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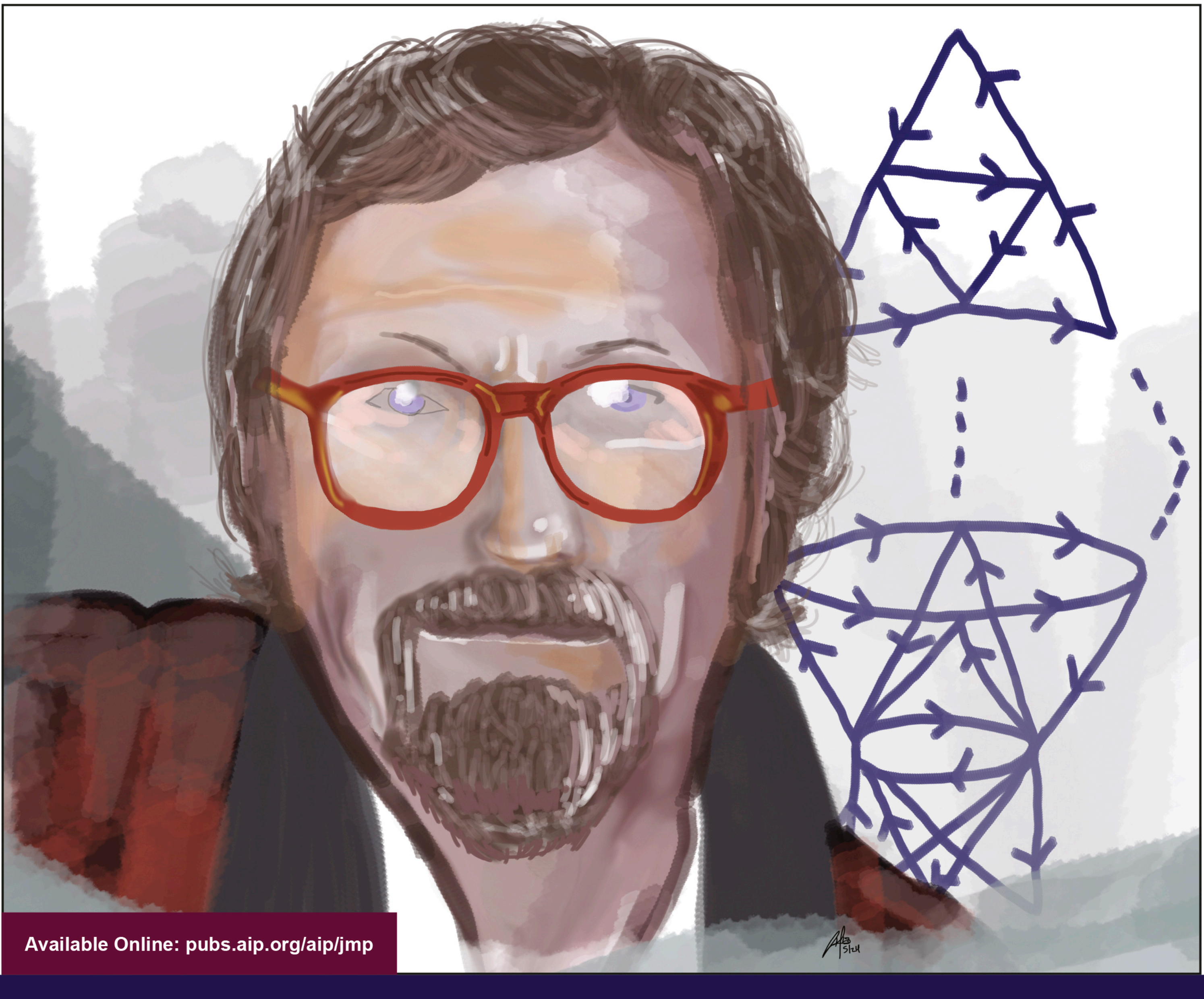

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ABSTRACT

This paper focuses on the global well-posedness of the Oberbeck–Boussinesq approximation for the unsteady motion of a drop in another bounded fluid separated by a closed interface with surface tension. We assume that the initial state of the drop is close to a ball B_R with the same volume as the drop, and that the boundary of the drop is a small perturbation of the boundary of B_R. To begin, we introduce the Hanzawa transformation with an added barycenter point to obtain the linearized Oberbeck–Boussinesq approximation in a fixed domain. From there, we establish time-weighted estimates of solutions for the shifted equation using maximal L^p–L^q regularities for the two-phase fluid motion of the linearized system, as obtained by Hao and Zhang [J. Differ. Equations **322**, 101–134 (2022)]. Using time decay estimates of the semigroup, we then obtain decay time-weighted estimates of solutions for the linearized problem. Additionally, we prove that these estimates are less than the sum of the initial value and its own square and cube by estimating the corresponding non-linear terms. Finally, the existence and uniqueness of solutions in the finite time interval $(0, T)$ was proven by Hao and Zhang [Commun. Pure Appl. Anal. **22**(7), 2099–2131 (2023)]. After that, we demonstrate that the solutions can be extended beyond T by analyzing the properties of the roots of algebraic equations.

I. INTRODUCTION

When studying natural or mixed convection flows, the most commonly used theoretical approach is the Oberbeck–Boussinesq approximation. This approximation provides an approximate description of the thermo-mechanical behavior of linear viscous fluids, such as Navier–Stokes fluids or Newtonian fluids. It was first proposed by Oberbeck[11] and later developed by Boussinesq.[1] This approximation is applicable to a wide range of problems in astrophysics, geophysics, and oceanography (see, for example, Ref. 9). There have been numerous thorough and comprehensive analyses on the derivation of this approximation from the general formulation of the fluid's local balance equations for mass, momentum, and energy, including works by Feireisl–Novotný,[4] Rajagopal *et al.*[16] and Roubíček.[17]

In the present paper, we consider the unsteady two-phase fluid motion of a drop of one incompressible viscous fluid inside another one in the Oberbeck–Boussinesq approximation with surface tension:

$$\begin{cases} \rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \operatorname{Div}(\mu\mathbf{D}(\mathbf{u}) - p\mathbf{I}) = \mathbf{f}(x,t) - \alpha\mathbf{g}\Theta & \text{in } \dot{\Omega}_t, \\ \partial_t\Theta + \operatorname{div}(\mathbf{u}\Theta - \nu\nabla\Theta) = 0 & \text{in } \dot{\Omega}_t, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \dot{\Omega}_t, \\ [\![(\mu\mathbf{D}(\mathbf{u}) - p\mathbf{I})\mathbf{n}_t]\!] = \sigma H(\Gamma_t), \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma_t, \\ [\![\nu\nabla\Theta\cdot\mathbf{n}_t]\!] = 0, \quad [\![\Theta]\!] = 0 & \text{on } \Gamma_t, \\ V_n = \mathbf{u}\cdot\mathbf{n}_t & \text{on } \Gamma_t, \\ \mathbf{u} = 0, \quad \nabla\Theta_-\cdot\mathbf{n}_- + \beta\Theta_- = b(x,t) & \text{on } \Gamma, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \Theta|_{t=0} = \Theta_0 & \text{in } \dot{\Omega}, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1(x,t),\ldots,u_N(x,t))^\top$, $p = p(x,t)$ and $\Theta(x,t)$ denote the velocity field, the pressure and the deviation from the average temperature, respectively. $\mathbf{f}$ is a given vector function of mass forces, and $\mathbf{g} = g(0,\ldots,0,1)^\top$ is a constant vector with the gravity constant g. Let Ω be a bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ with a rigid boundary Γ being a compact hypersurface of class C^2. Let S_R be the sphere of radius R centered at the origin and assume $\operatorname{dist}(\Gamma, S_R) = \inf\{|x-y| : x\in\Gamma, y\in S_R\} \geq 3d$ for some constant $d > 0$. The domain Ω_+ occupied by the drop is close to the ball B_R whose volume equals the volume of the drop, where B_R denotes the ball in $\mathbb{R}^N$ of radius R centered at the origin. Denote $\Omega_- := \Omega\backslash\overline{\Omega_+}$. Let Ω_{t+} be the evolution of Ω_+ and Γ_t be the boundary of Ω_{t+}, both of which depend on the time $t > 0$. Set $\Omega_{t-} := \Omega\backslash(\Omega_{t+}\cup\Gamma_t)$ with $\Omega_{0+} = \Omega_+$, $\Omega_{0-} = \Omega_-$ and Γ_0 being a normal perturbation of S_R. Let $\mathbf{n}_t$ be the normal to Γ_t oriented from Ω_{t+} into Ω_{t-}, $\mathbf{n} = y/R$ for $y \in S_R$ and $\mathbf{n}_-$ be the unit outward normal to Γ. Denote $\dot{\Omega}_t := \Omega_{t+}\cup\Omega_{t-}$ and $\dot{\Omega} := \dot{\Omega}_0$. The piece-wise positive constants ρ, μ, α and ν correspond to the mass density, the kinematic viscosity, the temperature expansion coefficient and the thermal conductivity, respectively. Here, both the above functions $\mathbf{u}, p, \Theta, \mathbf{f}$ and the constants ρ, μ, α, ν are piece-wisely defined, for instance, $\mathbf{u} = \mathbf{u}_+\chi_{\Omega_+} + \mathbf{u}_-\chi_{\Omega_-}$, $\rho = \rho_+\chi_{\Omega_+} + \rho_-\chi_{\Omega_-}$, etc., where $\chi_{\Omega_\pm}$ are the characteristic function of $\Omega_\pm$. $\mathbf{D}(\mathbf{u})$ is the doubled deformation tensor with the $(i,j)^{\text{th}}$ component $\partial_i u_j + \partial_j u_i$, and $\mathbf{I}$ is the $N\times N$ identity matrix. $b(x,t)$ is a given function on the fixed boundary Γ, and $\beta \geq 0$ is a constant. $\dot{\Omega}$, $\mathbf{u}_0$ and Θ_0 are the prescribed initial data for $\dot{\Omega}_t$, $\mathbf{u}$ and Θ, respectively. V_n is the evolution velocity of Γ_t along $\mathbf{n}_t$. σ is a positive constant describing the coefficient of the surface tension and $H(\Gamma_t)$ is $(N-1)$ times the mean curvature of Γ_t. Moreover, for any function $f(x,t) = f_\pm(x,t)$ for $x\in\Omega_{t\pm}$ and $t \geq 0$, we denote the jump of f across Γ_t by

$$[\![f]\!](x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega_{t+}}} f_+(x) - \lim_{\substack{x\to x_0\\ x\in\Omega_{t-}}} f_-(x)$$

for every point $x_0 \in \Gamma_t$.

This paper focuses on the free boundary problem of two-phase incompressible viscous fluids. For the two-phase Navier–Stokes flow, Tanaka[25] obtained a global solution near an equilibrium state for small initial data and proved its unique solvability by assuming certain regularity for the coefficients and the free boundary with general initial data;[26] Takahashi[24] also established a global weak solution. For the two-phase Stokes flow, Giga and Takahashi[6] constructed a global weak solution under periodic boundary conditions.

Assuming that the initial liquid velocities were small and the initial configuration of the inner fluid was close to a ball, Denisova and Solonnikov[3] proved the global solvability for two incompressible fluids. Shibata proved the existence of unique solutions to the generalized resolvent problems for the Stokes operator in Ref. 20. Shibata *et al.* have made significant contributions to the resolvent problems, such as in Refs. and 10 21 –10 23, with the aid of the $\mathscr{R}$-bounded operator theory. Prüss and Simonett[13–15] contributed to the L^p approach for two-phase problems, particularly for the case of surface tension. More recently, for the linearized electromagnetic field equations, Frolova and Shibata developed maximal L^p–L^q regularity in Ref. 5. Additionally, in the framework of maximal L^{p}–^{q} regularity, Saito, Shibata, and Zhang[19] established the local and global existence for the two-phase Navier–Stokes equations without surface tension. Taking into account surface tension and gravity, Saito and Shibata[18] proved the global well-posedness of the free boundary Navier–Stokes equations in an unbounded domain.

The rest of this paper is structured as follows. Before stating the main theorem, we first need to establish some assumptions about the initial region Ω_+ in Sec. II. Specifically, Ω_+ must satisfy (2.2). Once these assumptions are in place, we can use the Hanzawa transformation to reduce the problem in a time-dependent domain $\Omega(t)$ to a problem in a fixed domain Ω. In Sec. III, we establish time-weighted estimates for solutions of the shifted Eq. (3.2). Moving on to Sec. IV, we then study the decay properties of the analytic semigroup associated with Eq. (3.1), where $\mathbf{F}_6 = 0$, $F_7 = 0$, and $b = 0$ in (3.1). This allows us to obtain decay estimates for solutions of the linearized problem (3.1). In Sec. V, we focus on estimating the nonlinear terms, resulting in (5.39). Finally, in Sec. VI, we analyze solutions of the algebraic equations $x^3 + x^2 + \mathscr{I} - C^{-1}x = 0$ to obtain the inequality

$$\mathscr{E}_T^\kappa(\mathbf{v},\theta,h) \leq x_0(\mathscr{I}) \text{ for any } T\in(0,T_0).$$

By repeating this argument, we can extend the solutions of $\mathbf{v}$, θ, and h to the time interval $(0,\infty)$.

II. HANZAWA TRANSFORM AND THE MAIN THEOREM

In this section, we will use the Hanzawa transform to convert (1.1) into a problem on a fixed domain. We will then provide a statement on the global well-posedness of Eq. (1.1) when Ω is a bounded domain. Unlike the transformation used in the proof of local well-posedness, we will consider Ω_+ with the following assumptions:

$$|\Omega_+| = |B_R|, \quad \int_{\Omega_+} x dx = 0, \quad \Gamma_0 = \{x = \xi + h_0(\xi)\mathbf{n} \mid \xi \in S_R\}. \tag{2.1}$$

where $h_0(\xi)$ is a given small function defined on S_R. Notice that $\mathbf{n} = R^{-1}\xi$ $(\xi \in S_R)$ is the unit outer normal to S_R. Let Γ_t be given by

$$\Gamma_t = \{x = \xi + h(\xi,t)R^{-1}\xi + \zeta(t) \mid \xi \in S_R\} \tag{2.2}$$

with an unknown function $h(\xi,t)$ with $h(\xi,0) = h_0(\xi)$ and $\zeta(t)$ is the barycenter point of the domain Ω_{t+} defined by

$$\zeta(t) = \frac{1}{|\Omega_{t+}|}\int_{\Omega_{t+}} x dx. \tag{2.3}$$

By the assumptions mentioned above, it is clear that $\zeta(0) = 0$. In view of (2.1) and

$$\frac{d}{dt}\int_{\dot\Omega_t} \rho x_i dx = \int_{\dot\Omega_t} \rho \mathbf{u}_i dx,$$

we have

$$\zeta'(t) = \frac{1}{|B_R|}\int_{\Omega_{t+}} \mathbf{u}(x,t) dx.$$

Let Φ_h be a suitable extension of $h(\xi,t)$ such that $\Phi_h(\xi,t) = h(\xi,t)$ for $(\xi,t) \in S_R \times (0,T)$ and possesses the estimate

$$\begin{aligned} C_1\|h(\cdot,t)\|_{W_q^{k-1/q}(S_R)} &\le \|\Phi_h(\cdot,t)\|_{H_q^k(\dot\Omega)} \le C_2\|h(\cdot,t)\|_{W_q^{k-1/q}(S_R)}, \\ C_1\|\partial_t h(\cdot,t)\|_{W_q^{\ell-1/q}(S_R)} &\le \|\partial_t\Phi_h(\cdot,t)\|_{H_q^\ell(\dot\Omega)} \le C_2\|\partial_t h(\cdot,t)\|_{W_q^{\ell-1/q}(S_R)}, \end{aligned} \tag{2.4}$$

for $k = 1,2,3$ and $\ell = 1,2$. The definitions of $W_q^s(S_R)$ and $H_q^s(\dot\Omega)$ can be found in Appendix. Let $\Psi_h(\xi,t) = \chi(\xi)(\Phi_h(\xi,t)R^{-1}\xi + \zeta(t))$, where $\chi(\xi)$ is a $C^\infty(\mathbb{R}^N)$ function which equals 1 for $|\xi| < R + d$ and 0 for $|\xi| > R + 2d$. Then, we can use the following Hanzawa transform defined by

$$x = \xi + \Psi_h(\xi,t) \qquad \text{for} \quad \xi \in \Omega, \tag{2.5}$$

which was originally introduced by Hanzawa in Ref. 1 to treat classical solutions of the Stefan problem. In the following, we assume that

$$\sup_{t\in(0,T)} \|\Psi_h(\cdot,t)\|_{H^1_\infty(\Omega)} \le \varepsilon, \tag{2.6}$$

where ε is a small positive number. In fact, we can choose $\varepsilon \in (0,1)$, and then the Hanzawa transform defined above is an injective map. Let

$$\dot\Omega_t = \{x = \xi + \Psi_h(\xi,t) \mid \xi \in \dot\Omega\},$$

the Hanzawa transform maps $\dot\Omega$ into $\dot\Omega_t$ injectively. By (2.3) and the definition of $\chi(\xi)$, we can get $x = \xi + \Phi_h(\xi,t)R^{-1}\xi + \zeta(t)$ for $\xi \in S_R$ and $x = \xi$ for $\xi \in \Gamma$. Let $\frac{\partial x}{\partial \xi}$ be the Jacobean matrix of the transformation, that is, $\frac{\partial x}{\partial \xi} = \mathbf{I} + \nabla\Psi_h(\xi,t)$, where $\Psi_h(\xi,t) = (\Psi_1(\xi,t),\ldots,\Psi_N(\xi,t))^\top$, $\partial_i\Psi_j = \frac{\partial\Psi_j}{\partial\xi_i}$ and

$$\begin{aligned} \nabla\Psi_h &= \nabla\big(\chi(\xi)\Phi_h(\xi,t)R^{-1}\xi\big) + \nabla\chi(\xi)\zeta(t), && \text{in } \dot\Omega, \\ \nabla\Psi_h &= \nabla\big(\chi(\xi)h(\xi,t)R^{-1}\xi\big), && \text{on } S_R, \\ \partial_t\Psi_h &= \chi(\xi)\big(\partial_t\Phi_h(\xi,t)R^{-1}\xi + \xi'(t)\big) && \text{in } \Omega. \end{aligned} \tag{2.7}$$

Since ε is a small positive number, by (2.4) and (2.6) and the chain rule, we have

$$\frac{\partial\xi}{\partial x} = \left(\frac{\partial x}{\partial\xi}\right)^{-1} = \mathbf{I} + \sum_{k=1}^{\infty}(-\nabla\Psi_h(\xi,t))^k = \mathbf{I} + \mathbf{V}_0(\nabla\Psi_h), \tag{2.8}$$

where $\mathbf{V}_0(\mathbf{k})$ is an $N \times N$ matrix of analytic functions defined on $\{\mathbf{k} \mid |\mathbf{k}| < \varepsilon\}$ such that $\mathbf{V}_0(0) = 0$. Here and in the following, we will use the notation $\mathbf{k} = (k_{ij})$ with k_{ij} to represent the variables corresponding to $\partial_i \Psi_j$. Then we have

$$\nabla_x = (\mathbf{I} + \mathbf{V}_0(\mathbf{k}))\nabla_\xi, \quad \frac{\partial}{\partial x_\ell} = \frac{\partial}{\partial \xi_\ell} + \sum_{j=1}^N V_{0\ell j}(\mathbf{k})\frac{\partial}{\partial \xi_j}, \tag{2.9}$$

where $\nabla_z = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N}\right)^\top$ for $z = x$ or ξ. $V_{0\ell j}$ is the $(\ell, j)^{\text{th}}$ component of the $N \times N$ matrix $\mathbf{V}_0$. Let $\mathbf{u}$, p and Θ be solutions of (1.1), and we set

$$\mathbf{v}(\xi, t) = \mathbf{u}(\xi + \Psi_h(\xi, t), t), \quad q(\xi, t) = p(\xi + \Psi_h(\xi, t), t), \quad \theta(\xi, t) = \Theta(\xi + \Psi_h(\xi, t), t).$$

Noting that $x = \xi$ near Γ and $\zeta(0) = 0$, we have

$$\begin{aligned} &\mathbf{v} = 0, \quad \nabla\theta_- \cdot \mathbf{n}_- + \beta\theta_- = b(\xi, t) \quad \text{on } \Gamma \times (0, T), \\ &\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \dot{\Omega}; \quad \Phi_h|_{t=0} = h_0 \quad \text{on } S_R. \end{aligned} \tag{2.10}$$

Since $\zeta(t)$ is also an unknown function, by (2.5), we get $dx = dy + J(\mathbf{k})d\xi$ with

$$J(\mathbf{k}) = \det \begin{pmatrix} \dfrac{\partial \Psi_{h1}(\xi, t)}{\partial \xi_1} & \cdots & \dfrac{\partial \Psi_{h1}(\xi, t)}{\partial \xi_N} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \Psi_{hN}(\xi, t)}{\partial \xi_1} & \cdots & \dfrac{\partial \Psi_{hN}(\xi, t)}{\partial \xi_N} \end{pmatrix}.$$

In view of (2.3) and (2.5), we can obtain $\zeta'(t) = \tilde{\zeta}'(t)$ where $\tilde{\zeta}'(t)$ is defined by

$$\tilde{\zeta}'(t) := \frac{1}{|B_R|}\int_{B_R} \mathbf{v}(\xi, t)d\xi + \frac{1}{|B_R|}\int_{B_R} \mathbf{v}(\xi, t)\mathbf{J}(\mathbf{k})d\xi. \tag{2.11}$$

Since $\zeta(0) = 0$, we can define $\tilde{\zeta}(t)$ by setting

$$\tilde{\zeta}(t) = \int_0^t \tilde{\zeta}'(\tau)d\tau = \frac{1}{|B_R|}\int_0^t \int_{B_R} \mathbf{v}(\xi, \tau)d\xi d\tau + \frac{1}{|B_R|}\int_0^t \int_{B_R} \mathbf{v}(\xi, \tau)\mathbf{J}(\mathbf{k})d\xi d\tau. \tag{2.12}$$

In the following sections, although $\zeta(t)$ is an unknown function, we can change its estimate into the estimate of $\tilde{\zeta}(t)$. And then, by using the Hanzawa transform, we transform Eq. (1.1) to the following nonlinear equations:

$$\begin{cases} \rho\partial_t \mathbf{v} - \operatorname{Div}(\mu\mathbf{D}(\mathbf{v}) - q\mathbf{I}) = \mathbf{N}_1(\mathbf{v}, \mathbf{f}, \theta, \Psi_h) + \mathbf{f}(x(\xi, t), t) - \alpha\mathbf{g}\theta & \text{in } \dot{\Omega} \times (0, T), \\ \operatorname{div}\mathbf{v} = N_2(\mathbf{v}, \Psi_h) = \operatorname{div}\mathbf{N}_3(\mathbf{v}, \Psi_h) & \text{in } \dot{\Omega} \times (0, T), \\ \partial_t\theta - \nu\Delta\theta = N_4(\mathbf{v}, \theta, \Psi_h) & \text{in } \dot{\Omega} \times (0, T), \\ \partial_t h - \mathbf{n} \cdot \mathbf{P}\mathbf{v} = N_5(\mathbf{v}, \Psi_h) & \text{on } S_R \times (0, T), \\ [\![\mu\mathbf{D}(\mathbf{v})\mathbf{n}]\!]_\tau = [\![\mathbf{N}_6(\mathbf{v}, \Psi_h)]\!], \quad [\![\mathbf{v}]\!] = 0 & \text{on } S_R \times (0, T), \\ [\![(\mu\mathbf{D}(\mathbf{v})\mathbf{n}, \mathbf{n}) - q]\!] - \sigma\Delta_{S_R} h + \dfrac{N-1}{R^2} h = [\![N_7(\mathbf{v}, \Psi_h)]\!] & \text{on } S_R \times (0, T), \\ [\![\nu\nabla\theta \cdot \mathbf{n}]\!] = N_8(\theta, \Psi_h), [\![\theta]\!] = 0 & \text{on } S_R \times (0, T), \\ \mathbf{v} = 0, \quad \nabla\theta_- \cdot \mathbf{n}_- + \beta\theta_- = b(\xi, t) & \text{on } \Gamma \times (0, T), \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 \text{ in } \dot{\Omega}, \quad \theta|_{t=0} = \theta_0 \text{ in } \dot{\Omega}, \quad h|_{t=0} = h_0 \text{ on } S_R, \end{cases} \tag{2.13}$$

where $\mathbf{n} = R^{-1}\xi(\xi \in S_R)$, $\mathbf{P}\mathbf{v} = \mathbf{v} - \frac{1}{|B_R|}\int_{B_R} \mathbf{v}d\xi$, and Δ_{S_R} is the Laplace–Beltrami operator on S_R. Moreover, $\mathbf{N}_1(\mathbf{v}, \mathbf{f}, \theta, \Psi_h)$, $N_2(\mathbf{v}, \Psi_h)$, $\mathbf{N}_3(\mathbf{v}, \Psi_h)$, $N_4(\mathbf{v}, \theta, \Psi_h)$ and $\mathbf{N}_6(\mathbf{v}, \Psi_h)$ are the same as in Ref. 8, but the other nonlinear terms on boundary are slightly different from those in Ref. 8.

Thus, we need to reformulate the boundary conditions on S_R. Let $\boldsymbol{\omega} \in S_1$ be represented by $\boldsymbol{\omega} = \boldsymbol{\omega}(p_1, \ldots, p_{N-1})$ under a local coordinate $(p_1, \ldots, p_{N-1})$, and then for $x = (R + h)\boldsymbol{\omega} + \zeta(t) \in \Gamma_t$, we have

$$\frac{\partial x}{\partial p_j} = (R + h)\tau_j + \frac{\partial h}{\partial p_j}\omega, \tag{2.14}$$

where $\tau_j = \frac{\partial \omega}{\partial p_j}$ is a basis of the tangent space of S_1. From (2.7) and (2.14), we see that $\zeta(t)$ does not work in reformulation of boundary conditions.

Next, we give a representation formula of $\mathbf{n}_t$. We set

$$\mathbf{n}_t = a\bigg(\boldsymbol{\omega} + \sum_{i=1}^{N-1} b_i \tau_i\bigg),$$

where a and b_i's are unknown functions. Using the same method as in Ref. 8, we can obtain

$$\mathbf{n}_t = \boldsymbol{\omega} + \mathbf{V}_1(\bar{\nabla}\Phi_h)\bar{\nabla}\Phi_h, \tag{2.15}$$

where $\bar{\nabla} = (\mathbf{I}, \nabla)$, $\mathbf{V}_1(\bar{\nabla}\Phi_h)$ is an $N \times N$ matrix of analytic functions defined on $\{|\bar{\nabla}\Phi_h| < \varepsilon\}$ such that $\mathbf{V}_1(0) = 0$. By (2.15), we can find that $N_5(\mathbf{v}, \Psi_h)$, $\mathbf{N}_7(\mathbf{v}, \Psi_h)$ and $N_8(\theta, \Psi_h)$ only have some different C^∞ coefficient functions for the corresponding nonlinear terms in Ref. 8. Since C^∞ functions do not affect our estimation, we still use $N_5(\mathbf{v}, \Psi_h)$, $\mathbf{N}_7(\mathbf{v}, \Psi_h)$ and $N_8(\theta, \Psi_h)$ to represent the nonlinear terms in (2.13), and the formulas are the same as those in Ref. 8.

The main theorem about global well-posedness for (2.13) is stated as follows.

Theorem 2.1. *Let $2 < p < \infty$, $N < q < \infty$ and $2/p + N/q < 1$. Assume that Ω is a bounded domain, (2.1) holds and Γ is a compact hypersurface of class C^2. Let $0 < \kappa \le 1$ be a constant, $e^{\kappa t}\mathbf{f}(x(\xi,t),t) \in L^p((0,\infty), L^q(\dot{\Omega}))$ and $e^{\kappa t} b(\xi,t) \in L^p((0,\infty), H^1_q(\Omega_-)) \cap H^{1/2}_p((0,\infty), L^q(\Omega_-))$. Let $(\mathbf{u}_0, \theta_0) \in B^{2(1-1/p)}_{q,p}(\dot{\Omega})$ and $h_0 \in B^{1-1/p-1/q}_{q,p}(S_R)$ be the initial data for (2.13) and satisfy the smallness condition:*

$$\begin{aligned} &\|e^{\kappa t}\mathbf{f}\|_{L^p((0,\infty),L^q(\dot{\Omega}))} + \|e^{\kappa t} b\|_{L^p((0,\infty),H^1_q(\Omega_-))} + \|e^{\kappa t} b\|_{H^{1/2}_p((0,\infty),L^q(\Omega_-))} \\ &+ \|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})} + \|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot{\Omega})} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(S_R)} \le \varepsilon \end{aligned} \tag{2.16}$$

for some small number $\varepsilon > 0$. Assume that the compatibility conditions hold:

$$\begin{aligned} &\mathbf{v}_0 - \mathbf{N}_3\big(\mathbf{v}_0, \Psi_h|_{t=0}\big) \in \mathscr{D}(\Omega), \quad \operatorname{div}\mathbf{v}_0 = N_2\big(\mathbf{v}_0, \Psi_h|_{t=0}\big) && \text{in } \dot{\Omega}, \\ &[\![(\mu\mathbf{D}(\mathbf{v}_0)\mathbf{n})_\tau]\!] = [\![\big(\mathbf{N}_6\big(\mathbf{v}_0, \Psi_h|_{t=0}\big)\big)_\tau]\!], \quad [\![\mathbf{v}_0]\!] = 0 && \text{on } S_R, \\ &[\![\nu\nabla\theta_0 \cdot \mathbf{n}]\!] = N_8\big(\theta_0, \Psi_h|_{t=0}\big), \quad [\![\theta_0]\!] = 0 && \text{on } S_R, \\ &\mathbf{v}_0 = 0, \quad \nabla\theta_{0-} \cdot \mathbf{n}_- + \beta\theta_{0-} = b|_{t=0} && \text{on } \Gamma, \end{aligned}$$

where $\mathscr{D}(\Omega)$ can be found in the Appendix. Then, problem (2.13) with $T = \infty$ admits a unique solution which possesses the estimate

$$\begin{aligned} &\|(e^{\kappa t}\mathbf{v}, e^{\kappa t}\theta)\|_{L^p((0,\infty),H^2_q(\dot{\Omega}))} + \big\|(e^{\kappa t}\partial_t\mathbf{v}, e^{\kappa t}\partial_t\theta)\big\|_{L^p((0,\infty),L^q(\dot{\Omega}))} \\ &+ \big\|e^{\kappa t}\partial_t h\big\|_{L^p((0,\infty),W^{2-1/q}_q(\Omega))} + \big\|e^{\kappa t} h\big\|_{L^p((0,\infty),W^{3-1/p}_q(S_R))} \\ &+ \big\|e^{\kappa t}\partial_t h\big\|_{L^\infty((0,\infty),W^{1-1/q}_q(\Omega))} \lesssim \varepsilon, \end{aligned} \tag{2.17}$$

where the symbol "$\lesssim$" denotes "$\le C$" for some constant $C > 0$ independent of ε and κ.

III. ESTIMATES OF SOLUTIONS FOR THE SHIFTED EQUATIONS

In order to prove Theorem 2.1, we first consider the following linearized equations:

$$\begin{cases} \rho\partial_t\mathbf{v} - \operatorname{Div}(\mu\mathbf{D}(\mathbf{v}) - q\mathbf{I}) = \mathbf{F}_1 & \text{in } \dot{\Omega} \times (0,T), \\ \operatorname{div}\mathbf{v} = F_2 = \operatorname{div}\mathbf{F}_3 & \text{in } \dot{\Omega} \times (0,T), \\ \partial_t\theta - \nu\Delta\theta = F_4 & \text{in } \dot{\Omega} \times (0,T), \\ \partial_t h - \mathbf{n}\cdot\mathbf{P}\mathbf{v} = F_5 & \text{on } S_R \times (0,T), \\ [\![(\mu\mathbf{D}(\mathbf{v}) - q\mathbf{I})\mathbf{n}]\!] - \sigma\Big(\Delta_{S_R} + \dfrac{N-1}{R^2}\Big)h\mathbf{n} = [\![\mathbf{F}_6]\!], \quad [\![\mathbf{v}]\!] = 0 & \text{on } S_R \times (0,T), \\ [\![\nu\nabla\theta\cdot\mathbf{n}]\!] = F_7, \quad [\![\theta]\!] = 0 & \text{on } S_R \times (0,T), \\ \mathbf{v} = 0, \quad \nabla\theta_-\cdot\mathbf{n}_- + \beta\theta_- = b(\xi,t) & \text{on } \Gamma \times (0,T), \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \dot{\Omega}, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \dot{\Omega}, \quad h|_{t=0} = h_0 & \text{on } S_R, \end{cases} \tag{3.1}$$

where $\mathbf{F}_1, \ldots, F_7$ are given functions. Here and in the following, we denote $g^\kappa(t) = e^{\kappa t} g(t)$ for a function g, and it is clear that $g^\kappa|_{\kappa=0} = g(t)$ and $g^\kappa|_{t=0} = g(0)$. Let

$$\mathbb{E}_T^\kappa(\mathbf{v}, \theta, h) := \|(\mathbf{v}^\kappa, \theta^\kappa)\|_{L^p((0,T),H_q^2(\dot\Omega))} + \|(e^{\kappa t}\partial_t \mathbf{v}, e^{\kappa t}\partial_t\theta)\|_{L^p((0,T),L^q(\dot\Omega))}$$
$$+ \|e^{\kappa t}\partial_t h\|_{L^p((0,T),W_q^{2-1/q}(\Omega))} + \|h^\kappa\|_{L^p((0,T),W_q^{3-1/p}(S_R))},$$

and

$$E_T^\kappa(\mathbf{v}_0, \theta_0, h_0, \mathbf{F}_1, \ldots, F_7, b) := \|(\mathbf{v}_0, \theta_0)\|_{B_{q,p}^{2(1-1/p)}(\dot\Omega)} + \|h_0\|_{B_{q,p}^{3-1/p-1/q}(S_R)}$$
$$+ \left\|\left(\mathbf{F}_1^\kappa, F_4^\kappa\right)\right\|_{L^p((0,T),L^q(\dot\Omega))} + \left\|F_5^\kappa\right\|_{L^p((0,T),W_q^{2-1/q}(S_R))}$$
$$+ \|e^{\kappa t}\partial_t \mathbf{F}_3\|_{L^p(\mathbb{R},L^q(\dot\Omega))} + \left\|\left(F_2^\kappa, \mathbf{F}_6^\kappa, F_7^\kappa\right)\right\|_{L^p(\mathbb{R},H_q^1(\dot\Omega))} + \|b^\kappa\|_{L^p(\mathbb{R},H_q^1(\Omega_-))}$$
$$+ \left\|\left(F_2^\kappa, \mathbf{F}_6^\kappa, F_7^\kappa\right)\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))} + \|b^\kappa\|_{H_p^{1/2}(\mathbb{R},L^q(\Omega_-))}.$$

In order to get the decay estimates of solutions for the linearized Eq. (3.1), we need to consider the following shifted equations with $\lambda_1 > 0$:

$$\begin{cases}
\rho\partial_t \mathbf{v}_1 + \lambda_1 \mathbf{v}_1 - \mathrm{Div}(\mu\mathbf{D}(\mathbf{v}_1) - q_1\mathbf{I}) = \mathbf{F}_1 & \text{in } \dot\Omega \times (0,T),\\
\mathrm{div}\,\mathbf{v}_1 = F_2 = \mathrm{div}\,\mathbf{F}_3 & \text{in } \dot\Omega \times (0,T),\\
\partial_t\theta_1 + \lambda_1\theta_1 - \nu\Delta\theta_1 = F_4 & \text{in } \dot\Omega \times (0,T),\\
\partial_t h_1 + \lambda_1 h_1 - \mathbf{n}\cdot\mathbf{P}\mathbf{v}_1 = F_5 & \text{on } S_R \times (0,T),\\
[\![(\mu\mathbf{D}(\mathbf{v}_1) - q_1\mathbf{I})\mathbf{n}]\!] - \sigma\left(\Delta_{S_R} + \dfrac{N-1}{R^2}\right)h_1\mathbf{n} = [\![F_6]\!], \quad [\![\mathbf{v}_1]\!] = 0 & \text{on } S_R \times (0,T),\\
[\![\nu\nabla\theta_1\cdot\mathbf{n}]\!] = F_7, \quad [\![\theta_1]\!] = 0 & \text{on } S_R \times (0,T),\\
\mathbf{v}_1 = 0, \quad \nabla\theta_{1-}\cdot\mathbf{n}_- + \beta\theta_{1-} = b(\xi,t) & \text{on } \Gamma \times (0,T),\\
\mathbf{v}_1|_{t=0} = \mathbf{v}_0 \quad \text{in } \dot\Omega, \quad \theta_1|_{t=0} = \theta_0 \quad \text{in } \dot\Omega, \quad h_1|_{t=0} = h_0 & \text{on } S_R.
\end{cases} \tag{3.2}$$

Then, we devote to presenting the maximal regularity for the shifted Eq. (3.2). In fact, we will prove the following results.

Theorem 3.1. *Let $1 < p, q < \infty$ and $T > 0$. Assume that $2/p + 1/q \neq 1, 2$. Then, there exists a positive constant $\lambda_2 > 0$ such that if $\lambda_1 \geq \lambda_2$, then the following assertion holds: Let $\mathbf{v}_0, \theta_0 \in B_{q,p}^{2(1-1/p)}(\dot\Omega)$ and $h_0 \in B_{q,p}^{3-1/p-1/q}(S_R)$ be initial data for equations (3.2), and let $\mathbf{F}_1, \ldots, b$ be given functions on the right side of equations (3.2) satisfying for $0 < \kappa \leq \kappa_0 := \lambda_1 - \lambda_2$ that*

$$\mathbf{F}_1^\kappa, F_4^\kappa \in L^p\big((0,T), L^q(\dot\Omega)\big),\ F_5^\kappa \in L^p\Big((0,T), W_q^{2-1/q}(S_R)\Big),$$
$$\mathbf{F}_3^\kappa \in H_p^1\big(\mathbb{R}, L^q(\dot\Omega)\big),\ F_2^\kappa, \mathbf{F}_6^\kappa, F_7^\kappa \in H_p^1\big(\mathbb{R}, H_q^1(\dot\Omega)\big) \cap H_p^{1/2}\big(\mathbb{R}, L^q(\dot\Omega)\big),$$
$$b^\kappa \in L^p\big(\mathbb{R}, H_q^1(\Omega_-)\big) \cap H_p^{1/2}\big(\mathbb{R}, L^q(\Omega_-)\big).$$

Assume that the compatibility conditions hold:

$$\mathrm{div}\,\mathbf{v}_0 = F_2|_{t=0} \quad in \quad \dot\Omega, \quad \mathbf{v}_0 - \mathbf{F}_3|_{t=0} \in \mathscr{D}(\Omega),$$

and for $2/p + 1/q < 1$

$$[\![(\mu\mathbf{D}(\mathbf{v}_0)\mathbf{n})_\tau]\!] = [\![(\mathbf{F}_6)_\tau]\!]|_{t=0}, \quad [\![\nu\nabla\theta_0\cdot\mathbf{n}]\!] = F_7|_{t=0} \quad on\ S_R,$$
$$\nabla\theta_{0-}\cdot\mathbf{n}_- + \beta\theta_{0-} = b|_{t=0} \quad on\ \Gamma;$$

for $2/p + 1/q < 2$

$$[\![\mathbf{v}_0]\!] = 0, \quad [\![\theta_0]\!] = 0 \quad on\ S_R, \quad \mathbf{v}_0 = 0 \quad on\ \Gamma.$$

Then, problem (3.2) admits a unique solution possessing the estimate

$$\mathbb{E}_T^\kappa(\mathbf{v}_1, \theta_1, h_1) \lesssim E_T^\kappa(\mathbf{v}_0, \theta_0, h_0, \mathbf{F}_1, \ldots, F_7, b). \tag{3.3}$$

Proof. Let's first consider the corresponding resolvent problem of (3.2). Since the argument is the same as for the system with $\theta_1 = 0$, we only discuss the temperature equation with interface conditions, then the related results for $\mathbf{v}_1$ and h_1 can be obtained by a similar method. Let

$\tilde{F}_4$ be the zero extension of F_4 outside $(0,T)$, that is, $\tilde{F}_4(t) = F_4(t)$ for $t \in (0,T)$ and $\tilde{F}_4(t) = 0$ for $t \notin (0,T)$. Notice that $\tilde{F}_4(t), F_7(t)$ and $b(t)$ are defined on the whole line $\mathbb{R}$, we can apply the Laplace transform to temperature equation with interface conditions. Let $\mathscr{A}(\lambda)$ $(\lambda = \gamma + i\tau)$ be operators given in Ref. 7, and then, $\mathscr{A}(\lambda+\lambda_1)$ is $\mathscr{R}$-bounded solution operators for the generalized resolvent problem corresponding to the following equations:

$$\begin{cases} \lambda\hat{\theta}_1 + \lambda_1\hat{\theta}_1 - \nu\Delta\hat{\theta}_1 = \hat{\tilde{F}}_4 & \text{in } \dot{\Omega}, \\ [\![\nu\nabla\hat{\theta}_1\cdot\mathbf{n}]\!] = \hat{F}_7, \quad [\![\hat{\theta}_1]\!] = 0 & \text{on } S_R, \\ \nabla\hat{\theta}_{1-}\cdot\mathbf{n}_- + \beta\hat{\theta}_- = \hat{b} & \text{on } \Gamma, \end{cases} \tag{3.4}$$

provided that

$$\begin{aligned} &F_4 \in L^p\big((0,T), L^q(\dot{\Omega})\big), F_7 \in H^1_p\big(\mathbb{R}, H^1_q(\dot{\Omega})\big) \cap H^{1/2}_p\big(\mathbb{R}, L^q(\dot{\Omega})\big), \\ &b \in L^p\big(\mathbb{R}, H^1_q(\Omega_-)\big) \cap H^{1/2}_p\big(\mathbb{R}, L^q(\Omega_-)\big). \end{aligned} \tag{3.5}$$

In view of Ref. 7, Theorem 3.1, we have for $\tau = \operatorname{Im}\lambda$ and some positive constant r that

$$\mathscr{R}_{\mathscr{L}\left(\mathscr{X}_q(\dot{\Omega}), H^{2-k}_q(\dot{\Omega})\right)}\Big(\Big\{(\tau\partial_\tau)^\ell\Big((\lambda+\lambda_1)^{k/2}\mathscr{A}(\lambda+\lambda_1)\Big) : \lambda+\lambda_1 \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \le r \tag{3.6}$$

is valid for $\ell = 0,1$ and $k = 0,1,2$. Here,

$$\|(A_0, A_1, \ldots, A_4)\|_{\mathscr{X}_q(\dot{\Omega})} = \|(A_0, A_1)\|_{L^q(\dot{\Omega})} + \|A_2\|_{H^1_q(\dot{\Omega})} + \|A_3\|_{L^q(\Omega_-)} + \|A_4\|_{H^1_q(\Omega_-)},$$

where A_i's correspond to $\tilde{F}_4, (\lambda+\lambda_1)^{1/2}F_7, F_7, (\lambda+\lambda_1)^{1/2}b$ and b, respectively. As in Ref. 7, for $\tau \in \mathbb{R}$, the solution of Eq. (3.4) has the following expression:

$$\theta_1 = e^{\gamma t}\mathscr{F}^{-1}_\tau\big[\mathscr{A}((\lambda+\lambda_1))\mathscr{F}\big[e^{-\gamma t}G(t)\big](\tau)\big]$$

with $G(t) = \Big(\tilde{F}_4, \Lambda^{1/2}_\gamma F_7, F_7, \Lambda^{1/2}_\gamma b, b\Big)$ and

$$\Lambda^{1/2}_\gamma f := \mathscr{F}^{-1}_L\Big[(\lambda+\lambda_1)^{1/2}\mathscr{F}_L[f]\Big] = e^{\gamma t}\mathscr{F}^{-1}_\tau\Big[(\lambda+\lambda_1)^{1/2}\mathscr{F}\big[e^{-\gamma t}f\big]\Big]$$

where $\mathscr{F}^{-1}_L$ and $\mathscr{F}^{-1}_\tau$ can be found in the Appendix. In view of the properties of the Laplace transform, we have

$$\partial_t\theta_1 = e^{\gamma t}\mathscr{F}^{-1}_\tau\big[\lambda\mathscr{A}(\lambda+\lambda_1)\mathscr{F}\big[e^{-\gamma t}G(t)\big](\tau)\big].$$

Obviously, we can rewrite it as

$$\begin{aligned} \partial_t\theta_1 = {} & e^{\gamma t}\mathscr{F}^{-1}_\tau\big[(\lambda+\lambda_1)\mathscr{A}(\lambda+\lambda_1)\mathscr{F}\big[e^{-\gamma t}G(t)\big](\tau)\big] \\ & - \lambda_1 e^{\gamma t}\mathscr{F}^{-1}_\tau\big[\mathscr{A}(\lambda+\lambda_1)\mathscr{F}\big[e^{-\gamma t}G(t)\big](\tau)\big]. \end{aligned} \tag{3.7}$$

According to the maximal L^p–L^q regularity in Ref. 7 for the system of heat equations with interface conditions, by (3.6) and (3.7), we obtain

$$\begin{aligned} &\big\|e^{-\gamma t}\partial_t\theta_1\big\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))} + \big\|e^{-\gamma t}\theta_1\big\|_{L^p(\mathbb{R},H^2_q(\dot{\Omega}))} \\ &\lesssim \big\|e^{-\gamma t}\tilde{F}_4\big\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))} + \big\|e^{-\gamma t}F_7\big\|_{L^p\left(\mathbb{R},H^1_q(\dot{\Omega})\right)} + \Big\|e^{-\gamma t}\Lambda^{1/2}_\gamma F_7\Big\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))} \\ &\quad + \big\|e^{-\gamma t}b\big\|_{L^p(\mathbb{R},H^1_q(\Omega_-))} + \Big\|e^{-\gamma t}\Lambda^{1/2}_\gamma b\Big\|_{L^p(\mathbb{R},L^q(\Omega_-))}. \end{aligned} \tag{3.8}$$

Since $\Big|(\lambda+\lambda_1)^{1/2}\big(1+\tau^2\big)^{-1/4}\Big| \lesssim 1 + \lambda_1^{1/2} + \gamma^{1/2}$, we get

$$\begin{aligned} &\Big\|e^{-\gamma t}\Lambda^{1/2}_\gamma F_7\Big\|_{L^p(\mathbb{R},L^q(\dot{\Omega}))} + \Big\|e^{-\gamma t}\Lambda^{1/2}_\gamma b\Big\|_{L^p(\mathbb{R},L^q(\Omega_-))} \\ &\lesssim \Big(1+\lambda_1^{1/2}+\gamma^{1/2}\Big)\Big(\big\|e^{-\gamma t}F_7\big\|_{H^{1/2}_p(\mathbb{R},L^q(\dot{\Omega}))} + \big\|e^{-\gamma t}b\big\|_{H^{1/2}_p(\mathbb{R},L^q(\Omega_-))}\Big). \end{aligned} \tag{3.9}$$

Due to $|(\gamma+\lambda_1)(\lambda+\lambda_1)^{-1}| \le 1$, by (3.7)–(3.9), we have

$$\begin{aligned}
&\|e^{-\gamma t}\theta_1\|_{L^p(\mathbb{R},L^q(\dot\Omega))}\\
&\lesssim (\gamma+\lambda_1)^{-1}\big(\|e^{-\gamma t}\partial_t\theta_1\|_{L^p(\mathbb{R},L^q(\dot\Omega))} + \|e^{-\gamma t}\theta_1\|_{L^p(\mathbb{R},H^2_q(\dot\Omega))}\big)\\
&\lesssim (\gamma+\lambda_1)^{-1}\Big\{\|e^{-\gamma t}F_4\|_{L^p((0,T),L^q(\dot\Omega))} + \|e^{-\gamma t}F_7\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))} + \|e^{-\gamma t}b\|_{L^p(\mathbb{R},H^1_q(\Omega_-))}\\
&\quad + \Big(1+\lambda_1^{1/2}+\gamma^{1/2}\Big)\Big(\|e^{-\gamma t}F_7\|_{H^{1/2}_p(\mathbb{R},L^q(\dot\Omega))} + \|e^{-\gamma t}b\|_{H^{1/2}_p(\mathbb{R},L^q(\Omega_-))}\Big)\Big\}.
\end{aligned} \tag{3.10}$$

Thus, letting $\lambda_1 \to \infty$, we can get $\|\theta_1\|_{L^p((-\infty,0),L^q(\dot\Omega))} \le \|e^{-\gamma t}\theta_1\|_{L^p(\mathbb{R},L^q(\dot\Omega))} = 0$, which leads to $\theta_1 = 0$ a.e. for $t < 0$.

In Ref. 7, Theorem 2.1, $\gamma > 0$ is large enough to ensure $\theta_1 = 0$ a.e. for $t < 0$, but by (3.10), we can choose $\gamma = 0$ and $\lambda_1 \ge \lambda_2$, where λ_2 is sufficiently large position number, to lead to $\theta_1 = 0$ a.e. for $t < 0$. Thus, by (3.8)–(3.10) and choosing $\gamma = 0$, we have

$$\mathbb{E}^0_T(0,\theta_1,0) \lesssim E^0_T(0,\theta_0,0,0,\ldots,F_7,b).$$

Analogously, let $\tilde{\mathbf{F}}_1$ and $\tilde F_5$ be the zero extension of $\mathbf{F}_1$ and F_5 outside $(0, T)$, respectively, by the maximal regularity in Ref. 23 and choosing $\gamma = 0$ we have

$$\mathbb{E}^0_T(\mathbf{v}_1,\theta_1,h_1) \lesssim E^0_T(\mathbf{v}_0,\theta_0,h_0,\mathbf{F}_1,\ldots,F_7,b). \tag{3.11}$$

Then for any $\kappa > 0$, $\mathbf{v}_1^\kappa$, θ_1^κ and h_1^κ are solutions of the following shifted equations:

$$\begin{cases}
\rho\partial_t\mathbf{v}_1^\kappa + (\lambda_1-\kappa)\mathbf{v}_1^\kappa - \mathrm{Div}(\mu\mathbf{D}(\mathbf{v}_1^\kappa) - q_1^\kappa\mathbf{I}) = \mathbf{F}_1^\kappa & \text{in } \dot\Omega\times(0,T),\\
\mathrm{div}\,\mathbf{v}_1^\kappa = F_2^\kappa = \mathrm{div}\,\mathbf{F}_3^\kappa & \text{in } \dot\Omega\times(0,T),\\
\partial_t\theta_1^\kappa + (\lambda_1-\kappa)\theta_1^\kappa - \nu\Delta\theta_1^\kappa = F_4^\kappa & \text{in } \dot\Omega\times(0,T),\\
\partial_t h_1^\kappa + (\lambda_1-\kappa)h_1^\kappa - \mathbf{n}\cdot\mathbf{P}\mathbf{v}_1^\kappa = F_5^\kappa & \text{on } S_R\times(0,T),\\
[\![(\mu\mathbf{D}(\mathbf{v}_1^\kappa) - q_1^\kappa\mathbf{I})\mathbf{n}]\!] - \sigma\Big(\Delta_{S_R} + \dfrac{N-1}{R^2}\Big)h_1^\kappa\mathbf{n} = [\![F_6^\kappa]\!] & \text{on } S_R\times(0,T),\\
[\![\mathbf{v}_1^\kappa]\!] = 0,\ [\![\nu\nabla\theta_1^\kappa\cdot\mathbf{n}]\!] = F_7^\kappa,\ [\![\theta_1^\kappa]\!] = 0 & \text{on } S_R\times(0,T),\\
\mathbf{v}_1^\kappa = 0,\quad \nabla\theta_{1-}^\kappa\cdot\mathbf{n}_- + \beta\theta_{1-}^\kappa = b^\kappa & \text{on } \Gamma\times(0,T),\\
\mathbf{v}_1^\kappa|_{t=0} = \mathbf{v}_0 \quad \text{in } \dot\Omega,\quad \theta_1^\kappa\big|_{t=0} = \theta_0 \quad \text{in } \dot\Omega,\quad h_1^\kappa\big|_{t=0} = h_0 & \text{on } S_R.
\end{cases} \tag{3.12}$$

We can choose $0 < \kappa \le \kappa_0 = \lambda_1 - \lambda_2$, by (3.11) and $H^1_q(\dot\Omega) \subset L^q(\dot\Omega)$, then we have

$$\mathbb{E}^\kappa_T(\mathbf{v}_1,\theta_1,h_1) \lesssim \mathbb{E}^0_T(\mathbf{v}_1^\kappa,\theta_1^\kappa,h_1^\kappa) \lesssim E^\kappa_T(\mathbf{v}_0,\theta_0,h_0,\mathbf{F}_1,\ldots,F_7,b).$$

Therefore, we complete the Proof of Theorem 3.1. □

IV. DECAY ESTIMATES OF SOLUTIONS FOR THE LINEARIZED EQUATIONS

In this section, we study the resolvent problem corresponding to (3.1) in which we assume $F_2 = 0$ (i.e., $\mathrm{div}\,\mathbf{v} = 0$), $\mathbf{F}_6 = 0$, $F_7 = 0$ and $b = 0$. Since the pressure q in Eq. (3.1) has no evolution, we will eliminate q to formulate the problem in the semigroup setting. In view of Ref. 23, in fact, we shall reduce the corresponding resolvent equations to the following equivalent equations:

$$\begin{cases}
\rho\partial_t\mathbf{v} - \mathrm{Div}(\mu\mathbf{D}(\mathbf{v}) - K(\mathbf{v},h)\mathbf{I}) = \mathbf{F}_1 & \text{in } \dot\Omega\times(0,\infty),\\
\partial_t\theta - \nu\Delta\theta = F_4 & \text{in } \dot\Omega\times(0,T),\\
[\![\nu\nabla\theta\cdot\mathbf{n}]\!] = 0,\quad [\![\theta]\!] = 0 & \text{on } S_R\times(0,T),\\
\partial_t h - \mathbf{n}\cdot\mathbf{P}\mathbf{v} = F_5,\quad [\![\mathbf{v}]\!] = 0 & \text{on } S_R\times(0,\infty),\\
[\![(\mu\mathbf{D}(\mathbf{v}) - K(\mathbf{v},h)\mathbf{I})\mathbf{n}]\!] - \sigma\Big(\Delta_{S_R}h + \dfrac{N-1}{R^2}h\Big)\mathbf{n} = 0 & \text{on } S_R\times(0,\infty),\\
\mathbf{v} = 0,\quad \nabla\theta_-\cdot\mathbf{n}_- + \beta\theta_- = 0 & \text{on } \Gamma\times(0,\infty),\\
(\mathbf{v},\theta,h)|_{t=0} = (\mathbf{v}_0,\theta_0,h_0) & \text{on } \dot\Omega\times S_R.
\end{cases} \tag{4.1}$$

Here, let $K(\mathbf{v},h) \in H^1_q(\dot\Omega) + \hat H^1_q(\Omega)$ be a unique solution of the weak Dirichlet problem

$$(\nabla K(\mathbf{v},h), \nabla\varphi)_{\dot\Omega} = (\mathrm{Div}(\mu\mathbf{D}(\mathbf{v})) - \nabla\,\mathrm{div}\,\mathbf{v}, \nabla\varphi)_{\dot\Omega}$$

for any $\varphi \in \hat H^1_{q'}(\Omega)$, subject to

$$[\![K(\mathbf{v},h)]\!] = [\![(\mu\mathbf{D}(\mathbf{v})\mathbf{n},\mathbf{n})]\!] - \delta\Big(\Delta_{S_R}h + \frac{N-1}{R^2}h\Big) - [\![\mathrm{div}\,\mathbf{v}]\!] \quad \text{on } S_R.$$

Let

$$\mathbf{L}(\mathbf{v},\theta,h) = (\mathrm{Div}(\mu\mathbf{D}(\mathbf{v}) - K(\mathbf{v},h)\mathbf{I}), \nu\Delta\theta, (\mathbf{n}\cdot\mathbf{P}\mathbf{v})|_{S_R}).$$

For $(\mathbf{v},h) \in \mathscr{D}_q$ and $\theta \in \mathscr{J}_q$, when $\mathbf{F}_1 = 0$, $F_4 = 0$ and $F_5 = 0$, we see that (4.1) is formulated by

$$\partial_t(\mathbf{v},\theta,h) = \mathbf{L}(\mathbf{v},\theta,h) \quad \text{for } t > 0, \quad (\mathbf{v},\theta,h)|_{t=0} = (\mathbf{v}_0,\theta_0,h_0). \tag{4.2}$$

According to Ref. 23, Theorem 3.2.4 and Ref. 7, Theorem 3.1, we can obtain that $\mathbf{L}$ generates an analytic semigroup $S(t)$ on $(\mathbf{v},\theta,h) \in \mathscr{H}_q$ with the norm

$$\|(\mathbf{v},\theta,h)\|_{\mathscr{H}_q} = \|\mathbf{v}\|_{L^q(\dot\Omega)} + \|\theta\|_{L^q(\dot\Omega)} + \|h\|_{W^{2-1/q}_q(S_R)},$$

then, we have the following theorem.

Theorem 4.1. *Let $1 < q < \infty$. Then, $\{S(t)\}_{t\ge0}$ is exponentially stable, that is,*

$$\|S(t)(\mathbf{v},\theta,h)\|_{\mathscr{H}_q} \le Ce^{-\kappa_1 t}\|(\mathbf{v},\theta,h)\|_{\mathscr{H}_q},$$

for any $t > 0$ and $(\mathbf{v},\theta,h) \in \mathscr{H}_q$ with some positive constants C and κ_1.

According to the standard semigroup theory in Ref. 12, we consider the following resolvent equations:

$$(\lambda\mathbf{I} - \mathbf{L})(\mathbf{v},\theta,h) = U, \tag{4.3}$$

for $U = (\mathbf{U}_1, U_2, U_3) \in \mathscr{H}_q$. In fact, our task is to prove the following theorem.

Theorem 4.2. *Let $1 < q < \infty$ and $\Lambda = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \ge 0\}$. Then, for any $\lambda \in \Lambda$ and $U \in \mathscr{H}_q$, (4.3) admits a unique solution $(\mathbf{v},\theta,h) \in \mathscr{H}_q(\dot\Omega)$ with $(\mathbf{v},h) \in \mathscr{D}_q(\dot\Omega)$ and $\theta \in \mathscr{J}_q(\dot\Omega)$ possessing the estimate*

$$|\lambda|\|(\mathbf{v},\theta,h)\|_{\mathscr{H}_q} + \|(\mathbf{v},h)\|_{\mathscr{D}_q} + \|\theta\|_{\mathscr{J}_q} \le C\|U\|_{\mathscr{H}_q},$$

where $\|(\mathbf{v},h)\|_{\mathscr{D}_q} = \|\mathbf{v}\|_{H^2_q(\dot\Omega)} + \|h\|_{W^{3-1/q}_q(S_R)}$ and $\|\theta\|_{\mathscr{J}_q} = \|\theta\|_{H^2_q(\dot\Omega)}$.

Proof. Let $\Lambda_{\lambda_0} = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda \ge 0, \quad |\lambda| \le \lambda_0\}$, where λ_0 is a positive number in Theorem A.4. And then, we only consider the resolvent heat equation with the interface condition

$$\lambda\theta - \nu\Delta\theta = U_3, \quad \text{for} \quad \theta \in \mathscr{J}_q. \tag{4.4}$$

Since $\lambda_0 \in \Lambda_{\lambda_0}$, we can set $\lambda = \lambda_0 \in \Sigma_{\varepsilon_0,\lambda_0}$, we see that $(\lambda_0 I - \nu\Delta)^{-1}$ is a bounded linear operator by Theorem A.4, where I is the identity operator. Then for $\lambda \in \Lambda_{\lambda_0}$, we have

$$\lambda I - \nu\Delta = (\lambda - \lambda_0)I + \lambda_0 I - \nu\Delta = (I + (\lambda-\lambda_0)(\lambda_0 I - \nu\Delta)^{-1})(\lambda_0 I - \nu\Delta).$$

Therefore, if $\big(I + (\lambda-\lambda_0)(\lambda_0 I - \nu\Delta)^{-1}\big)^{-1}$ exists as a bounded linear operator, we have

$$\theta = (\lambda_0 I - \nu\Delta)^{-1}\big(I + (\lambda-\lambda_0)(\lambda_0 I - \nu\Delta)^{-1}\big)^{-1}U_3.$$

Since $\dot\Omega$ is a bounded domain, $\mathscr{J}_q$ is compactly embedded into L^q by the compactness theorem, then by Theorem A.4, $(\lambda_0 I - \nu\Delta)^{-1}$ is a compact operator from $\mathscr{H}_q$ into itself. Thus, by the Riesz–Schauder theorem, it suffices to prove the triviality of the kernel of $I + (\lambda-\lambda_0)(\lambda_0 I - \nu\Delta)^{-1}$ in order to prove the existence of the inverse operator $(I + (\lambda-\lambda_0)(\lambda_0 I - \nu\Delta)^{-1})^{-1}$. Notice that $f = -(\lambda-\lambda_0)(\lambda_0 I - \nu\Delta)^{-1}f \in \mathscr{J}_q$. Moreover,

$$(\lambda I - \nu\Delta)f = (\lambda_0 I - \nu\Delta + (\lambda-\lambda_0)I)f = -(\lambda-\lambda_0)f + (\lambda-\lambda_0)f = 0.$$

Our task is to prove $f = 0$. Namely, $f \in \mathscr{J}_q$ satisfies the homogeneous equations

$$\begin{cases} \lambda f - \nu\Delta f = 0 & \text{in } \dot{\Omega}, \\ [\![\nu\nabla f \cdot \mathbf{n}]\!] = 0, \quad [\![f]\!] = 0 & \text{on } S_R, \\ \nabla f_- \cdot \mathbf{n}_- + \beta f_- = 0 & \text{on } \Gamma, \end{cases} \tag{4.5}$$

then we devote to proving $f = 0$.

Firstly, we consider the case that $2 \le q < \infty$. Since $f \in \mathscr{J}_q$ and $\dot{\Omega}$ is a bounded domain, by the divergence theorem and (4.5), we have

$$\begin{aligned} 0 &= (\lambda f - \nu\Delta f, f)_{\dot{\Omega}} \\ &= \lambda\|f\|^2_{L^2(\dot{\Omega})} + \nu\|\nabla f\|^2_{L^2(\dot{\Omega})} - ([\![\nu\nabla f \cdot \mathbf{n}]\!], f)_{S_R} - (\nu\nabla f_- \cdot \mathbf{n}_-, f_-)_\Gamma. \end{aligned}$$

Thus, noticing the interface conditions in (4.5), we obtain

$$0 = \lambda\|f\|^2_{L^2(\dot{\Omega})} + \nu\|\nabla f\|^2_{L^2(\dot{\Omega})} + \frac{\beta}{\nu}\|f_-\|^2_{L_2(\Gamma)}. \tag{4.6}$$

Therefore, combined with $\operatorname{Re}\lambda \ge 0$, taking the real part of the (4.6) leads to $\nabla f = 0$ in $\dot{\Omega}$. Since $[\,[f]\,] = 0$ on S_R and $f_+ = f_- = 0$ on Γ, we have $f = 0$ in $\dot{\Omega}$. Namely, problem (4.4) admits a unique solution $\theta \in \mathscr{J}_q$ possessing the estimate

$$|\lambda|\|\theta\|_{L^q(\dot{\Omega})} + \|\theta\|_{\mathscr{J}_q(\dot{\Omega})} \le C\|U_3\|_{L^q(\dot{\Omega})} \tag{4.7}$$

with some constant $C > 0$, $2 \le q < \infty$ and $\lambda \in \Lambda_{\lambda_0}$.

Next, we consider the case that $1 < q < 2$ and for any $\omega \in L^{q'}$. Let $g \in \mathscr{J}_{q'}$ be a solution of the following equations:

$$\begin{cases} \bar{\lambda} g - \nu\Delta g = \omega & \text{in } \dot{\Omega}, \\ [\![\nu\nabla g \cdot \mathbf{n}]\!] = 0, \quad [\![g]\!] = 0 & \text{on } S_R, \\ \nabla g_- \cdot \mathbf{n}_- + \beta g_- = 0 & \text{on } \Gamma. \end{cases} \tag{4.8}$$

Since $\lambda, \bar{\lambda} \in \Lambda_{\lambda_0}$ and $2 < q' < \infty$, by the fact proved above we know the unique existence of $g \in \mathscr{J}_{q'}$. By (4.5) and (4.8) and the divergence theorem, we have

$$\begin{aligned} 0 &= (\lambda f - \nu\Delta f, g)_{\dot{\Omega}} \\ &= \lambda(f, g)_{\dot{\Omega}} + \nu(\nabla f, \nabla g)_{\dot{\Omega}} - ([\![\nabla f \cdot \mathbf{n}]\!], g)_{S_R} - (\nabla f_- \cdot \mathbf{n}_-, g_-)_\Gamma, \\ (f, \omega)_{\dot{\Omega}} &= (f, \bar{\lambda} g)_{\dot{\Omega}} + \nu(\nabla f, \nabla g)_{\dot{\Omega}} - ([\![\nabla g \cdot \mathbf{n}]\!], f)_{S_R} - (\nabla g_- \cdot \mathbf{n}_-, f_-)_\Gamma. \end{aligned}$$

By the boundary condition on Γ, we obtain

$$(f, \omega)_{\dot{\Omega}} = 0 \quad \text{for any } \omega \in L^{q'}(\dot{\Omega}),$$

which yields $f = 0$. Thus, problem (4.4) admits a unique solution $\theta \in \mathscr{J}_q(\dot{\Omega})$ possessing the estimate (4.7) when $1 < q < 2$ and $\lambda \in \Lambda_{\lambda_0}$.

Finally, we consider $\lambda \in \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \ge 0, \quad |\lambda| \ge \lambda_0\}$. It is evident that the result in Theorem A.4 can be obtained. The estimation of $\mathbf{v}$ and h is similar to the treatment of temperature θ, and one can refer to Ref. 23, therefore, we omit the proof details. In summary, we have completed the Proof of Theorem 4.2. □

Now, we can get the following decay estimates for linearized Eq. (3.1).

Theorem 4.3. *Let $1 < p, q < \infty$ and $T > 0$. Assume that $2/p + 1/q \ne 1, 2$. Assume that Γ is a nonempty compact hypersurface of class C^2. Let $0 < \kappa \le 1$, $\mathbf{u}_0, \theta_0 \in B^{2(1-1/p)}_{q,p}(\dot{\Omega})$ and $h_0 \in B^{1-1/p-1/q}_{q,p}(S_R)$ be initial data for (3.1), and let $\mathbf{F}_1, \ldots, b$ be given functions on the right side of (3.1) satisfying:*

$$\begin{aligned} &(\mathbf{F}_1^\kappa, F_4^\kappa) \in L^p\big((0,T), L^q(\dot{\Omega})\big),\ F_5^\kappa \in L^p\Big((0,T), W_q^{2-1/q}(S_R)\Big), \\ &\mathbf{F}_3^\kappa \in H_p^1\big(\mathbb{R}, L^q(\dot{\Omega})\big), \\ &F_2^\kappa, F_6^\kappa, F_7^\kappa \in H_p^1\big(\mathbb{R}, H_q^1(\dot{\Omega})\big) \cap H_p^{1/2}\big(\mathbb{R}, L^q(\dot{\Omega})\big) \\ &b^\kappa \in L^p\big(\mathbb{R}, H_q^1(\Omega_-)\big) \cap H_p^{1/2}\big(\mathbb{R}, L^q(\Omega_-)\big). \end{aligned}$$

Assume that the compatibility conditions hold:

$$\operatorname{div}\mathbf{v}_0 = F_2|_{t=0} \quad in \quad \dot{\Omega}, \quad \mathbf{v}_0 - \mathbf{F}_3|_{t=0} \in \mathscr{D}(\Omega),$$

where $\mathscr{D}(\Omega)$ *can be found in the Appendix, and for* $2/p + 1/q < 1$

$$\begin{aligned}&[\![(\mu\mathbf{D}(\mathbf{v}_0)\mathbf{n})_\tau]\!] = [\![(\mathbf{F}_6)_\tau]\!]|_{t=0}, \quad [\![\nu\nabla\theta_0\cdot\mathbf{n}]\!] = F_7|_{t=0} \quad on\, S_R,\\ &\nabla\theta_{0-}\cdot\mathbf{n}_- + \beta\theta_{0-} = b|_{t=0} \quad on\,\Gamma;\end{aligned}$$

for $2/p + 1/q < 2$

$$[\![\mathbf{v}_0]\!] = 0, \quad [\![\theta_0]\!] = 0 \quad on\, S_R, \quad \mathbf{v}_0 = 0 \quad on\,\Gamma.$$

Then, problem (3.1) admits a unique solution possessing the estimate

$$\mathbb{E}_T^\kappa(\mathbf{v},\theta,h) \lesssim E_T^\kappa(\mathbf{v}_0,\theta_0,h_0,\mathbf{F}_1,\ldots,F_7,b).$$

Proof. We consider the solutions $\mathbf{v}$, θ, q and h of Eq. (3.1) of the form: $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, $\theta = \theta_1 + \theta_2$, $q = q_1 + q_2$ and $h = h_1 + h_2$, where $\mathbf{v}_1$, θ_1, q_1 and h_1 are solutions of the shifted Eq. (3.2), and then $\mathbf{v}_2$, θ_2, q_2 and h_2 satisfy the equations:

$$\begin{cases}\rho\partial_t\mathbf{v}_2 - \operatorname{Div}(\mu\mathbf{D}(\mathbf{v}_2) - q_2\mathbf{I}) = \lambda_1\mathbf{v}_1, \quad \operatorname{div}\mathbf{v}_2 = 0 & \text{in } \dot{\Omega}\times(0,T),\\ \partial_t\theta_2 - \nu\Delta\theta_2 = \lambda_1\theta_1 & \text{in } \dot{\Omega}\times(0,T),\\ \partial_t h_2 - \mathbf{n}\cdot\mathbf{P}\mathbf{v}_2 = \lambda_1 h_1 & \text{on } S_R\times(0,T),\\ [\![(\mu\mathbf{D}(\mathbf{v}_2) - q_2\mathbf{I})\mathbf{n}]\!] - \sigma\Big(\Delta_{S_R}h_2 + \dfrac{N-1}{R^2}h_2\Big)\mathbf{n} = 0, \ [\![\mathbf{v}_2]\!] = 0 & \text{on } S_R\times(0,T),\\ [\![\nu\nabla\theta_2\cdot\mathbf{n}]\!] = 0, \quad [\![\theta_2]\!] = 0 & \text{on } S_R\times(0,T),\\ \mathbf{v}_2 = 0, \quad \nabla\theta_{2-}\cdot\mathbf{n}_- + \beta\theta_{2-} = 0 & \text{on } \Gamma\times(0,T),\\ (\mathbf{v}_2,\theta_2,h_2)|_{t=0} = (0,0,0) & \text{on } \dot{\Omega}\times S_R.\end{cases} \tag{4.9}$$

In fact, the estimations of (4.9) can be reduced to the study of (4.1). Then, we can replace $\mathbf{F}_1, F_4, F_5$ with $\lambda_1\mathbf{v}_1$, $\lambda_1\theta_1$ and $\lambda_1 h_1$, respectively. Thus, in view of (4.2), $\mathbf{L}$ generates an analytic semigroup $S(t)$ on $(\mathbf{v}_2,\theta_2,h_2) \in \mathscr{H}_q$. Then by the Duhamel principle, it yields

$$(\mathbf{v}_2,\theta_2,h_2)(\cdot,\tau) = \int_0^\tau S(\tau-s)(\lambda_1\mathbf{v}_1(\cdot,s),\lambda_1\theta_1(\cdot,s),\lambda_1 h_1(\cdot,s))ds.$$

Owing to Theorem 4.1, we have

$$\begin{aligned}\|(\mathbf{v}_2,\theta_2,h_2)(\cdot,\tau)\|_{\mathscr{H}_q} &\lesssim \int_0^\tau e^{-\kappa_1(\tau-s)}\|(\mathbf{v}_1(\cdot,s),\theta_1(\cdot,s),h_1(\cdot,s))\|_{\mathscr{H}_q}ds\\ &\lesssim \Big(\int_0^\tau e^{-\kappa_1(\tau-s)}ds\Big)^{1/p'}\Big(\int_0^\tau e^{-\kappa_1(\tau-s)}\|(\mathbf{v}_1(\cdot,s),\theta_1(\cdot,s),h_1(\cdot,s))\|_{\mathscr{H}_q}^p ds\Big)^{1/p}.\end{aligned}$$

Thus, by the above inequality we obtain

$$\begin{aligned}&\int_0^t\Big(e^{\kappa\tau}\|(\mathbf{v}_2(\cdot,\tau),\theta_2(\cdot,\tau),h_2(\cdot,\tau))\|_{\mathscr{H}_q}\Big)^p d\tau\\ &\quad\lesssim \int_0^t\Big(\int_0^\tau e^{-(\kappa_1-p\kappa)(\tau-s)}\Big(e^{\kappa s}\|((\mathbf{v}_1(\cdot,s),\theta_1(\cdot,s),h_1(\cdot,s)))\|_{\mathscr{H}_q}\Big)^p ds\Big)d\tau\\ &\quad\lesssim (\kappa_1-p\kappa)^{-1}\int_0^T\Big(e^{\kappa s}\|(\mathbf{v}_1(\cdot,s),\theta_1(\cdot,s),h_1(\cdot,s))\|_{\mathscr{H}_q}\Big)^p ds.\end{aligned}$$

Choosing $0 < \kappa < 1$ such that $0 < \kappa p < \kappa_1$, and combining with (3.3), for any $t \in (0,T)$, we have

$$\big\|e^{\kappa\tau}(\mathbf{v}_2,\theta_2,h_2)\big\|_{L^p((0,T),\mathscr{H}_q)} \lesssim E_T^\kappa(\mathbf{v}_0,\theta_0,h_0,\mathbf{F}_1,\ldots,F_7,b). \tag{4.10}$$

By (4.9), $(\mathbf{v}_2, \theta_2, h_2)$ are solutions of the shifted equations:

$$\begin{cases} \rho\partial_t \mathbf{v}_2 + \lambda_1 \mathbf{v}_2 - \mathrm{Div}(\mu \mathbf{D}(\mathbf{v}_2) - q_2 \mathbf{I}) = \lambda_1 \mathbf{v}_1 + \lambda_1 \mathbf{v}_2 & \text{in } \dot{\Omega} \times (0,T), \\ \partial_t \theta_2 + \lambda_1 \theta_2 - \nu\Delta\theta_2 = \lambda_1\theta_1 + \lambda_1\theta_2 & \text{in } \dot{\Omega} \times (0,T), \\ \partial_t h_2 + \lambda_1 h_2 - \mathbf{n}\cdot\mathbf{P}\mathbf{v}_2 = \lambda_1 h_1 + \lambda_1 h_2 & \text{on } S_R \times (0,T), \\ [\![(\mu\mathbf{D}(\mathbf{v}_2) - q_2\mathbf{I})\mathbf{n}]\!] - \sigma\Big(\Delta_{S_R} h_2 + \dfrac{N-1}{R^2} h_2\Big)\mathbf{n} = 0, \quad [\![\mathbf{v}_2]\!] = 0 & \text{on } S_R \times (0,T), \\ [\![\nu\nabla\theta_2\cdot\mathbf{n}]\!] = 0, \quad [\![\theta_2]\!] = 0 & \text{on } S_R \times (0,T), \\ \mathbf{v}_2 = 0, \quad \nabla\theta_{2-}\cdot\mathbf{n}_- + \beta\theta_{2-} = 0 & \text{on } \Gamma \times (0,T), \\ (\mathbf{v}_2, \theta_2, h_2)|_{t=0} = (0,0,0) & \text{on } \dot{\Omega} \times S_R. \end{cases} \tag{4.11}$$

Using Theorem 3.1, (3.3) and (4.10), we have

$$\begin{aligned} \mathbb{E}_T^{\kappa}(\mathbf{v}_2, \theta_2, h_2) &\lesssim E_T^{\kappa}(0,0,0,\lambda_1\mathbf{v}_1 + \lambda_1\mathbf{v}_2, \lambda_1\theta_1 + \lambda_1\theta_2, \lambda_1 h_1 + \lambda_1 h_2, 0\ldots,0) \\ &\lesssim \big\| e^{\kappa s}(\mathbf{v}_1,\theta_1,h_1)\big\|_{L^p((0,T),\mathscr{H}_q)} + \big\| e^{\kappa s}(\mathbf{v}_2,\theta_2,h_2)\big\|_{L^p((0,T),\mathscr{H}_q)} \\ &\lesssim E_T^{\kappa}(\mathbf{v}_0,\theta_0,h_0,\mathbf{F}_1,\ldots,F_7,b). \end{aligned} \tag{4.12}$$

Thus, $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, $\theta = \theta_1 + \theta_2$ and $h = h_1 + h_2$ are the required solutions of Eq. (3.1). And by (3.3) and (4.12), we obtain

$$\mathbb{E}_T^{\kappa}(\mathbf{v},\theta,h) \lesssim \mathbb{E}_T^{\kappa}(\mathbf{v}_1,\theta_1,h_1) + \mathbb{E}_T^{\kappa}(\mathbf{v}_2,\theta_2,h_2) \lesssim E_T^{\kappa}(\mathbf{v}_0,\theta_0,h_0,\mathbf{F}_1,\ldots,F_7,b), \tag{4.13}$$

which completes the Proof of Theorem 4.3. □

V. ESTIMATES OF NONLINEAR TERMS

Let $T_0 > 1$ be a given positive number such that $T \in (0, T_0]$, then assume that the initial data are so small enough that for some $\kappa > 0$ and small number ε,

$$\begin{aligned} \mathscr{I} := \|\mathbf{f}^{\kappa}\|_{L^p((0,\infty),L^q(\dot{\Omega}))} + \|b^{\kappa}\|_{L^p((0,\infty),H_q^1(\Omega_-))} + \|b^{\kappa}\|_{H_p^{1/2}((0,\infty),L^q(\Omega_-))} \\ + \|\mathbf{v}_0\|_{B_{q,p}^{2-2/p}(\dot{\Omega})} + \|\theta_0\|_{B_{q,p}^{2-2/p}(\dot{\Omega})} + \|h_0\|_{W_{q,p}^{3-1/p-1/q}(S_R)} \le \varepsilon. \end{aligned} \tag{5.1}$$

Denote

$$\mathscr{E}_T^{\kappa}(\mathbf{v},\theta,h) = \mathbb{E}_T^{\kappa}(\mathbf{v},\theta,h) + \|e^{\kappa t}\partial_t h\|_{L^\infty((0,T),W_q^{1-1/q}(S_R))}.$$

In what follows, we shall use the following inequalities

$$\begin{aligned} \sup_{t\in(0,T)} \|(\mathbf{v}^{\kappa}(\cdot,t),\theta^{\kappa}(\cdot,t))\|_{B_{q,p}^{2(1-1/p)}(\dot{\Omega})} &\lesssim \mathscr{I} + \mathscr{E}_T^{\kappa}(\mathbf{v},\theta,h), \\ \sup_{t\in(0,T)} \|h^{\kappa}(\cdot,t)\|_{B_{q,p}^{3-1/p-1/q}(S_R)} &\lesssim \mathscr{I} + \mathscr{E}_T^{\kappa}(\mathbf{v},\theta,h), \end{aligned} \tag{5.2}$$

which will be proved later. Since $p > 2$ and $1/p + 1/q < 1$, combined with (2.4) and (5.2), they lead to

$$\begin{aligned} \sup_{t\in(0,T)} \|(\mathbf{v}^{\kappa}(\cdot,t),\theta^{\kappa}(\cdot,t))\|_{H_q^1(\dot{\Omega})} &\lesssim \sup_{t\in(0,T)} \|(\mathbf{v}^{\kappa}(\cdot,t),\theta^{\kappa}(\cdot,t))\|_{B_{q,p}^{2(1-1/p)}(\dot{\Omega})} \lesssim \mathscr{I} + \mathscr{E}_T^{\kappa}(\mathbf{v},\theta,h), \\ \sup_{t\in(0,T)} \|h^{\kappa}(\cdot,t)\|_{W_q^{2-1/q}(S_R)} &\lesssim \sup_{t\in(0,T)} \|h^{\kappa}(\cdot,t)\|_{B_{q,p}^{3-1/p-1/q}(S_R)} \lesssim \mathscr{I} + \mathscr{E}_T^{\kappa}(\mathbf{v},\theta,h). \end{aligned} \tag{5.3}$$

In fact, we also need to assume

$$\sup_{t\in(0,T_0)} \|\Psi_h(\cdot,t)\|_{H_\infty^1(\Omega)} \le \varepsilon, \tag{5.4}$$

by (2.7) and the definition of $\tilde{\zeta}(t)$, we have

$$\|\Psi_h(\cdot,t)\|_{H_\infty^1(\dot{\Omega})} \le C_0\Big(\|\Phi_h(\cdot,t)\|_{H_\infty^1(\dot{\Omega})} + |\tilde{\zeta}(t)|\Big),$$

where C_0 is a positive constant independent of T. From this point of view, instead of (5.4), we assume that

$$\|\Phi_h(\cdot,t)\|_{H^1_\infty(\dot\Omega)} \le \varepsilon_1, \quad |\tilde\zeta(t)| \le \varepsilon_1, \tag{5.5}$$

where ε_1 is a small number such that $2C_0\varepsilon_1 \le \varepsilon$. We need to guarantee the justification of the *a priori* assumption of (5.5). By (2.11) and Sobolev's inequality, we have

$$|\mathbf{J}(\mathbf{k})| \lesssim \|\Phi_h(\cdot,t)\|_{H^1_\infty(\dot\Omega)}, \quad |\tilde\zeta'(t)| \le C\|\mathbf{v}(\cdot,t)\|_{L^q(B_R)}, \tag{5.6}$$

provided that $\|\Phi_h(\cdot,t)\|_{H^1_\infty(\dot\Omega)}$ is small enough. Thus, by (2.12) and (5.6) and Hölder's inequality, we get

$$\begin{aligned}\sup_{t\in(0,T_0)} |\tilde\zeta(t)| &\le C_R\int_0^{T_0} \|\mathbf{v}(\cdot,\tau)\|_{L^q(B_R)}d\tau \\ &\lesssim \left(\int_0^{T_0} e^{-\kappa\tau p'}d\tau\right)^{1/p'} \|e^{\kappa t}\mathbf{v}\|_{L^p((0,T_0),L^q(B_R))} \\ &\lesssim (\kappa p')^{1/p'}\|e^{\kappa t}\mathbf{v}\|_{L^p((0,T_0),L^q(B_R))}.\end{aligned} \tag{5.7}$$

Then by (2.11), (5.3), and (5.7) and Sobolev's inequality, we can get that

$$\begin{aligned}&\sup_{t\in(0,T_0)} \|\Phi_h(\cdot,t)\|_{H^1_\infty(\dot\Omega)} \lesssim \mathscr{I} + \mathscr{E}^\kappa_{T_0}(\mathbf{v},\theta,h), \\ &\sup_{t\in(0,T_0)} |\tilde\zeta(t)| \lesssim \mathscr{I} + \mathscr{E}^\kappa_{T_0}(\mathbf{v},\theta,h), \quad \sup_{t\in(0,T_0)} e^{\kappa t}|\tilde\zeta'(t)| \lesssim \mathscr{I} + \mathscr{E}^\kappa_{T_0}(\mathbf{v},\theta,h).\end{aligned} \tag{5.8}$$

Thus, when $\mathscr{I}$ and $\mathscr{E}^\kappa_{T_0}(\mathbf{v},\theta,h)$ are small enough, (5.5) is obviously reasonable, we complete the justification of (5.5).

Now, for the decay estimates of solutions for linearized Eq. (3.1), from (4.13), we need to estimate $E^\kappa_T(\mathbf{v}_0,\theta_0,h_0,\mathbf{F}_1,\ldots,F_7,b)$. Thus, each nonlinear term can be directly replaced with the corresponding term on the right side of (2.13). Firstly, we will replace $\mathbf{F}_1$ in (3.1) with $\mathbf{N}_1 + \mathbf{f} - \alpha\mathbf{g}\theta$. By Theorem 4.3, we need to estimate $\|(\mathbf{f}^\kappa,\theta^\kappa)\|_{L^p((0,T),L^q(\dot\Omega))}$ with a given function $\mathbf{f}$. Then, in fact, we can get the following estimate of θ by Theorem 4.3 as follows:

$$\begin{aligned}\|\theta^\kappa\|_{L^p((0,T),L^q(\dot\Omega))} \lesssim& \|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \|N^\kappa_4\|_{L^p((0,T),L^q(\dot\Omega))} + \|N^\kappa_8\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))} \\ &+ \|b^\kappa\|_{L^p(\mathbb{R},H^1_q(\Omega_-))} + \|N^\kappa_8\|_{H^{1/2}_p(\mathbb{R},L^q(\dot\Omega))} + \|b^\kappa\|_{H^{1/2}_p(\mathbb{R},L^q(\Omega_-))}.\end{aligned}$$

Thus, we only need to consider the nonlinear term $\mathbf{N}_1(\mathbf{v},\mathbf{f},\theta,\Psi_h)$. From (2.4) and (2.7) and the formula of $\mathbf{N}_1(\mathbf{v},\mathbf{f},\theta,\Psi_h)$ in Ref. 8, we get

$$\begin{aligned}&\|\mathbf{N}_1(\mathbf{v},\mathbf{f},\theta,\Psi_h)\|_{L^q(\dot\Omega)} \\ \lesssim&(\|\mathbf{v}\|_{L^\infty(\dot\Omega)} + \|\partial_t\Phi_h\|_{L^\infty(\dot\Omega)} + |\tilde\zeta'(t)|)\|\nabla\mathbf{v}\|_{L^q(\dot\Omega)} \\ &+ (\|\Phi_h\|_{H^1_\infty(\dot\Omega)} + |\tilde\zeta(t)|)(\|\partial_t\mathbf{v}\|_{L^q(\dot\Omega)} + \|\nabla^2\mathbf{v}\|_{L^q(\dot\Omega)} + \|\mathbf{f}\|_{L^q(\dot\Omega)} + \|\theta\|_{L^q(\dot\Omega)}) \\ &+ (\|\Phi_h\|_{H^2_q(\dot\Omega)} + |\tilde\zeta(t)|)\|\nabla\mathbf{v}\|_{L^\infty(\dot\Omega)}.\end{aligned} \tag{5.9}$$

Therefore, in view of (5.3), (5.8), and (5.9), $N < q$ and Sobolev's inequality, we get

$$\begin{aligned}&\|\mathbf{N}^\kappa_1(\mathbf{v},\theta,\mathbf{f},\Psi_h)\|_{L^p((0,T),L^q(\dot\Omega))} \\ &\lesssim \left(\|\mathbf{v}^\kappa\|_{L^\infty((0,T),H^1_q(\dot\Omega))} + \|\partial_t h^\kappa\|_{L^\infty((0,T),W^{1-1/q}_q(S_R))} + \sup_{t\in(0,T)} e^{\kappa t}|\tilde\zeta'(t)|\right) \\ &\quad \|\mathbf{v}^\kappa\|_{L^p((0,T),H^1_q(\dot\Omega))} + \left(\|h^\kappa\|_{L^\infty((0,T),W^{2-1/q}_q(S_R))} + \sup_{t\in(0,T)}|\tilde\zeta(t)|\right) \\ &\quad \Big(\|\mathbf{v}^\kappa\|_{L^p((0,T),H^2_q(\dot\Omega))} + \|\partial_t\mathbf{v}^\kappa\|_{L^p((0,T),L^q(\dot\Omega))} + \|\theta^\kappa\|_{L^p((0,T),L^q(\dot\Omega))} \\ &\quad + \|\mathbf{f}^\kappa\|_{L^p((0,T),L^q(\dot\Omega))}\Big) + \left(\|h^\kappa\|_{L^\infty((0,T),W^{2-1/q}_q(S_R))} + \sup_{t\in(0,T)}|\tilde\zeta(t)|\right) \\ &\quad \times \|\mathbf{v}^\kappa\|_{L^p((0,T),H^2_q(\dot\Omega))} \\ &\lesssim \left(\mathscr{I} + \mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\right)^2.\end{aligned} \tag{5.10}$$

Let $N_4(\mathbf{v},\theta,\Psi_h)$ be a nonlinear term given in Ref. 8, by (2.7), we have

$$\begin{aligned}
&\|N_4(\mathbf{v},\theta,\Psi_h)\|_{L^q(\dot\Omega)}\\
&\lesssim(\|\mathbf{v}\|_{L^\infty(\dot\Omega)}+\|\partial_t\Phi_h\|_{L^\infty(\dot\Omega)}+|\tilde\zeta'(t)|)\|\nabla\theta\|_{L^q(\dot\Omega)}\\
&\quad+(\|\Phi_h\|_{H^1_\infty(\dot\Omega)}+|\tilde\zeta(t)|)\|\nabla^2\theta\|_{L^q(\dot\Omega)}+(\|\Phi_h\|_{H^2_q(\dot\Omega)}+|\tilde\zeta(t)|)\|\nabla\theta\|_{L^\infty(\dot\Omega)}.
\end{aligned} \tag{5.11}$$

Employing a similar argument as that in proving (5.10), we have

$$\begin{aligned}
&\left\|N_4^\kappa(\mathbf{v},\theta,\Psi_h)\right\|_{L^p((0,T),L^q(\dot\Omega))}\\
&\lesssim\left(\left\|\mathbf{v}^\kappa\right\|_{L^\infty\left((0,T),H^1_q(\dot\Omega)\right)}+\left\|\partial_t h^\kappa\right\|_{L^\infty\left((0,T),W^{1-1/q}_q(S_R)\right)}+\sup_{t\in(0,T)}\left|\tilde\zeta'(t)\right|\right)\\
&\left\|\theta^\kappa\right\|_{L^p\left((0,T),H^1_q(\dot\Omega)\right)}+\left(\left\|h^\kappa\right\|_{L^\infty\left((0,T),W^{2-1/q}_q(S_R)\right)}+\sup_{t\in(0,T)}|\tilde\zeta(t)|\right)\left\|\theta^\kappa\right\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)}\\
&+\left(\left\|\partial_t h^\kappa\right\|_{L^\infty\left((0,T),W^{2-1/q}_q(S_R)\right)}+\sup_{t\in(0,T)}|\tilde\zeta(t)|\right)\left\|\theta^\kappa\right\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)}\\
&\lesssim\left(\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\right)\mathscr{E}^\kappa_T(\mathbf{v},\theta,h).
\end{aligned} \tag{5.12}$$

We next consider $N_5(\mathbf{v},\Psi_h)$ given in Ref. 8. Since the detailed proof was given in Ref. 23, we omit the details. Then we have

$$\begin{aligned}
\left\|N_5^\kappa(\mathbf{v},\Psi_h)\right\|_{L^\infty\left((0,T),W^{1-1/q}_q(S_R)\right)}&\lesssim\left(\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\right)\mathscr{E}^\kappa_T(\mathbf{v},\theta,h),\\
\left\|N_5^\kappa(\mathbf{v},\Psi_h)\right\|_{L^p\left((0,T),W^{2-1/q}_q(S_R)\right)}&\lesssim\left(\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\right)\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\\
&\quad+\left(\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\right)^2\mathscr{E}^\kappa_T(\mathbf{v},\theta,h).
\end{aligned} \tag{5.13}$$

According to Theorem 4.3, we have to extend other nonlinear terms to the whole time interval $\mathbb{R}$. Before turning to the extension of these functions, we recall some definitions and estimations in Ref. 8. Let $T_\mathbf{v}(t)$ be the given analytic semigroup defined in Ref. 8, we can set $T_\mathbf{v}(t)\mathbf{v}_0 = T_{\mathbf{v}\pm}(t)\mathbf{v}_0|_{\Omega_\pm}$ for $x\in\Omega_\pm$, then $T_\mathbf{v}(0)\mathbf{v}_0=\mathbf{v}_0$ in $\dot\Omega$. By the analytic semigroup theory and a standard real-interpolation method, we directly have

$$\begin{aligned}
&\left\|e^tT_\mathbf{v}(\cdot)\mathbf{v}_0\right\|_{D(\dot\Omega)}\lesssim\|\mathbf{v}_0\|_{D(\dot\Omega)}\qquad\text{for }D(\dot\Omega)\in\left\{H^k_q(\dot\Omega),B^{2(1-1/p)}_{q,p}(\dot\Omega)\right\},\\
&\left\|e^tT_\mathbf{v}(\cdot)\mathbf{v}_0\right\|_{H^1_p\left((0,\infty),L^q(\dot\Omega)\right)}+\left\|e^tT_\mathbf{v}(\cdot)\mathbf{v}_0\right\|_{L^p\left((0,\infty),H^2_q(\dot\Omega)\right)}\lesssim\|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}.
\end{aligned} \tag{5.14}$$

We can also define a similar operator for $T_\theta(t)\theta_0$. Setting $T_h(t)h_0$ to be the solution of the given shifted equations in Ref. 8, we have $T_h(0)h_0 = \Psi_{h_0}$ in $\dot\Omega$, $T_h(0)h_0=h_0$ on S_R and

$$\begin{aligned}
&\left\|e^tT_h(\cdot)h_0\right\|_{L^\infty((0,\infty),B^{3-1/p}_{q,p}(\dot\Omega))}+\left\|e^t\partial_tT_h(\cdot)h_0\right\|_{L^\infty((0,\infty),H^1_q(\dot\Omega))}\\
&\quad+\left\|e^tT_h(\cdot)h_0\right\|_{L^p((0,\infty),H^3_q(\dot\Omega))}+\left\|e^t\partial_tT_h(\cdot)h_0\right\|_{L^p((0,\infty),H^2_q(\dot\Omega))}\\
&\lesssim\|h_0\|_{B^{3-1/p-1/q}_{q,p}(S_R)}.
\end{aligned} \tag{5.15}$$

Then, given a function $\varphi(t)$ defined on $(0,T)$, the extension $E_T[\varphi]$ of φ is defined by

$$E_T[\varphi](t)=\begin{cases}0 & \text{for } t<0,\\ \varphi(t) & \text{for } 0<t<T,\\ \varphi(2T-t) & \text{for } T<t<2T,\\ 0 & \text{for } t>2T.\end{cases} \tag{5.16}$$

Obviously, $E_T[\varphi](t) = \varphi(t)$ for $t \in (0,T)$ and $E_T[\varphi](t) = 0$ for $t \notin (0,2T)$. Let $\psi(t)$ be a function in $C^\infty(\mathbb{R})$ which equals 1 for $t > -1$ and 0 for $t < -2$. Under these preparations, we now define the extensions $\mathscr{E}_1[\mathbf{v}^\kappa]$, $\mathscr{E}_2[\theta^\kappa]$ and $\mathscr{E}_3[\Psi_h]$ of $\mathbf{v}^\kappa$, θ^κ and Ψ_h, respectively, to $\mathbb{R}$ by

$$\begin{aligned}
\mathscr{E}_1[\mathbf{v}^\kappa] &= E_T[\mathbf{v}^\kappa - T_{\mathbf{v}}(t)\mathbf{v}_0] + \psi(t)T_{\mathbf{v}}(|t|)\mathbf{v}_0,\\
\mathscr{E}_2[\theta^\kappa] &= E_T[\theta^\kappa - T_\theta(t)\theta_0] + \psi(t)T_\theta(|t|)\theta_0,\\
\mathscr{E}_3[\Psi_h] &= \chi(y)\big(R^{-1}(E_T[\Phi_h(y,t) - T_h(t)h_0] + \psi(t)T_h(t)h_0)y + E_T[\tilde\zeta](t)\big).
\end{aligned} \tag{5.17}$$

Since $\tilde\zeta(0) = 0$, we have

$$\mathscr{E}_1[\mathbf{v}^\kappa] = \mathbf{v}^\kappa, \quad \mathscr{E}_2[\theta^\kappa] = \theta^\kappa, \quad \mathscr{E}_3[\Psi_h] = \Psi_h \quad \text{in } \dot\Omega \times (0,T), \tag{5.18}$$

and then, applying the Hanzawa transform $x = \xi + \mathscr{E}_3[\Psi_h]$ instead of $x = \xi + \Psi_h$, by (5.5), (5.15), and (5.17), choosing ε_1 smaller if necessary, we may assume that

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \|\mathscr{E}_3[\Psi_\rho]\|_{H^1_\infty(\dot\Omega)} &\lesssim \sup_{t\in(0,T)} \|\Phi_h\|_{H^1_\infty(\dot\Omega)} + \|T_h(\cdot)h_0\|_{L^\infty((0,\infty),H^2_q(\dot\Omega))} + \sup_{t\in(0,T)}|\tilde\zeta(t)|\\
&\lesssim \varepsilon.
\end{aligned} \tag{5.19}$$

Now, in view of (2.9) and (2.15), $N_8(\theta,\Psi_h)$ is given by

$$N_8(\theta,\Psi_h) = [\![\nu\nabla\theta\cdot(\mathbf{n}-\mathbf{n}_t)]\!] - [\![\nu\mathbf{V}_0(\mathbf{k})\nabla\theta\cdot\mathbf{n}_t]\!] = \mathbf{V}_8(\bar{\mathbf{k}})\bar{\mathbf{k}}\otimes[\![\nu\nabla\theta]\!]. \tag{5.20}$$

Here, $\bar{\mathbf{k}}$ is the variable corresponding to $\bar\nabla\Psi_h = (\Psi_h, \nabla\Psi_h)$ and $\partial_{\bar{\mathbf{k}}}$ denotes the partial derivative with respect to variable $\bar{\mathbf{k}}$. $\mathbf{V}_8(\bar{\mathbf{k}}) = \mathbf{V}_8(\cdot,\bar{\mathbf{k}})$ is a set of matrices of functions consisting of products of elements of $\mathbf{n} = \xi/|R|$ and smooth functions defined for $|\mathbf{k}| < \varepsilon$, possessing the estimate

$$\sup_{|\bar{\mathbf{k}}|\le\varepsilon} \big\|\big(\mathbf{V}_8(\cdot,\bar{\mathbf{k}}), \partial_{\bar{\mathbf{k}}}\mathbf{V}_8(\cdot,\bar{\mathbf{k}})\big)\big\|_{L^\infty(\dot\Omega)} \le C. \tag{5.21}$$

We extend $N_8^\kappa(\theta,\Psi_h)$ to the whole time interval $\mathbb{R}$. Let

$$\tilde N_8^\kappa(\theta,\Psi_h) = \mathbf{V}_8(\bar\nabla\mathscr{E}_3[\Psi_h])\bar\nabla\mathscr{E}_3[\Psi_h]\otimes[\![\nu\nabla\mathscr{E}_2[\theta^\kappa]]\!]. \tag{5.22}$$

Obviously, we have

$$\tilde N_8^\kappa(\theta,\Psi_h) = N_8^\kappa(\theta,\Psi_h) \qquad \text{in } \dot\Omega \times (0,T).$$

Let $\mathscr{Z}_\mp$ be an extension map acting on $\theta_\pm \in H^2_q(\Omega_\pm)$ satisfying the properties: $\mathscr{Z}_\mp(\theta_\pm) \in H^2_q(\Omega)$, $\mathscr{Z}_-(\theta_+) = \theta_-$ in Ω_-, $\mathscr{Z}_+(\theta_-) = \theta_+$ in Ω_+, we have

$$\big(\partial_x^\alpha\mathscr{Z}_\mp(\theta_\pm)\big)(x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega_\pm}} \partial_x^\alpha\theta_\pm(x), \quad [\![\partial_x^\alpha\theta]\!] = \partial_x^\alpha\mathscr{Z}_-(\theta_+)\big|_{S_R} - \partial_x^\alpha\mathscr{Z}_+(\theta_-)\big|_{S_R}, \tag{5.23}$$

for $x_0 \in S_R$, $|\alpha| \le 1$ and for $i = 0,1,2$.

$$\|\mathscr{Z}_\mp(\theta_\pm)\|_{H^i_q(\Omega)} \le C_{i,q}\Big(\|\theta_+\|_{H^i_q(\Omega)} + \|\theta_-\|_{H^i_q(\Omega)}\Big) = C_{i,q}\|\theta\|_{H^i_q(\dot\Omega)}. \tag{5.24}$$

Thus, by (5.23), we obtain

$$\tilde N_8^\kappa(\theta,\Psi_h) = \mathbf{V}_8(\bar\nabla\mathscr{E}_3[\Psi_h])\bar\nabla\mathscr{E}_3[\Psi_h]\otimes\nabla\mathscr{E}_2\Big[\nu_-\mathscr{Z}_-\theta_+^\kappa\big|_{S_R} - \nu_+\mathscr{Z}_+\theta_-^\kappa\big|_{S_R}\Big]. \tag{5.25}$$

Moreover, since

$$\begin{aligned}
\partial_t(\mathbf{V}_8(\bar\nabla\mathscr{E}_3[\Psi_h])\bar\nabla\mathscr{E}_3[\Psi_h]) &= \mathbf{V}_8(\bar\nabla\mathscr{E}_3[\Psi_h])\partial_t\bar\nabla\mathscr{E}_3[\Psi_h]\\
&\quad + \mathbf{V}_8'(\bar\nabla\mathscr{E}_3[\Psi_h])\partial_t\bar\nabla\mathscr{E}_3[\Psi_h]\bar\nabla\mathscr{E}_3[\Psi_h],
\end{aligned} \tag{5.26}$$

where $\mathbf{V}_8'$ denotes the derivative of $\mathbf{V}_8(\bar{\mathbf{k}})$ with respect to $\bar{\mathbf{k}}$, by (2.4), (5.5), (5.14), (5.15), (5.17), and (5.21) and the fact that $\chi(\xi) = 1$ on S_R, we have the following estimates:

$$\begin{aligned}
&\|\mathbf{V}_8(\bar\nabla\mathscr{E}_3[\Psi_h])\bar\nabla\mathscr{E}_3[\Psi_h]\|_{L^\infty(\mathbb{R},H^1_q(\dot\Omega))}\\
&\lesssim \|T_h(|t|)h_0\|_{L^\infty((-2,\infty),H^2_q(\dot\Omega))} + \|\Psi_h\|_{L^\infty((0,T),H^2_q(\dot\Omega))}\\
&\lesssim \|h_0\|_{W^{3-1/p-1/q}_{q,p}(S_R)} + \|h\|_{L^\infty((0,T),W^{2-1/q}_q(\dot\Omega))} \lesssim \mathscr{I} + \mathscr{E}_T^\kappa(\mathbf{v},\theta,h),
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}&\|\partial_t(\mathbf{V}_8(\bar\nabla\mathscr{E}_3[\Psi_h])\bar\nabla\mathscr{E}_3[\Psi_h])\|_{L^\infty(\mathbb{R},H^1_q(\dot\Omega))}\\&\lesssim\|\partial_t h\|_{L^\infty((0,T),W_q^{1-1/q}(S_R))}+\|\partial_t T_h(|t|)h_0\|_{L^\infty((-2,\infty),H^1_q(\dot\Omega))}\\&\lesssim\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h).\end{aligned}\tag{5.28}$$

From Lemma A.2, (5.27) and (5.28), it follows

$$\begin{aligned}&\left\|\bar N_8^\kappa(\theta,\Psi_h)\right\|_{L^p\left(\mathbb{R},H^1_q(\dot\Omega)\right.}+\left\|\bar N_8^\kappa(\theta,\Psi_h)\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}\\&\lesssim\left(\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\right)\left(\left\|\nabla\mathscr{E}_2\left[\theta^\kappa\right]\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}+\left\|\nabla\mathscr{E}_2\left[\theta^\kappa\right]\right\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))}\right).\end{aligned}\tag{5.29}$$

Applying Lemma A.3, (5.14) and (5.17), we have

$$\begin{aligned}&\left\|\nabla\mathscr{E}_2\left[\theta^\kappa\right]\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}+\left\|\nabla\mathscr{E}_3\left[\theta^\kappa\right]\right\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))}\\&\lesssim\mathscr{I}+\left\|\partial_t\theta^\kappa\right\|_{L^p((0,T),L^q(\dot\Omega))}+\left\|\theta^\kappa\right\|_{L^p((0,T),H^2_q(\dot\Omega))}.\end{aligned}\tag{5.30}$$

Thus, combining with (5.29) and (5.30), we have

$$\left\|\bar N_8^\kappa(\theta,\Psi_h)\right\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))}+\left\|\bar N_8^\kappa(\theta,\Psi_h)\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}\lesssim(\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h))^2.\tag{5.31}$$

Moreover, we extend $N_2(\mathbf{v},\Psi_h)$, $\mathbf{N}_3(\mathbf{v},\Psi_h)$, $\mathbf{N}_6(\mathbf{v},\Psi_h)$ and $N_7(\mathbf{v},\Psi_h)$ to the whole time interval $\mathbb{R}$. In fact, $\zeta(t)$ does not play a role in $\mathbf{N}_6(\mathbf{v},\Psi_h)$ and $N_7(\mathbf{v},\Psi_h)$, we just need to estimate Φ_h. By (5.17), (Refs. 8 and 23), let

$$\begin{aligned}\bar N_2^\kappa(\mathbf{v},\Psi_h)&=\mathbf{V}_2(\nabla\mathscr{E}_3[\Psi_h])\nabla\mathscr{E}_3[\Psi_h]\otimes\nabla\mathscr{E}_1[\mathbf{v}^\kappa],\\\bar{\mathbf{N}}_3^\kappa(\mathbf{v},\Psi_h)&=\mathbf{V}_3(\nabla\mathscr{E}_3[\Psi_h])\nabla\mathscr{E}_3[\Psi_h]\otimes\mathscr{E}_1[\mathbf{v}^\kappa],\\\bar{\mathbf{N}}_6^\kappa(\mathbf{v},\Psi_h)&=\mathbf{V}_6(\bar\nabla\mathscr{E}_3[\Phi_h])\bar\nabla\mathscr{E}_3[\Phi_h]\otimes\nabla\mathscr{E}_1[\mathbf{v}^\kappa],\\\bar N_7^\kappa(\mathbf{v},\Psi_h)&=\mathbf{V}_7(\bar\nabla\mathscr{E}_3[\Phi_h])\bar\nabla\mathscr{E}_3[\Phi_h]\otimes\nabla\mathscr{E}_1[\mathbf{v}^\kappa]\\&\quad+\sigma\mathbf{V}_5(\bar\nabla\mathscr{E}_3[\Psi_h])\bar\nabla\mathscr{E}_3[\Phi_h]\otimes\bar\nabla^2\mathscr{E}_3[\Phi_h^\kappa],\end{aligned}\tag{5.32}$$

where for $i=2,3,5,6,7$, $\mathbf{V}_i(\bar{\mathbf{k}})$ are some matrix of C^1 functions consisting of products of elements of $\mathbf{n}=\xi/|R|$ defined on $\dot\Omega\times\{\bar{\mathbf{k}}\mid|\bar{\mathbf{k}}|<\varepsilon\}$ possessing the estimate

$$\sup_{|\bar{\mathbf{k}}|\le\varepsilon}\left\|\left(\mathbf{V}_i(\cdot,\bar{\mathbf{k}}),\partial_{\bar{\mathbf{k}}}\mathbf{V}_i(\cdot,\bar{\mathbf{k}})\right)\right\|_{H^1_\infty(\dot\Omega)}\le C.\tag{5.33}$$

By the fact $H^1_p(\mathbb{R},L^q(\dot\Omega))\subset H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))$, (2.4), (5.3), (5.15), and (5.17), we have

$$\begin{aligned}&\left\|\bar\nabla^2\mathscr{E}_3\left[\Psi_h^\kappa\right]\right\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))}+\left\|\bar\nabla^2\mathscr{E}_3\left[\Psi_h^\kappa\right]\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega)}\Big)\\&\lesssim\left\|\mathscr{E}_3\left[\Psi_h^\kappa\right]\right\|_{H^1_p\left(\mathbb{R},H^2_q(\dot\Omega)\right)}+\left\|\mathscr{E}_3\left[\Psi_h^\kappa\right]\right\|_{L^p(\mathbb{R},H^3_q(\dot\Omega))}\\&\lesssim\mathscr{I}+\left\|\Phi_h^\kappa\right\|_{H^1_p((0,T),H^2_q(\dot\Omega))}+\left\|\Phi_h^\kappa\right\|_{L^p((0,T),H^3_q(\dot\Omega))}.\\&\lesssim\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h).\end{aligned}\tag{5.34}$$

Then, by (5.24), $[\![\bar{\mathbf{N}}_6^\kappa]\!]$ and $[\![\bar N_7^\kappa]\!]$ can replace with $\bar{\mathbf{N}}_6^\kappa$ and $\bar N_7^\kappa$, respectively, employing the same argument as in proving (5.31), we have

$$\begin{aligned}&\left\|\left(\bar N_2^\kappa(\mathbf{v},\Psi_h),\bar{\mathbf{N}}_6^\kappa(\mathbf{v},\Psi_h),\bar N_7^\kappa(\mathbf{v},\Psi_h)\right)\right\|_{H_p^{1/2}(\mathbb{R},L^q(\dot\Omega))}+\left\|\left(\bar{\mathbf{N}}_3^\kappa(\mathbf{v},\Psi_h)\right)\right\|_{H^1_p(\mathbb{R},L^q(\dot\Omega))}\\&\quad+\left\|\left(\bar N_2^\kappa(\mathbf{v},\Psi_h),\bar{\mathbf{N}}_6^\kappa(\mathbf{v},\Psi_h),\bar N_7^\kappa(\mathbf{v},\Psi_h)\right)\right\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))}\\&\lesssim(\mathscr{I}+\mathscr{E}^\kappa_T(\mathbf{v},\theta,h))^2.\end{aligned}\tag{5.35}$$

Finally, we consider the interface condition on Γ. Noting that $x = \xi$ near $\Gamma \times (0,T)$, then, we consider the following equation for $t \in \mathbb{R}$

$$\nabla \mathscr{E}_2[\theta_-^\kappa]\cdot \mathbf{n}_- + \beta \mathscr{E}_2[\theta_-^\kappa] = \bar{b}^\kappa(\xi,t)$$

with

$$\bar{b}^\kappa = \begin{cases} 0 & \text{for } t < -2, \\ \psi(t)\nabla T_{\theta_-}(|t|)\theta_0\cdot\mathbf{n}_- + \beta\psi(t)T_{\theta_-}(|t|)\theta_0 & \text{for } -2 < t < -1, \\ \nabla T_{\theta_-}(|t|)\theta_0\cdot\mathbf{n}_- + \beta T_{\theta_-}(|t|)\theta_0 & \text{for } -1 < t < 0, \\ e^{\kappa t} b(\xi,t) & \text{for } 0 < t < T, \\ e^{\kappa(2T-t)} b(\xi,2T-t) & \text{for } T < t < 2T, \\ \nabla T_{\theta_-}(t)\theta_0\cdot\mathbf{n}_- + \beta T_{\theta_-}(t)\theta_0 & \text{for } t > 2T. \end{cases}$$

Obviously, we have

$$\nabla \mathscr{E}_2[\theta_-^\kappa]\cdot\mathbf{n}_- + \beta\mathscr{E}_2[\theta_-^\kappa] = \nabla\theta_-^\kappa\cdot\mathbf{n}_- + \beta\theta_-^\kappa = b^\kappa(\xi,t) \qquad \text{on } \Gamma\times(0,T).$$

By Lemma A.2, (5.14) and (5.17) and $\|\mathbf{n}_-\|_{H^1_\infty} \le C$, we have

$$\begin{aligned} &\|\bar{b}^\kappa\|_{L^p(\mathbb{R},H^1_q(\Omega_-))} + \|\bar{b}^\kappa\|_{H_p^{1/2}(\mathbb{R},L^q(\Omega_-))} \\ &\lesssim \|T_{\theta_-}(|t|)\theta_0\|_{L^p((-2,\infty),H^2_q(\Omega_-))} + \|T_{\theta_-}(|t|)\theta_0\|_{H^1_p((-2,\infty),L^q(\Omega_-))} \\ &\quad + \|b^\kappa\|_{L^p\left((0,T),H^1_q(\Omega_-)\right)} + \|b^\kappa\|_{H_p^{1/2}((0,T),L^q(\Omega_-))} \\ &\lesssim \|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \|b^\kappa\|_{L^p((0,T),H^1_q(\Omega_-))} + \|b^\kappa\|_{H_p^{1/2}((0,T),L^q(\Omega_-))} \lesssim \mathscr{I}. \end{aligned} \tag{5.36}$$

Applying Theorem 4.3 to Eq. (2.13) and using (5.10), (5.12), (5.13), (5.31), (5.35), and (5.36), we have

$$\mathbb{E}_T^\kappa(\mathbf{v},\theta,h) \le C(\mathscr{I} + (\mathscr{E}_T^\kappa(\mathbf{v},\theta,h))^2 + (\mathscr{E}_T^\kappa(\mathbf{v},\theta,h))^3) \tag{5.37}$$

for some positive constant C. Moreover, by the fourth equation in Eqs. (2.13) and (5.37), it yields

$$\|e^{\kappa t}\partial_t h\|_{L^\infty((0,T),W_q^{1-1/q}(S_R))} \le C(\mathscr{I} + (\mathscr{E}_T^\kappa(\mathbf{v},\theta,h))^2 + (\mathscr{E}_T^\kappa(\mathbf{v},\theta,h))^3). \tag{5.38}$$

Combined with (5.37) and (5.38),

$$\mathscr{E}_T^\kappa(\mathbf{v},\theta,h) \le C(\mathscr{I} + (\mathscr{E}_T^\kappa(\mathbf{v},\theta,h))^2 + (\mathscr{E}_T^\kappa(\mathbf{v},\theta,h))^3) \tag{5.39}$$

holds for any $T \in (0, T_0]$, and for some positive constants C independent of T and T_0.

Finally, we should prove (5.2). We just consider $\mathbf{v}^\kappa$, let $\mathscr{E}_1[\mathbf{v}^\kappa]$ be the function given in (5.17), then by Lemma A.1, we have

$$\begin{aligned}\|\mathbf{v}^\kappa\|_{L^\infty((0,T),B^{2(1-1/p)}_{q,p}(\dot\Omega))} &\le \|\mathscr{E}_1[\mathbf{v}^\kappa]\|_{L^\infty((0,\infty),B^{2(1-1/p)}_{q,p}(\dot\Omega))}\\ &\le C\Big\{\|\mathscr{E}_1[\mathbf{v}^\kappa]\|_{L^p((0,\infty),H^2_q(\dot\Omega))}+\|\partial_t\mathscr{E}_1[\mathbf{v}^\kappa]\|_{L^p((0,\infty),L^q(\dot\Omega))}\Big\}\end{aligned}$$

which, combined with (5.14), leads to inequality (5.2).

VI. COMPLETION OF THE PROOF OF THEOREM 2.1

From Theorem A.5, when $\mathscr{I}$ is small enough, problem (2.13) admits a unique solution in $(0,T_0)$. Thus, we shall prove that $\mathbf{v}, q, \theta$ and h can be prolonged beyond T_0 keeping condition (5.5) provided that $\varepsilon_1 > 0$ is small enough. From Sec. V, we know that

$$\mathscr{E}^\kappa_T(\mathbf{v},\theta,h) \le C\big(\mathscr{I} + (\mathscr{E}^\kappa_T(\mathbf{v},\theta,h))^2 + (\mathscr{E}^\kappa_T(\mathbf{v},\theta,h))^3\big)$$

holds for any $T \in (0,T_0)$ with some constant $C>0$ independent of ε, ε_1, T and T_0. In fact, we need to consider the properties of the roots of an algebraic equation $x^3+x^2-C^{-1}x+\mathscr{I}=0$. Using Veda's theorem and changing C larger in (5.39) if necessary, since we choose $\mathscr{I}\le\varepsilon$ enough small, we can make the algebraic equation have three different real solutions $x_0(\mathscr{I})$ and $x_\pm(\mathscr{I})$:

$$\begin{aligned}&x_0(\mathscr{I}) = C\mathscr{I} + O(\mathscr{I}^2), x_+(\mathscr{I}) = C^{-1} + O(C^{-2}) + O(\mathscr{I}),\\ &x_-(\mathscr{I}) = -1 - C^{-1} + O(C^{-2}) + O(\mathscr{I}),\end{aligned} \tag{6.1}$$

as $C\to\infty$ and $\mathscr{I}\to 0$. Since $\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\ge 0 > x_-(\mathscr{I})$, by (5.39), one of the following cases holds:

$$\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\le x_0(\mathscr{I}),\qquad \mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\ge x_+(\mathscr{I}).$$

In view of Theorem A.5, when $\mathscr{I}\le\varepsilon$, Eq. (2.13) admits a unique solution $(\mathbf{v},\theta,h)$ satisfying:

$$\mathscr{E}^0_{T_0}(\mathbf{v},\theta,h)\le\delta, \tag{6.2}$$

where $\delta > 0$ is small enough such that

$$e\delta < 1/2C.$$

By (6.2), we have $\mathscr{E}^\kappa_{\kappa^{-1}}(\mathbf{v},\theta,h)\le e\mathscr{E}^0_{T_0}(\mathbf{v},\theta,h) < 1/2C < x_+(\mathscr{I})$, and therefore,

$$\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\le x_0(\mathscr{I})\quad \text{for } T\in(0,\kappa^{-1}).$$

But $\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)$ is a continuous function with respect to $T\in(0,T_0)$ which yields that

$$\mathscr{E}^\kappa_T(\mathbf{v},\theta,h)\le x_0(\mathscr{I})\quad \text{for any } T\in(0,T_0). \tag{6.3}$$

In particular, setting $\tilde T = T_0 - 1/2$, by (5.2) and (6.3), we have

$$\|\mathbf{v}(\cdot,\tilde T)\|_{B^{2-2/p}_{q,p}(\dot\Omega)} + \|\theta(\cdot,\tilde T)\|_{B^{2-2/p}_{q,p}(\dot\Omega)} + \|h(\cdot,\tilde T)\|_{W^{3-1/p-1/q}_{q,p}(S_R)}\le x_0(\mathscr{I}).$$

Thus, choosing $\mathscr{I}$ sufficiently small and using the same argument as that in proving Theorem A.5, we see that there exists a unique solution $(\tilde{\mathbf{v}},\tilde\theta,\tilde q,\tilde h)$ of the following equations

$$\begin{cases}\rho\partial_t\tilde{\mathbf{v}} - \mathrm{Div}(\mu\mathbf{D}(\tilde{\mathbf{u}}) - \tilde q\mathbf{I}) = \mathbf{N}_1(\tilde{\mathbf{v}},\mathbf{f},\tilde\theta,\Psi_{\tilde h}) + \mathbf{f}(x(\xi,t),t) - \alpha\mathbf{g}\tilde\theta & \text{in } \dot\Omega\times(\tilde T,\tilde T+1),\\ \mathrm{div}\,\tilde{\mathbf{v}} = N_2(\tilde{\mathbf{v}},\Psi_{\tilde h}) = \mathrm{div}\,\mathbf{N}_3(\tilde{\mathbf{v}},\Psi_{\tilde h}) & \text{in } \dot\Omega\times(\tilde T,\tilde T+1),\\ \partial_t\tilde\theta - \nu\Delta\tilde\theta = N_4(\tilde{\mathbf{v}},\tilde\theta,\Psi_{\tilde h}) & \text{in } \dot\Omega\times(\tilde T,\tilde T+1),\\ \partial_t\tilde h - \mathbf{n}\cdot\mathbf{P}\mathbf{v} = N_5(\tilde{\mathbf{v}},\Psi_{\tilde h}) & \text{on } S_R\times(\tilde T,\tilde T+1),\\ [\![\mu\mathbf{D}(\tilde{\mathbf{v}})\mathbf{n}]\!]_\tau = [\![\mathbf{N}_6(\tilde{\mathbf{v}},\Psi_{\tilde h})]\!],\quad [\![\tilde{\mathbf{v}}]\!]=0 & \text{on } S_R\times(\tilde T,\tilde T+1),\\ [\![(\mu\mathbf{D}(\tilde{\mathbf{v}})\mathbf{n},\mathbf{n}) - \tilde q]\!] - \sigma\Delta_{S_R}\tilde h + \dfrac{N-1}{R^2}\tilde h = [\![N_7(\tilde{\mathbf{v}},\Psi_{\tilde h})]\!] & \text{on } S_R\times(\tilde T,\tilde T+1),\\ \tilde{\mathbf{v}} = 0,\quad \nabla\tilde\theta\cdot\mathbf{n}_- + \beta\tilde\theta = b(\xi,t) & \text{on } \Gamma\times(\tilde T,\tilde T+1),\\ \tilde{\mathbf{v}}|_{t=0} = \mathbf{v}(\cdot,\tilde T)\ \text{in } \dot\Omega,\quad \tilde\theta\big|_{t=\tilde T} = \theta(\cdot,\tilde T)\ \text{in } \dot\Omega,\quad \tilde h\big|_{t=\tilde T} = h(\cdot,\tilde T) & \text{on } S_R,\end{cases} \tag{6.4}$$

which satisfies the condition:

$$\sup_{\tilde{T}<t<\tilde{T}+1} \|\Phi_{\tilde{h}}(\cdot,t)\|_{H^1_\infty(\dot{\Omega})} \le \varepsilon_1/2, \quad \sup_{\tilde{T}<t<\tilde{T}+1} |\tilde{\zeta}(t,\tilde{\mathbf{v}},\tilde{h})| \le \varepsilon_1/2, \quad \mathscr{E}^0_{(\tilde{T},\tilde{T}+1)}(\tilde{\mathbf{v}},\tilde{\theta},\tilde{h}) \lesssim \delta, \tag{6.5}$$

and

$$\begin{aligned}\mathscr{E}^0_{(\tilde{T},\tilde{T}+1)}(\tilde{\mathbf{v}},\tilde{\theta},\tilde{h}) = &\|(\tilde{\mathbf{v}},\tilde{\theta})\|_{L^p((\tilde{T},\tilde{T}+1),H^2_q(\dot{\Omega}))} + \|(\partial_t\tilde{\mathbf{v}},\partial_t\tilde{\theta})\|_{L^p((\tilde{T},\tilde{T}+1),L^q(\dot{\Omega}))} \\ &+ \|\partial_t\tilde{h}\|_{L^p((\tilde{T},\tilde{T}+1),W^{2-1/q}_q(\dot{\Omega}))} + \|\tilde{h}\|_{L^p((\tilde{T},\tilde{T}+1),W^{3-1/p}_q(S_R))} \\ &+ \|\partial_t\tilde{h}\|_{L^\infty((\tilde{T},\tilde{T}+1),W^{1-1/q}_q(S_R))}.\end{aligned}$$

Let

$$\mathbf{v}_1 = \begin{cases} \mathbf{v} & 0<t\le\tilde{T}, \\ \tilde{\mathbf{v}} & \tilde{T}<t<\tilde{T}+1, \end{cases} \quad q_1 = \begin{cases} q & 0<t\le\tilde{T}, \\ \tilde{q} & \tilde{T}<t<\tilde{T}+1, \end{cases}$$

$$\theta_1 = \begin{cases} \theta & 0<t\le\tilde{T}, \\ \tilde{\theta} & \tilde{T}<t<\tilde{T}+1, \end{cases} \quad h_1 = \begin{cases} h & 0<t\le\tilde{T}, \\ \tilde{h} & \tilde{T}<t<\tilde{T}+1, \end{cases}$$

and then $(\mathbf{v}_1,\theta_1,q_1,h_1)$ satisfies the condition:

$$\sup_{0<t<\tilde{T}+1} \|\Phi_{h_1}(\cdot,t)\|_{H^1_\infty(\dot{\Omega})} \le \varepsilon_1, \quad \sup_{0<t<\tilde{T}+1} |\tilde{\zeta}(t,\mathbf{v}_1,h_1)| \le \varepsilon_1, \quad \mathscr{E}^0_{\tilde{T}+1}(\mathbf{v}_1,\theta_1,h_1) \lesssim \delta. \tag{6.6}$$

Since $\tilde{T}+1 = T_0+1/2$, we can prolong the solutions. Repeating this argument, we can prolong $\mathbf{v}$, θ and h to the time interval $(0,\infty)$ satisfying (2.17). This completes the Proof of Theorem 2.1.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Wei Zhang: Formal analysis (equal); Methodology (equal); Resources (equal); Writing – original draft (equal); Writing – review & editing (equal). **Jie Fu**: Formal analysis (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Chengchun Hao**: Formal analysis (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Siqi Yang**: Formal analysis (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: NOTATIONS AND USEFUL RESULTS

For any scalar function $a = a(x)$ and N-vector function $\mathbf{b} = (b_1(x),\dots,b_N(x))^\top$, we write

$$\nabla a = (\partial_1 a(x),\dots,\partial_N a(x)), \quad \nabla\mathbf{b} = (\nabla b_1(x),\dots,\nabla b_N(x)),$$

$$\operatorname{div}\mathbf{b} = \sum_{j=1}^N \partial_j b_j(x), \quad \nabla^2 a = (\partial_i\partial_j a)_{i,j=1}^N, \quad \nabla^2\mathbf{b} = (\nabla^2 b_1,\dots,\nabla^2 b_N).$$

For any n-vector $\mathbf{X} = (x_1, \ldots, x_n)^\top$ and m-vector $\mathbf{Y} = (y_1, \ldots, y_m)^\top$, $\mathbf{Y} \otimes \mathbf{X}$ denotes an (n, m)-matrix whose $(i, j)^{\text{th}}$ component is $\mathbf{X}_i \mathbf{Y}_j$. For any (mn, N)-matrix $\mathbf{A} = (A_{ij,k})$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, N$, $\mathbf{A}\mathbf{X} \otimes \mathbf{Y}$ denotes an N column vector whose i^{th} component is the quantity: $\sum_{j=1}^n \sum_{k=1}^m A_{jk,i} x_j y_k$.

Let $L^q(\Omega)$, $H_q^s(\Omega)$ and $B_{q,p}^s(\Omega)$ denote Lebesgue's spaces, Sobolev's spaces and Besov's spaces, respectively, on an open set Ω of $\mathbb{R}^N$ with $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Let

$$D(\dot{\Omega}) = \{f : f|_{\Omega_\pm} \in D(\Omega_\pm)\}, \quad \|f\|_{D(\dot{\Omega})} = \|f|_{\Omega_+}\|_{D(\Omega_+)} + \|f|_{\Omega_-}\|_{D(\Omega_-)},$$

for $D \in \{L^q, H_q^s, B_{q,p}^s\}$. For simplicity, we write $\|\mathbf{g}\|_{X(\dot{\Omega})^N} =: \|\mathbf{g}\|_{X(\dot{\Omega})}$. In this paper, for boundary S_R, we write $W_q^s(S_R) := B_{q,q}^s(S_R)$ with the norm written by $\|\cdot\|_{W_q^s(S_R)}$. $|X|$ denotes the Lebesgue measure of a Lebesgue measurable set X in $\mathbb{R}^N$. For any two Banach spaces E and H, $\mathscr{L}(E, H)$ denotes the set of all bounded linear operators from E into H. Let $\mathscr{D}(\mathbb{R}, E)$ denote the space of E-valued distributions. $\mathscr{S}(\mathbb{R}, E)$ denotes the space of E-valued Schwartz functions and $\mathscr{S}'(\mathbb{R}, E) = \mathscr{L}(\mathscr{S}(\mathbb{R}, E), \mathbb{R})$ is the space of E-valued tempered distributions. For a domain U in $\mathbb{C}$, let $\mathrm{Hol}(U, \mathscr{L}(X, Y))$ be the set of all $\mathscr{L}(X, Y)$-valued holomorphic functions defined on U, where $\mathbb{C}$ denotes the set of all complex numbers. Set

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C} \backslash \{0\} \mid |\arg \lambda| \le \pi - \varepsilon\}, \quad \Sigma_{\varepsilon, \lambda_0} = \{\lambda \in \Sigma_\varepsilon \mid |\lambda| \ge \lambda_0\}. \tag{A1}$$

For any N-vectors $\mathbf{a}$ and $\mathbf{b}$, we set $\mathbf{a} \cdot \mathbf{b} = (\mathbf{a}, \mathbf{b}) = \sum_{j=1}^N a_i b_i$, and the tangential component $\mathbf{a}_\tau$ of $\mathbf{a}$ with respect to the normal $\mathbf{n}$ is defined by $\mathbf{a}_\tau = \mathbf{a} - (\mathbf{a}, \mathbf{n})\mathbf{n}$. For complex-valued functions f and g, we set $(f, g)_\Omega = \int_\Omega f(x)\overline{g(x)}dx$ where $\overline{g(x)}$ is the complex conjugate of $g(x)$, and for any two N-vector functions $\mathbf{f}$ and $\mathbf{g}$, denote $(\mathbf{f}, \mathbf{g})_\Omega = \sum_{j=1}^N (f_j, g_j)_\Omega$. Let $1 < q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$, we introduce the following spaces

$$\begin{aligned}
J_q(\dot{\Omega}) &:= \left\{\mathbf{v} \in L^q(\dot{\Omega})^N \mid (\mathbf{v}, \nabla\varphi)_{\dot{\Omega}} = 0 \quad \text{for any } \varphi \in \hat{H}_{q'}^1(\Omega)\right\},\\
\hat{H}_q^1(\Omega) &:= \left\{v \in L^{q,\mathrm{loc}}(\Omega) \mid \nabla v \in L^q(\Omega)\right\},\\
\mathscr{H}_q &:= \left\{(\mathbf{v}, \theta, h) \mid \mathbf{v} \in J_q(\dot{\Omega}), \quad \theta \in L^q(\dot{\Omega}), \quad h \in W_q^{2-1/q}(S_R)\right\},\\
\mathscr{D}_q &:= \Big\{(\mathbf{v}, h) \mid \mathbf{v} \in J_q(\dot{\Omega}) \cap H_q^2(\dot{\Omega}), \quad h \in W_q^{3-1/q}(S_R),\\
&\qquad [\![(\mu\mathbf{D}(\mathbf{v})\mathbf{n})_\tau]\!] = 0, \quad [\![\mathbf{v}]\!] = 0 \text{ on } S_R, \quad \mathbf{v} = 0 \quad \text{on } \Gamma\Big\},\\
\mathscr{J}_q &:= \left\{\theta \in H_q^2(\dot{\Omega}) \mid [\![\nu\nabla\theta\cdot\mathbf{n}]\!] = 0, \quad [\![\theta]\!] = 0 \text{ on } S_R, \quad \nabla\theta_-\cdot\mathbf{n}_- + \beta\theta_- = 0 \text{ on } \Gamma\right\}.
\end{aligned}$$

Definition A.1. $\mathscr{F}[f]$ and $\mathscr{F}_\xi^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively, given by

$$\mathscr{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)dx, \quad \mathscr{F}_\xi^{-1}[g(\xi)](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)d\xi.$$

Let $\mathscr{F}_L$ and $\mathscr{F}_L^{-1}$ be the Laplace transform and the inverse Laplace transform, respectively, defined by

$$\hat{f}(\lambda) = \mathscr{F}_L[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)dt, \quad \mathscr{F}_L^{-1}[g(\lambda)](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\tau)d\tau,$$

for $\lambda = \gamma + i\tau \in \mathbb{C}$. Obviously,

$$\begin{aligned}
\mathscr{F}_L[f](\lambda) &= \int_{-\infty}^{\infty} e^{-i\tau t}e^{-\gamma t} f(t)dt = \mathscr{F}\left[e^{-\gamma t} f\right](\tau),\\
\mathscr{F}_L^{-1}[g](t) &= e^{\gamma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t} g(\tau)d\tau = e^{\gamma t}\mathscr{F}^{-1}[g](t), \quad \mathscr{F}_L\mathscr{F}_L^{-1} = \mathscr{F}_L^{-1}\mathscr{F}_L = \mathbf{I}.
\end{aligned}$$

Definition A.2. Let both X and Y be Banach spaces. A family of operators $\mathscr{T} \subset \mathscr{L}(X, Y)$ is called to be $\mathscr{R}$-bounded on $\mathscr{L}(X, Y)$, if there exist some constants $C > 0$ and $p \in [1, \infty)$ such that for each $n \in \mathbb{N}$, $T_j \in \mathscr{L}(X, Y)$ and $f_j \in X (j = 1, \ldots, n)$, we have

$$\left\|\sum_{j=1}^n r_j T_j f_j\right\|_{L^p((0,1), Y)} \le C\left\|\sum_{j=1}^n r_j f_j\right\|_{L^p((0,1), X)}.$$

Here, the Rademacher functions $\{r_j\}_{j=1}^n$ are defined from [0, 1] into $\{-1, 1\}$. The smallest of such C's is called the $\mathscr{R}$-bound of $\mathscr{T}$ on $\mathscr{L}(X, Y)$, and denoted by $\mathscr{R}_{\mathscr{L}(X,Y)}\mathscr{T}$.

Lemma A.1 (cf. Ref. 2). If X is a dense subspace of Y and the embedding $X \subset Y$ is continuous, we have

$$L^p(0,\infty;X) \cap H^1_p(0,\infty;Y) \subset C\big([0,\infty);(X,Y)_{1/p,p}\big)$$

and

$$\sup_{t\in[0,\infty)} \|u(t)\|_{(X,Y)_{1/p,p}} \le \Big(\|u\|^p_{L^p(0,\infty;X)} + \|u\|^p_{H^1_p(0,\infty;Y)}\Big)^{1/p}$$

for each $p \in (1,\infty)$.

Lemma A.2 (cf. Ref. 23). Let $1 < p < \infty$ and $N < q < \infty$. Let

$$f \in L^\infty\big(\mathbb{R}, H^1_q(\dot\Omega)\big) \cap H^1_\infty\big(\mathbb{R}, L^q(\dot\Omega)\big),$$
$$g \in H^{1/2}_p\big(\mathbb{R}, L^q(\dot\Omega)\big) \cap L^p\big(\mathbb{R}, H^1_q(\dot\Omega)\big).$$

Then, we have

$$\begin{aligned}&\|fg\|_{H^{1/2}_p(\mathbb{R},L^q(\dot\Omega))} + \|fg\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))}\\ &\le C\Big(\|f\|_{L^\infty(\mathbb{R},H^1_q(\dot\Omega))} + \|f\|_{H^1_\infty(\mathbb{R},L^q(\dot\Omega))}\Big) \times \Big(\|g\|_{H^{1/2}_p(\mathbb{R},L^q(\dot\Omega))} + \|g\|_{L^p(\mathbb{R},H^1_q(\dot\Omega))}\Big).\end{aligned} \tag{A2}$$

Lemma A.3 (cf. Ref. 23). Let $1 < p, q < \infty$. Then,

$$H^1_p\big(\mathbb{R}, L^q(\dot\Omega)\big) \cap L^p\big(\mathbb{R}, H^2_q(\dot\Omega)\big) \subset H^{1/2}_p\big(\mathbb{R}, H^1_q(\dot\Omega)\big)$$

and

$$\|u\|_{H^{1/2}_p(\mathbb{R},H^1_q(\dot\Omega))} \le C\Big\{\|u\|_{L^p(\mathbb{R},H^2_q(\dot\Omega))} + \|\partial_t u\|_{L^p(\mathbb{R},L^q(\dot\Omega))}\Big\}. \tag{A3}$$

Theorem A.4 *(cf. Ref. 7). Let $1 < q < \infty$ and $0 < \varepsilon_0 < \pi/2$. Then, there exists a $\lambda_0 > 0$ such that for any $\lambda \in \Sigma_{\varepsilon_0,\lambda_0}$ and $(\mathbf{v},\theta,h) \in \mathscr{H}_q$, equations (4.3) admits a unique solution $(\mathbf{v},h) \in \mathscr{D}_q$ and $\theta \in \mathscr{J}_q$ possessing the estimate:*

$$|\lambda|\|(\mathbf{v},\theta,h)\|_{\mathscr{H}_q} + \|(\mathbf{v},h)\|_{\mathscr{D}_q} + \|\theta\|_{\mathscr{J}_q} \le C\|U\|_{\mathscr{H}_q}.$$

Theorem A.5 *(cf. Ref. 8). Let $2 < p < \infty, N < q < \infty$, $2/p + N/q < 1$ and $T > 0$. Assume that Ω is a bounded domain, (2.1) holds and Γ is a compact hypersurface of class C^2. Let $\mathbf{f}(x(\xi,t),t) \in L^p((0,T),L^q(\dot\Omega))$ and $b(\xi,t) \in L^p\big((0,T),H^1_q(\Omega_-)\big) \cap H^{1/2}_p\big((0,T),L^q(\Omega_-)\big)$. Then, let $(\mathbf{u}_0,\theta_0) \in B^{2(1-1/p)}_{q,p}(\dot\Omega)$ and $h_0 \in B^{1-1/p-1/q}_{q,p}(S_R)$ be initial data for (2.13) and satisfying the smallness condition:*

$$\begin{aligned}&\|\mathbf{f}\|_{L^p((0,T),L^q(\dot\Omega))} + \|b\|_{L^p((0,T),H^1_q(\Omega_-))} + \|b\|_{H^{1/2}_p((0,T),L^q(\Omega_-))}\\ &+ \|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \|\theta_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)} + \|h_0\|_{B^{3-1/p-1/q}_{q,p}(S_R)} \le \varepsilon_1^2\end{aligned} \tag{A4}$$

for some small number $\varepsilon_1 > 0$. Let $\sup_{t\in(0,T)} \|\Psi_h(\cdot,t)\|_{H^1_\infty(\dot\Omega)} \le \varepsilon_1$. Assume that the compatibility conditions hold:

$$\begin{aligned}&\mathbf{v}_0 - \mathbf{N}_3\big(\mathbf{v}_0, \Psi_h|_{t=0}\big) \in \mathscr{D}(\Omega), \quad \operatorname{div}\mathbf{v}_0 = N_2\big(\mathbf{v}_0, \Psi_h|_{t=0}\big) && \text{in } \dot\Omega,\\ &[\![(\mu\mathbf{D}(\mathbf{v}_0)\mathbf{n})_\tau]\!] = [\![\big(\mathbf{N}_6\big(\mathbf{v}_0, \Psi_h|_{t=0}\big)\big)_\tau]\!], \quad [\![\mathbf{v}_0]\!] = 0 && \text{on } S_R,\\ &[\![\nu\nabla\theta_0\cdot\mathbf{n}]\!] = N_8\big(\theta_0, \Psi_h|_{t=0}\big), \quad [\![\theta_0]\!] = 0 && \text{on } S_R,\\ &\mathbf{v}_0 = 0, \quad \nabla\theta_{0-}\cdot\mathbf{n}_- + \beta\theta_{0-} = b|_{t=0} && \text{on } \Gamma,\end{aligned}$$

Then, problem (2.13) admits a unique solution possessing the estimate

$$\begin{aligned}&\|(\mathbf{v},\theta)\|_{L^p((0,T),H^2_q(\dot\Omega))} + \|(\partial_t\mathbf{v},\partial_t\theta)\|_{L^p((0,T),L^q(\dot\Omega))} + \|\partial_t h\|_{L^p((0,T),W^{2-1/q}_q(S_R))}\\ &+ \|h\|_{L^p((0,T),W^{3-1/p}_q(S_R))} + \|\partial_t h\|_{L^\infty((0,T),W^{1-1/q}_q(S_R))} \lesssim \varepsilon_1.\end{aligned} \tag{A5}$$

Proof. Different from Ref. 8, Theorem 2.1, we do not require time to be sufficiently small. Meanwhile, we need to assume that $\|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}(\dot\Omega)}$ is small enough. Employing a similar argument to that in the proof of Ref. 8, Theorem 2.1, we have

$$\begin{aligned}&\|\mathbf{v}\|_{L^p\left((0,T),H^2_q(\dot\Omega)\right)} + \|\partial_t\mathbf{v}\|_{L^p((0,T),L^q(\dot\Omega))} + \|h\|_{L^p\left((0,T),W^{3-1/q}_q(S_R)\right)}\\ &+ \|\partial_t h\|_{L^p((0,T),W^{2-1/q}_q(S_R))} + \|\partial_t h\|_{L^\infty((0,T),W^{1-1/q}_q(S_R))} \le Ce^{\gamma T}\varepsilon_1^2\end{aligned} \tag{A6}$$

for some positive constants C and γ. Thus, we can choose ε_1 so small that $Ce^{\gamma T}\varepsilon_1 \le 1$, then use the contraction mapping principle to complete the proof. So, we may omit its detailed proof. □

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