on the estimate of d_1^f and d_4^f and the others are similar.

For $\|d_1^f\|_{L^2H^{s-1}}$, from the hypotheses $\tilde{X}^{(n)} \in B(X)$, $\tilde{G}^{(n)} \in B(G)$ and applying Lemmas 4.3, A.1 and A.3, we have

$$\begin{aligned}
\|d_1^f\|_{L^2H^{s-1}} &= \|(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0)\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})(\tilde{G}^{(n)} - \tilde{G}_0)\|_{L^2H^{s-1}} \\
&\le \|\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0\|_{L^2H^{s-1}}\|\tilde{\zeta}^{(n)}\|_{L^\infty H^{s-1}}\|J(\tilde{X}^{(n)})\|_{L^\infty H^{s-1}}\|\tilde{G}^{(n)} - \tilde{G}_0\|_{L^\infty H^{s-1}} \\
&\le T^{\frac{1}{2}+2\delta}C(N, \tilde{v}_0, \tilde{G}_0)
\end{aligned}$$

for some $\delta > 0$ small enough. For $\|d_4^f\|_{L^2H^{s-1}}$, we apply Lemmas A.1 and A.3 to obtain

$$\begin{aligned}
\|d_4^f\|_{L^2H^{s-1}} &\le \|\nabla\tilde{G}_0\tilde{\zeta}^{(n)}(J(\tilde{X}^{(n)}) - J)\tilde{G}_0\|_{L^2H^{s-1}} \\
&\le \|\nabla\tilde{G}_0\|_{L^2H^{s-1}}\|\tilde{\zeta}^{(n)}\|_{L^\infty H^{s-1}}\|J(\tilde{X}^{(n)}) - J\|_{L^\infty H^{s-1}}\|\tilde{G}_0\|_{L^\infty H^{s-1}} \\
&\le T^{\frac{1}{2}}C(N, \tilde{v}_0, \tilde{G}_0).
\end{aligned}$$

Next, we estimate $\|\tilde{f}_G^{(n)} - \tilde{f}_{G_0}\|_{H_{(0)}^{\frac{s-1}{2}}L^2}$. A different way to write $\tilde{f}_G^{(n)} - \tilde{f}_{G_0}$ is:

$$\begin{aligned}
&\tilde{f}_G^{(n)} - \tilde{f}_{G_0} \\
&= \nabla\tilde{G}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)} - \nabla\tilde{G}_0J\tilde{G}_0 \\
&= [(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0) + \nabla\tilde{G}_0][(\tilde{\zeta}^{(n)} - \mathcal{I}) + \mathcal{I}][(J(\tilde{X}^{(n)}) - J) + J][(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad - \nabla\tilde{G}_0J\tilde{G}_0
\end{aligned}$$

$$=: \sum_{i=1}^{15} \bar{d}_i^f,$$

where we expand $\nabla \tilde{G}^{(n)} \tilde{\zeta}^{(n)} J(\tilde{X}^{(n)}) \tilde{G}^{(n)}$ into 16 terms and $\nabla \tilde{G}_0 J \tilde{G}_0$ has been eliminated. We only focus on

$$\bar{d}_1^f = (\nabla \tilde{G}^{(n)} - \nabla \tilde{G}_0)(\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J)(\tilde{G}^{(n)} - \tilde{G}_0),$$

and the other terms are similar or easier.

From the hypotheses $\tilde{X}^{(n)} \in B(X)$, $\tilde{G}^{(n)} \in B(G)$ and applying Lemmas 4.2, 4.3, A.1, A.3, A.7 and A.10, we obtain

$$\begin{aligned}
&\|\bar{d}_1^f\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \\
&\le \|(\nabla \tilde{G}^{(n)} - \nabla \tilde{G}_0)(\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J)(\tilde{G}^{(n)} - \tilde{G}_0)\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \\
&\le \|(\nabla \tilde{G}^{(n)} - \nabla \tilde{G}_0)(\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J)\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\le \|\nabla \tilde{G}^{(n)} - \nabla \tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \|(\tilde{\zeta}^{(n)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J)\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\le \|\tilde{\zeta}^{(n)} - \mathcal{I}\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \|J(\tilde{X}^{(n)}) - J\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \|\tilde{G}^{(n)} - \tilde{G}_0\|^2_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\le \|\tilde{X}^{(n)} - \tilde{\omega}\|_{H_{(0)}^{\frac{s-1}{2}} H^{2+\eta}} C(N, \tilde{v}_0, \tilde{G}_0)(\|\tilde{X}^{(n)} - \hat{X}\|_{H_{(0)}^{\frac{s-1}{2}+\delta} H^{1+\eta}} + T) T^{2\delta} \\
&\le T^{3\delta} C(N, \tilde{v}_0, \tilde{G}_0)(\|\tilde{X}^{(n)} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}} + T) \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{3\delta}
\end{aligned}$$

for some $\delta, \eta > 0$ small enough.

We conclude that $\|d_i^f\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \le C(N, \tilde{v}_0, \tilde{G}_0) T^{\delta_i}$ for $\delta_i > 0, i = 1, \cdots, 15$.

Combined with the above calculations, it follows that

$$\|\tilde{f}_w^{(n)}\|_{\mathcal{K}_{(0)}^{s-1}} + \|\tilde{f}_\phi^{(n)}\|_{\mathcal{K}_{(0)}^{s-1}} + \|\tilde{f}_q^{(n)}\|_{\mathcal{K}_{(0)}^{s-1}} + \|\tilde{f}_G^{(n)} - \tilde{f}_{G_0}\|_{\mathcal{K}_{(0)}^{s-1}} \le T^\delta C(N, \tilde{v}_0, \tilde{G}_0) \tag{4.20}$$

for some $\delta > 0$.

Estimate for $\bar{g}^{(n)}$.

As in [4], we rewrite $\bar{g}^{(n)}$:

$$\begin{aligned}
\bar{g}^{(n)} &= \mathrm{Tr}(\nabla(\tilde{w}^{(n)} + \phi)J) - \mathrm{Tr}(\nabla(\tilde{w}^{(n)} + \phi)\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)}) + \mathrm{Tr}(\nabla\phi\tilde{\zeta}_\phi J_\phi) - \mathrm{Tr}(\nabla\phi J)) \\
&= \mathrm{Tr}(\nabla\tilde{w}^{(n)}(J - J(\tilde{X}^{(n)}))) + \mathrm{Tr}(\nabla\tilde{w}^{(n)}(\mathcal{I} - \tilde{\zeta}^{(n)}) J(\tilde{X}^{(n)})) \\
&\quad + \mathrm{Tr}(\nabla\phi\tilde{\zeta}_\phi(J_\phi - J(\tilde{X}^{(n)}))) + \mathrm{Tr}(\nabla\phi(\tilde{\zeta}_\phi - \tilde{\zeta}^{(n)}) J(\tilde{X}^{(n)})).
\end{aligned}$$

We refer to [4] for the details and the final estimate depends on both $\tilde{v}_0$ and $\tilde{G}_0$, i.e.,

$$\|\bar{g}^{(n)}\|_{\bar{\mathcal{K}}^s_{(0)}} \le T^\theta C(N, \tilde{v}_0, \tilde{G}_0) \tag{4.21}$$

for some $\theta > 0$.

Estimate for $\tilde{h}^{(n)}$.

$\tilde{h}^{(n)}$ can be expanded as follows:

$$\begin{aligned}
\tilde{h}^{(n)} &= -\tilde{q}^{(n)} J^{-1}\tilde{n}_0 + \tilde{q}^{(n)} J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + ((\nabla(\tilde{w}^{(n)} + \phi)J) + (\nabla(\tilde{w}^{(n)} + \phi)J)^\top)J^{-1}\tilde{n}_0 \\
&\quad - ((\nabla(\tilde{w}^{(n)} + \phi)\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)})) \\
&\quad + (\nabla(\tilde{w}^{(n)} + \phi)\tilde{\zeta}^{(n)} J(\tilde{X}^{(n)}))^\top) J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad - \tilde{G}^{(n)}\tilde{G}^{(n)\top} J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&= \nabla\tilde{w}^{(n)}\tilde{n}_0 - \nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + (\nabla\tilde{w}^{(n)} J)^\top J^{-1}\tilde{n}_0 - (\nabla\tilde{w}^{(n)} J(\tilde{X}^{(n)}))^\top J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + \nabla\phi\tilde{n}_0 - \nabla\tilde{w}^{(n)}\tilde{\zeta}^{(n)}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + (\nabla\phi J)^\top J^{-1}\tilde{n}_0 - (\nabla\phi J(\tilde{X}^{(n)}))^\top J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + (-\tilde{q}^{(n)} J^{-1}\tilde{n}_0 + \tilde{q}^{(n)} J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0) \\
&\quad + (-\tilde{G}^{(n)}\tilde{G}^{(n)\top} J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0) \\
&=: \tilde{h}^{(n)}_w + \tilde{h}^{(n)}_{w^\top} + \tilde{h}^{(n)}_\phi + \tilde{h}^{(n)}_{\phi^\top} + \tilde{h}^{(n)}_q + \tilde{h}^{(n)}_G.
\end{aligned} \tag{4.22}$$

The estimates of $\tilde{h}^{(n)}_w, \tilde{h}^{(n)}_{w^\top}, \tilde{h}^{(n)}_\phi, \tilde{h}^{(n)}_{\phi^\top}$ and $\tilde{h}^{(n)}_q$ in the above are the same as in [4]. Therefore, we only focus on the estimate of $\tilde{h}^{(n)}_G - \tilde{h}_{G_0}$ and write $\tilde{h}_{G_0} - \tilde{h}^{(n)}_G$ in a different way

$$\begin{aligned}
\tilde{h}_{G_0} - \tilde{h}^{(n)}_G &= \tilde{G}^{(n)}\tilde{G}^{(n)\top} J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 - \tilde{G}_0\tilde{G}_0^{(n)\top} J^{-1}\tilde{n}_0 \\
&= (\tilde{G}^{(n)} - \tilde{G}_0)(\tilde{G}^{(n)} - \tilde{G}_0)^\top J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + \tilde{G}_0(\tilde{G}^{(n)} - \tilde{G}_0)^\top J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + (\tilde{G}^{(n)} - \tilde{G}_0)\tilde{G}_0^\top J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + \tilde{G}_0\tilde{G}_0(J(\tilde{X}^{(n)})^{-1} - J^{-1})\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + \tilde{G}_0\tilde{G}_0^\top J^{-1}\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&=: \sum_{i=1}^5 d_i^h.
\end{aligned}$$

We only concentrate on the most difficult term d_1^h. It is worth noting that these products involve the inverse matrix of J, i.e.,

$$J(\tilde{X})^{-1} = \begin{pmatrix} 2\tilde{X}^1 & -2\tilde{X}^2 \\ 2\tilde{X}^2 & 2\tilde{X}^1 \end{pmatrix}. \tag{4.23}$$

This, together with the hypotheses $\tilde{X}^{(n)} \in B(X)$, $\tilde{G}^{(n)} \in B(G)$ and applying Lemmas 4.2, 4.3 and Theorem A.11, we obtain

$$\begin{aligned}
&\|d_1^h\|_{L^2 H^{s-\frac{1}{2}}} \\
&= \|(\tilde{G}^{(n)} - \tilde{G}_0)(\tilde{G}^{(n)} - \tilde{G}_0)^\top J(\tilde{X}^{(n)})^{-1} \nabla_\Lambda \tilde{X}^{(n)} \tilde{n}_0\|_{L^2 H^{s-\frac{1}{2}}} \\
&\leq \|\tilde{G}^{(n)} - \tilde{G}_0\|_{L^2 H^{s-\frac{1}{2}}} \|(\tilde{G}^{(n)} - \tilde{G}_0)^\top\|_{L^\infty H^{s-\frac{1}{2}}} \|J(\tilde{X}^{(n)})^{-1}\|_{L^\infty H^{s-\frac{1}{2}}} \|\nabla_\Lambda \tilde{X}^{(n)} \tilde{n}_0\|_{L^\infty H^{s-\frac{1}{2}}} \\
&\leq CT^{\frac{1}{2}} \|\tilde{G}^{(n)} - \tilde{G}_0\|^2_{L^\infty H^{s-\frac{1}{2}}} \|J(\tilde{X}^{(n)})^{-1}\|_{L^\infty H^{s-\frac{1}{2}}} \|\nabla_\Lambda \tilde{X}^{(n)} \tilde{n}_0\|_{L^\infty H^{s-\frac{1}{2}}} \\
&\leq CT^{\frac{1}{2}} \|\tilde{G}^{(n)} - \tilde{G}_0\|^2_{L^\infty H^s} \|\tilde{X}^{(n)}\|^2_{L^\infty H^s} \\
&\leq C(N, \tilde{v}_0, \tilde{G}_0)T.
\end{aligned}$$

Similarly,

$$\sum_{i=2}^{5} \|d_i^h\|_{L^2 H^{s-\frac{1}{2}}} \leq C(N, \tilde{v}_0, \tilde{G}_0) T^{\frac{3}{4}}.$$

For $\|\tilde{h}_{G_0} - \tilde{h}_G^{(n)}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} L^2}$, we use the following expansion:

$$\begin{aligned}
\tilde{h}_{G_0} - \tilde{h}_G^{(n)} &= \tilde{G}^{(n)} \tilde{G}^{(n)\top} J(\tilde{X}^{(n)})^{-1} \nabla_\Lambda \tilde{X}^{(n)} \tilde{n}_0 - \tilde{G}_0 \tilde{G}_0^{(n)\top} J^{-1} \tilde{n}_0 \\
&= [(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0][(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0]^\top [(J(\tilde{X}^{(n)})^{-1} - J^{-1}) + J^{-1}] \\
&\quad \cdot [(\nabla_\Lambda \tilde{X}^{(n)} - \mathcal{I}) + \mathcal{I}]\tilde{n}_0 - \tilde{G}_0 \tilde{G}_0^{(n)\top} J^{-1} \tilde{n}_0 \\
&=: \sum_{i=1}^{15} \bar{d}_i^h.
\end{aligned}$$

We focus on the most difficult term $\bar{d}_1^h$. From the hypotheses $\tilde{X}^{(n)} \in B(X)$, $\tilde{G}^{(n)} \in B(G)$, (4.23) and applying Lemmas 4.2, 4.3, A.7, A.8 and Theorem A.11, we obtain

$$\begin{aligned}
&\|\bar{d}_1^h\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} L^2} \\
&= \|(\tilde{G}^{(n)} - \tilde{G}_0)(\tilde{G}^{(n)} - \tilde{G}_0)^\top (J(\tilde{X}^{(n)})^{-1} - J^{-1})(\nabla_\Lambda \tilde{X}^{(n)} - \mathcal{I})\tilde{n}_0\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} L^2} \\
&\leq \|(\tilde{G}^{(n)} - \tilde{G}_0)(\tilde{G}^{(n)} - \tilde{G}_0)^\top\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} L^2} \\
&\quad \cdot \|(J(\tilde{X}^{(n)})^{-1} - J^{-1})(\nabla_\Lambda \tilde{X}^{(n)} - \mathcal{I})\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{\frac{1}{2}+\eta}} \\
&\leq \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{\frac{1}{2}+\eta}} \|(\tilde{G}^{(n)} - \tilde{G}_0)^\top\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{\frac{1}{2}-\eta}} \|J(\tilde{X}^{(n)})^{-1} - J^{-1}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{\frac{1}{2}+\eta}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \|\nabla_\Lambda \tilde{X}^{(n)} - \mathcal{I}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{\frac{1}{2}+\eta}} \\
&\le \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{1+\eta}} \|(\tilde{G}^{(n)} - \tilde{G}_0)^\top\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{1-\eta}} \|J(\tilde{X}^{(n)})^{-1} - J^{-1}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{1+\eta}} \\
&\cdot \|\nabla_\Lambda \tilde{X}^{(n)} - \mathcal{I}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{1+\eta}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{2\delta} \|\tilde{X}^{(n)} - \tilde{\omega}\|^2_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} H^{2+\eta}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{4\delta}
\end{aligned}
$$

for $\eta, \delta > 0$ small enough. Similarly,

$$
\|\bar{d}_i^h\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}} L^2} \le C(N, \tilde{v}_0, \tilde{G}_0) T^{\delta_i}
$$

for some $\delta_i > 0, i = 2, 3, \cdots, 15$, and we conclude that

$$
\begin{aligned}
&\|\tilde{h}_w^{(n)}\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}} + \|\tilde{h}_{w^\top}^{(n)}\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}} + \|\tilde{h}_\phi^{(n)}\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}} + \|\tilde{h}_{\phi^\top}^{(n)}\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}} + \|\tilde{h}_q^{(n)}\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}} \\
&\qquad + \|\tilde{h}_G^{(n)} - \tilde{h}_{G_0}\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}} \le C(N, \tilde{v}_0, \tilde{G}_0) T^\beta
\end{aligned} \tag{4.24}
$$

for some $\beta > 0$.

Combining the estimates (4.20), (4.21) and (4.24), we complete the proof of **Part 1.**

Part 2.

Recalling

$$
L(\tilde{w}^{(n+1)} - \tilde{w}^{(n)}, \tilde{q}_w^{(n+1)} - \tilde{q}_w^{(n)}) = (\tilde{f}^{(n)} - \tilde{f}^{(n-1)}, \bar{g}^{(n)} - \bar{g}^{(n-1)}, \tilde{h}^{(n)} - \tilde{h}^{(n-1)})
$$

and applying Lemma 4.1, we have

$$
\begin{aligned}
&\|(\tilde{w}^{(n+1)} - \tilde{w}^{(n)}, \tilde{q}_w^{(n+1)} - \tilde{q}_w^{(n)})\|_{X_0} \\
&= \|L^{-1}(\tilde{f}^{(n)} - \tilde{f}^{(n-1)}, \bar{g}^{(n)} - \bar{g}^{(n-1)}, \tilde{h}^{(n)} - \tilde{h}^{(n-1)})\|_{X_0} \\
&\le C(\|\tilde{f}^{(n)} - \tilde{f}^{(n-1)}\|_{\mathcal{K}_{(0)}^{s-1}} + \|\bar{g}^{(n)} - \bar{g}^{(n-1)}\|_{\bar{\mathcal{K}}_{(0)}^{s}} + \|\tilde{h}^{(n)} - \tilde{h}^{(n-1)}\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}}).
\end{aligned}
$$

Therefore, it is sufficient to estimate $\|\tilde{f}^{(n)} - \tilde{f}^{(n-1)}\|_{\mathcal{K}_{(0)}^{s-1}}$, $\|\bar{g}^{(n)} - \bar{g}^{(n-1)}\|_{\bar{\mathcal{K}}_{(0)}^{s}}$ and $\|\tilde{h}^{(n)} - \tilde{h}^{(n-1)}\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}}$.

Estimate for $\tilde{f}^{(n)} - \tilde{f}^{(n-1)}$.

In (4.19), we expand $\tilde{f}^{(n)}$ into four terms, i.e., $\tilde{f}^{(n)} = \tilde{f}_w^{(n)} + \tilde{f}_\phi^{(n)} + \tilde{f}_q^{(n)} + \tilde{f}_G^{(n)}$. The study for $\tilde{f}^{(n)} - \tilde{f}^{(n-1)}$ involves estimates of $\tilde{f}_w^{(n)} - \tilde{f}_w^{(n-1)}$, $\tilde{f}_\phi^{(n)} - \tilde{f}_\phi^{(n-1)}$ and $\tilde{f}_q^{(n)} - \tilde{f}_q^{(n-1)}$ which can be found in [4, Proposition 5.4]. Therefore, we only concentrate on the estimates $\|\tilde{f}_G^{(n)} - \tilde{f}_G^{(n-1)}\|_{L^2H^{s-1}}$ and $\|\tilde{f}_G^{(n)} - \tilde{f}_G^{(n-1)}\|_{H_{(0)}^{\frac{s-1}{2}}L^2}$.

For the first term, we write it as follows:

$$
\begin{aligned}
\tilde{f}_G^{(n)} - \tilde{f}_G^{(n-1)} &= \nabla\tilde{G}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)} \\
&= \nabla\tilde{G}^{(n)}[(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)}) + \tilde{\zeta}^{(n-1)}][(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})) + J(\tilde{X}^{(n)})]\tilde{G}^{(n)} \\
&= \nabla\tilde{G}^{(n)}[(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J(\tilde{X}^{(n)}) + \tilde{\zeta}^{(n-1)}(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})) \\
&\quad + \tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})]\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)} \\
&= \nabla\tilde{G}^{(n)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J(\tilde{X}^{(n)})\tilde{G}^{(n)} + \nabla\tilde{G}^{(n)}\tilde{\zeta}^{(n-1)}(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})) \\
&\quad \tilde{G}^{(n)} + \nabla\tilde{G}^{(n)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)} \\
&= [(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0) + \nabla\tilde{G}_0](\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J(\tilde{X}^{(n)})[(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + [(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0) + \nabla\tilde{G}_0]\tilde{\zeta}^{(n-1)}(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))[(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + [(\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0) + \nabla\tilde{G}_0]\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})(\tilde{G}^{(n)} - \tilde{G}^{(n-1)}) \\
&\quad + (\nabla\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)})\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})[(\tilde{G}^{(n-1)} - \tilde{G}_0) + \tilde{G}_0] \\
&=: \sum_{i=1}^{4}\bar{d}_i^f + \sum_{i=5}^{8}\bar{d}_i^f + \sum_{i=9}^{10}\bar{d}_i^f + \sum_{i=11}^{12}\bar{d}_i^f.
\end{aligned}
$$

In this process, we first avoid multiplying $(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})$ and $(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))$. Next, for the difference $\tilde{G}^{(n)} - \tilde{G}^{(n-1)}$, we leave just one term as the difference between the iterative steps $n-1$ and n in each product. Finally, to apply the estimates, we subtract their initial values.

We only focus on $\bar{d}_1^f$ and $\bar{d}_9^f$ and the others are similar. For $\bar{d}_1^f$, from $\tilde{X}^{(n)}, \tilde{X}^{(n-1)} \in B(X)$, $\tilde{G}^{(n)} \in B(G)$ and applying Lemmas 4.3, A.1 and A.4, we obtain

$$
\begin{aligned}
&\|\bar{d}_1^f\|_{L^2H^{s-1}} \\
&= \|(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0)(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J(\tilde{X}^{(n)})(\tilde{G}^{(n)} - \tilde{G}_0)\|_{L^2H^{s-1}} \\
&\le \|\nabla\tilde{G}^{(n)} - \nabla\tilde{G}_0\|_{L^\infty H^{s-1}}\|\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)}\|_{L^\infty H^{s-1}}\|J(\tilde{X}^{(n)})\|_{L^\infty H^{s-1}}\|\tilde{G}^{(n)} - \tilde{G}_0\|_{L^2H^{s-1}} \\
&\le T^{\frac{1}{2}}\|\tilde{G}^{(n)} - \tilde{G}_0\|_{L^\infty H^s}^2\|\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)}\|_{L^\infty H^{s-1}}\|J(\tilde{X}^{(n)})\|_{L^\infty H^{s-1}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0)T\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{L^\infty H^s} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0)T^{\frac{5}{4}}\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{L^\infty_{\frac{1}{4}}H^{s+1}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0)T^{\frac{5}{4}}\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}}.
\end{aligned}
$$

Similarly, applying Lemmas 4.3, A.1 and A.3, it follows that

$$
\begin{aligned}
&\|\bar{d}_9^f\|_{L^2H^{s-1}} \\
&= \|(\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0)\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})(\tilde{G}^{(n)} - \tilde{G}^{(n-1)})\|_{L^2H^{s-1}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \|\nabla\tilde{G}^{(n)}-\nabla\tilde{G}_0\|_{L^\infty H^{s-1}}\|\tilde{\zeta}^{(n-1)}\|_{L^\infty H^{s-1}}\|J(\tilde{X}^{(n-1)})\|_{L^\infty H^{s-1}}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{L^2H^{s-1}}\\
&\le T^{\frac12}\|\tilde{G}^{(n)}-\tilde{G}_0\|_{L^\infty H^{s}}\|\tilde{\zeta}^{(n-1)}\|_{L^\infty H^{s-1}}\|J(\tilde{X}^{(n-1)})\|_{L^\infty H^{s-1}}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{L^\infty H^{s-1}}\\
&\le C(N,\tilde{v}_0,\tilde{G}_0)T^{\frac34}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{L^\infty H^{s-1}}\\
&\le C(N,\tilde{v}_0,\tilde{G}_0)T\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{i=2}^{8}\|\bar{d}_i^f\|_{L^2H^{s-1}}\le C(N,\tilde{v}_0,\tilde{G}_0)T^{\delta_1}\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}},\\
&\sum_{i=10}^{12}\|\bar{d}_i^f\|_{L^2H^{s-1}}\le C(N,\tilde{v}_0,\tilde{G}_0)T^{\delta_2}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}},
\end{aligned}
$$

for some $\delta_1,\delta_2>0$.

It remains to show the estimates in $H_{(0)}^{\frac{s-1}{2}}L^2$. We expand $\tilde{f}_G^{(n)}-\tilde{f}_G^{(n-1)}$ as follows:

$$
\begin{aligned}
\tilde{f}_G^{(n)}-\tilde{f}_G^{(n-1)}&=\nabla\tilde{G}^{(n)}\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}-\nabla\tilde{G}^{(n-1)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})\tilde{G}^{(n-1)}\\
&=(\nabla\tilde{G}^{(n)}-\nabla\tilde{G}^{(n-1)})\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}\\
&\quad+\nabla\tilde{G}^{(n-1)}(\tilde{\zeta}^{(n)}-\tilde{\zeta}^{(n-1)})J(\tilde{X}^{(n)})\tilde{G}^{(n)}\\
&\quad+\nabla\tilde{G}^{(n-1)}\tilde{\zeta}^{(n-1)}(J(\tilde{X}^{(n)})-J(\tilde{X}^{(n-1)}))\tilde{G}^{(n)}\\
&\quad+\nabla\tilde{G}^{(n-1)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})(\tilde{G}^{(n)}-\tilde{G}^{(n-1)})\\
&=:\sum_{i=1}^{4}I_i.
\end{aligned}
$$

We only concentrate on the estimates of I_1 and I_3. We write I_1 as follows:

$$
\begin{aligned}
I_1&=(\nabla\tilde{G}^{(n)}-\nabla\tilde{G}^{(n-1)})\tilde{\zeta}^{(n)}J(\tilde{X}^{(n)})\tilde{G}^{(n)}\\
&=(\nabla\tilde{G}^{(n)}-\nabla\tilde{G}^{(n-1)})[(\tilde{\zeta}^{(n)}-\mathcal{I})+\mathcal{I}][(J(\tilde{X}^{(n)})-J)+J][(\tilde{G}^{(n)}-\tilde{G}_0)+\tilde{G}_0]\\
&=:\sum_{i=1}^{8}d_{1,i}^f.
\end{aligned}
$$

To control $d_{1,1}^f=(\nabla\tilde{G}^{(n)}-\nabla\tilde{G}^{(n-1)})(\tilde{\zeta}^{(n)}-\mathcal{I})(J(\tilde{X}^{(n)})-J)(\tilde{G}^{(n)}-\tilde{G}_0)$, we recall the hypotheses $\tilde{X}^{(n-1)},\tilde{X}^{(n)}\in B(X)$, $\tilde{G}^{(n-1)},\tilde{G}^{(n)}\in B(G)$ and apply Lemmas 4.2, 4.3, A.1, A.3 and A.10 to obtain

$$
\|d_{1,1}^f\|_{H_{(0)}^{\frac{s-1}{2}}L^2}
$$

$$
\begin{aligned}
&= \|(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)})(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J)(\tilde{G}^{(n)} - \tilde{G}_0)\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \\
&\le \|(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)})(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J)\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\le \|\nabla\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \|(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J)\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\quad \cdot \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\le \|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s-1}{2}} H^1} \|\tilde{\zeta}^{(n-1)} - \mathcal{I}\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \|J(\tilde{X}^{(n)}) - J\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\quad \cdot \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\le T^{3\delta} \|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}},
\end{aligned}
$$

for $\delta, \eta > 0$ small enough. For $d_{1,8}^f = (\nabla\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)})J\tilde{G}_0$, applying Lemmas A.6 and A.10, we have

$$
\begin{aligned}
\|d_{1,8}^f\|_{H_{(0)}^{\frac{s-1}{2}} L^2} &= \|(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)})J\tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \\
&\le \|(\nabla\tilde{G}^{(n)} - \nabla\tilde{G}^{(n-1)})\|_{H_{(0)}^{\frac{s-1}{2}} L^2} \|J\tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\le C(\tilde{G}_0)\|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s-1}{2}} H^1} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) \left\| \partial_t \int_0^t \tilde{G}^{(n)} - \tilde{G}^{(n-1)} d\tau \right\|_{H_{(0)}^{\frac{s-1}{2}+\delta_1-\delta_1} H^1} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{\delta_1} \|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s-1}{2}+\delta_1} H^1} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{\delta_1} \|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}},
\end{aligned}
$$

for $\delta_1, \eta > 0$ small enough.

Next, we write I_3 as follows:

$$
\begin{aligned}
I_3 &= \nabla\tilde{G}^{(n-1)}\tilde{\zeta}^{(n-1)}(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\tilde{G}^{(n)} \\
&= [(\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0) + \nabla\tilde{G}_0][(\tilde{\zeta}^{(n-1)} - \mathcal{I}) + \mathcal{I}] \\
&\quad \cdot (J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))[(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0] \\
&=: \sum_{i=1}^{8} d_{3,i}^f.
\end{aligned}
$$

For $d^f_{3,1} = (\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0)(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))(\tilde{G}^{(n)} - \tilde{G}_0)$, we apply Lemmas 4.2, 4.3, A.2, A.3 and A.10 to obtain

$$
\begin{aligned}
&\|d^f_{3,1}\|_{H^{\frac{s-1}{2}}_{(0)} L^2} \\
&= \|(\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0)(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))(\tilde{G}^{(n)} - \tilde{G}_0)\|_{H^{\frac{s-1}{2}}_{(0)} L^2} \\
&\le \|(\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0)(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\|_{H^{\frac{s-1}{2}}_{(0)} L^2} \\
&\quad \cdot \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \\
&\le \|\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} L^2} \|(\tilde{\zeta}^{(n-1)} - \mathcal{I})(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \\
&\quad \cdot \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \\
&\le \|\tilde{G}^{(n-1)} - \tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} H^1} \|\tilde{\zeta}^{(n-1)} - \mathcal{I}\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \|J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \\
&\quad \cdot \|\tilde{G}^{(n)} - \tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \\
&\le T^{3\delta} \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}},
\end{aligned}
$$

for $\delta, \eta > 0$ small enough. To control $d^f_{3,8} = \nabla\tilde{G}_0(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\tilde{G}_0$, we apply Lemmas A.2, A.6 and A.10 that

$$
\begin{aligned}
\|d^f_{3,8}\|_{H^{\frac{s-1}{2}}_{(0)} L^2} &= \|\nabla\tilde{G}_0(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} L^2} \\
&\le \|\nabla\tilde{G}_0(J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)}))\|_{H^{\frac{s-1}{2}}_{(0)} L^2} \|\tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \\
&\le \|\nabla\tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} L^2} \|J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \|\tilde{G}_0\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \\
&\le C(\tilde{G}_0) \|J(\tilde{X}^{(n)}) - J(\tilde{X}^{(n-1)})\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\eta}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) \left\| \partial_t \int_0^t \tilde{X}^{(n)} - \tilde{X}^{(n-1)} d\tau \right\|_{H^{\frac{s-1}{2}+\delta_1-\delta_1}_{(0)} H^{1+\eta}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{\delta_1} \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{H^{\frac{s-1}{2}+\delta_1}_{(0)} H^{1+\eta}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{\delta_1} \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}},
\end{aligned}
$$

for $\delta_1, \eta > 0$ small enough.

Similarly, I_2 and I_4 can be expanded as follows:

$$\begin{aligned}I_2 &= \nabla\tilde{G}^{(n-1)}(\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})J(\tilde{X}^{(n)})\tilde{G}^{(n)}\\ &= [(\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0) + \nabla\tilde{G}_0](\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)})\\ &\quad\cdot [(J(\tilde{X}^{(n)} - J) + J)][(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0],\\ I_4 &= \nabla\tilde{G}^{(n-1)}\tilde{\zeta}^{(n-1)}J(\tilde{X}^{(n-1)})(\tilde{G}^{(n)} - \tilde{G}^{(n-1)})\\ &= [(\nabla\tilde{G}^{(n-1)} - \nabla\tilde{G}_0) + \nabla\tilde{G}_0][(\tilde{\zeta}^{(n-1)} - \mathcal{I}) + \mathcal{I}]\\ &\quad\cdot [(J(\tilde{X}^{(n)} - J) + J)](\tilde{G}^{(n)} - \tilde{G}^{(n-1)}),\end{aligned}$$

and we conclude that

$$\begin{aligned}\|I_1\|_{H_{(0)}^{\frac{s-1}{2}}L^2} &\le C(N, \tilde{v}_0, \tilde{G}_0)T^{\delta_1}\|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}},\\ \|I_2\|_{H_{(0)}^{\frac{s-1}{2}}L^2} &\le C(N, \tilde{v}_0, \tilde{G}_0)T^{\delta_2}\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}},\\ \|I_3\|_{H_{(0)}^{\frac{s-1}{2}}L^2} &\le C(N, \tilde{v}_0, \tilde{G}_0)T^{\delta_3}\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}},\\ \|I_4\|_{H_{(0)}^{\frac{s-1}{2}}L^2} &\le C(N, \tilde{v}_0, \tilde{G}_0)T^{\delta_4}\|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}},\end{aligned}$$

for some $\delta_i > 0$.

Estimate for $\tilde{g}^{(n)} - \tilde{g}^{(n-1)}$.

From [4, Proposition 5.4], we have

$$\|\tilde{g}^{(n)} - \tilde{g}^{(n-1)}\|_{\bar{\mathcal{K}}^s_{(0)}} \le C(N, \tilde{v}_0, \tilde{G}_0)T^{\theta}(\|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}} + \|\tilde{w}^{(n)} - \tilde{w}^{(n-1)}\|_{\mathcal{K}^{s+1}_{(0)}}).$$

Estimate for $\tilde{h}^{(n)} - \tilde{h}^{(n-1)}$.

In (4.22), we expand $\tilde{h}^{(n)}$ into six terms $\tilde{h}^{(n)} = \tilde{h}^{(n)}_w + \tilde{h}^{(n)}_{w^\top} + \tilde{h}^{(n)}_\phi + \tilde{h}^{(n)}_{\phi^\top} + \tilde{h}^{(n)}_q + \tilde{h}^{(n)}_G$. We only focus on the estimates of $\tilde{h}^{(n)}_G - \tilde{h}^{(n-1)}_G$ in $L^2H^{s-\frac{1}{2}}$ and $H_{(0)}^{\frac{s}{2}-\frac{1}{4}}L^2$ and the other estimates can be found in [4, Proposition 5.4].

We expand $\|\tilde{h}^{(n-1)}_G - \tilde{h}^{(n)}_G\|_{L^2H^{s-\frac{1}{2}}}$ as follows:

$$\begin{aligned}&\tilde{h}^{(n-1)}_G - \tilde{h}^{(n)}_G\\ &= \tilde{G}^{(n)}\tilde{G}^{(n)\top}J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0 - \tilde{G}^{(n-1)}\tilde{G}^{(n-1)\top}J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda\tilde{X}^{(n-1)}\tilde{n}_0\\ &= \tilde{G}^{(n)}\tilde{G}^{(n)\top}(J(\tilde{X}^{(n)})^{-1} - J(\tilde{X}^{(n-1)})^{-1})\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0\\ &\quad + \tilde{G}^{(n)}\tilde{G}^{(n)\top}J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0\\ &\quad - \tilde{G}^{(n-1)}\tilde{G}^{(n-1)\top}J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda\tilde{X}^{(n-1)}\tilde{n}_0\\ &= [(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0][(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0]^\top(J(\tilde{X}^{(n)})^{-1} - J(\tilde{X}^{(n-1)})^{-1})\nabla_\Lambda\tilde{X}^{(n)}\tilde{n}_0\\ &\quad + \tilde{G}^{(n)}\tilde{G}^{(n)\top}J(\tilde{X}^{(n-1)})^{-1}[(\nabla_\Lambda\tilde{X}^{(n)} - \nabla_\Lambda\tilde{X}^{(n-1)})]\tilde{n}_0\end{aligned}$$

$$
\begin{aligned}
&+ [\tilde{G}^{(n)}\tilde{G}^{(n)\top} - \tilde{G}^{(n-1)}\tilde{G}^{(n-1)\top}] J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda \tilde{X}^{(n-1)}\tilde{n}_0 \\
&= [(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0][(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0]^\top (J(\tilde{X}^{(n)})^{-1} - J(\tilde{X}^{(n-1)})^{-1})\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0 \\
&\quad + [(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0][(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0]^\top J(\tilde{X}^{(n-1)})^{-1}[(\nabla_\Lambda \tilde{X}^{(n)} - \nabla_\Lambda \tilde{X}^{(n-1)})]\tilde{n}_0 \\
&\quad + [(\tilde{G}^{(n)} - \tilde{G}_0) + \tilde{G}_0](\tilde{G}^{(n)\top} - \tilde{G}^{(n-1)\top}) J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda \tilde{X}^{(n-1)}\tilde{n}_0 \\
&\quad + (\tilde{G}^{(n)} - \tilde{G}^{(n-1)})[(\tilde{G}^{(n-1)} - \tilde{G}_0) + \tilde{G}_0]^\top J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda \tilde{X}^{(n-1)}\tilde{n}_0 \\
&=: \sum_{i=1}^{4} \bar{d}_i^h + \sum_{i=4}^{8} \bar{d}_i^h + \sum_{i=9}^{10} \bar{d}_i^h + \sum_{i=11}^{12} \bar{d}_i^h.
\end{aligned}
$$

To obtain the above expression, we first deal with $(J(\tilde{X}^{(n)}))^{-1}$ and $(J(\tilde{X}^{(n-1)}))^{-1}$. Then, we expand $\tilde{G}^{(n)} - \tilde{G}^{(n-1)}$ in each product and subtract the initial values.

We only focus on the estimates of $\bar{d}_1^h$ and $\bar{d}_9^h$ and the others can be deduced similarly. For $\bar{d}_1^h$, from $\tilde{X}^{(n-1)}, \tilde{X}^{(n)} \in B(X)$, $\tilde{G}^{(n)} \in B(G)$, (4.23) and applying Lemmas 4.2 and 4.3, it follows that

$$
\begin{aligned}
&\|\bar{d}_1^h\|_{L^2 H^{s-\frac{1}{2}}} \\
&= \|(\tilde{G}^{(n)} - \tilde{G}_0)(\tilde{G}^{(n)} - \tilde{G}_0)^\top (J(\tilde{X}^{(n)})^{-1} - J(\tilde{X}^{(n-1)})^{-1})\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0\|_{L^2 H^{s-\frac{1}{2}}} \\
&\le \|\tilde{G}^{(n)} - \tilde{G}_0\|_{L^2 H^{s-\frac{1}{2}}} \|(\tilde{G}^{(n)} - \tilde{G}_0)^\top\|_{L^\infty H^{s-\frac{1}{2}}} \|J(\tilde{X}^{(n)})^{-1} - J(\tilde{X}^{(n-1)})^{-1}\|_{L^\infty H^{s-\frac{1}{2}}} \\
&\quad \|\nabla_\Lambda \tilde{X}^{(n)}\tilde{n}_0\|_{L^\infty H^{s-\frac{1}{2}}} \\
&\le CT^{\frac{1}{2}} \|\tilde{G}^{(n)} - \tilde{G}_0\|^2_{L^\infty H^{s-\frac{1}{2}}} \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{L^\infty H^{s-\frac{1}{2}}} \|\tilde{X}^{(n)}\|_{L^\infty H^{s+\frac{1}{2}}} \\
&\le CT^{\frac{1}{2}} \|\tilde{G}^{(n)} - \tilde{G}_0\|^2_{L^\infty H^s} \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{L^\infty H^s} \|\tilde{X}^{(n)}\|_{L^\infty H^{s+1}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{L^\infty H^s} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{\frac{5}{4}} \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{L^\infty_{\frac{1}{4}} H^{s+1}} \\
&\le C(N, \tilde{v}_0, \tilde{G}_0) T^{\frac{5}{4}} \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}}.
\end{aligned}
$$

To control $\bar{d}_9^h$, applying (4.23), Lemmas 4.2 and 4.3, we have

$$
\begin{aligned}
\|\bar{d}_9^h\|_{L^2 H^{s-\frac{1}{2}}} &= \|(\tilde{G}^{(n)} - \tilde{G}_0)(\tilde{G}^{(n)\top} - \tilde{G}^{(n-1)\top}) J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda \tilde{X}^{(n-1)}\tilde{n}_0\|_{L^2 H^{s-\frac{1}{2}}} \\
&\le \|\tilde{G}^{(n)} - \tilde{G}_0\|_{L^2 H^{s-\frac{1}{2}}} \|(\tilde{G}^{(n)} - \tilde{G}^{(n-1)})^\top\|_{L^\infty H^{s-\frac{1}{2}}} \|J(\tilde{X}^{(n-1)})^{-1}\|_{L^\infty H^{s-\frac{1}{2}}} \\
&\quad \cdot \|\nabla_\Lambda \tilde{X}^{(n)}\|_{L^\infty H^{s-\frac{1}{2}}} \\
&\le CT^{\frac{1}{2}} \|\tilde{G}^{(n)} - \tilde{G}_0\|_{L^\infty H^{s-\frac{1}{2}}} \|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{L^\infty H^{s-\frac{1}{2}}} \|\tilde{X}^{(n)}\|^2_{L^\infty H^{s+\frac{1}{2}}} \\
&\le CT^{\frac{1}{2}} \|\tilde{G}^{(n)} - \tilde{G}_0\|_{L^\infty H^s} \|\tilde{G}^{(n)} - \tilde{G}^{(n-1)}\|_{L^\infty H^s} \|\tilde{X}^{(n)}\|_{L^\infty H^{s+1}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C(N,\tilde v_0,\tilde G_0)T^{\frac34}\|\tilde G^{(n)}-\tilde G^{(n-1)}\|_{L^\infty H^s}\\
&\le C(N,\tilde v_0,\tilde G_0)T\|\tilde G^{(n)}-\tilde G^{(n-1)}\|_{L^\infty_{\frac14}H^s}\\
&\le C(N,\tilde v_0,\tilde G_0)T\|\tilde G^{(n)}-\tilde G^{(n-1)}\|_{\mathcal{A}^{s,\gamma}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_{i=2}^{8}\|\bar d_i^h\|_{L^2H^{s-\frac12}}\le C(N,\tilde v_0,\tilde G_0)T^{\delta_1}\|\tilde G^{(n)}-\tilde G^{(n-1)}\|_{\mathcal{A}^{s,\gamma}},\\
&\sum_{i=10}^{12}\|\bar d_i^h\|_{L^2H^{s-\frac12}}\le C(N,\tilde v_0,\tilde G_0)T^{\delta_2}\|\tilde X^{(n)}-\tilde X^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}},
\end{aligned}
$$

for some $\delta_1,\delta_2>0$.

It remains to estimate $\|\tilde h_G^{(n-1)}-\tilde h_G^{(n)}\|_{H_{(0)}^{\frac{s}{2}-\frac14}L^2}$. In each product, we leave only one term as the difference of the iterative steps $n-1$ and n. Then, for the remaining terms, we subtract the initial values. Finally, we expand $\tilde h_G^{(n-1)}-\tilde h_G^{(n)}$ as follows:

$$
\begin{aligned}
&\tilde h_G^{(n-1)}-\tilde h_G^{(n)}\\
&=\tilde G^{(n)}\tilde G^{(n)\top}J(\tilde X^{(n)})^{-1}\nabla_\Lambda\tilde X^{(n)}\tilde n_0-\tilde G^{(n-1)}\tilde G^{(n-1)\top}J(\tilde X^{(n-1)})^{-1}\nabla_\Lambda\tilde X^{(n-1)}\tilde n_0\\
&=[(\tilde G^{(n)}-\tilde G^{(n-1)})]\tilde G^{(n)\top}J(\tilde X^{(n)})^{-1}\nabla_\Lambda\tilde X^{(n)}\tilde n_0+\tilde G^{(n-1)}\tilde G^{(n)\top}J(\tilde X^{(n)})^{-1}\nabla_\Lambda\tilde X^{(n)}\tilde n_0\\
&\quad-\tilde G^{(n-1)}\tilde G^{(n-1)\top}J(\tilde X^{(n-1)})^{-1}\nabla_\Lambda\tilde X^{(n-1)}\tilde n_0\\
&=[(\tilde G^{(n)}-\tilde G^{(n-1)})][(\tilde G^{(n)\top}-\tilde G_0^\top)+\tilde G_0^\top][(J(\tilde X^{(n)})^{-1}-J^{-1})+J^{-1}]\\
&\qquad\cdot[(\nabla_\Lambda\tilde X^{(n)}-\mathcal{I})+\mathcal{I}]\tilde n_0\\
&\quad+\tilde G^{(n-1)}(\tilde G^{(n)\top}J(\tilde X^{(n)})^{-1}\nabla_\Lambda\tilde X^{(n)}\tilde n_0-\tilde G^{(n-1)\top}J(\tilde X^{(n-1)})^{-1}\nabla_\Lambda\tilde X^{(n-1)}\tilde n_0)\\
&=[(\tilde G^{(n)}-\tilde G^{(n-1)})][(\tilde G^{(n)\top}-\tilde G_0^\top)+\tilde G_0^\top][(J(\tilde X^{(n)})^{-1}-J^{-1})+J^{-1}]\\
&\qquad\cdot[(\nabla_\Lambda\tilde X^{(n)}-\mathcal{I})+\mathcal{I}]\tilde n_0\\
&\quad+[(\tilde G^{(n-1)}-\tilde G_0)+\tilde G_0](\tilde G^{(n)}-\tilde G^{(n-1)})^\top[(J(\tilde X^{(n)})^{-1}-J^{-1})+J^{-1}]\\
&\qquad\cdot[(\nabla_\Lambda\tilde X^{(n)}-\mathcal{I})+\mathcal{I}]\tilde n_0\\
&\quad+[(\tilde G^{(n-1)}-\tilde G_0)+\tilde G_0][(\tilde G^{(n-1)}-\tilde G_0)+\tilde G_0]^\top(J(\tilde X^{(n)})^{-1}-J(\tilde X^{(n-1)})^{-1})\\
&\qquad\cdot[(\nabla_\Lambda\tilde X^{(n)}-\mathcal{I})+\mathcal{I}]\tilde n_0\\
&\quad+[(\tilde G^{(n-1)}-\tilde G_0)+\tilde G_0][(\tilde G^{(n-1)}-\tilde G_0)+\tilde G_0]^\top[(J(\tilde X^{(n-1)})^{-1}-J^{-1})+J^{-1}]\\
&\qquad(\nabla_\Lambda\tilde X^{(n)}-\nabla_\Lambda\tilde X^{(n-1)})\tilde n_0\\
&=:\sum_{i=1}^{8}\dot d_i^h+\sum_{i=9}^{16}\dot d_i^h+\sum_{i=17}^{24}\dot d_i^h+\sum_{i=25}^{32}\dot d_i^h
\end{aligned}
$$

by applying

$$\begin{aligned}
&\tilde{G}^{(n)\top}J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda\tilde{X}^{(n)}-\tilde{G}^{(n-1)\top}J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda\tilde{X}^{(n-1)}\\
=&(\tilde{G}^{(n)\top}-\tilde{G}^{(n-1)\top})J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda\tilde{X}^{(n)}+\tilde{G}^{(n-1)\top}J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda\tilde{X}^{(n)}\\
&-\tilde{G}^{(n-1)\top}J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda\tilde{X}^{(n-1)}\\
=&(\tilde{G}^{(n)\top}-\tilde{G}^{(n-1)\top})[(J(\tilde{X}^{(n)})^{-1}-J^{-1})+J^{-1}][(\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I})+\mathcal{I}]\\
&+\tilde{G}^{(n-1)\top}(J(\tilde{X}^{(n)})^{-1}\nabla_\Lambda\tilde{X}^{(n)}-J(\tilde{X}^{(n-1)})^{-1}\nabla_\Lambda\tilde{X}^{(n-1)})\\
=&(\tilde{G}^{(n)\top}-\tilde{G}^{(n-1)\top})[(J(\tilde{X}^{(n)})^{-1}-J^{-1})+J^{-1}][(\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I})+\mathcal{I}]\\
&+\tilde{G}^{(n-1)\top}(J(\tilde{X}^{(n)})^{-1}-J(\tilde{X}^{(n-1)})^{-1})\nabla_\Lambda\tilde{X}^{(n)}\\
&+\tilde{G}^{(n-1)\top}J(\tilde{X}^{(n-1)})^{-1}(\nabla_\Lambda\tilde{X}^{(n)}-\nabla_\Lambda\tilde{X}^{(n-1)}).
\end{aligned}$$

To control $\dot{d}_1^h$, from $\tilde{X}^{(n)}\in B(X)$, $\tilde{G}^{(n-1)},\tilde{G}^{(n)}\in B(G)$, (4.23) and applying Lemmas 4.2, 4.3, A.6, A.8, A.10 and Theorem A.11, we obtain

$$\begin{aligned}
&\|\dot{d}_1^h\|_{H_{(0)}^{\frac{s}{2}-\frac14}L^2}\\
=&\|(\tilde{G}^{(n)}-\tilde{G}^{(n-1)})(\tilde{G}^{(n)\top}-\tilde{G}_0^\top)(J(\tilde{X}^{(n)})^{-1}-J^{-1})(\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I})\tilde{n}_0\|_{H_{(0)}^{\frac{s}{2}-\frac14}L^2}\\
\le&\|(\tilde{G}^{(n)}-\tilde{G}^{(n-1)})(\tilde{G}^{(n)\top}-\tilde{G}_0^\top)\|_{H_{(0)}^{\frac{s}{2}-\frac14}L^2}\\
&\cdot\|(J(\tilde{X}^{(n)})^{-1}-J^{-1})(\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I})\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{\frac12+\eta}}\\
\le&\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{\frac12-\mu}}\|(\tilde{G}^{(n)\top}-\tilde{G}_0^\top)\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{\frac12+\mu}}\\
&\cdot\|J(\tilde{X}^{(n)})^{-1}-J^{-1}\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{\frac12+\eta}}\|\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I}\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{\frac12+\eta}}\\
\le&\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{1-\mu}}\|\tilde{G}^{(n)}-\tilde{G}_0\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{1+\mu}}\|J(\tilde{X}^{(n)})^{-1}-J^{-1}\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{1+\eta}}\\
&\cdot\|\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I}\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{1+\eta}}\\
\le&C\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{1-\mu}}\|\tilde{G}^{(n)}-\tilde{G}_0\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{1+\mu}}\|\tilde{X}^{(n)}-\tilde{\omega}\|^2_{H_{(0)}^{\frac{s}{2}-\frac14}H^{2+\eta}}\\
\le&C(N,\tilde{v}_0,\tilde{G}_0)T^{3\delta}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s}{2}-\frac14}H^{1-\mu}}\\
\le&C(N,\tilde{v}_0,\tilde{G}_0)T^{3\delta}T^{\delta}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{H_{(0)}^{\frac{s}{2}-\frac14+\delta}H^{1-\mu}}\\
\le&C(N,\tilde{v}_0,\tilde{G}_0)T^{4\delta}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}}
\end{aligned}$$

for δ,η and $\mu>0$ small enough.

$\|\dot{d}_{17}^h\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}L^2}$ can be estimated from a similar argument by applying the same lemmas and we obtain

$$
\begin{aligned}
&\|\dot{d}_{17}^h\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}L^2} \\
&= \|(\tilde{G}^{(n-1)}-\tilde{G}_0)(\tilde{G}^{(n-1)}-\tilde{G}_0)^\top(J(\tilde{X}^{(n)})^{-1}-J(\tilde{X}^{(n-1)})^{-1}) \\
&\qquad \cdot(\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I})\tilde{n}_0\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}L^2} \\
&\le \|(\tilde{G}^{(n-1)}-\tilde{G}_0)(\tilde{G}^{(n-1)\top}-\tilde{G}_0^\top)\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}L^2} \\
&\quad \cdot\|(J(\tilde{X}^{(n)})^{-1}-J(\tilde{X}^{(n-1)})^{-1})(\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I})\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{\frac{1}{2}+\eta}} \\
&\le \|\tilde{G}^{(n-1)}-\tilde{G}_0\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{\frac{1}{2}-\mu}}\|\tilde{G}^{(n-1)}-\tilde{G}_0\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{\frac{1}{2}+\mu}} \\
&\quad \cdot\|J(\tilde{X}^{(n)})^{-1}-J(\tilde{X}^{(n-1)})^{-1}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{\frac{1}{2}+\eta}}\|\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{\frac{1}{2}+\eta}} \\
&\le C\|\tilde{G}^{(n-1)}-\tilde{G}_0\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{1-\mu}}\|\tilde{G}^{(n-1)}-\tilde{G}_0\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{1+\mu}}\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{1+\eta}} \\
&\quad \cdot\|\nabla_\Lambda\tilde{X}^{(n)}-\mathcal{I}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{1+\eta}} \\
&\le C(N,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\|\tilde{X}^{(n)}-\tilde{\omega}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{2+\eta}}\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{1+\eta}} \\
&\le C(N,\tilde{v}_0,\tilde{G}_0)T^{3\delta}\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}H^{1+\eta}} \\
&\le C(N,\tilde{v}_0,\tilde{G}_0)T^{3\delta}\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}}
\end{aligned}
$$

for δ,η and $\mu>0$ small enough.

Similarly,

$$
\begin{aligned}
&\sum_{i=2}^{16}\|\dot{d}_i^h\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}L^2}\le C(N,\tilde{v}_0,\tilde{G}_0)T^{\delta_1}\|\tilde{G}^{(n)}-\tilde{G}^{(n-1)}\|_{\mathcal{A}^{s,\gamma}}, \\
&\sum_{i=18}^{32}\|\dot{d}_i^h\|_{H_{(0)}^{\frac{s}{2}-\frac{1}{4}}L^2}\le C(N,\tilde{v}_0,\tilde{G}_0)T^{\delta_2}\|\tilde{X}^{(n)}-\tilde{X}^{(n-1)}\|_{\mathcal{A}^{s+1,\gamma+1}}
\end{aligned}
$$

for some $\delta_1,\delta_2>0$.

Combining the estimates together, we complete the proof of **Part 2**. □

With Propositions 4.8 and 4.9, let us now explain how to prove Proposition 4.6 and Theorem 4.4.

First, the initial iterative sequence meets the hypotheses of **Part 1** in Lemma 4.7, Propositions 4.8 and 4.9. For $T>0$ small enough, we conclude that all the iterative sequences belong to the corresponding spaces and especially, the predetermined balls. Next, the hypotheses of **Part 2** in these lemmas hold for any $n \geq 1$ automatically. As a result, we obtain the estimates as desired. It is clear that for $T > 0$ small enough, the iterative sequence is Cauchy and the same estimates hold when n tends to infinity. In particular, the limit of the iterative sequence solves system (3.5).

5. Stability estimates

In this section, we prove the stability of system (3.5).

Recall $\tilde{\Omega}_0 = P(\Omega_0)$ and let $(\tilde{w}, \tilde{q}_w, \tilde{X}, \tilde{G})$ be a solution in $\tilde{\Omega}_0$ with initial data $\tilde{v}_0$ and $\tilde{G}_0$ as in Theorem 4.4. To prove the stability, we choose a family of initial data $\tilde{v}'_{\varepsilon,0}$, $\tilde{G}'_{\varepsilon,0}$ and define

$$\tilde{\Omega}_\varepsilon(0) = \tilde{\Omega}_0 + \varepsilon b,$$

where $|b| = 1$ is a constant vector such that $P^{-1}(\tilde{\Omega}_\varepsilon(0))$ is not a self-intersecting domain as in Fig. 3 and Fig. 6.

Let $(\tilde{w}'_\varepsilon, \tilde{q}'_{w,\varepsilon}, \tilde{X}'_\varepsilon, \tilde{G}'_\varepsilon)$ be a solution in $\tilde{\Omega}_\varepsilon(0)$ with initial data $\tilde{v}'_{\varepsilon,0}$ and $\tilde{G}'_{\varepsilon,0}$ as in Theorem 4.4. Let N_ε be the constant defined in (4.14). Clearly, to apply Theorem 4.4, we should define a new ϕ'_ε as follows:

$$\phi'_\varepsilon := \tilde{v}'_{\varepsilon,0} + t\hat{\phi}'_\varepsilon := \tilde{v}'_{\varepsilon,0} + t(Q^2\Delta\tilde{v}'_{\varepsilon,0} - J^\top\nabla\tilde{q}'_{\phi,\varepsilon} + \nabla\tilde{G}'_{\varepsilon,0}J\tilde{G}'_{\varepsilon,0}),$$

where $\tilde{q}'_{\phi,\varepsilon}$ solves

$$\begin{cases} -Q^2\Delta\tilde{q}'_{\phi,\varepsilon} = \mathrm{Tr}(\nabla\tilde{v}'_{\varepsilon,0}J\nabla\tilde{v}'_{\varepsilon,0}J) - \mathrm{Tr}(\nabla\tilde{G}'_{\varepsilon,0}J\nabla\tilde{G}'_{\varepsilon,0}J), & \text{in } \tilde{\Omega}_\varepsilon(0), \\ \tilde{q}'_{\phi,\varepsilon}J^{-1}\tilde{n}_0 = (\nabla\tilde{v}'_{\varepsilon,0}J + (\nabla\tilde{v}'_{\varepsilon,0}J)^\top + \tilde{G}'_{\varepsilon,0}\tilde{G}'^{\top}_{\varepsilon,0})J^{-1}\tilde{n}_0, & \text{on } \partial\tilde{\Omega}_\varepsilon(0). \end{cases}$$

To compare the solutions, we first shift the solutions from $\tilde{\Omega}_\varepsilon(0)$ to $\tilde{\Omega}_0$ and define

$$(\tilde{w}_\varepsilon, \tilde{q}_{w,\varepsilon}, \tilde{X}_\varepsilon, \tilde{G}_\varepsilon)(t, \tilde{\omega}) := (\tilde{w}'_\varepsilon, \tilde{q}'_{w,\varepsilon}, \tilde{X}'_\varepsilon, \tilde{G}'_\varepsilon)(t, \tilde{\omega} + \varepsilon b), \quad \tilde{\omega} \in \tilde{\Omega}_0.$$

Similarly, we define ϕ_ε, $\tilde{v}_\varepsilon$ and $\tilde{q}_\varepsilon$. Meanwhile, we choose initial data $\tilde{v}'_{\varepsilon,0}$, $\tilde{G}'_{\varepsilon,0}$ such that

$$(\tilde{v}'_{\varepsilon,0}, \tilde{G}'_{\varepsilon,0})(\tilde{\omega} + \varepsilon b) = (\tilde{v}_0, \tilde{G}_0)(\tilde{\omega}), \quad \tilde{\omega} \in \tilde{\Omega}_0.$$

Therefore, we have

$$(\tilde{v}_\varepsilon, \tilde{G}_\varepsilon)(0, \tilde{\omega}) = (\tilde{v}, \tilde{G})(0, \tilde{\omega}).$$

From Theorem 4.4, letting n tend to infinity in (4.11), (4.12) and (4.13), the solution $(\tilde{w}, \tilde{q}_w, \tilde{X}, \tilde{G})$ solves

$$\begin{cases} \partial_t\tilde{w} - Q^2\Delta\tilde{w} + J^\top\nabla\tilde{q}_w = \tilde{f} + \tilde{f}^L_\phi, \\ \mathrm{Tr}(\nabla\tilde{w}J) = \tilde{g} + \tilde{g}^L_\phi, \\ [-\tilde{q}_w\mathcal{I} + ((\nabla\tilde{w}J) + (\nabla\tilde{w}J)^\top)]J^{-1}\tilde{n}_0 = \tilde{h} + \tilde{h}^L_\phi, \\ \tilde{w}(0,\cdot) = 0, \end{cases}$$

where

$$\begin{aligned}
\tilde{f} =& -Q^2\Delta(\tilde{w}+\phi)+J^\top\nabla(\tilde{q}_w+\tilde{q}_\phi)+Q^2(\tilde{X})\nabla(\nabla(\tilde{w}+\phi)\tilde{\zeta})\tilde{\zeta}\\
&-J(\tilde{X})^\top\tilde{\zeta}^\top\nabla(\tilde{q}_w+\tilde{q}_\phi)+\nabla\tilde{G}\tilde{\zeta}J(\tilde{X})\tilde{G},\\
\tilde{f}^L_\phi =& -\partial_t\phi+Q^2\Delta\phi-J^\top\nabla\tilde{q}_\phi,\\
\tilde{g} =& \operatorname{Tr}(\nabla(\tilde{w}+\phi)J)-\operatorname{Tr}(\nabla(\tilde{w}+\phi)\tilde{\zeta}J(\tilde{X})),\\
\tilde{g}^L_\phi =& -\operatorname{Tr}(\nabla\phi J),\\
\tilde{h} =& -(\tilde{q}_w+\tilde{q}_\phi)J^{-1}\tilde{n}_0+(\tilde{q}_w+\tilde{q}_\phi)(J(\tilde{X}))^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&+((\nabla(\tilde{w}+\phi)J)+(\nabla(\tilde{w}+\phi)J)^\top)J^{-1}\tilde{n}_0\\
&-((\nabla(\tilde{w}+\phi)\tilde{\zeta}J(\tilde{X}))+(\nabla(\tilde{w}+\phi)\tilde{\zeta}J(\tilde{X}))^\top)J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&-\tilde{G}\tilde{G}^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0,\\
\tilde{h}^L_\phi =& \tilde{q}_\phi J^{-1}\tilde{n}_0-(\nabla\phi J+(\nabla\phi J)^\top)J^{-1}\tilde{n}_0,\\
\tilde{G}(t,\tilde{\omega}) =& \tilde{G}_0+\int_0^t\nabla(\tilde{w}(\tau,\tilde{\omega})+\phi(\tau,\tilde{\omega}))\tilde{\zeta}(\tilde{\omega})J(\tilde{X}(\tau,\tilde{\omega}))\tilde{G}(\tau,\tilde{\omega})d\tau,
\end{aligned}$$

and

$$\tilde{X}(t,\tilde{\omega})=\tilde{\omega}+\int_0^t J(\tilde{X}(\tau,\tilde{\omega}))(\tilde{w}(\tau,\tilde{\omega})+\phi(\tau,\tilde{\omega}))d\tau.$$

Similarly, the shifted solution $(\tilde{w}_\varepsilon,\tilde{q}_{w,\varepsilon})$ solves

$$\begin{cases}
\partial_t\tilde{w}_\varepsilon-Q_\varepsilon^2\Delta\tilde{w}_\varepsilon+J_\varepsilon^\top\nabla\tilde{q}_{w,\varepsilon}=\tilde{f}_\varepsilon+\tilde{f}^L_{\phi,\varepsilon},\\
\operatorname{Tr}(\nabla\tilde{w}_\varepsilon J_\varepsilon)=\tilde{g}_\varepsilon+\tilde{g}^L_{\phi,\varepsilon},\\
[-\tilde{q}_{w,\varepsilon}\mathcal{I}+((\nabla\tilde{w}_\varepsilon J_\varepsilon)+(\nabla\tilde{w}_\varepsilon J_\varepsilon)^\top)]J_\varepsilon^{-1}\tilde{n}_0=\tilde{h}_\varepsilon+\tilde{h}^L_{\phi,\varepsilon},\\
\tilde{w}_\varepsilon(0,\cdot)=0,
\end{cases}$$

in $\tilde{\Omega}_0$, where

$$\begin{aligned}
\tilde{f}_\varepsilon =& -Q_\varepsilon^2\Delta(\tilde{w}_\varepsilon+\phi_\varepsilon)+J_\varepsilon^\top\nabla(\tilde{q}_{w,\varepsilon}+\tilde{q}_{\phi,\varepsilon})+Q_\varepsilon^2(\tilde{X}_\varepsilon)\nabla(\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)\tilde{\zeta}_\varepsilon)\tilde{\zeta}_\varepsilon\\
&-J_\varepsilon(\tilde{X}_\varepsilon)^\top\tilde{\zeta}_\varepsilon^\top\nabla(\tilde{q}_{w,\varepsilon}+\tilde{q}_{\phi,\varepsilon})+\nabla\tilde{G}_\varepsilon\tilde{\zeta}_\varepsilon J_\varepsilon(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon,\\
\tilde{f}^L_{\phi,\varepsilon} =& -\partial_t\phi_\varepsilon+Q_\varepsilon^2\Delta\phi_\varepsilon-J_\varepsilon^\top\nabla\tilde{q}_{\phi,\varepsilon},\\
\tilde{g}_\varepsilon =& \operatorname{Tr}(\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)J_\varepsilon)-\operatorname{Tr}(\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)\tilde{\zeta}_\varepsilon J_\varepsilon(\tilde{X}_\varepsilon)),\\
\tilde{g}^L_{\phi,\varepsilon} =& -\operatorname{Tr}(\nabla\phi_\varepsilon J_\varepsilon),\\
\tilde{h}_\varepsilon =& -(\tilde{q}_{w,\varepsilon}+\tilde{q}_{\phi,\varepsilon})J_\varepsilon^{-1}\tilde{n}_0+(\tilde{q}_{w,\varepsilon}+\tilde{q}_{\phi,\varepsilon})(J_\varepsilon(\tilde{X}_\varepsilon))^{-1}\nabla_\Lambda\tilde{X}_\varepsilon\tilde{n}_0\\
&+((\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)J_\varepsilon)+(\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)J_\varepsilon)^\top)J_\varepsilon^{-1}\tilde{n}_0
\end{aligned}$$

$$
\begin{aligned}
&\quad - ((\nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J_\varepsilon(\tilde{X}_\varepsilon)) + (\nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J_\varepsilon(\tilde{X}_\varepsilon))^\top) J_\varepsilon(\tilde{X}_\varepsilon)^{-1}\nabla_\Lambda \tilde{X}_\varepsilon \tilde{n}_0 \\
&\quad - \tilde{G}_\varepsilon \tilde{G}_\varepsilon^\top J_\varepsilon(\tilde{X}_\varepsilon)^{-1}\nabla_\Lambda \tilde{X}_\varepsilon \tilde{n}_0, \\
\tilde{h}^L_{\phi,\varepsilon} &= \tilde{q}_{\phi,\varepsilon} J_\varepsilon^{-1}\tilde{n}_0 - (\nabla\phi_\varepsilon J_\varepsilon + (\nabla\phi_\varepsilon J_\varepsilon)^\top) J_\varepsilon^{-1}\tilde{n}_0, \\
J_\varepsilon(\tilde{\omega}) &:= J(\tilde{\omega} + \varepsilon b), \\
Q^2_\varepsilon(\tilde{\omega}) &:= Q^2(\tilde{\omega} + \varepsilon b),
\end{aligned}
$$

and

$$
\phi_\varepsilon := \tilde{v}_0 + t\hat{\phi}_\varepsilon := \tilde{v}_0 + t(Q^2_\varepsilon \Delta\tilde{v}_0 - J_\varepsilon^\top \nabla\tilde{q}_{\phi,\varepsilon} + \nabla\tilde{G}_0 J_\varepsilon \tilde{G}_0),
$$

where $\tilde{q}_{\phi,\varepsilon}$ solves

$$
\begin{cases}
-Q^2_\varepsilon \Delta\tilde{q}_{\phi,\varepsilon} = \mathrm{Tr}(\nabla\tilde{v}_0 J_\varepsilon \nabla\tilde{v}_0 J_\varepsilon) - \mathrm{Tr}(\nabla\tilde{G}_0 J_\varepsilon \nabla\tilde{G}_0 J_\varepsilon), & \text{in } \tilde{\Omega}_0, \\
\tilde{q}_{\phi,\varepsilon} J_\varepsilon^{-1}\tilde{n}_0 = (\nabla\tilde{v}_0 J_\varepsilon + (\nabla\tilde{v}_0 J_\varepsilon)^\top + \tilde{G}_0\tilde{G}_0^\top) J_\varepsilon^{-1}\tilde{n}_0, & \text{on } \partial\tilde{\Omega}_0.
\end{cases}
$$

Next, we consider the difference $(\tilde{w} - \tilde{w}_\varepsilon, \tilde{q}_w - \tilde{q}_{w,\varepsilon}, \tilde{X} - \tilde{X}_\varepsilon, \tilde{G} - \tilde{G}_\varepsilon)$.
$(\tilde{w} - \tilde{w}_\varepsilon, \tilde{q}_w - \tilde{q}_{w,\varepsilon})$ solves

$$
\begin{cases}
\partial_t(\tilde{w} - \tilde{w}_\varepsilon) - Q^2\Delta(\tilde{w} - \tilde{w}_\varepsilon) + J^\top\nabla(\tilde{q}_w - \tilde{q}_{w,\varepsilon}) = \tilde{F}_\varepsilon, \\
\mathrm{Tr}(\nabla(\tilde{w} - \tilde{w}_\varepsilon)J) = \tilde{K}_\varepsilon, \\
[-(\tilde{q}_w - \tilde{q}_{w,\varepsilon})\mathcal{I} + ((\nabla(\tilde{w} - \tilde{w}_\varepsilon)J) + (\nabla(\tilde{w} - \tilde{w}_\varepsilon)J)^\top)]J^{-1}\tilde{n}_0 = \tilde{H}_\varepsilon, \\
(\tilde{w} - \tilde{w}_\varepsilon)(0,\cdot) = 0,
\end{cases}
$$

in $\tilde{\Omega}_0$, where

$$
\begin{aligned}
\tilde{F}_\varepsilon &= \tilde{f} - \tilde{f}_\varepsilon + \tilde{f}^L_\phi - \tilde{f}^L_{\phi,\varepsilon} + (Q^2 - Q^2_\varepsilon)\Delta\tilde{w}_\varepsilon + (J_\varepsilon^\top - J^\top)\nabla\tilde{q}_{w,\varepsilon}, \\
\tilde{K}_\varepsilon &= \tilde{g} - \tilde{g}_\varepsilon + \tilde{g}^L_\phi - \tilde{g}^L_{\phi,\varepsilon} + \mathrm{Tr}(\nabla\tilde{w}_\varepsilon(J_\varepsilon - J)), \\
\tilde{H}_\varepsilon &= \tilde{h} - \tilde{h}_\varepsilon + \tilde{h}^L_\phi - \tilde{h}^L_{\phi,\varepsilon} + \tilde{q}_{w,\varepsilon}(J^{-1} - J_\varepsilon^{-1})\tilde{n}_0 \\
&\quad + (\nabla\tilde{w}_\varepsilon J_\varepsilon)^\top J_\varepsilon^{-1}\tilde{n}_0 - (\nabla\tilde{w}_\varepsilon J)^\top J^{-1}\tilde{n}_0.
\end{aligned}
$$

For the flux and magnetic field, we have

$$
\begin{cases}
\dfrac{d}{dt}\tilde{X}_\varepsilon(t,\tilde{\omega}) = J(\tilde{X}_\varepsilon(t,\tilde{\omega}))\tilde{v}_\varepsilon(t,\tilde{\omega}), \\
\tilde{X}_\varepsilon(0,\tilde{\omega}) = \tilde{\omega} + \varepsilon b, \quad \text{in } \tilde{\Omega}_0,
\end{cases}
$$

and

$$
\begin{cases}
\partial_t\tilde{G}_\varepsilon(t,\tilde{\omega}) = \nabla\tilde{v}_\varepsilon(t,\tilde{\omega})\tilde{\zeta}_\varepsilon(\tilde{\omega})J(\tilde{X}_\varepsilon(t,\tilde{\omega}))\tilde{G}_\varepsilon(t,\tilde{\omega}), \\
\tilde{G}_\varepsilon(0,\tilde{\omega}) = \tilde{G}_0(\tilde{\omega}), \quad \text{in } \tilde{\Omega}_0.
\end{cases}
$$

It follows that

$$\tilde{G}_\varepsilon = \tilde{G}_0 + \int_0^t \nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon d\tau,$$

$$\tilde{X}_\varepsilon = \tilde{\omega} + \varepsilon b + \int_0^t J(\tilde{X}_\varepsilon)(\tilde{w}_\varepsilon + \phi_\varepsilon) d\tau,$$

and we conclude that $(\tilde{X} - \tilde{X}_\varepsilon, \tilde{G} - \tilde{G}_\varepsilon)$ solves

$$\tilde{G} - \tilde{G}_\varepsilon = \int_0^t [\nabla(\tilde{w} + \phi)\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon] d\tau,$$

$$\tilde{X} - \tilde{X}_\varepsilon = -\varepsilon b + \int_0^t [J(\tilde{X})(\tilde{w} + \phi) - J(\tilde{X}_\varepsilon)(\tilde{w}_\varepsilon + \phi_\varepsilon)] d\tau.$$

We state the main result in the following stability theorem:

Theorem 5.1. *Let $2 < s < \frac{5}{2}$, $1 < \gamma < s - 1$. If $0 < T < C^{-\frac{1}{\delta}}$ for some $\delta > 0$, then*

$$\|\tilde{X} - \tilde{X}_\varepsilon\|_{L^\infty H^{s+1}} \leq C\varepsilon$$

where the constant C depends on the initial data. In particular, we have

$$\operatorname{dist}(\partial\tilde{\Omega}(t), \partial\tilde{\Omega}_\varepsilon(t)) \leq C\varepsilon \tag{5.1}$$

for $t > 0$ small enough.

Thanks to Theorem 4.4, the solution belongs to the predetermined balls whose radii depend on the initial data. As a result, Theorem 5.1 is a consequence of the following results:

Lemma 5.2. *Let $2 < s < \frac{5}{2}$, $1 < \gamma < s - 1$ and $\delta_1 > 0$, it follows that*

(1) $\|J - J_\varepsilon\|_{H^r} \leq C\varepsilon$ *and* $\|Q^2 - Q_\varepsilon^2\|_{H^r} \leq C\varepsilon$ *for* $r \geq 0$.
(2) $\|\phi - \phi_\varepsilon\|_{L^\infty H^{s+1}} \leq C\varepsilon$ *and* $\|\phi - \phi_\varepsilon\|_{H^1_{(0)} H^r} \leq C\varepsilon$ *for smooth initial data* $\tilde{v}_0, \tilde{G}_0$ *and* $r \leq s + 1$.
(3) $\|\tilde{q}_\phi - \tilde{q}_{\phi,\varepsilon}\|_{H^{r+1}} \leq C\varepsilon$ *for* $r \geq 0$.
(4)

$$\begin{aligned}&\|\tilde{X} - \tilde{X}_\varepsilon + \varepsilon b - t(J - J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}} \\ &\leq C\varepsilon + CT^{\delta_1}(\|\tilde{X} - \tilde{X}_\varepsilon + \varepsilon b - (J - J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}} + \|\tilde{w} - \tilde{w}_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}})\end{aligned} \tag{5.2}$$

for some constant C depending on the initial data.

Proof. The idea of the proof is very similar to the one in [4, Lemma 6.1]. It is sufficient to make minor changes to the definitions of ϕ and ϕ_ε, which now rely on both $\tilde{v}_0$ and $\tilde{G}_0$. $\square$

5.1. Estimates for the magnetic field

In this subsection, we estimate the magnetic field $\tilde{G} - \tilde{G}_\varepsilon$ and prove the following proposition:

Proposition 5.3. *Let* $2 < s < \frac{5}{2}$, $1 < \gamma < s - 1$ *and* $\delta_2 > 0$, *we have*

$$\begin{aligned} &\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}} \\ &\le CT^{\delta_2}(\|\tilde{w} - \tilde{w}_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}} + \|\tilde{X} - \tilde{X}_\varepsilon + \varepsilon b - t(J - J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}} \\ &\quad + \|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}}) + C\varepsilon, \end{aligned} \tag{5.3}$$

where the constant C depends on the initial data.

Proof. First, we estimate $\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{L^\infty_{\frac{1}{4}}H^s}$ as follows:

$$\begin{aligned} &\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{L^\infty_{\frac{1}{4}}H^s} \\ &= \sup_{t\in[0,T]} t^{-\frac{1}{4}} \left\| \int_0^t \nabla(\tilde{w} + \phi)\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0 d\tau \right\|_{H^s} \\ &\le \sup_{t\in[0,T]} t^{\frac{1}{4}} \|\nabla(\tilde{w} + \phi)\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{L^2([0,t];H^s)} \\ &\le T^{\frac{1}{4}} \|\nabla(\tilde{w} + \phi)\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{L^2([0,T];H^s)}. \end{aligned}$$

We expand $\nabla(\tilde{w} + \phi)\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0$ into 25 terms:

$$\begin{aligned} &\nabla(\tilde{w} + \phi)\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla(\tilde{w}_\varepsilon + \phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0 \\ &= [\nabla\tilde{w}\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon] + [t\nabla\hat{\phi}\tilde{\zeta} J(\tilde{X})\tilde{G} - t\nabla\hat{\phi}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon] \\ &\quad + [\nabla\tilde{v}_0\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0] \\ &=: \sum_{i=1}^{25} I_i. \end{aligned}$$

More precisely, we leave just one term as the difference of the iterative step $n-1$ and n in each product. Then, we subtract the initial values as before, i.e.,

$$\begin{aligned} &\nabla\tilde{w}\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\ &= \nabla\tilde{w}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon) + (J - J_\varepsilon) + J(\tilde{X}_\varepsilon)]\tilde{G} - \nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\ &= \nabla\tilde{w}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon) + (J - J_\varepsilon)]\tilde{G} + \nabla\tilde{w}[(\tilde{\zeta} - \tilde{\zeta}_\varepsilon) + \tilde{\zeta}_\varepsilon]J(\tilde{X}_\varepsilon)\tilde{G} \end{aligned}$$

$$
\begin{aligned}
&\quad - \nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\
&= \nabla\tilde{w}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)]\tilde{G} + \nabla\tilde{w}\tilde{\zeta}(J - J_\varepsilon)\tilde{G} + \nabla\tilde{w}[(\tilde{\zeta} - \tilde{\zeta}_\varepsilon)]J(\tilde{X}_\varepsilon)\tilde{G} \\
&\quad + \nabla[(\tilde{w} - \tilde{w}_\varepsilon) + \tilde{w}_\varepsilon]\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G} - \nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\
&= \nabla\tilde{w}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)][(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla\tilde{w}\tilde{\zeta}(J - J_\varepsilon)[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla\tilde{w}(\tilde{\zeta} - \tilde{\zeta}_\varepsilon)J(\tilde{X}_\varepsilon)[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla(\tilde{w} - \tilde{w}_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)(\tilde{G} - \tilde{G}_\varepsilon) \\
&=: \sum_{i=1}^{2} I_i + \sum_{i=3}^{4} I_i + \sum_{i=5}^{6} I_i + \sum_{i=7}^{8} I_i + I_9,
\end{aligned}
$$

$$
\begin{aligned}
&\quad t\nabla\hat{\phi}\tilde{\zeta} J(\tilde{X})\tilde{G} - t\nabla\hat{\phi}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\
&= t\nabla\hat{\phi}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon) + (J - J_\varepsilon) + J(\tilde{X}_\varepsilon)]\tilde{G} - t\nabla\hat{\phi}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\
&= t\nabla\hat{\phi}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon) + (J - J_\varepsilon)]\tilde{G} + t\nabla\hat{\phi}[(\tilde{\zeta} - \tilde{\zeta}_\varepsilon) + \tilde{\zeta}_\varepsilon]J(\tilde{X}_\varepsilon)\tilde{G} \\
&\quad - t\nabla\hat{\phi}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\
&= t\nabla\hat{\phi}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)]\tilde{G} + t\nabla\hat{\phi}\tilde{\zeta}(J - J_\varepsilon)\tilde{G} + t\nabla\hat{\phi}(\tilde{\zeta} - \tilde{\zeta}_\varepsilon)J(\tilde{X}_\varepsilon)\tilde{G} \\
&\quad + t\nabla[(\hat{\phi} - \hat{\phi}_\varepsilon) + \hat{\phi}_\varepsilon]\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G} - t\nabla\hat{\phi}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\
&= t\nabla\hat{\phi}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)][(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + t\nabla\hat{\phi}\tilde{\zeta}(J - J_\varepsilon)[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + t\nabla\hat{\phi}(\tilde{\zeta} - \tilde{\zeta}_\varepsilon)J(\tilde{X}_\varepsilon)[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + t\nabla(\hat{\phi} - \hat{\phi}_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + t\nabla\hat{\phi}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)(\tilde{G} - \tilde{G}_\varepsilon) \\
&=: \sum_{i=10}^{11} I_i + \sum_{i=12}^{13} I_i + \sum_{i=14}^{15} I_i + \sum_{i=16}^{17} I_i + I_{18},
\end{aligned}
$$

and

$$
\begin{aligned}
&\quad \nabla\tilde{v}_0\tilde{\zeta} J(\tilde{X})\tilde{G} - \nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0 \\
&= \nabla\tilde{v}_0\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon) + (J - J_\varepsilon) + J(\tilde{X}_\varepsilon)]\tilde{G} \\
&\quad - \nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0 \\
&= \nabla\tilde{v}_0\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon) + (J - J_\varepsilon)]\tilde{G} + \nabla\tilde{v}_0[(\tilde{\zeta} - \tilde{\zeta}_\varepsilon) + \tilde{\zeta}_\varepsilon]J(\tilde{X}_\varepsilon)\tilde{G} \\
&\quad - \nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0 \\
&= \nabla\tilde{v}_0\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)]\tilde{G} + \nabla\tilde{v}_0\tilde{\zeta}(J - J_\varepsilon)\tilde{G} + \nabla\tilde{v}_0(\tilde{\zeta} - \tilde{\zeta}_\varepsilon)J(\tilde{X}_\varepsilon)\tilde{G}
\end{aligned}
$$

$$
\begin{aligned}
&+\nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}-\nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon-\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\\
=&\nabla\tilde{v}_0\tilde{\zeta}[(J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon)][(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&+\nabla\tilde{v}_0\tilde{\zeta}(J-J_\varepsilon)[(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]-\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\\
&+\nabla\tilde{v}_0(\tilde{\zeta}-\tilde{\zeta}_\varepsilon)J(\tilde{X}_\varepsilon)[(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&+\nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)(\tilde{G}-\tilde{G}_\varepsilon)\\
=&\nabla\tilde{v}_0\tilde{\zeta}[(J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon)][(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&+\nabla\tilde{v}_0\tilde{\zeta}(J-J_\varepsilon)(\tilde{G}-\tilde{G}_0)\\
&+\nabla\tilde{v}_0(\tilde{\zeta}-\tilde{\zeta}_\varepsilon)J(\tilde{X}_\varepsilon)[(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&+\nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)(\tilde{G}-\tilde{G}_\varepsilon)\\
&+\nabla\tilde{v}_0(\tilde{\zeta}-\mathcal{I})(J-J_\varepsilon)\tilde{G}_0\\
=:&\sum_{i=19}^{20}I_i+I_{21}+\sum_{i=22}^{23}I_i+I_{24}+I_{25}.
\end{aligned}
$$

We only focus on the estimates of I_1, I_7 and I_{18}. The others are similar since $\tilde{v}_0, \hat{\phi}$ and $\hat{\phi}_\varepsilon$ only depend on the initial data.

To control $T^{\frac{1}{4}}\|I_1\|_{L^2H^s}$, applying Theorem 4.4, Lemmas 4.3, 5.2, A.2 and A.3, we obtain

$$
\begin{aligned}
&T^{\frac{1}{4}}\|I_1\|_{L^2H^s}\\
=&T^{\frac{1}{4}}\|\nabla\tilde{w}\tilde{\zeta}[(J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon)](\tilde{G}-\tilde{G}_0)\|_{L^2H^s}\\
\le&T^{\frac{1}{4}}\|\nabla\tilde{w}\|_{L^2H^s}\|\tilde{\zeta}\|_{L^\infty H^s}\|J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon\|_{L^\infty H^s}\|\tilde{G}-\tilde{G}_0\|_{L^\infty H^s}\\
\le&C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}\|J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon\|_{L^\infty H^s}\\
\le&C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}\\
&\cdot(\|J(\tilde{X}+\varepsilon b)-J(\tilde{X}_\varepsilon)\|_{L^\infty H^s}+\|J(\tilde{X})-J(\tilde{X}+\varepsilon b)+J_\varepsilon-J\|_{L^\infty H^s})\\
\le&C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}\\
&\cdot(\|\tilde{X}+\varepsilon b-\tilde{X}_\varepsilon\|_{L^\infty H^s}+\|J(\tilde{X})-J(\tilde{X}+\varepsilon b)\|_{L^\infty H^s}+\|J_\varepsilon-J\|_{L^\infty H^s})\\
\le&C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}(\|\tilde{X}-\tilde{X}_\varepsilon+\varepsilon b\|_{L^\infty H^s}+C\varepsilon)\\
\le&C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}(\|\tilde{X}-\tilde{X}_\varepsilon+\varepsilon b-t(J-J_\varepsilon)\tilde{v}_0\|_{L^\infty H^s}+C(\tilde{v}_0)\varepsilon)\\
\le&C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}\|\tilde{X}-\tilde{X}_\varepsilon+\varepsilon b-t(J-J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}}+C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{\frac{1}{2}}\varepsilon.
\end{aligned}
$$

For $T^{\frac{1}{4}}\|I_7\|_{L^2H^s}$, applying Theorem 4.4, Lemmas 4.3, A.1 and A.3, it follows that

$$
\begin{aligned}
T^{\frac{1}{4}}\|I_7\|_{L^2H^s}&=T^{\frac{1}{4}}\|\nabla(\tilde{w}-\tilde{w}_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)(\tilde{G}-\tilde{G}_0)\|_{L^2H^s}\\
&\le T^{\frac{1}{4}}\|\tilde{w}-\tilde{w}_\varepsilon\|_{L^2H^{s+1}}\|\tilde{\zeta}_\varepsilon\|_{L^\infty H^s}\|J(\tilde{X}_\varepsilon)\|_{L^\infty H^s}\|(\tilde{G}-\tilde{G}_0)\|_{L^\infty H^s}
\end{aligned}
$$

$$\leq C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{\frac{1}{2}}\|\tilde{w}-\tilde{w}_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}}.$$

$T^{\frac{1}{4}}\|I_{18}\|_{L^2H^s}$ can be estimated by using Theorem 4.4, Lemmas 5.2, A.1 and A.3

$$\begin{aligned}
T^{\frac{1}{4}}\|I_{18}\|_{L^2H^s} &= T^{\frac{1}{4}}\|t\nabla\hat{\phi}_\varepsilon\|_{L^2H^s}\|\tilde{\zeta}_\varepsilon\|_{L^\infty H^s}\|J(\tilde{X}_\varepsilon)\|_{L^\infty H^s}(\|(\tilde{G}-\tilde{G}_\varepsilon)-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{L^\infty H^s}\\
&\quad+\|t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{L^\infty H^s})\\
&\leq C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{\frac{7}{4}}(\|(\tilde{G}-\tilde{G}_\varepsilon)-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}}+\varepsilon)\\
&\leq C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{\frac{7}{4}}\|(\tilde{G}-\tilde{G}_\varepsilon)-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}}.
\end{aligned}$$

Next, we control $\|\tilde{G}-\tilde{G}_\varepsilon-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{H^2_{(0)}H^\gamma}$. Applying Lemma A.6, we obtain

$$\begin{aligned}
&\|\tilde{G}-\tilde{G}_\varepsilon-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{H^2_{(0)}H^\gamma}\\
&=\left\|\int_0^t \nabla(\tilde{w}+\phi)\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon-\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0 d\tau\right\|_{H^2_{(0)}H^\gamma}\\
&\leq\|\nabla(\tilde{w}+\phi)\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon-\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{H^1_{(0)}H^\gamma}.
\end{aligned}$$

Then, in a different way, we write $\nabla(\tilde{w}+\phi)\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon-\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0$ as follows:

$$\begin{aligned}
&\nabla(\tilde{w}+\phi)\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla(\tilde{w}_\varepsilon+\phi_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon-\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\\
&=(\nabla\tilde{w}\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon)+(t\nabla\hat{\phi}\tilde{\zeta}J(\tilde{X})\tilde{G}-t\nabla\hat{\phi}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon)\\
&\quad+(\nabla\tilde{v}_0\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla\tilde{v}_0\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon-\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0).
\end{aligned}$$

We only focus on $\nabla\tilde{w}\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon$ and the others are similar since $\tilde{v}_0$, $\hat{\phi}$ and $\hat{\phi}_\varepsilon$ depend on the initial data.

In fact, we expand $\nabla\tilde{w}\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon$ into 24 terms as follows:

$$\begin{aligned}
&\nabla\tilde{w}\tilde{\zeta}J(\tilde{X})\tilde{G}-\nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon\\
&=\nabla\tilde{w}\tilde{\zeta}[(J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon)][(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]+\nabla\tilde{w}\tilde{\zeta}(J-J_\varepsilon)[(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&\quad+\nabla\tilde{w}[(\tilde{\zeta}-\tilde{\zeta}_\varepsilon)]J(\tilde{X}_\varepsilon)[(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]+\nabla(\tilde{w}-\tilde{w}_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)[(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&\quad+\nabla\tilde{w}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)(\tilde{G}-\tilde{G}_\varepsilon)\\
&=\nabla\tilde{w}[(\tilde{\zeta}-\mathcal{I})+\mathcal{I}][(J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon)][(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&\quad+\nabla\tilde{w}[(\tilde{\zeta}-\mathcal{I})+\mathcal{I}](J-J_\varepsilon)[(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&\quad+\nabla\tilde{w}(\tilde{\zeta}-\tilde{\zeta}_\varepsilon)[(J(\tilde{X}_\varepsilon)-J_\varepsilon)+J_\varepsilon][(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]\\
&\quad+\nabla(\tilde{w}-\tilde{w}_\varepsilon)[(\tilde{\zeta}_\varepsilon-\mathcal{I})+\mathcal{I}][(J(\tilde{X}_\varepsilon)-J_\varepsilon)+J_\varepsilon][(\tilde{G}-\tilde{G}_0)+\tilde{G}_0]
\end{aligned}$$

$$
\begin{aligned}
&+\nabla\tilde{w}_\varepsilon[(\tilde{\zeta}_\varepsilon-\mathcal{I})+\mathcal{I}][(J(\tilde{X}_\varepsilon)-J_\varepsilon)+J_\varepsilon](\tilde{G}-\tilde{G}_\varepsilon)\\
=:&\sum_{i=1}^{4}I\!I_i+\sum_{i=5}^{8}I\!I_i+\sum_{i=9}^{12}I\!I_i+\sum_{i=13}^{20}I\!I_i+\sum_{i=21}^{24}I\!I_i.
\end{aligned}
$$

We concentrate on the estimates of $\|I\!I_1\|_{H^1_{(0)}H^\gamma}$ and $\|I\!I_{24}\|_{H^1_{(0)}H^\gamma}$. For $\|I\!I_1\|_{H^1_{(0)}H^\gamma}$, applying Lemmas 4.2, 4.3, 5.2, A.2, A.3, A.10, and Theorem 4.4, we have

$$
\begin{aligned}
&\|I\!I_1\|_{H^1_{(0)}H^\gamma}\\
=&\|\nabla\tilde{w}\|_{H^1_{(0)}H^\gamma}\|\tilde{\zeta}-\mathcal{I}\|_{H^1_{(0)}H^\gamma}\|J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon\|_{H^1_{(0)}H^\gamma}\|\tilde{G}-\tilde{G}_0\|_{H^1_{(0)}H^\gamma}\\
\le& C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\|J(\tilde{X})-J(\tilde{X}_\varepsilon)-J+J_\varepsilon\|_{H^1_{(0)}H^\gamma}\\
\le& C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\\
&\cdot(\|J(\tilde{X}+\varepsilon b)-J(\tilde{X}_\varepsilon)\|_{H^1_{(0)}H^\gamma}+\|J(\tilde{X})-J(\tilde{X}+\varepsilon b)+J_\varepsilon-J\|_{H^1_{(0)}H^\gamma})\\
\le& C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\\
&\cdot(\|\tilde{X}+\varepsilon b-\tilde{X}_\varepsilon\|_{H^1_{(0)}H^\gamma}+\|J(\tilde{X})-J(\tilde{X}+\varepsilon b)\|_{H^1_{(0)}H^\gamma}+\|J_\varepsilon-J\|_{H^1_{(0)}H^\gamma})\\
\le& C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}(\|\tilde{X}+\varepsilon b-\tilde{X}_\varepsilon-t(J-J_\varepsilon)\tilde{v}_0+t(J-J_\varepsilon)\tilde{v}_0\|_{H^1_{(0)}H^\gamma})\\
&+C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\varepsilon\\
\le& C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}(\|\tilde{X}+\varepsilon b-\tilde{X}_\varepsilon-t(J-J_\varepsilon)\tilde{v}_0\|_{H^1_{(0)}H^\gamma}+\|t(J-J_\varepsilon)\tilde{v}_0\|_{H^1_{(0)}H^\gamma})\\
&+C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\varepsilon\\
\le& C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\|\tilde{X}+\varepsilon b-\tilde{X}_\varepsilon-t(J-J_\varepsilon)\tilde{v}_0\|_{H^1_{(0)}H^\gamma}+C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\varepsilon\\
\le& C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\|\tilde{X}+\varepsilon b-\tilde{X}_\varepsilon-t(J-J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}}\\
&+C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)T^{2\delta}\varepsilon,
\end{aligned}
$$

for $\delta>0$ small enough. $\|I\!I_{24}\|_{H^1_{(0)}H^\gamma}$ can be estimated by using Theorem 4.4, Lemmas 5.2, A.1 and A.10

$$
\begin{aligned}
\|I\!I_{24}\|_{H^1_{(0)}H^\gamma}&=\|\nabla\tilde{w}_\varepsilon\mathcal{I}J_\varepsilon(\tilde{G}-\tilde{G}_\varepsilon)\|_{H^1_{(0)}H^\gamma}\\
&\le\|\nabla\tilde{w}_\varepsilon\|_{H^1_{(0)}H^\gamma}\|J_\varepsilon\|_{H^1_{(0)}H^\gamma}\|\tilde{G}-\tilde{G}_\varepsilon\|_{H^1_{(0)}H^\gamma}\\
&\le C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\|\tilde{G}-\tilde{G}_\varepsilon-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{H^1_{(0)}H^\gamma}\\
&\quad+C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\|t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{H^1_{(0)}H^\gamma}\\
&\le C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\left\|\int_0^t\partial_\tau(\tilde{G}-\tilde{G}_\varepsilon-\tau\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0)d\tau\right\|_{H^{1+\eta-\delta}_{(0)}H^\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&+C(N,N_\varepsilon,\tilde v_0,\tilde G_0)\varepsilon\\
&\le C(N,N_\varepsilon,\tilde v_0,\tilde G_0)T^\delta\|\tilde G-\tilde G_\varepsilon-t\nabla\tilde v_0(J-J_\varepsilon)\tilde G_0\|_{\mathcal{A}^{s,\gamma}}+C(N,N_\varepsilon,\tilde v_0,\tilde G_0)\varepsilon.
\end{aligned}
$$

We can control the other terms similarly.

Combining the above calculations, we complete the proof. □

5.2. Estimates for the velocity field and pressure

In this subsection, we estimate the velocity and pressure $(\tilde w-\tilde w_\varepsilon,\tilde q_w-\tilde q_{w,\varepsilon})$ and prove the following proposition:

Proposition 5.4. *Let $2<s<\frac{5}{2}$, $1<\gamma<s-1$ and $\delta_3>0$, we have*

$$
\begin{aligned}
\|\tilde w-\tilde w_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}}+\|\tilde q_w-\tilde q_{w,\varepsilon}\|_{\mathcal{K}^s_{pr}(0)}\le C\varepsilon+CT^{\delta_3}(&\|\tilde w-\tilde w_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}}+\|\tilde q_w-\tilde q_{w,\varepsilon}\|_{\mathcal{K}^s_{pr}(0)}\\
&+\|\tilde X-\tilde X_\varepsilon+\varepsilon b-t(J-J_\varepsilon)\tilde v_0\|_{\mathcal{A}^{s+1,\gamma+1}}\\
&+\|\tilde G-\tilde G_\varepsilon-t\nabla\tilde v_0(J-J_\varepsilon)\tilde G_0\|_{\mathcal{A}^{s,\gamma}}),
\end{aligned}
\tag{5.4}
$$

where the constant C depends on the initial data.

Proof. Due to

$$
(\tilde w-\tilde w_\varepsilon,\tilde q_w-\tilde q_{w,\varepsilon})=L^{-1}(\tilde F_\varepsilon,\tilde K_\varepsilon,\tilde H_\varepsilon,0),
$$

from Lemma 4.1, it follows that

$$
\|\tilde w-\tilde w_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}}+\|\tilde q_w-\tilde q_{w,\varepsilon}\|_{\mathcal{K}^s_{pr}(0)}\le C(\|\tilde F_\varepsilon\|_{\mathcal{K}^{s-1}_{(0)}}+\|\tilde K_\varepsilon\|_{\bar{\mathcal{K}}^s_{(0)}}+\|\tilde H_\varepsilon\|_{\mathcal{K}^{s-\frac12}_{(0)}}).
$$

Therefore, it is sufficient to estimate $\|\tilde F_\varepsilon\|_{\mathcal{K}^{s-1}_{(0)}}$, $\|\tilde K_\varepsilon\|_{\bar{\mathcal{K}}^s_{(0)}}$ and $\|\tilde H_\varepsilon\|_{\mathcal{K}^{s-\frac12}_{(0)}}$.

Estimate for $\tilde F_\varepsilon$.

Recall that

$$
\tilde F_\varepsilon=\tilde f-\tilde f_\varepsilon+\tilde f^L_\phi-\tilde f^L_{\phi,\varepsilon}+(Q^2-Q^2_\varepsilon)\Delta\tilde w_\varepsilon+(J_\varepsilon^\top-J^\top)\nabla\tilde q_{w,\varepsilon}
$$

and we analyze $(Q^2-Q^2_\varepsilon)\Delta\tilde w_\varepsilon+(J_\varepsilon^\top-J^\top)\nabla\tilde q_{w,\varepsilon}$ first.

Applying Lemma 5.2, we have

$$
\begin{aligned}
&\|(Q^2-Q^2_\varepsilon)\Delta\tilde w_\varepsilon\|_{L^2H^{s-1}}\le\|Q^2-Q^2_\varepsilon\|_{L^\infty H^{s-1}}\|\tilde w_\varepsilon\|_{L^2H^{s+1}}\le C\varepsilon,\\
&\|(J^\top-J_\varepsilon^\top)\nabla\tilde q_{w,\varepsilon}\|_{L^2H^{s-1}}\le\|J^\top-J_\varepsilon^\top\|_{L^\infty H^{s-1}}\|\tilde q_{w,\varepsilon}\|_{L^2H^s}\le C\varepsilon.
\end{aligned}
$$

Then, for $\|(Q^2-Q^2_\varepsilon)\Delta\tilde w_\varepsilon\|_{H^{\frac{s-1}{2}}_{(0)}L^2}$ and $\|(J^\top-J_\varepsilon^\top)\nabla\tilde q_{w,\varepsilon}\|_{H^{\frac{s-1}{2}}_{(0)}L^2}$, we apply Lemmas A.5, A.7 and Theorem 4.4 to obtain

$$
\begin{aligned}
\|(Q^2 - Q_\varepsilon^2)\Delta\tilde{w}_\varepsilon\|_{H_{(0)}^{\frac{s-1}{2}}L^2} &\le \|Q^2 - Q_\varepsilon^2\|_{H^{1+\eta}}\|\Delta\tilde{w}_\varepsilon\|_{H_{(0)}^{\frac{s-1}{2}}L^2} \\
&\le C\varepsilon\|\tilde{w}_\varepsilon\|_{H_{(0)}^{\frac{s-1}{2}}H^2} \le C\varepsilon, \\
\|(J^\top - J_\varepsilon^\top)\nabla\tilde{q}_{w,\varepsilon}\|_{H_{(0)}^{\frac{s-1}{2}}L^2} &\le \|J^\top - J_\varepsilon^\top\|_{H^{1+\eta}}\|\nabla\tilde{q}_{w,\varepsilon}\|_{H_{(0)}^{\frac{s-1}{2}}L^2} \le C\varepsilon.
\end{aligned}
$$

Next, we pay attention to the estimate of $\tilde{f} - \tilde{f}_\varepsilon + \tilde{f}_\phi^L - \tilde{f}_{\phi,\varepsilon}^L$. First, we rewrite $\tilde{f}_\phi^L - \tilde{f}_{\phi,\varepsilon}^L$ as follows:

$$
\begin{aligned}
\tilde{f}_\phi^L - \tilde{f}_{\phi,\varepsilon}^L &= \partial_t\phi_\varepsilon - \partial_t\phi + (Q^2\Delta\phi - Q_\varepsilon^2\Delta\phi_\varepsilon) + (J_\varepsilon^\top\nabla\tilde{q}_{\phi,\varepsilon} - J^\top\nabla\tilde{q}_\phi) \\
&= Q_\varepsilon^2\Delta\tilde{v}_0 + \nabla\tilde{G}_0 J_\varepsilon\tilde{G}_0 - Q^2\Delta\tilde{v}_0 - \nabla\tilde{G}_0 J\tilde{G}_0 + (Q^2\Delta\phi - Q_\varepsilon^2\Delta\phi_\varepsilon) \\
&= \nabla\tilde{G}_0 J_\varepsilon\tilde{G}_0 - \nabla\tilde{G}_0 J\tilde{G}_0 + t(Q^2\Delta\hat{\phi} - Q_\varepsilon^2\Delta\hat{\phi}_\varepsilon) \\
&= \nabla\tilde{G}_0(J_\varepsilon - J)\tilde{G}_0 + t(Q^2 - Q_\varepsilon^2)\Delta\hat{\phi} + tQ_\varepsilon^2(\Delta\hat{\phi} - \Delta\hat{\phi}_\varepsilon).
\end{aligned}
$$

Then, we expand $\tilde{f} - \tilde{f}_\varepsilon + \tilde{f}_\phi^L - \tilde{f}_{\phi,\varepsilon}^L$ into 4 terms

$$
\begin{aligned}
\tilde{f} - \tilde{f}_\varepsilon + \tilde{f}_\phi^L - \tilde{f}_{\phi,\varepsilon}^L &= \tilde{f}_w - \tilde{f}_{w,\varepsilon} + \tilde{f}_\phi - \tilde{f}_{\phi,\varepsilon} + \tilde{f}_q - \tilde{f}_{q,\varepsilon} + \tilde{f}_G - \tilde{f}_{G,\varepsilon} + \tilde{f}_\phi^L - \tilde{f}_{\phi,\varepsilon}^L \\
&= (\tilde{f}_w - \tilde{f}_{w,\varepsilon} + \tilde{f}_\phi - \tilde{f}_{\phi,\varepsilon} + \tilde{f}_q - \tilde{f}_{q,\varepsilon}) + (\tilde{f}_G - \tilde{f}_{G,\varepsilon} + \nabla\tilde{G}_0(J_\varepsilon - J)\tilde{G}_0) \\
&\quad + t(Q^2 - Q_\varepsilon^2)\Delta\hat{\phi} + tQ_\varepsilon^2(\Delta\hat{\phi} - \Delta\hat{\phi}_\varepsilon) \\
&=: \sum_{i=1}^4 I_i.
\end{aligned}
$$

We point out that I_1 has already been studied in [4, Lemma 6.2]. For I_3 and I_4, all the terms depend on the initial data and we have a t in front of these terms. Thus, for $T > 0$ small enough, from Lemma 5.2, it follows that

$$
\|I_3\|_{\mathcal{K}_{(0)}^{s-1}} + \|I_4\|_{\mathcal{K}_{(0)}^{s-1}} \le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)\varepsilon.
$$

To control $\|I_2\|_{L^2H^{s-1}}$, from (4.19), we have

$$
I_2 = \nabla\tilde{G}\tilde{\zeta}J(\tilde{X})\tilde{G} - \nabla\tilde{G}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon + \nabla\tilde{G}_0(J_\varepsilon - J)\tilde{G}_0
$$

and we expand I_2 as follows:

$$
\begin{aligned}
I_2 &= \nabla\tilde{G}\tilde{\zeta}J(\tilde{X})\tilde{G} - \nabla\tilde{G}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon + \nabla\tilde{G}_0(J_\varepsilon - J)\tilde{G}_0 \\
&= \nabla\tilde{G}\tilde{\zeta}[(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon) + (J - J_\varepsilon) + J(\tilde{X}_\varepsilon)]\tilde{G} \\
&\quad - \nabla\tilde{G}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon + \nabla\tilde{G}_0(J_\varepsilon - J)\tilde{G}_0 \\
&= \nabla\tilde{G}\tilde{\zeta}(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)\tilde{G} + \nabla\tilde{G}\tilde{\zeta}(J - J_\varepsilon)\tilde{G} \\
&\quad + \nabla\tilde{G}\tilde{\zeta}J(\tilde{X}_\varepsilon)\tilde{G} - \nabla\tilde{G}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon + \nabla\tilde{G}_0(J_\varepsilon - J)\tilde{G}_0
\end{aligned}
$$

$$
\begin{aligned}
&= \nabla\tilde{G}\tilde{\zeta}(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)\tilde{G} + \nabla\tilde{G}\tilde{\zeta}(J - J_\varepsilon)\tilde{G} + \nabla\tilde{G}\tilde{\zeta}J(\tilde{X}_\varepsilon)(\tilde{G} - \tilde{G}_\varepsilon) \\
&\quad + \nabla\tilde{G}\tilde{\zeta}J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon - \nabla\tilde{G}_\varepsilon\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon + \nabla\tilde{G}_0(J_\varepsilon - J)\tilde{G}_0 \\
&= \nabla\tilde{G}\tilde{\zeta}(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)\tilde{G} + \nabla\tilde{G}\tilde{\zeta}(J - J_\varepsilon)\tilde{G} - \nabla\tilde{G}_0(J - J_\varepsilon)\tilde{G}_0 \\
&\quad + \nabla\tilde{G}\tilde{\zeta}J(\tilde{X}_\varepsilon)(\tilde{G} - \tilde{G}_\varepsilon) + \nabla\tilde{G}(\tilde{\zeta} - \tilde{\zeta}_\varepsilon)J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon + \nabla(\tilde{G} - \tilde{G}_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)\tilde{G}_\varepsilon \\
&= \nabla[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0]\tilde{\zeta}(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon)[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0]\tilde{\zeta}(J - J_\varepsilon)[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0] - \nabla\tilde{G}_0\tilde{\zeta}(J - J_\varepsilon)\tilde{G}_0 \\
&\quad + \nabla\tilde{G}_0(\tilde{\zeta} - \mathcal{I})(J - J_\varepsilon)\tilde{G}_0 \\
&\quad + \nabla[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0]\tilde{\zeta}J(\tilde{X}_\varepsilon)(\tilde{G} - \tilde{G}_\varepsilon) \\
&\quad + \nabla[(\tilde{G} - \tilde{G}_0) + \tilde{G}_0](\tilde{\zeta} - \tilde{\zeta}_\varepsilon)J(\tilde{X}_\varepsilon)[(\tilde{G}_\varepsilon - \tilde{G}_0) + \tilde{G}_0] \\
&\quad + \nabla(\tilde{G} - \tilde{G}_\varepsilon)\tilde{\zeta}_\varepsilon J(\tilde{X}_\varepsilon)[(\tilde{G}_\varepsilon - \tilde{G}_0) + \tilde{G}_0] \\
&=: \sum_{i=1}^{4} I_{2,i} + \sum_{i=5}^{7} I_{2,i} + I_{2,8} + \sum_{i=9}^{10} I_{2,i} + \sum_{i=11}^{14} I_{2,i} + \sum_{i=15}^{16} I_{2,i}.
\end{aligned}
$$

To obtain the above expression, we leave just one term as the difference of the iterative steps in each product and we subtract the initial values.

For $\|I_{2,1}\|_{L^2H^{s-1}}$, applying Lemmas 4.3, 5.2, A.2 and A.3, we have

$$
\begin{aligned}
&\|I_{2,1}\|_{L^2H^{s-1}} \\
&\le \|\nabla(\tilde{G} - \tilde{G}_0)\|_{L^\infty H^{s-1}}\|\tilde{\zeta}\|_{L^\infty H^{s-1}}\|J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon\|_{L^\infty H^{s-1}}\|\tilde{G} - \tilde{G}_0\|_{L^2H^{s-1}} \\
&\le T^{\frac{1}{2}}\|\tilde{G} - \tilde{G}_0\|_{L^\infty H^s}\|\tilde{\zeta}\|_{L^\infty H^{s-1}}\|J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon\|_{L^\infty H^{s-1}}\|\tilde{G} - \tilde{G}_0\|_{L^\infty H^{s-1}} \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T\|J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon\|_{L^\infty H^{s-1}} \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T[\|J(\tilde{X} + \varepsilon b) - J(\tilde{X}_\varepsilon)\|_{L^\infty H^{s-1}} \\
&\quad + \|J(\tilde{X}) - J(\tilde{X}_\varepsilon + \varepsilon b) - J + J_\varepsilon\|_{L^\infty H^{s-1}}] \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T[\|\tilde{X} + \varepsilon b - \tilde{X}_\varepsilon\|_{L^\infty H^{s-1}} + \|J(\tilde{X}) - J\|_{L^\infty H^{s-1}} \\
&\quad + \|J(\tilde{X}_\varepsilon + \varepsilon b) - J_\varepsilon\|_{L^\infty H^{s-1}}] \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T\|\tilde{X} + \varepsilon b - \tilde{X}_\varepsilon\|_{L^\infty H^{s-1}} + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T\varepsilon \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T[\|\tilde{X} + \varepsilon b - \tilde{X}_\varepsilon - t(J - J_\varepsilon)\tilde{v}_0\|_{L^\infty H^{s-1}} + \|t(J - J_\varepsilon)\tilde{v}_0\|_{L^\infty H^{s-1}}] \\
&\quad + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T\varepsilon \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T\|\tilde{X} + \varepsilon b - \tilde{X}_\varepsilon - t(J - J_\varepsilon)\tilde{v}_0\|_{L^\infty H^{s-1}} + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T\varepsilon \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{\frac{5}{4}}\|\tilde{X} + \varepsilon b - \tilde{X}_\varepsilon - t(J - J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}} + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T\varepsilon.
\end{aligned}
$$

For $\|I_{2,15}\|_{L^2H^{s-1}}$, we apply Lemmas 4.3, 5.2, A.1 and A.3 to obtain

$$
\begin{aligned}
&\|I_{2,15}\|_{L^2H^{s-1}}\\
&\le \|\nabla(\tilde G-\tilde G_\varepsilon)\|_{L^\infty H^{s-1}}\|\tilde\zeta_\varepsilon\|_{L^\infty H^{s-1}}\|J(\tilde X_\varepsilon)\|_{L^\infty H^{s-1}}\|\tilde G_\varepsilon-\tilde G_0\|_{L^2H^{s-1}}\\
&\le T^{\frac14}\|\tilde G-\tilde G_\varepsilon\|_{L^\infty H^s}\|\tilde\zeta_\varepsilon\|_{L^\infty H^{s-1}}\|J(\tilde X_\varepsilon)\|_{L^\infty H^{s-1}}\|\tilde G_\varepsilon-\tilde G_0\|_{L^\infty H^{s-1}}\\
&\le C(N,N_\varepsilon,\tilde v_0,\tilde G_0)T^{\frac12}[\|\tilde G-\tilde G_\varepsilon-t\nabla\tilde v_0(J-J_\varepsilon)\tilde G_0\|_{L^\infty H^s}+\|t\nabla\tilde v_0(J-J_\varepsilon)\tilde G_0\|_{L^\infty H^s}]\\
&\le C(N,N_\varepsilon,\tilde v_0,\tilde G_0)T^{\frac34}\|\tilde G-\tilde G_\varepsilon-t\nabla\tilde v_0(J-J_\varepsilon)\tilde G_0\|_{\mathcal{A}^{s,\gamma}}+C(N,N_\varepsilon,\tilde v_0,\tilde G_0)T^{\frac12}\varepsilon.
\end{aligned}
$$

The other terms are estimated similarly.

To control $\|I_2\|_{H_{(0)}^{\frac{s-1}{2}}L^2}$, we write I_2 as follows:

$$
\begin{aligned}
I_2&=\nabla\tilde G\tilde\zeta J(\tilde X)\tilde G-\nabla\tilde G_\varepsilon\tilde\zeta_\varepsilon J(\tilde X_\varepsilon)\tilde G_\varepsilon+\nabla\tilde G_0(J_\varepsilon-J)\tilde G_0\\
&=\nabla\tilde G\tilde\zeta J(\tilde X)(\tilde G-\tilde G_\varepsilon)+\nabla\tilde G(\tilde\zeta-\tilde\zeta_\varepsilon)J(\tilde X)\tilde G_\varepsilon+\nabla(\tilde G-\tilde G_\varepsilon)\tilde\zeta_\varepsilon J(\tilde X)\tilde G_\varepsilon\\
&\quad+[\nabla\tilde G_\varepsilon\tilde\zeta_\varepsilon(J(\tilde X)-J(\tilde X_\varepsilon))\tilde G_\varepsilon-\nabla\tilde G_0(J-J_\varepsilon)\tilde G_0]\\
&=:\sum_{i=1}^4 I'_{2,i}.
\end{aligned}
$$

These terms need to be estimated separately and we show the main idea. We expand $I'_{2,1}$ into 8 terms:

$$
\begin{aligned}
I'_{2,1}&=\nabla\tilde G\tilde\zeta J(\tilde X)(\tilde G-\tilde G_\varepsilon)\\
&=\nabla[(\tilde G-\tilde G_0)+\tilde G_0][(\tilde\zeta-\mathcal{I})+\mathcal{I}][(J(\tilde X)-J)+J](\tilde G-\tilde G_\varepsilon)\\
&=:\sum_{i=1}^8 I'_{2,1,i}
\end{aligned}
$$

For $I'_{2,1,1}$, we apply Lemmas 4.2, 4.3, 5.2, A.1, A.3, A.7, A.8 and A.10 to obtain

$$
\begin{aligned}
&\|I'_{2,1,1}\|_{H_{(0)}^{\frac{s-1}{2}}L^2}\\
&=\|\nabla(\tilde G-\tilde G_0)(\tilde\zeta-\mathcal{I})(J(\tilde X)-J)(\tilde G-\tilde G_\varepsilon)\|_{H_{(0)}^{\frac{s-1}{2}}L^2}\\
&\le\|\nabla(\tilde G-\tilde G_0)(\tilde\zeta-\mathcal{I})\|_{H_{(0)}^{\frac{s-1}{2}}L^2}\|(J(\tilde X)-J)(\tilde G-\tilde G_\varepsilon)\|_{H_{(0)}^{\frac{s-1}{2}}H^{1+\eta}}\\
&\le\|\nabla(\tilde G-\tilde G_0)\|_{H_{(0)}^{\frac{s-1}{2}}H^{1-\mu}}\|\tilde\zeta-\mathcal{I}\|_{H_{(0)}^{\frac{s-1}{2}}H^{1+\mu}}\\
&\quad\cdot\|J(\tilde X)-J\|_{H_{(0)}^{\frac{s-1}{2}}H^{1+\eta}}\|\tilde G-\tilde G_\varepsilon\|_{H_{(0)}^{\frac{s-1}{2}}H^{1+\eta}}\\
&\le\|\tilde G-\tilde G_0\|_{H_{(0)}^{\frac{s-1}{2}}H^{1+(1-\mu)}}\|\tilde\zeta-\mathcal{I}\|_{H_{(0)}^{\frac{s-1}{2}}H^{1+\mu}}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \|J(\tilde{X}) - J\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \|\tilde{G} - \tilde{G}_\varepsilon\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\leq C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{2\delta} \Big[\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla \tilde{v}_0 (J - J_\varepsilon)\tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \\
&\qquad + \|t\nabla \tilde{v}_0 (J - J_\varepsilon)\tilde{G}_0\|_{H_{(0)}^{\frac{s-1}{2}} H^{1+\eta}} \Big] \\
&\leq C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{2\delta} \|\tilde{G} - \tilde{G}_\varepsilon - t\nabla \tilde{v}_0 (J - J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}} + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{2\delta} \varepsilon
\end{aligned}
$$

and we control the other terms similarly.

We expand $I'_{2,2}$, $I'_{2,3}$ and $I'_{2,4}$ as follows:

$$
\begin{aligned}
I'_{2,2} &= \nabla \tilde{G} (\tilde{\zeta} - \tilde{\zeta}_\varepsilon) J(\tilde{X}) \tilde{G}_\varepsilon \\
&= [(\nabla \tilde{G} - \nabla \tilde{G}_0) + \nabla \tilde{G}_0](\tilde{\zeta} - \tilde{\zeta}_\varepsilon)[(J(\tilde{X}) - J) + J][(\tilde{G}_\varepsilon - \tilde{G}_0) + \tilde{G}_0] \\
&=: \sum_{i=1}^{8} I'_{2,2,i}, \\
I'_{2,3} &= \nabla (\tilde{G} - \tilde{G}_\varepsilon) \tilde{\zeta}_\varepsilon J(\tilde{X}) \tilde{G}_\varepsilon \\
&= \nabla (\tilde{G} - \tilde{G}_\varepsilon)[(\tilde{\zeta}_\varepsilon - \mathcal{I}) + \mathcal{I}][(J(\tilde{X}) - J) + J][(\tilde{G}_\varepsilon - \tilde{G}_0) + \tilde{G}_0] \\
&=: \sum_{i=1}^{8} I'_{2,3,i}, \\
I'_{2,4} &= \nabla \tilde{G}_\varepsilon \tilde{\zeta}_\varepsilon (J(\tilde{X}) - J(\tilde{X}_\varepsilon)) \tilde{G}_\varepsilon - \nabla \tilde{G}_0 (J - J_\varepsilon) \tilde{G}_0 \\
&= \nabla [(\tilde{G}_\varepsilon - \tilde{G}_0) + \tilde{G}_0][(\tilde{\zeta}_\varepsilon - \mathcal{I}) + \mathcal{I}][(J(\tilde{X}) - J(\tilde{X}_\varepsilon) - J + J_\varepsilon) + (J - J_\varepsilon)] \\
&\qquad \cdot [(\tilde{G}_\varepsilon - \tilde{G}_0) + \tilde{G}_0] \\
&\quad - \nabla \tilde{G}_0 (J - J_\varepsilon) \tilde{G}_0 \\
&=: \sum_{i=1}^{15} I'_{2,4,i}.
\end{aligned}
$$

Combining the above calculations, we conclude that

$$
\begin{aligned}
\sum_{i,j} \|I'_{2,i,j}\|_{H_{(0)}^{\frac{s-1}{2}} L^2} &\leq C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{\delta_1} \|\tilde{X} - \tilde{X}_\varepsilon - t\nabla \tilde{v}_0 (J - J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s+1,\gamma+1}} \\
&\quad + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{\delta_2} \|\tilde{G} - \tilde{G}_\varepsilon - t\nabla \tilde{v}_0 (J - J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}} \\
&\quad + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)\varepsilon
\end{aligned}
$$

for $\delta_1, \delta_2 > 0$.

Estimate for $\tilde{K}_\varepsilon$.

It can be found in [4, Lemma 6.2] and we omit the details.

Estimate for $\tilde{H}_\varepsilon$.

We rewrite $\tilde{H}_\varepsilon$ as

$$\begin{aligned}\tilde{H}_\varepsilon &= \tilde{h} - \tilde{h}_\varepsilon + \tilde{h}^L_\phi - \tilde{h}^L_{\phi,\varepsilon} + \tilde{q}_{w,\varepsilon}(J^{-1} - J_\varepsilon^{-1})\tilde{n}_0 + (\nabla\tilde{w}_\varepsilon J_\varepsilon)^\top J_\varepsilon^{-1}\tilde{n}_0 - (\nabla\tilde{w}_\varepsilon J)^\top J^{-1}\tilde{n}_0\\ &= \tilde{h} - \tilde{h}_\varepsilon + \tilde{h}^L_\phi - \tilde{h}^L_{\phi,\varepsilon} + \bar{H}_\varepsilon.\end{aligned}$$

To begin with, we write $\bar{H}_\varepsilon$ as follows:

$$\begin{aligned}\bar{H}_\varepsilon &= \tilde{q}_{w,\varepsilon}(J^{-1} - J_\varepsilon^{-1})\tilde{n}_0 + (\nabla\tilde{w}_\varepsilon J_\varepsilon)^\top J_\varepsilon^{-1}\tilde{n}_0 - (\nabla\tilde{w}_\varepsilon J)^\top J^{-1}\tilde{n}_0\\ &= \tilde{q}_{w,\varepsilon}(J^{-1} - J_\varepsilon^{-1})\tilde{n}_0 + [(\nabla\tilde{w}_\varepsilon J_\varepsilon)^\top J_\varepsilon^{-1} - (\nabla\tilde{w}_\varepsilon J)^\top J_\varepsilon^{-1}]\tilde{n}_0\\ &\quad + [(\nabla\tilde{w}_\varepsilon J)^\top J_\varepsilon^{-1} - (\nabla\tilde{w}_\varepsilon J)^\top J^{-1}]\tilde{n}_0\\ &= \tilde{q}_{w,\varepsilon}(J^{-1} - J_\varepsilon^{-1})\tilde{n}_0 + (\nabla\tilde{w}_\varepsilon (J_\varepsilon - J))^\top J_\varepsilon^{-1}\tilde{n}_0 + (\nabla\tilde{w}_\varepsilon J)^\top (J_\varepsilon^{-1} - J^{-1})\tilde{n}_0\\ &=: \sum_{i=1}^{3} \bar{I}_i.\end{aligned}$$

Applying Lemma 5.2, we have

$$\|\bar{I}_1\|_{\mathcal{K}^{s-\frac{1}{2}}_{(0)}} + \|\bar{I}_2\|_{\mathcal{K}^{s-\frac{1}{2}}_{(0)}} + \|\bar{I}_3\|_{\mathcal{K}^{s-\frac{1}{2}}_{(0)}} \le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)\varepsilon.$$

To control $\tilde{h} - \tilde{h}_\varepsilon + \tilde{h}^L_\phi - \tilde{h}^L_{\phi,\varepsilon}$, from the definition of $\tilde{q}_\phi$ and $\tilde{q}_{\phi,\varepsilon}$, we have

$$\begin{aligned}&\tilde{h} - \tilde{h}_\varepsilon + \tilde{h}^L_\phi - \tilde{h}^L_{\phi,\varepsilon}\\ &= \tilde{h}_w + \tilde{h}_{w^\top} + \tilde{h}_\phi + \tilde{h}_{\phi^\top} + \tilde{h}_q + \tilde{h}_G - (\tilde{h}_{w,\varepsilon} + \tilde{h}_{w^\top,\varepsilon} + \tilde{h}_{\phi,\varepsilon} + \tilde{h}_{\phi^\top,\varepsilon} + \tilde{h}_{q,\varepsilon} + \tilde{h}_{G,\varepsilon})\\ &\quad + \tilde{q}_\phi J^{-1}\tilde{n}_0 - (\nabla\phi J + (\nabla\phi J)^\top)J^{-1}\tilde{n}_0\\ &\quad - (\tilde{q}_{\phi,\varepsilon} J_\varepsilon^{-1}\tilde{n}_0 - (\nabla\phi_\varepsilon J_\varepsilon + (\nabla\phi_\varepsilon J_\varepsilon)^\top)J_\varepsilon^{-1}\tilde{n}_0)\\ &= \tilde{h}_w - \tilde{h}_{w,\varepsilon} + \tilde{h}_{w^\top} - \tilde{h}_{w^\top,\varepsilon} + \tilde{h}_\phi - \tilde{h}_{\phi,\varepsilon} + \tilde{h}_{\phi^\top} - \tilde{h}_{\phi^\top,\varepsilon} + \tilde{h}_q - \tilde{h}_{q,\varepsilon}\\ &\quad + \tilde{q}_\phi J^{-1}\tilde{n}_0 - (\nabla\phi J + (\nabla\phi J)^\top)J^{-1}\tilde{n}_0\\ &\quad - (\tilde{q}_{\phi,\varepsilon} J_\varepsilon^{-1}\tilde{n}_0 - (\nabla\phi_\varepsilon J_\varepsilon + (\nabla\phi_\varepsilon J_\varepsilon)^\top)J_\varepsilon^{-1}\tilde{n}_0)\\ &\quad + \tilde{h}_G - \tilde{h}_{G,\varepsilon}\\ &= \tilde{h}_w - \tilde{h}_{w,\varepsilon} + \tilde{h}_{w^\top} - \tilde{h}_{w^\top,\varepsilon} + \tilde{h}_\phi - \tilde{h}_{\phi,\varepsilon} + \tilde{h}_{\phi^\top} - \tilde{h}_{\phi^\top,\varepsilon} + \tilde{h}_q - \tilde{h}_{q,\varepsilon}\\ &\quad + (\nabla\tilde{v}_0 J + (\nabla\tilde{v}_0 J)^\top + \tilde{G}_0\tilde{G}_0^\top)J^{-1}\tilde{n}_0 - (\nabla\phi J + (\nabla\phi J)^\top)J^{-1}\tilde{n}_0\\ &\quad - ((\nabla\tilde{v}_0 J_\varepsilon + (\nabla\tilde{v}_0 J_\varepsilon)^\top + \tilde{G}_0\tilde{G}_0^\top)J_\varepsilon^{-1}\tilde{n}_0 - (\nabla\phi_\varepsilon J_\varepsilon + (\nabla\phi_\varepsilon J_\varepsilon)^\top)J_\varepsilon^{-1}\tilde{n}_0)\\ &\quad + \tilde{h}_G - \tilde{h}_{G,\varepsilon}\\ &= \tilde{h}_w - \tilde{h}_{w,\varepsilon} + \tilde{h}_{w^\top} - \tilde{h}_{w^\top,\varepsilon} + \tilde{h}_\phi - \tilde{h}_{\phi,\varepsilon} + \tilde{h}_{\phi^\top} - \tilde{h}_{\phi^\top,\varepsilon} + \tilde{h}_q - \tilde{h}_{q,\varepsilon}\\ &\quad + \tilde{G}_0\tilde{G}_0^\top J^{-1}\tilde{n}_0 - t(\nabla\hat{\phi} J + (\nabla\hat{\phi} J)^\top)J^{-1}\tilde{n}_0\\ &\quad - (\tilde{G}_0\tilde{G}_0^\top J_\varepsilon^{-1}\tilde{n}_0 - t(\nabla\hat{\phi}_\varepsilon J_\varepsilon + (\nabla\hat{\phi}_\varepsilon J_\varepsilon)^\top)J_\varepsilon^{-1}\tilde{n}_0)\end{aligned}$$

$$
\begin{aligned}
&+\tilde{h}_G-\tilde{h}_{G,\varepsilon}\\
=&\tilde{h}_w-\tilde{h}_{w,\varepsilon}+\tilde{h}_{w^\top}-\tilde{h}_{w^\top,\varepsilon}+\tilde{h}_\phi-\tilde{h}_{\phi,\varepsilon}+\tilde{h}_{\phi^\top}-\tilde{h}_{\phi^\top,\varepsilon}+\tilde{h}_q-\tilde{h}_{q,\varepsilon}\\
&+t\nabla(\hat{\phi}_\varepsilon-\hat{\phi})\tilde{n}_0\\
&+t[(\nabla\hat{\phi}_\varepsilon J_\varepsilon)^\top J_\varepsilon^{-1}-(\nabla\hat{\phi}J)^\top J^{-1}]\tilde{n}_0\\
&+\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\tilde{n}_0-\tilde{G}\tilde{G}^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top J(\tilde{X}_\varepsilon)^{-1}\nabla_\Lambda\tilde{X}_\varepsilon\tilde{n}_0\\
=:&\sum_{i=1}^4\tilde{I}_i.
\end{aligned}
$$

For $\tilde{I}_1$, the estimates of $\tilde{h}_w-\tilde{h}_{w,\varepsilon}$, $\tilde{h}_{w^\top}-\tilde{h}_{w^\top,\varepsilon}$, $\tilde{h}_\phi-\tilde{h}_{\phi,\varepsilon}$, $\tilde{h}_{\phi^\top}-\tilde{h}_{\phi^\top,\varepsilon}$ and $\tilde{h}_q-\tilde{h}_{q,\varepsilon}$ have already been studied in [4, Lemma 6.2]. For $\tilde{I}_2$ and $\tilde{I}_3$, we apply Lemma 5.2 to obtain

$$
\|\tilde{I}_2\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}}+\|\tilde{I}_3\|_{\mathcal{K}_{(0)}^{s-\frac{1}{2}}}\le C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\varepsilon.
$$

Finally, we expand $\tilde{I}_4$ as follows:

$$
\begin{aligned}
\tilde{I}_4=&\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\tilde{n}_0-\tilde{G}\tilde{G}^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top J(\tilde{X}_\varepsilon)^{-1}\nabla_\Lambda\tilde{X}_\varepsilon\tilde{n}_0\\
=&\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\tilde{n}_0+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top(J(\tilde{X}_\varepsilon)^{-1}-J(\tilde{X})^{-1})\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0-\tilde{G}\tilde{G}^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top J(\tilde{X}_\varepsilon)^{-1}\nabla_\Lambda\tilde{X}_\varepsilon\tilde{n}_0-\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top J(\tilde{X}_\varepsilon)^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
=&\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\tilde{n}_0+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top(J(\tilde{X}_\varepsilon)^{-1}-J(\tilde{X})^{-1})\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top J(\tilde{X}_\varepsilon)^{-1}(\nabla_\Lambda\tilde{X}_\varepsilon-\nabla_\Lambda\tilde{X})\tilde{n}_0+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&-\tilde{G}\tilde{G}_\varepsilon^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0+\tilde{G}\tilde{G}_\varepsilon^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0-\tilde{G}\tilde{G}^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
=&\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\tilde{n}_0+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top(J(\tilde{X}_\varepsilon)^{-1}-J(\tilde{X})^{-1})\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&+\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top J(\tilde{X}_\varepsilon)^{-1}(\nabla_\Lambda\tilde{X}_\varepsilon-\nabla_\Lambda\tilde{X})\tilde{n}_0\\
&+(\tilde{G}_\varepsilon-\tilde{G})\tilde{G}_\varepsilon^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&+\tilde{G}(\tilde{G}_\varepsilon^\top-\tilde{G}^\top)J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
=:&\sum_{i=1}^4\tilde{I}_{4,i}.
\end{aligned}\tag{5.5}
$$

We only focus on $\|\tilde{I}_{4,1}\|_{L^2H^{s-\frac{1}{2}}}$ and $\|\tilde{I}_{4,3}\|_{L^2H^{s-\frac{1}{2}}}$. The others are similar or easier. We expand $\tilde{I}_{4,1}$ as follows:

$$
\begin{aligned}
&\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top(J(\tilde{X})^{-1}-J(\tilde{X}_\varepsilon)^{-1})\nabla_\Lambda\tilde{X}\tilde{n}_0-\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\tilde{n}_0\\
=&\tilde{G}_\varepsilon\tilde{G}_\varepsilon^\top(J(\tilde{X})^{-1}-J(\tilde{X}_\varepsilon)^{-1})\nabla_\Lambda\tilde{X}\tilde{n}_0-\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\nabla_\Lambda\tilde{X}\tilde{n}_0
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\nabla_\Lambda\tilde{X}\tilde{n}_0-\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\tilde{n}_0\\
=&[(\tilde{G}_\varepsilon-\tilde{G}_0)+\tilde{G}_0][(\tilde{G}_\varepsilon-\tilde{G}_0)+\tilde{G}_0]^\top[(J(\tilde{X})^{-1}-J(\tilde{X}_\varepsilon)^{-1})-(J^{-1}-J_\varepsilon^{-1})\\
&+(J^{-1}-J_\varepsilon^{-1})]\nabla_\Lambda\tilde{X}\tilde{n}_0-\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&+\tilde{G}_0\tilde{G}_0^\top(J^{-1}-J_\varepsilon^{-1})(\nabla_\Lambda\tilde{X}-\mathcal{I})\tilde{n}_0\\
=:&\sum_{i=1}^{7}\tilde{I}_{4,1,i}+\tilde{I}_{4,1,8}.
\end{aligned}
$$

For $\|\tilde{I}_{4,1,1}\|_{L^2H^{s-\frac{1}{2}}}$, from Theorem A.11, Lemmas 4.2, 4.3, 5.2 and A.2, it follows that

$$
\begin{aligned}
&\|\tilde{I}_{4,1,1}\|_{L^2H^{s-\frac{1}{2}}}\\
=&\|(\tilde{G}_\varepsilon-\tilde{G}_0)(\tilde{G}_\varepsilon-\tilde{G}_0)[(J(\tilde{X})^{-1}-J(\tilde{X}_\varepsilon)^{-1})-(J^{-1}-J_\varepsilon^{-1})]\nabla_\Lambda\tilde{X}\tilde{n}_0\|_{L^2H^{s-\frac{1}{2}}}\\
\le&\|\tilde{G}_\varepsilon-\tilde{G}_0\|_{L^2H^{s-\frac{1}{2}}}\|\tilde{G}_\varepsilon-\tilde{G}_0\|_{L^\infty H^{s-\frac{1}{2}}}\|(J(\tilde{X})^{-1}-J(\tilde{X}_\varepsilon)^{-1})-(J^{-1}-J_\varepsilon^{-1})\|_{L^\infty H^{s-\frac{1}{2}}}\\
&\cdot\|\nabla_\Lambda\tilde{X}\tilde{n}_0\|_{L^\infty H^{s-\frac{1}{2}}}\\
\le&T^{\frac{1}{2}}\|\tilde{G}_\varepsilon-\tilde{G}_0\|_{L^\infty H^s}\|\tilde{G}_\varepsilon-\tilde{G}_0\|_{L^\infty H^s}\|(J(\tilde{X})^{-1}-J(\tilde{X}_\varepsilon)^{-1})-(J^{-1}-J_\varepsilon^{-1})\|_{L^\infty H^s}\\
&\cdot\|\nabla_\Lambda\tilde{X}\tilde{n}_0\|_{L^\infty H^s}\\
\le&TC(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\|(J(\tilde{X})^{-1}-J(\tilde{X}_\varepsilon)^{-1})-(J^{-1}-J_\varepsilon^{-1})\|_{L^\infty H^s}\\
\le&TC(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\|J(\tilde{X}+\varepsilon b)^{-1}-J(\tilde{X}_\varepsilon)^{-1}\|_{L^\infty H^s}\\
&+TC(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)(\|J(\tilde{X})^{-1}-J(\tilde{X}+\varepsilon b)^{-1}\|_{L^\infty H^s}+\|J^{-1}-J_\varepsilon^{-1}\|_{L^\infty H^s})\\
\le&TC(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\|\tilde{X}+\varepsilon b-\tilde{X}_\varepsilon\|_{L^\infty H^s}+TC(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\varepsilon\\
\le&T^{\frac{5}{4}}C(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\|\tilde{X}+\varepsilon b-\tilde{X}_\varepsilon-t(J-J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}}+TC(N,N_\varepsilon,\tilde{v}_0,\tilde{G}_0)\varepsilon.
\end{aligned}
$$

We expand $\tilde{I}_{4,3}$ as follows:

$$
\begin{aligned}
\tilde{I}_{4,3}&=(\tilde{G}_\varepsilon-\tilde{G})\tilde{G}_\varepsilon^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&=(\tilde{G}_\varepsilon-\tilde{G})(\tilde{G}_\varepsilon-\tilde{G}_0)^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0+(\tilde{G}_\varepsilon-\tilde{G})\tilde{G}_0^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\\
&=:\tilde{I}_{4,3,1}+\tilde{I}_{4,3,2}.
\end{aligned}
$$

From Lemmas 4.2, 4.3, 5.2, A.1 and Theorem A.11, we obtain

$$
\begin{aligned}
&\|\tilde{I}_{4,3,1}\|_{L^2H^{s-\frac{1}{2}}}\\
=&\|(\tilde{G}_\varepsilon-\tilde{G})(\tilde{G}_\varepsilon-\tilde{G}_0)^\top J(\tilde{X})^{-1}\nabla_\Lambda\tilde{X}\tilde{n}_0\|_{L^2H^{s-\frac{1}{2}}}\\
\le&\|\tilde{G}_\varepsilon-\tilde{G}\|_{L^\infty H^{s-\frac{1}{2}}}\|(\tilde{G}_\varepsilon-\tilde{G}_0)^\top\|_{L^2H^{s-\frac{1}{2}}}\|J(\tilde{X})^{-1}\|_{L^\infty H^{s-\frac{1}{2}}}\|\nabla_\Lambda\tilde{X}\|_{L^\infty H^{s-\frac{1}{2}}}\\
\le&T^{\frac{1}{2}}\|\tilde{G}_\varepsilon-\tilde{G}\|_{L^\infty H^s}\|(\tilde{G}_\varepsilon-\tilde{G}_0)^\top\|_{L^\infty H^s}\|J(\tilde{X})^{-1}\|_{L^\infty H^s}\|\nabla_\Lambda\tilde{X}\|_{L^\infty H^s}
\end{aligned}
$$

$$
\begin{aligned}
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{\frac{3}{4}} \|\tilde{G}_\varepsilon - \tilde{G}\|_{L^\infty H^s} \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{\frac{3}{4}} (\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{L^\infty H^s} + \|t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{L^\infty H^s}) \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T \|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}} + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{\frac{3}{4}} \varepsilon.
\end{aligned}
$$

$\tilde{I}_{4,1,i}$ and $\tilde{I}_{4,3,j}$ can be controlled similarly. Likewise, the estimates of $\tilde{I}_{4,2}$ and $\tilde{I}_{4,4}$ can be achieved in the same way.

For the estimates in $H^{\frac{s}{2}-\frac{1}{4}}_{(0)} L^2$, we apply (5.5). We only concentrate on $\|\tilde{I}_{4,2}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} L^2}$ and $\|\tilde{I}_{4,4}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} L^2}$. We expand $\tilde{I}_{4,2}$ as follows:

$$
\begin{aligned}
\tilde{I}_{4,2} &= [(\tilde{G}_\varepsilon - \tilde{G}_0) + \tilde{G}_0][(\tilde{G}_\varepsilon - \tilde{G}_0) + \tilde{G}_0]^\top [(J(\tilde{X}_\varepsilon)^{-1} - J^{-1}) + J^{-1}](\nabla_\Lambda \tilde{X}_\varepsilon - \nabla_\Lambda \tilde{X})\tilde{n}_0 \\
&=: \sum_{i=1}^{8} \tilde{I}_{4,2,i}.
\end{aligned}
$$

For $\|\tilde{I}_{4,2,1}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} L^2}$, applying Lemmas 4.3, 5.2, A.1, A.7, A.8 and A.10, we have

$$
\begin{aligned}
&\|\tilde{I}_{4,2,1}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} L^2} \\
&= \|(\tilde{G}_\varepsilon - \tilde{G}_0)(\tilde{G}_\varepsilon - \tilde{G}_0)^\top (J(\tilde{X}_\varepsilon)^{-1} - J^{-1})(\nabla_\Lambda \tilde{X}_\varepsilon - \nabla_\Lambda \tilde{X})\tilde{n}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} L^2} \\
&\le \|(\tilde{G}_\varepsilon - \tilde{G}_0)(\tilde{G}_\varepsilon - \tilde{G}_0)^\top\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{\frac{1}{2}+\eta}} \\
&\quad \cdot \|(J(\tilde{X}_\varepsilon)^{-1} - J^{-1})(\nabla_\Lambda \tilde{X}_\varepsilon - \nabla_\Lambda \tilde{X})\tilde{n}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} L^2} \\
&\le \|\tilde{G}_\varepsilon - \tilde{G}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{\frac{1}{2}+\eta}} \|(\tilde{G}_\varepsilon - \tilde{G}_0)^\top\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{\frac{1}{2}+\eta}} \|J(\tilde{X}_\varepsilon)^{-1} - J^{-1}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{\frac{1}{2}+\mu}} \\
&\quad \cdot \|(\nabla_\Lambda \tilde{X}_\varepsilon - \nabla_\Lambda \tilde{X})\tilde{n}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{\frac{1}{2}-\mu}} \\
&\le \|\tilde{G}_\varepsilon - \tilde{G}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{1+\eta}} \|(\tilde{G}_\varepsilon - \tilde{G}_0)^\top\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{1+\eta}} \|J(\tilde{X}_\varepsilon)^{-1} - J^{-1}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{1+\mu}} \\
&\quad \cdot \|(\nabla_\Lambda \tilde{X}_\varepsilon - \nabla_\Lambda \tilde{X})\tilde{n}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{1-\mu}} \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{3\delta} \|\tilde{X}_\varepsilon - \tilde{X}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{1+(1-\mu)}} \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{3\delta} \|\tilde{X}_\varepsilon - \tilde{X} - \varepsilon b - t(J_\varepsilon - J)\tilde{v}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)} H^{1+(1-\mu)}} \\
&\quad + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{3\delta} \varepsilon \\
&\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0) T^{3\delta} \|\tilde{X}_\varepsilon - \tilde{X} - \varepsilon b - t(J_\varepsilon - J)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}}
\end{aligned}
$$

$$+ C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{3\delta}\varepsilon$$

for some positive numbers η, δ and μ small enough.

Then, we expand the other term into 8 terms:

$$\begin{aligned}\tilde{I}_{4,4} &= [(\tilde{G} - \tilde{G}_0) + \tilde{G}_0](\tilde{G}_\varepsilon^\top - \tilde{G}^\top)[(J(\tilde{X})^{-1} - J^{-1}) + J^{-1}][(\nabla_\Lambda \tilde{X} - \mathcal{I}) + \mathcal{I}]\tilde{n}_0 \\ &=: \sum_{i=1}^{8} \tilde{I}_{4,4,i}.\end{aligned}$$

For the first term, applying Lemmas 4.3, 5.2, A.1, A.7, A.8 and A.10, we obtain

$$\begin{aligned}&\|\tilde{I}_{4,4,1}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}L^2} \\ &= \|(\tilde{G} - \tilde{G}_0)(\tilde{G}_\varepsilon^\top - \tilde{G}^\top)(J(\tilde{X})^{-1} - J^{-1})(\nabla_\Lambda \tilde{X} - \mathcal{I})\tilde{n}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}L^2} \\ &\le \|(\tilde{G} - \tilde{G}_0)(\tilde{G}_\varepsilon - \tilde{G})^\top\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{\frac{1}{2}+\eta}} \|(J(\tilde{X})^{-1} - J^{-1})(\nabla_\Lambda \tilde{X} - \mathcal{I})\tilde{n}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}L^2} \\ &\le \|\tilde{G} - \tilde{G}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{\frac{1}{2}+\eta}} \|\tilde{G}_\varepsilon - \tilde{G}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{\frac{1}{2}+\eta}} \|J(\tilde{X})^{-1} - J^{-1}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{\frac{1}{2}+\mu}} \\ &\quad \|(\nabla_\Lambda \tilde{X} - \mathcal{I})\tilde{n}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{\frac{1}{2}-\mu}} \\ &\le \|\tilde{G} - \tilde{G}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{1+\eta}} \|\tilde{G}_\varepsilon - \tilde{G}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{1+\eta}} \|J(\tilde{X})^{-1} - J^{-1}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{1+\mu}} \\ &\quad \|(\nabla_\Lambda \tilde{X} - \mathcal{I})\tilde{n}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{1-\mu}} \\ &\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{3\delta}\|\tilde{G}_\varepsilon - \tilde{G}\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{1+\eta}} \\ &\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{3\delta}\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{H^{\frac{s}{2}-\frac{1}{4}}_{(0)}H^{1+\eta}} \\ &\quad + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{3\delta}\varepsilon \\ &\le C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{3\delta}\|\tilde{G} - \tilde{G}_\varepsilon - t\nabla\tilde{v}_0(J - J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}} + C(N, N_\varepsilon, \tilde{v}_0, \tilde{G}_0)T^{3\delta}\varepsilon\end{aligned}$$

for some η, δ and $\mu > 0$ small enough. The other terms $\tilde{I}_{4,2,i}$ and $\tilde{I}_{4,4,j}$ can be controlled similarly. Analogously, the estimates of $\tilde{I}_{4,1}$ and $\tilde{I}_{4,3}$ can be achieved in the same manner.

Combining the estimates $\|\tilde{F}_\varepsilon\|_{\mathcal{K}^{s-1}_{(0)}}$, $\|\tilde{K}_\varepsilon\|_{\bar{\mathcal{K}}^{s}_{(0)}}$ and $\|\tilde{H}_\varepsilon\|_{\mathcal{K}^{s-\frac{1}{2}}_{(0)}}$, we complete the proof of Proposition 5.4. □

Finally, we prove Theorem 5.1.

Proof of Theorem 5.1. From (5.2), (5.3) and (5.4), we conclude that

$$\begin{aligned}
&\|\tilde{X}-\tilde{X}_\varepsilon+\varepsilon b-t(J-J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}}+\|\tilde{w}-\tilde{w}_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}}+\|\tilde{q}_w-\tilde{q}_{w,\varepsilon}\|_{\mathcal{K}^s_{pr}(0)}\\
&+\|\tilde{G}-\tilde{G}_\varepsilon-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}}\\
\le{}& C\varepsilon+CT^\delta(\|\tilde{X}-\tilde{X}_\varepsilon+\varepsilon b-t(J-J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}}+\|\tilde{w}-\tilde{w}_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}}\\
&+\|\tilde{q}_w-\tilde{q}_{w,\varepsilon}\|_{\mathcal{K}^s_{pr}(0)}+\|\tilde{G}-\tilde{G}_\varepsilon-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}}),
\end{aligned}$$

where $\delta=\min\{\delta_1,\delta_2,\delta_3\}$ and C depends only on the initial data.

If $0<T<(\frac{C}{2})^{-\frac{1}{\delta}}$, it follows that

$$\begin{aligned}
&\|\tilde{X}-\tilde{X}_\varepsilon+\varepsilon b-t(J-J_\varepsilon)\tilde{v}_0\|_{\mathcal{A}^{s+1,\gamma+1}}+\|\tilde{w}-\tilde{w}_\varepsilon\|_{\mathcal{K}^{s+1}_{(0)}}+\|\tilde{q}_w-\tilde{q}_{w,\varepsilon}\|_{\mathcal{K}^s_{pr}(0)}\\
&\quad+\|\tilde{G}-\tilde{G}_\varepsilon-t\nabla\tilde{v}_0(J-J_\varepsilon)\tilde{G}_0\|_{\mathcal{A}^{s,\gamma}}\le 2C\varepsilon.
\end{aligned}$$

In particular, we obtain

$$\|\tilde{X}-\tilde{X}_\varepsilon\|_{L^\infty H^{s+1}}\le C\varepsilon,$$

and therefore, we complete the proof of Theorem 5.1. □

6. Existence of splash singularity

In this section, we prove the existence of the splash singularity.

It is crucial to choose some appropriate initial data. Accordingly, we are looking for some initial data such that the compatibility conditions (1.2) hold and the inner product of the velocity and the outer normal is positive as in Fig. 4. Following that, the evolution of the splash curve is demonstrated in Fig. 5.

For our case, we extend the analysis for the initial velocity choice already performed in [4, Section 7]. We recall the arguments in [4] for the reader's convenience.

In the following discussion, the typical domain is denoted by Ω for simplicity and the boundary $\partial\Omega$ is parametrized as follows:

$$z(r)=(z_1(r),z_2(r)),\quad |\dot{z}(r)|=1. \tag{6.1}$$

Let U be a sufficiently small neighborhood of $\partial\Omega$ and we define the coordinates

$$\begin{aligned}
x(r,\lambda)&:=z(r)+\lambda\dot{z}(r)^\perp,\\
(x_1(r,\lambda),x_2(r,\lambda))&:=(z_1(r)-\lambda\dot{z}_2(r),z_2(r)+\lambda\dot{z}_1(r)).
\end{aligned}$$

Then, the stream function $\psi(x_1,x_2)$ is defined as follows:

$$\begin{aligned}
\bar{\psi}(r,\lambda)&:=\psi_0(r)+\psi_1(r)\lambda+\frac{1}{2}\psi_2(r)\lambda^2,\\
\psi(x(r,\lambda))&:=\bar{\psi}(r,\lambda),
\end{aligned} \tag{6.2}$$

and we extend ψ to Ω smoothly.

We define $u_0(x_1, x_2) := \nabla^\perp \psi(x_1, x_2)$ and notice that u_0 is divergence-free, i.e.,

$$\operatorname{div} u_0 = \partial_1(-\partial_2 \psi) + \partial_2(\partial_1 \psi) = 0, \quad \text{in } \Omega$$

Recalling the compatibility conditions (1.2), the initial velocity field u_0 must satisfy

$$n^\perp((\nabla u_0 + \nabla u_0^\top) + H_0 \otimes H_0)n = 0, \quad \text{on } \partial\Omega$$

where $n^\perp = (-n^2, n^1)$ and $n = (n^1, n^2)^\top$ are the tangential and normal vectors, respectively. We denote by T and N the extension of $n^\perp$ and n to the neighborhood U, respectively. Then, it follows that

$$(T(\nabla u_0 + \nabla u_0^\top)N)|_{\partial\Omega} = -(TH_0 \otimes H_0 N)|_{\partial\Omega}. \tag{6.3}$$

From (6.1), we know that

$$\ddot{z}_2 \dot{z}_2 + \ddot{z}_1 \dot{z}_1 = 0,$$

and we define

$$\begin{aligned} T(r,\lambda) &:= \partial_r x(r,\lambda) = \dot{z}(r) + \lambda \ddot{z}^\perp(r) = (1 - \lambda\kappa(r))\dot{z}(r), \\ N(r,\lambda) &:= \partial_\lambda x(r,\lambda) = \dot{z}^\perp(r), \end{aligned}$$

where $\kappa(r) := \ddot{z}(r) \cdot \dot{z}^\perp(r)$. Also, we notice that $(\ddot{z}_2, -\ddot{z}_1) = \kappa(\dot{z}_1, \dot{z}_2)$.

From (6.1) and (6.2), we have

$$(\nabla\psi)(x(r,\lambda)) = \frac{1}{1-\kappa(r)\lambda}\partial_r \bar{\psi}(r,\lambda)\dot{z}(r) + \partial_\lambda \bar{\psi}(r,\lambda)\dot{z}^\perp(r).$$

Let $\bar{u}_0(r,\lambda) := u_0(x(r,\lambda))$. Clearly,

$$\bar{u}_0(r,\lambda) = (\nabla\psi)^\perp(x(r,\lambda)) = \frac{1}{1-\kappa(r)\lambda}\partial_r \bar{\psi}(r,\lambda)\dot{z}^\perp(r) + \partial_\lambda \bar{\psi}(r,\lambda)\dot{z}(r).$$

Then, it follows that

$$\begin{aligned} (\partial_r \bar{u}_0^j)(r,\lambda) &= T^i(r,\lambda)(\partial_i u_0^j)(x(r,\lambda)), \\ (\partial_\lambda \bar{u}_0^j)(r,\lambda) &= N^i(r,\lambda)(\partial_i u_0^j)(x(r,\lambda)). \end{aligned}$$

With this observation, the left-hand side of (6.3) becomes

$$\begin{aligned} T(r,\lambda)\nabla u_0(x(r,\lambda))N(r,\lambda) &= T^i(r,\lambda)(\partial_i u_0^j)(x(r,\lambda))N^j(r,\lambda) = (\partial_r \bar{u}_0^j)(r,\lambda)N^j(r,\lambda), \\ T(r,\lambda)\nabla u_0^\top(x(r,\lambda))N(r,\lambda) &= T^i(r,\lambda)(\partial_j u_0^i)(x(r,\lambda))N^j(r,\lambda) = (\partial_\lambda \bar{u}_0^i)(r,\lambda)T^i(r,\lambda), \end{aligned}$$

i.e.,

$$T\nabla u_0(x(r,\lambda))N = \partial_r(\bar{u}_0 \cdot N) - \bar{u}_0 \cdot \partial_r N,$$
$$T\nabla u_0^\top(x(r,\lambda))N = \partial_\lambda(\bar{u}_0 \cdot T) - \bar{u}_0 \cdot \partial_\lambda T.$$

Note that

$$\begin{cases} \bar{u}_0 \cdot N = \frac{1}{1-\lambda\kappa}\partial_r \bar{\psi}, \\ \bar{u}_0 \cdot T = -(1-\lambda\kappa)\partial_\lambda \bar{\psi}, \\ \partial_r N = \ddot{z}^\perp = -\kappa \dot{z}, \\ \partial_\lambda T = \ddot{z}^\perp = -\kappa \dot{z}, \end{cases}$$

and it follows that

$$\begin{cases} \partial_r(\bar{u}_0 \cdot N)|_{\lambda=0} = \partial_r^2 \bar{\psi}(r,0), \\ \partial_\lambda(\bar{u}_0 \cdot T)|_{\lambda=0} = \kappa \partial_\lambda \bar{\psi}(r,0) - \partial_\lambda^2 \bar{\psi}(r,0), \\ (\bar{u}_0 \cdot \partial_r N)|_{\lambda=0} = \kappa \partial_\lambda \bar{\psi}(r,0), \\ (\bar{u}_0 \cdot \partial_\lambda T)|_{\lambda=0} = \kappa \partial_\lambda \bar{\psi}(r,0). \end{cases}$$

Therefore, the left-hand side of (6.3) becomes

$$(T\nabla u_0(x(r,\lambda))N + T\nabla u_0^\top(x(r,\lambda))N)|_{\lambda=0} = \partial_r^2 \bar{\psi}(r,0) - \kappa \partial_\lambda \bar{\psi}(r,0) - \partial_\lambda^2 \bar{\psi}(r,0),$$

i.e.,

$$\partial_r^2 \bar{\psi}(r,0) - \kappa \partial_\lambda \bar{\psi}(r,0) - \partial_\lambda^2 \bar{\psi}(r,0) = -(TH_0 \otimes H_0 N)|_{\lambda=0}.$$

We conclude that

$$\partial_r^2 \psi_0(r) - \kappa \psi_1(r) - \psi_2(r) = -(TH_0 \otimes H_0 N)|_{\lambda=0}. \tag{6.4}$$

Therefore, for any ψ_0 and H_0 such that $u_0 \cdot n|_{\partial\Omega} = \partial_r \psi_0 > 0$ and $\operatorname{div} H_0 = 0$, there exist ψ_1 and ψ_2, satisfying (6.4). In particular, the initial data satisfy the compatibility condition (6.3). For simplicity, we may choose $\psi_1 = 0$ and $\psi_2 = \partial_r^2 \psi_0 + (TH_0 \otimes H_0 N)|_{\partial\Omega}$.

With the above discussion, we state the following proposition:

Proposition 6.1. *Let $\psi_0 \in C^2(U)$, such that $\partial_r \psi_0 > 0$. Let $H_0 \in C^1(U)$ be the initial magnetic field and suppose $\operatorname{div} H_0 = 0$. Then, there exists a stream function as in (6.2) with $\psi_1 = 0$ and $\psi_2 = \partial_r^2 \psi_0 + (TH_0 \otimes H_0 N)|_{\partial\Omega}$ such that the initial velocity defined by $u_0(x_1, x_2) = \nabla^\perp \psi(x_1, x_2)$ is divergence-free and the compatibility conditions (1.2) hold. Moreover, we have $u_0 \cdot n > 0$ on $\partial\Omega$.*

Thanks to Proposition 6.1 and recalling (local existence) Theorem 4.4, the choice of the initial velocity $\tilde{v}_0$ allows us to obtain domain $\tilde{\Omega}(t)$ with $0 < t < T$ for T small enough. Then, from $\tilde{v}_0 \cdot \tilde{n}_0 > 0$ on $\partial\tilde{\Omega}_0$, we can choose $\bar{t} \in (0, T)$ small enough such that $P^{-1}(\partial\tilde{\Omega}(\bar{t}))$ is a self-intersecting domain. Following that, for ε small enough, (5.1) in (stability) Theorem 5.1 ensures that $P^{-1}(\partial\tilde{\Omega}_\varepsilon(\bar{t}))$ is also self-intersecting as in Fig. 7. Note that $P^{-1}(\partial\tilde{\Omega}_\varepsilon(0))$ is regular as

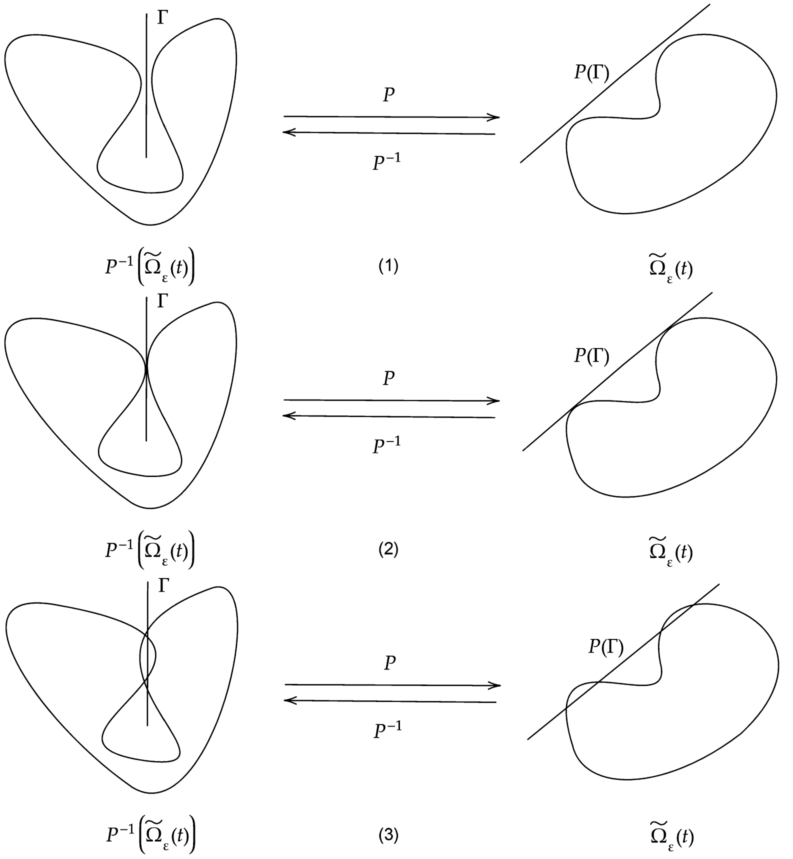

Fig. 9. Possibilities at time $t > 0$.

in Fig. 6 and end up in a self-intersecting domain later on. Finally, the continuity argument guarantees the existence of a splash singularity in finite time.

More precisely, we fix $\varepsilon > 0$ and define

$$t_\varepsilon^* := \inf\{t \in [0, \bar{t}] : P^{-1}(\partial\tilde{\Omega}_\varepsilon(t)) \text{ and } \partial\tilde{\Omega}_\varepsilon(t) \text{ are as in Fig. 9 (2) or (3)}\}.$$

It follows that $0 < t_\varepsilon^* < \bar{t}$ and $P^{-1}(\partial\tilde{\Omega}_\varepsilon(t_\varepsilon^*))$ is as in Fig. 9 (2). Moreover, for $0 \le t < t_\varepsilon^*$, $P^{-1}(\partial\tilde{\Omega}_\varepsilon(t_\varepsilon^*))$ is as in Fig. 9 (1). Therefore, $(\tilde{\Omega}_\varepsilon(t), \tilde{w}'_\varepsilon, \tilde{q}'_\varepsilon, \tilde{X}'_\varepsilon, \tilde{G}'_\varepsilon)$ solves the viscous MHD equation for $t \in \left[0, t_\varepsilon^*\right)$. Furthermore, when $t = t_\varepsilon^*$, the interface $\partial\Omega(t_\varepsilon^*)$ self-intersects in at least one point which creates a splash singularity.

Finally, our result is stated in the following theorem:

Theorem 6.2. *There exists a bounded domain $\Omega_0 = P^{-1}(\partial\tilde{\Omega}_\varepsilon(0))$ with a sufficiently smooth boundary, as in the above discussion, such that for any divergence-free $H_0 \in H^k(\Omega_0)$ with the integer k large enough, we can construct a suitable initial velocity $u_0 \in H^k(\Omega_0)$, and there exists a solution (u, p, H) to viscous MHD equations (1.1) in $\left[0, t_\varepsilon^*\right)$ for $t_\varepsilon^* > 0$, such that $(\tilde{w}, \tilde{q}_w, \tilde{G} -$*

$\hat{G}) \in \mathcal{K}^{s+1}_{(0)} \times \mathcal{K}^{s}_{pr(0)} \times \mathcal{A}^{s,\gamma}$ for $2 < s < \frac{5}{2}$ and $1 < \gamma < s-1$. Moreover, the interface $\partial\Omega(t^*_\varepsilon)$ remains regular but self-intersects in at least one point which creates a splash singularity.

Data availability

No data was used for the research described in the article.

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Appendix A. Further estimates and key lemmas

Lemma A.1 *([4, Lemmas 3.10 and 3.11], [10, Lemma A.1]). Let $2 < s < \frac{5}{2}$, $1 < \gamma < s-1$ and $\tilde{X} - \hat{X} \in \mathcal{A}^{s+1,\gamma+1}$. Let $\delta, \mu > 0$ be sufficiently small. Then, for $T > 0$ small enough, we have*

$$\begin{aligned}
&\|J(\tilde{X})\|_{L^\infty H^{s+1}} \le C(M, \tilde{v}_0, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}),\\
&\|J(\tilde{X}) - J\|_{L^\infty H^{s+1}} \le C(M, \tilde{v}_0, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}})(\|\tilde{X} - \hat{X}\|_{L^\infty H^{s+1}} + \|tJ\tilde{v}_0\|_{L^\infty H^{s+1}}),\\
&\|J(\tilde{X}) - J\|_{H^1_{(0)} H^{\gamma+1}} \le C(M, \tilde{v}_0, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}})\|\tilde{X} - \tilde{\omega}\|_{H^1_{(0)} H^{\gamma+1}},\\
&\|J(\tilde{X}) - J\|_{H^{\frac{s-1}{2}}_{(0)} H^{1+\mu}} \le C(M, \tilde{v}_0, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}})(\|\tilde{X} - \hat{X}\|_{H^{\frac{s-1}{2}+\delta}_{(0)} H^{1+\mu}} + T),
\end{aligned}$$

where

$$M = \frac{1}{\inf_{\tilde{\omega}} |\tilde{\omega}| - C(\tilde{v}_0)T - T^{\frac{1}{4}}\|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}}.$$

Lemma A.2 *([4, Lemma 3.12]). Let $2 < s < \frac{5}{2}$, $1 < \gamma < s-1$ and $\tilde{X} - \hat{X}, \tilde{Y} - \hat{X} \in \mathcal{A}^{s+1,\gamma+1}$. Let $\delta, \mu > 0$ be sufficiently small. Then, for $T > 0$ small enough, we have*

$$\begin{aligned}
&\|J(\tilde{X}) - J(\tilde{Y})\|_{L^\infty H^{s+1}} \le C(M, \tilde{v}_0, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}, \|\tilde{Y} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}})\|\tilde{X} - \tilde{Y}\|_{L^\infty H^{s+1}},\\
&\|J(\tilde{X}) - J(\tilde{Y})\|_{H^1_{(0)} H^{\gamma+1}} \le C(M, \tilde{v}_0, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}, \|\tilde{Y} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}})\|\tilde{X} - \tilde{Y}\|_{H^1_{(0)} H^{\gamma+1}},
\end{aligned}$$

where

$$\begin{aligned}
M = \max\Bigg\{&\frac{1}{\inf|\tilde{\omega}| - C(\tilde{v}_0)T - T^{\frac{1}{4}}\|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}},\\
&\frac{1}{\inf|\tilde{\omega}| - C(\tilde{v}_0)T - T^{\frac{1}{4}}\|\tilde{Y} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}}\Bigg\}.
\end{aligned}$$

Lemma A.3 *([4, Lemma 3.13], [10, Lemma A.4]). Let* $2 < s < \frac{5}{2}, 1 < \gamma < s-1, \tilde{X} - \hat{X} \in \mathcal{A}^{s+1,\gamma+1}$ *and* $\tilde{\zeta} = (\nabla \tilde{X})^{-1}$*. Let* $\delta, \mu > 0$ *be sufficiently small. Then for* $T > 0$ *small enough, we have*

$$\|\tilde{\zeta}\|_{L^\infty H^s} + \sum_{i=1}^{2} \|\partial_i \tilde{\zeta}\|_{L^\infty H^s} \leq C(M, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}),$$

$$\|\tilde{\zeta} - \mathcal{I}\|_{L^\infty H^s} \leq C(M, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}) \|\tilde{X} - \tilde{\omega}\|_{L^\infty H^{s+1}},$$

$$\|\tilde{\zeta} - \mathcal{I}\|_{H_{(0)}^{\frac{s-1}{2}+\delta} H^{1+\mu}} \leq C(M, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}) \|\tilde{X} - \tilde{\omega}\|_{H_{(0)}^{\frac{s-1}{2}+\delta} H^{2+\mu}},$$

$$\|\tilde{\zeta} - \mathcal{I}\|_{H^1_{(0)} H^\gamma} \leq C(M, \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}}) \|\tilde{X} - \tilde{\omega}\|_{H^1_{(0)} H^{\gamma+1}},$$

where

$$M = \frac{1}{1 - C(\tilde{v}_0)T - CT^{\frac{1}{4}} \|\tilde{X} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}} - CT^{\frac{1}{2}} \|\tilde{X} - \hat{X}\|^2_{\mathcal{A}^{s+1,\gamma+1}}}.$$

Lemma A.4 *([4, Lemma 3.15], [10, Lemma A.5]). Let* $2 < s < \frac{5}{2}, 1 < \gamma < s-1, \tilde{X}^{(n)} - \hat{X}, \tilde{X}^{(n-1)} - \hat{X} \in \mathcal{A}^{s+1,\gamma+1}, \tilde{\zeta}^{(n)} = (\nabla \tilde{X}^{(n)})^{-1}$ *and* $\tilde{\zeta}^{(n-1)} = (\nabla \tilde{X}^{(n-1)})^{-1}$*. Let* $\delta, \mu > 0$ *be sufficiently small. Then, for* $T > 0$ *small enough, we have*

$$\|\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)}\|_{H_{(0)}^{\frac{s-1}{2}+\delta} H^{1+\mu}} \leq C(M, \tilde{v}_0) \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{H_{(0)}^{\frac{s-1}{2}+\delta} H^{2+\mu}},$$

$$\|\tilde{\zeta}^{(n)} - \tilde{\zeta}^{(n-1)}\|_{H^1_{(0)} H^\gamma} \leq C(M, \tilde{v}_0) \|\tilde{X}^{(n)} - \tilde{X}^{(n-1)}\|_{H^1_{(0)} H^{\gamma+1}},$$

where

$$M = \max_{m=n-1,n} \frac{1}{1 - C(\tilde{v}_0)T - CT^{\frac{1}{4}} \|\tilde{X}^{(m)} - \hat{X}\|_{\mathcal{A}^{s+1,\gamma+1}} - CT^{\frac{1}{2}} \|\tilde{X}^{(m)} - \hat{X}\|^2_{\mathcal{A}^{s+1,\gamma+1}}}.$$

Lemma A.5 *([2, Lemma 2.3]). Suppose* $0 \leq r \leq 4$.

(1) The Identity extends to a bounded operator

$$\mathcal{K}^r((0,T);\Omega) \to H^p(0,T) H^{r-2p}(\Omega)$$

for $p \leq \frac{r}{2}$.

(2) If r is not an odd integer, the restriction of this operator to the subspace with $\partial_t^k v(0) = 0, 0 \leq k < \frac{r-1}{2}$ *is bounded independently on* T*, indeed*

$$\|v\|_{H^p_{(0)} H^{r-2p}} \leq C \|v\|_{\mathcal{K}^r_{(0)}}.$$

Lemma A.6 *([10, Lemma 3.3]). Let* $\bar{T} > 0$ *be arbitrary, B a Hilbert space and choose* $T \leq \bar{T}$.

(1) For $v \in L^2((0,T);B)$, we define $V \in H^1((0,T);B)$ by

$$V(t) = \int_0^t v(\tau)d\tau.$$

For $0 < s < \frac{1}{2}$ and $0 \le \varepsilon < s$, then the map $v \to V$ is a bounded operator from $H^s((0,T);B)$ to $H^{s+1-\varepsilon}((0,T);B)$, and

$$\|V\|_{H^{s+1-\varepsilon}((0,T);B)} \le C_0 T^{\varepsilon} \|v\|_{H^s((0,T);B)},$$

where C_0 is independent of T for $0 < T \le \bar{T}$.

(2) For $\frac{1}{2} < s < 1$, we impose $v(0) = 0$ and $0 \le \varepsilon < s$. Then $v \to V$ is a bounded operator from $H^s_{(0)}((0,T);B)$ to $H^{s+1-\varepsilon}_{(0)}((0,T);B)$ and

$$\|V\|_{H^{s+1-\varepsilon}_{(0)}((0,T);B)} \le C_0 T^{\varepsilon} \|v\|_{H^s_{(0)}((0,T);B)},$$

where C_0 is independent of T for $0 < T < \bar{T}$.

Lemma A.7. *Suppose $\Omega \in \mathbb{R}^n$, $r > \frac{n}{2}$ and $r \ge s \ge 0$. If $v \in H^r(\Omega)$ and $w \in H^s(\Omega)$, then $vw \in H^s(\Omega)$ and*

$$\|vw\|_{H^s(\Omega)} \le C\|v\|_{H^r(\Omega)}\|w\|_{H^s(\Omega)}.$$

Lemma A.8 *([4, Lemma 3.6]). If $v \in H^{\frac{1}{q}}$ and $w \in H^{\frac{1}{p}}$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$ then*

$$\|vw\|_{L^2} \le C\|v\|_{H^{\frac{1}{q}}}\|w\|_{H^{\frac{1}{p}}}.$$

Lemma A.9 *([2, Lemma 2.6]). Suppose B, Y, Z are Hilbert spaces, and $M : B \times Y \to Z$ is a bounded, bilinear operator. Suppose $w \in H^s((0,T);B)$ and $v \in H^s, ((0,T);Y)$, where $s > \frac{1}{2}$. If vw is defined by $M(v,w)$, then $vw \in H^s((0,T);Z)$ and the following hold:*

(1) $\|vw\|_{H^s((0,T);Z)} \le C\|v\|_{H^s((0,T);Y)}\|w\|_{H^s((0,T);B)}$.

(2) In addition, if $s \le 2$ and $\partial_t^k v(0) = \partial_t^k w(0) = 0$, $0 \le k < s - \frac{1}{2}$ and $s - \frac{1}{2}$ is not an integer, then the constant C in (1) can be chosen independently on T. Indeed,

$$\|vw\|_{H^s_{(0)}((0,T);Z)} \le C\|v\|_{H^s_{(0)}((0,T);Y)}\|w\|_{H^s_{(0)}((0,T);B)}.$$

Lemma A.10 *([4, Lemma 3.8], [10, Lemma 3.7]). Let $2 < s < \frac{5}{2}$, $\varepsilon, \delta > 0$ small enough and $v \in \mathcal{A}^{s+1,\gamma}$, the following estimates hold:*

(1) $\|v\|_{H^{\frac{s+1}{2}}_{(0)} H^{1-\varepsilon}} \le C\|v\|_{\mathcal{A}^{s+1,\gamma}}$,

(2) $\|v\|_{H^{\frac{s+1}{2}+\varepsilon}_{(0)} H^{1+\delta}} \le C\|v\|_{\mathcal{A}^{s+1,\gamma}}$,

(3) $\|v\|_{H^{\frac{s-1}{2}+\varepsilon}_{(0)} H^{2+\delta}} \le C\|v\|_{\mathcal{A}^{s+1,\gamma}}$,
(4) $\|v\|_{H^{\frac{s-1}{2}+\varepsilon}_{(0)} H^{1+\delta}} \le C\|v\|_{\mathcal{A}^{s,\gamma-1}}$,
(5) $\|v\|_{H^{\frac{s}{2}-\frac{1}{4}+\varepsilon}_{(0)} H^{2+\delta}} \le C\|v\|_{\mathcal{A}^{s+1,\gamma}}$,
(6) $\|v\|_{H^{\frac{s}{2}-\frac{1}{4}+\varepsilon}_{(0)} H^{1+\delta}} \le C\|v\|_{\mathcal{A}^{s,\gamma-1}}$,
(7) $\|v\|_{H^1_{(0)} H^{s-1}} \le C\|v\|_{\mathcal{A}^{s+1,\gamma}}$,
(8) $\|v\|_{H^{\frac{1}{2}+2\varepsilon}_{(0)} H^{s}} \le C\|v\|_{\mathcal{A}^{s+1,\gamma}}$.

Lemma A.11 *([2, Lemma 2.1]). Let Ω be a bounded set with a sufficiently smooth boundary, then the following trace theorems hold:*

(1) Suppose $\frac{1}{2} < s \le 5$. The mapping $v \to \partial_n^j v$ extends to a bounded operator $\mathcal{K}^s([0,T];\Omega) \to \mathcal{K}^{s-j-\frac{1}{2}}([0,T];\partial\Omega)$, where j is an integer $0 \le j < s - \frac{1}{2}$. The mapping $v \to \partial_t^k v(\alpha, 0)$ extends to a bounded operator $\mathcal{K}^s([0,T];\Omega) \to H^{s-2k-1}(\Omega)$, if k is an integer $0 \le k < \frac{1}{2}(s-1)$.
(2) Suppose $\frac{3}{2} < s < 5, s \ne 3$ and $s - \frac{1}{2}$ not an integer. Let

$$\mathcal{W}^s = \prod_{0\le j\le s-\frac{1}{2}} \mathcal{K}^{s-j-\frac{1}{2}}([0,T];\partial\Omega) \times \prod_{0\le k<\frac{s-1}{2}} H^{s-2k-1}(\Omega),$$

and let $\mathcal{W}_0^s$ the subspace consisting of $\{a_j, w_k\}$, which are the traces described in the previous point, so that $\partial_t^k a_j(\alpha,0) = \partial_n^j w_k(\alpha), \alpha \in \partial\Omega$, for $j + 2k < s - \frac{3}{2}$. Then the traces in the previous point form a bounded operator $\mathcal{K}^s([0,T];\Omega) \to \mathcal{W}_0^s$ and this operator has a bounded right inverse.

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