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ON THE MOTION OF THE CLOSED FREE SURFACE IN THREE-DIMENSIONAL INCOMPRESSIBLE IDEAL MHD WITH SURFACE TENSION

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ABSTRACT. We consider the three-dimensional free boundary incompressible ideal magnetohydrodynamics (MHD) equations with surface tension in a bounded domain. The moving closed surface is parameterized by the height function, which is defined on a smooth and compact reference hypersurface. In Eulerian coordinates, we establish a priori estimates for smooth solutions in the C^∞-class without losing any regularity. This approach allows us to avoid dealing with the spatial regularity of the flow map in Lagrangian coordinates, which may lack maximal regularity and the geometric characteristics, such as the curvature and normal velocity, of the evolutionary domain. The scaling $\partial_t \sim \nabla^{\frac{3}{2}}$ motivates us to formulate the energy functional, and the regularity estimates are driven by the curvature bound and the regularity of the pressure. In the spirit of the Beale-Kato-Majda criterion, we propose a set of the a priori assumptions to guarantee the possibility of extending the solution for a short period while preserving the initial regularity within the time interval of existence. As far as we know, this is the first result involving the blow-up of the free boundary incompressible ideal MHD equations with surface tension. It is worth noting that the velocity and magnetic fields remain bounded in Sobolev's space H^6, while the second fundamental form remains bounded in H^5, throughout the time interval $[0, T_0]$. The value of T_0 depends only on the H^6 norm of the initial velocity and magnetic fields, the initial domain, and the H^5 norm of the mean curvature.

CONTENTS

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1. Introduction

In this paper, we consider the following three-dimensional free boundary incompressible ideal magnetohydrodynamics (MHD) equations with surface tension in a bounded domain:

$$\begin{cases} \mathcal{D}_t v + \nabla p = H \cdot \nabla H, & \text{in } \Omega_t, \\ \mathcal{D}_t H = H \cdot \nabla v, & \text{in } \Omega_t, \\ \operatorname{div} v = 0, \quad \operatorname{div} H = 0, & \text{in } \Omega_t, \\ H \cdot \nu = 0, \quad p = \mathcal{A}_{\Gamma_t}, \quad v_n = V, & \text{on } \Gamma_t, \\ v(0,\cdot) = v_0, \quad H(0,\cdot) = H_0, & \text{in } \Omega_0, \end{cases} \tag{1.1}$$

where the time $t > 0$, $v = (v^1, v^2, v^3)^\top$ is the velocity field, $H = (H^1, H^2, H^3)^\top$ is the magnetic field, p is the scalar total pressure, and the domain $\Omega_t \subset \mathbb{R}^3$ is bounded with a closed surface $\Gamma_t := \partial\Omega_t$. $\nu = (\nu^1, \nu^2, \nu^3)^\top$ represents the unit outer normal to Γ_t, $\mathcal{A}_{\Gamma_t}$ denotes the mean curvature of Γ_t, $v_n = v \cdot \nu$ is the normal velocity, and V represents the normal velocity of Γ_t. Additionally, $H \cdot \nabla$ and $v \cdot \nabla$ are the directional derivatives, and $\mathcal{D}_t := \partial_t + v \cdot \nabla$ represents the material derivative. Ω_0, v_0 and H_0 are the prescribed initial data. We denote $\Gamma_0 := \partial\Omega_0$, and have assumed that the coefficient of surface tension is 1 for simplicity.

In recent decades, there has been significant interest in studying the free boundary incompressible Euler equations, and substantial advancements have been made. Extensive research has been achieved for the irrotational case, especially, the water wave equations. We refer readers to [Lan05, Wu09, Wu11, GMS12, IP15] and the references therein. If the fluid flow exhibits vorticity, one may refer to [CL00, Lin03, Lin05, CS07, SZ08a, ZZ08, DE14, Sch05, DE16, DKT19] for results on a priori estimates, local well-posedness with or without surface tension, the zero surface tension limit, the large surface tension limit, and more.

The investigation of free boundary problems for MHD equations has emerged relatively recently in comparison to the study of the Euler equations, mainly because of the strong interactions between the magnetic and velocity fields. We focus on the incompressible MHD equations in this paper. Hao and Luo [HL14] obtained a priori estimates for free boundary problems of the incompressible ideal MHD without surface tension under the Taylor-type sign condition. They considered the case where the initial domain is bounded and homeomorphic to a ball. They also showed the ill-posedness of the problem if the Taylor-type sign condition is violated in the two-dimensional case [HL20]. Luo and Zhang [LZ20] derived a priori estimates for the lower regular initial data in the initial domain of sufficiently small volume. In [GW19], a local existence result was given, with a detailed proof provided in an initial flat domain $\mathbb{T}^2 \times (0,1)$. The local well-posedness for the incompressible ideal MHD equations with surface tension is established by Gu, Luo, and Zhang in [GLZ23], in the same initial domain setting, namely, the flat domain $\mathbb{T}^2 \times (0,1)$. For the problem of the current-vortex sheet, the nonlinear stability of the current-vortex sheet in the incompressible MHD equations was solved by Sun, Wang and Zhang [SWZ18] under the Syrovatskij stability condition, assuming that the free boundaries are graphs in $\mathbb{T}^2 \times (-1,1)$. Wang and Xin [WX21] established the global well-posedness of a free interface problem for the incompressible inviscid resistive MHD, taking into account magnetic diffusion, under similar assumptions regarding the graph. We also refer to some related works [SWZ19, HL21, Lee17, FHYZ23, HLZ23, HY24, GLZ22, MTT08, LZ23, Tra05, Tra09, TW21] on the topics of well-posedness, current-vortex sheets problem, breakdown criterion, viscous splash singularity, zero surface tension limit, and compressible MHD.

It may be possible to reduce the problem of a general free boundary to the case of a graph by selecting local coordinates that flatten the boundary near a point. However, this reduction will be technically complicated with challenging difficulties. In the presence of surface tension, if we only select a portion of the free boundary and flatten it near a point, there is a risk of losing certain geometric characteristics of the free boundary, such as the evolution of its curvature. We discovered that the blow-up of the curvature would result in the breakdown of a smooth solution in Sobolev spaces. Additionally, the Lagrangian coordinate can be utilized to transform a moving domain into a fixed one and the aforementioned well-posedness results for MHD equations are mainly derived from this methodology. Nevertheless, as indicated in [SZ08a, SZ11], the Lagrangian map lacks maximal regularity because all the variables are defined on an evolutionary domain. In fact, the moving surface can also be described using alternative methods, such as the study of the Euler equations with surface tension [Sch05], the fluid interface problem [SZ11], the surface diffusion flow with elasticity [FJM20], the motion of charged liquid drop [JLM22], and the plasma-vacuum problem [LX23b], among others.

Different from the local well-posedness result in [GLZ23] for the flat domain employing Lagrangian coordinates, we investigate system (1.1) in an arbitrary bounded domain by parameterizing the moving boundary with the height function defined on a smooth and compact reference hypersurface. Additionally, we eliminate the requirement for the initial velocity on the boundary ($v_0 \in H^{4.5}(\Omega_0) \cap H^5(\Gamma)$ is assumed in [GLZ23]). We establish a distinct energy functional by preserving the material derivative, which avoids destroying the structure of system (1.1) when separating the time derivative ∂_t. Also, we mention that the spatial-temporal scaling is different, i.e., $\partial_t \sim \nabla^{\frac{3}{2}}$ and the energy estimates are driven by the second fundamental form together with the pressure, which is different from the strategy used in [GLZ23]. For example, we cannot define the fractional derivative using the Fourier transform, which makes calculating the time derivative of the energy challenging.

The curvature of the moving boundary is crucial for well-posedness. Roughly speaking, the blow-up of curvature ($\|B\|_{H^{1+\delta}} \to \infty$ with $\delta > 0$ small) will result in the breakdown of a smooth solution in $H^6(\Omega_t)$. We provide the a priori assumptions and prove higher-order energy estimates to ensure the extension of a smooth solution without any loss of regularity. Moreover, we should avoid the self-intersection of the free boundary, as indicated in [CS14].

1.1. Motivation for the Construction of Energy Functionals.

Given any smooth initial domain Ω_0 and any divergence-free initial velocity and magnetic fields $v_0, H_0 \in C^\infty(\Omega_0)$ such that $H_0 \cdot \nu_{\Gamma_0} = 0$ on Γ_0, we assume that system (1.1) has a smooth solution that exists for a short time interval throughout the paper. In particular, the solution (v, H, p) is well-defined on Γ_t, allowing us to define $\nabla v, \nabla H, \nabla p$, etc. on Γ_t by taking limits inside the domain.

It is well-known that the physical energy is conserved. Indeed, applying Lemmas A.2 and A.3 (see, e.g., [LZ21]), one can verify that

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega_t}\left[|v(x,t)|^2 + |H(x,t)|^2\right]dx + \int_{\Gamma_t} 1 dS\right) = 0,$$

where dS denotes the measure on Γ_t.

Motivated by [SZ08a, JLM22], we construct the higher-order energy functional as

$$\begin{aligned} e_l(t) := \frac{1}{2}\left(\int_{\Omega_t}\left[|\mathcal{D}_t^{l+1}v|^2 + |\mathcal{D}_t^{l+1}H|^2\right]dx + \int_{\Gamma_t}|\bar{\nabla}(\mathcal{D}_t^l v\cdot\nu)|^2 dS\right) \\ + \frac{1}{2}\int_{\Omega_t}\left[|\nabla^{\lfloor\frac{3l+1}{2}\rfloor}\operatorname{curl} v|^2 + |\nabla^{\lfloor\frac{3l+1}{2}\rfloor}\operatorname{curl} H|^2\right]dx, \end{aligned} \tag{1.2}$$

for any $l \geq 1$, and we denote the sum of the first three by $\bar{e}(t)$, i.e.,

$$\begin{aligned} \bar{e}(t) = \frac{1}{2}\sum_{k=1}^{3}\left(\int_{\Omega_t}\left[|\mathcal{D}_t^{k+1}v|^2 + |\mathcal{D}_t^{k+1}H|^2\right]dx + \int_{\Gamma_t}|\bar{\nabla}(\mathcal{D}_t^k v\cdot\nu)|^2 dS\right) \\ + \frac{1}{2}\sum_{k=2,3,5}\int_{\Omega_t}\left[|\nabla^k \operatorname{curl} v|^2 + |\nabla^k \operatorname{curl} H|^2\right]dx. \end{aligned} \tag{1.3}$$

Above, $\lfloor\cdot\rfloor$ denotes the integer part of a given number, and $\bar{\nabla}$ denotes the tangential derivative. We also define the following energy functional, taking into account the spatial regularity:

$$\begin{aligned} E_l(t) := \sum_{k=0}^{l}\left(\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|\mathcal{D}_t^{l+1-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_t)}\right) + \|v\|^2_{H^{\lfloor\frac{3l+3}{2}\rfloor}(\Omega_t)} \\ + \|H\|^2_{H^{\lfloor\frac{3l+3}{2}\rfloor}(\Omega_t)} + \|\bar{\nabla}(\mathcal{D}_t^l v\cdot\nu)\|^2_{L^2(\Gamma_t)} + 1, \end{aligned} \tag{1.4}$$

for any $l \geq 1$. As before, we define

$$\bar{E}(t) := \sum_{k=0}^{3}\left(\|\mathcal{D}_t^{4-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|\mathcal{D}_t^{4-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_t)}\right) + \|v\|^2_{H^6(\Omega_t)} + \|H\|^2_{H^6(\Omega_t)} + \sum_{k=1}^{3}\|\bar{\nabla}(\mathcal{D}_t^k v\cdot\nu)\|^2_{L^2(\Gamma_t)} + 1, \tag{1.5}$$

and we observe that $\sum_{i=1}^{3} E_i(t) \leq C\bar{E}(t) \leq C\sum_{i=1}^{3} E_i(t)$.

Remark. Note that we define the energy functional preserving the material derivative in contrast to the energy in [GLZ23].

The energy $e_l(t)$ is specifically designed to cancel the leading terms on the free boundary Γ_t when computing its time derivative. Since the evolution of the boundary contributes to the energy estimates at higher orders, it is necessary to consider the divergence-free condition and the fact that the normal component of the magnetic field vanishes on the free boundary. In addition, we need to include the term $\frac{1}{2}\|\bar{\nabla}(\mathcal{D}_t^l v\cdot\nu)\|^2_{L^2(\Gamma_t)}$ in the energy. This term helps us eliminate the dominant terms that arise from the material derivatives of the pressure on Γ_t (cf. Lemma 2.13). We exclude the term $\frac{1}{2}\|\mathcal{D}_t^l v\cdot\nu\|^2_{L^2(\Gamma_t)}$ since it can be controlled by applying either the divergence theorem or the trace theorem (provided that the a priori assumptions defined below hold). With this simplification, we will avoid the tedious proof required when we close the energy estimates. It is worth mentioning that we define the energy starting from $\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)}$ and $\|\mathcal{D}_t^2 H\|^2_{L^2(\Omega_t)}$. We make this choice because $\|\mathcal{D}_t v\|^2_{L^2(\Omega_t)}$ can be controlled by the pressure (cf. Section 4 and Proposition 6.1), and $\|\mathcal{D}_t H\|^2_{L^2(\Omega_t)}$ can be estimated by substituting $\mathcal{D}_t H = H\cdot\nabla v$. The curl part in $e_l(t)$ is used to control the energy $E_l(t)$ when applying the div-curl estimates. Furthermore, it is essential to note that the region under consideration is neither a periodic region nor the entire space. Consequently, it is not possible to define the fractional derivative using the Fourier transform. Therefore, we shall choose the integer part of $\frac{3l}{2}+\frac{1}{2}$, which is a significant difference.

The scaling $\frac{3}{2}$ is revealed in [SZ08b] that a second-order time derivative can be roughly equated to a third-order spatial differentiation. In other words, one-order time derivative ∂_t is associated with spatial regularity of $\frac{3}{2}$-order, indicating the regularizing effect of the surface tension. In the absence of surface tension, however, two-order time derivatives are similar to one-order spatial differentiation, i.e., $\partial_t \sim \nabla^{\frac{1}{2}}$. This point will be consistently used throughout the paper. Nevertheless, the improvement in regularity of the free boundary Γ_t is geometric, as it is connected to the regularity of the mean curvature (cf. Lemma A.4), and is not completely evident in the Lagrangian coordinates. Therefore, we do not adopt the strategy of fixing the boundary, which has to deal with the spatial regularity of the flow map. Instead, we choose a reference hypersurface to serve as a representation of the free boundary by utilizing the height function. Then, it is more convenient to control the mean curvature using the height function. The free boundary Γ_t is specifically parameterized by a smooth and compact hypersurface $\Gamma = \partial\Omega$. Here, Ω is a smooth and compact subset of $\mathbb{R}^3$ that satisfies the interior and exterior ball condition with a radius $\mathcal{R} > 0$. For any $t \geq 0$, we represent the free boundary Γ_t as $\Gamma_t = \{x + h(x,t)\nu_\Gamma(x) : x \in \Gamma\}$ using the height function $h(\cdot,t) : \Gamma \to \mathbb{R}$ with $\|h(\cdot,t)\|_{L^\infty(\Gamma)} < \mathcal{R}$. Note that the time derivative of h is equal to the normal velocity of the free boundary, i.e., $\partial_t h = v_n$. Given a solution (v, H, p, Ω_t) to system (1.1), whose time interval of existence is $[0,T)$, we define the following quantity

$$\mathcal{M}_T := \mathcal{R} - \sup_{0\leq t<T}\|h(\cdot,t)\|_{L^\infty(\Gamma)}. \tag{1.6}$$

It is obvious that the height function is well-defined in the interval $[0,T)$ as long as $\mathcal{M}_T > 0$.

From the perspective of system (1.1), the scaling suggests that we can reduce "$\frac{1}{2}$-order" spatial regularity if we substitute $\mathcal{D}_t v = -\nabla p + H\cdot\nabla H$ and $\mathcal{D}_t H = H\cdot\nabla v$. In this sense, we can also reduce "$\frac{1}{2}$-order" spatial regularity when the operators $\mathcal{D}_t$ and curl are combined (cf. Lemma 2.5). These observations are crucial in deriving the optimal expressions for $\operatorname{div}\mathcal{D}_t^l v, \operatorname{curl}\mathcal{D}_t^l v$, etc. based on the principle of "reducing derivatives" (see, e.g., Lemmas 2.8 and 2.10 in Section 2 for details). These lemmas will play a vital role in our application of the div-curl estimates to control the higher-order energy (cf. Lemma 6.2).

1.2. **A Priori Assumptions.** Having introduced the quantity $\mathcal{M}_T$ to ensure the well-definedness of the height function, we then define the following quantity to ensure that the solution can be extended and controlled by the initial data. Taking $\delta > 0$ to be sufficiently small, we define

$$\mathcal{N}_T := \sup_{0\le t<T} (\|h(\cdot,t)\|_{H^{3+\delta}(\Gamma)} + \|v(\cdot,t)\|_{H^4(\Omega_t)} + \|H(\cdot,t)\|_{H^4(\Omega_t)} + \|v_n(\cdot,t)\|_{H^4(\Gamma_t)}). \tag{1.7}$$

It should be noted that in order to simplify the a priori assumptions, neither the pressure nor the time derivative are included in $\mathcal{N}_T$. Estimating certain pressure-related terms may become challenging as a consequence. For instance, in Section 4, it is not possible to directly control $\|\nabla \mathcal{D}_t p\|_{L^2(\Omega_t)}$ using $\mathcal{N}_T$ and the initial data. Instead, we need to utilize $\Delta \mathcal{D}_t p$ in Ω_t along with $\mathcal{D}_t p$ on Γ_t to control it. Furthermore, the requirement for the height function h and the normal velocity v_n is related to the fact that the boundary Γ_t is moving and will be clarified in Section 4. The former will ensure that the second fundamental form B_{Γ_t} is uniformly bounded, i.e., $\|B\|_{L^\infty(\Gamma_t)} \le C$, provided $\mathcal{N}_T \le C$. Moreover, these two terms will be constrained by the curvature bound in Section 7.

In contrast to the Euler equations [JLM22], the presence of the magnetic field necessitates an enhancement in the regularity of the velocity and magnetic fields in both the a priori assumptions and the initial data. It is challenging to assume that the initial velocity and magnetic fields belong to $H^3(\Omega_0)$ even for the flat domain (see, e.g., [LZ21]). For this reason, in order to demonstrate that the initial data in $H^s(\Omega_0)$ controls the energy of the same order, we can assume that $v_0 \in H^4(\Omega_0)$ and $H_0 \in H^4(\Omega_0)$ in order to estimate $E_2(t)$. However, recalling that we select integer-order Sobolev spaces in the energy functional to compute the time derivative, the floor function ($\lfloor \frac{9}{2} \rfloor = 4$) suggests that we do not have a control over the first two terms in $E_2(t)$. To address this issue, we attempted to enhance the regularity of $E_1(t)$. However, we encountered a challenging problem as the commutators consistently generated uncontrollable terms of the highest order.

Ultimately, we are able to control the energy $\bar{E}(t)$ for arbitrary initial data in $H^6(\Omega_0)$, provided the following the a priori assumptions hold:

$$\mathcal{N}_T < \infty, \text{ and } \mathcal{M}_T > 0.$$

We expect the requirement of $\|v\|_{H^4(\Omega_t)}$ and $\|H\|_{H^4(\Omega_t)}$ in the a priori assumptions to be crucial as it allows us to control $\|v\|_{H^6(\Omega_t)}$ and $\|H\|_{H^6(\Omega_t)}$.

1.3. **Main Results.** The following is the main result of this paper.

Theorem 1.1. *Let $\Omega_0 \subset \mathbb{R}^3$ be a smooth domain, and assume that the boundary $\Gamma_0 = \partial\Omega_0$ can be represented as $\Gamma_0 = \{x + h_0(x)\nu_\Gamma(x) : x \in \Gamma\}$ with $\|h_0\|_{L^\infty(\Gamma)} < \mathcal{R}$. Let $v_0, H_0 \in C^\infty(\Omega_0; \mathbb{R}^3)$ be any divergence-free initial velocity and magnetic fields such that $H_0 \cdot \nu_{\Gamma_0} = 0$ on Γ_0. Assume that system (1.1) has a smooth solution in $[0,T)$ for some time $T > 0$, and the following the a priori assumptions hold*

$$\mathcal{N}_T < \infty, \text{ and } \mathcal{M}_T > 0. \tag{1.8}$$

Then, we have the following results:

(1) *There exists a positive number τ, which depends on $T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}$, $\|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$, such that the solution exists in the time interval $[0, T+\tau)$.*
(2) *We have the following lower-order quantitative regularity estimates:*

$$\sup_{0\le t<T+\tau} \left(\bar{E}(t) + \sum_{k=0}^{3} \|\mathcal{D}_t^{3-k} p\|_{H^{\frac{3}{2}k+1}(\Omega_t)} + \|B_{\Gamma_t}\|_{H^5(\Gamma_t)} \right) \le \bar{C},$$

where the constant $\bar{C}$ depends on $T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$.
(3) *For $l \ge 4$, we also have the higher-order regularity estimates:*

$$\sup_{0\le t<T+\tau} E_l(t) \le C_l,$$

where the constant C_l depends on $l, T, \mathcal{N}_T, \mathcal{M}_T$, and $E_l(0)$.

In particular, the smooth solution of system (1.1) can be extended and remains smooth (with respect to both t and x), as long as the a priori assumptions (1.8) are satisfied.

Moreover, there exists some $T_0 > 0$, which depends only on the quantities $\mathcal{M}_0$, $\|v_0\|_{H^6(\Omega_0)}$, $\|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$, such that the a priori assumptions (1.8) hold for $T = T_0$.

We will make a few remarks about the Main Theorem.

The first result is similar in spirit to the Beale-Kato-Majda criterion, which dates back to the remarkable work [BKM84]. Roughly speaking, in order to ensure that the solution remains bounded in H^3, the L^∞-norm of the vorticity of the flow must be integrated over the entire time interval of existence. However, the assumptions here regarding the velocity, magnetic field, and height function are stronger in order to guarantee that the velocity and magnetic fields remain bounded in $H^6(\Omega_t)$. The a priori assumption $\mathcal{M}_T > 0$ is technical. If $\mathcal{M}_T = 0$, we will select a different reference surface to parameterize the free boundary. It is worth noting that we can always select the reference surface unless the boundary self-intersects in at least one point (splash or splat singularity, see, e.g., [CS14]). The hypothesis of the height function is roughly related to the singularity of the free boundary. In other words, the second fundamental form remains uniformly bounded. The singularity $\|B\|_{L^\infty(\Gamma_t)} = \infty$ suggests that the curvature of the free boundary will blow up. Moreover, the volume-preserving property suggests that the domain becomes exceedingly narrow as $|h(\cdot,t)|$ increases. The requirement for the normal velocity of the free boundary is reasonable because we do not fix the boundary using Lagrangian coordinates, and it also ensures the boundedness of the pressure.

The last statement highlights that the a priori estimates remain bounded until time $T_0 > 0$, which is determined by the initial height function, velocity, magnetic field, and the mean curvature, i.e.,

$$T_0 = T_0\left(\mathcal{M}_0, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}, \|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}\right).$$

On the time interval $(0, T_0)$, our lower-order energy estimates yield

$$\sup_{0\le t<T_0}\left(\bar{E}(t) + \sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|_{H^{\frac{3}{2}k+1}(\Omega_t)} + \|B_{\Gamma_t}\|_{H^5(\Gamma_t)}\right) \le C,$$

where the constant $C = C\left(\mathcal{M}_0, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}, \|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}\right)$. In particular, we bound the velocity and magnetic fields in $H^6(\Omega_t)$ and the second fundamental form in $H^5(\Gamma_t)$ by the initial data $\mathcal{M}_0$, $\|v_0\|_{H^6(\Omega_0)}$, $\|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$ without any loss of regularity. It is worth noting that we only need the initial mean curvature to control the second fundamental form of the same order. We also included the pressure $\|p\|_{H^{\frac{11}{2}}(\Omega_t)}$, as well as the time derivatives of the velocity, magnetic field, and pressure.

In fact, from the definitions of the material derivative ($\mathcal{D}_t = \partial_t + v\cdot\nabla$) and $\bar{E}(t)$, it is easy to verify that

$$\sup_{0\le t<T_0}\sum_{k=1}^{3}\left(\|\partial_t^{3-k}v\|^2_{H^{\frac{3}{2}(k+1)}(\Omega_t)} + \|\partial_t^{3-k}H\|^2_{H^{\frac{3}{2}(k+1)}(\Omega_t)}\right) \le C,$$
$$\sup_{0\le t<T_0}\sum_{k=1}^{3}\|\partial_t^{3-k}p\|^2_{H^{\frac{3}{2}k+1}(\Omega_t)} \le C,$$

with the constant $C = C\left(\mathcal{M}_0, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}, \|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}\right)$ defined above.

Furthermore, the higher-order energy $E_l(t)$ can be controlled by its initial value $E_l(0)$ as long as the a priori assumptions hold. The higher-order energy estimates also include the estimates for $\partial_t^i v$, $\partial_t^j H$, $\partial_t^k p$, and the second fundamental form in appropriate Sobolev spaces. Therefore, the solution can be extended while remaining smooth with respect to both the time and spatial variables, provided that the a priori assumptions (1.8) hold.

1.4. Outline of the Proofs and the Structure of the Paper.

1.4.1. *Reduce the derivatives.* The foundation of our energy estimates lies in the reduction of the derivatives, which helps in formulating the error terms, $\operatorname{div} \mathcal{D}_t^l v$, $\operatorname{curl} \mathcal{D}_t^l v$, $\Delta \mathcal{D}_t^l p$, $\mathcal{D}_t^l p$, etc. We will explain this in detail below.

(1) Based on the structure of system (1.1), the equation $\mathcal{D}_t H = H \cdot \nabla v$ suggests that we can replace the material derivative of the magnetic field by the gradient of the velocity. To be more specific, the formulas (2.9) and (2.10) in Lemma 2.6 demonstrate that for any $i, j \in \mathbb{N}$, it is true that

$$\mathcal{D}_t^j H = \sum_{1 \le m \le j} \sum_{|\beta| \le j-m} \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_m} v \star H,$$

and

$$\nabla^i \mathcal{D}_t^j H = \sum_{1 \le m \le j} \sum_{\substack{|\alpha| \le i \\ |\beta| \le j-m}} \nabla^{1+\alpha_1} \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_m} \mathcal{D}_t^{\beta_m} v \star \nabla^{\alpha_{m+1}} H,$$

where $\star$ denotes the contraction of certain indices of tensors with constant coefficients (see, e.g., [Ham82]). In the above, at least one-order material derivative is substituted by the spatial derivative at the cost of the expression containing more product terms. These enable us to convert certain estimates for the magnetic field into those for the velocity field (e.g., Propositions 4.7, 6.1 and 6.2).

(2) When the operators $\mathcal{D}_t$ and curl join together, we can reduce spatial regularity by $\frac{1}{2}$-order. For example, a straightforward calculation yields

$$\operatorname{curl} \mathcal{D}_t v = (\nabla H)^\top \operatorname{curl} H + \operatorname{curl} H \nabla H + (H \cdot \nabla)(\operatorname{curl} H),$$

where $\operatorname{curl} H := \nabla H - (\nabla H)^\top$. This demonstrates that the material derivative $\mathcal{D}_t$ is replaced by the spatial derivative ∇, while $\partial_t \sim \nabla^{\frac{3}{2}}$. This fact is essential for formulating the error terms (e.g., Lemmas 2.8 and 2.10) to control the energy. It also indicates that we should reserve the curl operator (rather than simply ∇) for the highest order term when seeking strategies to reduce the derivatives.

(3) To compute $-\Delta \mathcal{D}_t^{k+1} p$, we will calculate the divergence of $\mathcal{D}_t^k (H \cdot \nabla H)$ (cf. Lemma 2.7). The divergence-free condition ensures that when taking the divergence of $\mathcal{D}_t^k (H \cdot \nabla H)$, the order of derivatives does not increase and even "decreases by $\frac{1}{2}$-order".

1.4.2. *Estimates for the product of functions.* To establish the energy estimates, we will estimate the time derivative of the energy functional using the Kato-Ponce inequalities. We start with the estimates

$$\frac{d}{dt} \bar{e}(t) \le C \left(1 + \|\nabla p\|^2_{H^2(\Omega_t)} \right) \bar{E}(t), \tag{1.9}$$

and

$$\frac{d}{dt} e_l(t) \le C E_l(t), \quad l \ge 4, \tag{1.10}$$

for any $0 < t < T$, provided that the a priori assumptions (1.8) hold for the time $T > 0$.

To estimate the time derivative of the higher-order energy $e_l(t)$ (i.e., $l \ge 4$), we primarily rely on induction arguments. However, when working with the estimates for $\frac{d}{dt}\bar{e}(t)$, it is crucial to accurately estimate the product ($\star$ product) of functions in the error terms. Our strategy involves controlling the most challenging terms (at most two) in the product using pressure and energy $\bar{E}(t)$, while imposing the a priori assumptions on the remaining terms. This helps us distribute the derivatives. In the end, we only require $\|\nabla p\|_{H^2(\Omega_t)}$ in (1.9) to control $\frac{d}{dt}\bar{e}(t)$.

At the end of the proof of (1.9) in Section 3, we could not simply integrate by parts when estimating the integral of $\int_{\Omega_t} \operatorname{div} \mathcal{D}_t^{l+1} (H \cdot \nabla H) u dx$ because the leading term in $\mathcal{D}_t^{l+1}(H \cdot \nabla H)$, i.e., $\nabla^2 \mathcal{D}_t^l v$, is out of control (cf. Lemma 2.6). Fortunately, as Lemmas 2.6 and 2.7 show, the divergence-free condition facilitates the substitution of one order material derivative with one order spatial derivative. Roughly speaking, we reduce the $\frac{1}{2}$-order derivative. Consequently, we can control $\int_{\Omega_t} \operatorname{div} \mathcal{D}_t^{l+1} (H \cdot \nabla H) u dx$ using Lemma 2.7 and $H \cdot \nu = 0$.

1.4.3. *Auxiliary functions to control the pressure.* Since (1.10) is established using induction arguments, it is necessary to demonstrate that

$$\frac{d}{dt}\bar{e}(t) \le C\bar{E}(t),$$

where C shall depend only on the initial data. For this purpose, we need to estimate the pressure in (1.9), and we apply an ad-hoc argument to show that

$$\sup_{t\in[0,T]} \|p\|_{H^3(\Omega_t)}^2 \le C\left(T, \mathcal{N}_T, \mathcal{M}_T, \|\bar{\nabla}^2 p\|_{L^2(\Gamma_0)}, \|\nabla p\|_{H^2(\Omega_0)}\right). \tag{1.11}$$

In fact, to prove (1.11), we compute the evolution of the pressure in the domain Ω_t, and obtain

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|\nabla p\|_{L^2(\Omega_t)}^2 &\le C\left(1 + \|\nabla p\|_{L^2(\Omega_t)}^2\right),\\
\frac{d}{dt}\frac{1}{2}\|\nabla^2 p\|_{L^2(\Omega_t)}^2 &\le C\left(1 + \|\nabla p\|_{H^1(\Omega_t)}^2 + \|p\|_{H^2(\Gamma_t)}^2\right),\\
\frac{d}{dt}\frac{1}{2}\|\nabla^3 p\|_{L^2(\Omega_t)}^2 &\le C\left(1 + \|\nabla p\|_{H^2(\Omega_t)}^2 + \|p\|_{H^3(\Gamma_t)}^2\right).
\end{aligned}$$

As a result, we need to additionally consider the pressure (e.g., $\bar{\nabla}^2 p$ and $\bar{\nabla}^3 p$) on the free boundary. We study the following functions

$$\int_{\Gamma_t} \bar{\nabla} p \cdot \bar{\nabla}(\nabla v\nu \cdot \nu) dS, \text{ and } \int_{\Gamma_t} \bar{\nabla}^2 p : \bar{\nabla}^2(\nabla v\nu \cdot \nu) dS + \varepsilon \int_{\Gamma_t} |\bar{\nabla}^2 p|^2 dS,$$

with $\varepsilon > 0$ sufficiently small, which are designed based on the formula of the Beltrami-Laplace operator, i.e.,

$$\Delta_B v = \Delta u - (\nabla^2 v\nu \cdot \nu) - \mathcal{A}\partial_\nu v.$$

To estimate their time derivatives, we apply the estimates

$$\begin{aligned}
\|\bar{\nabla}^2 p\|_{L^2(\Gamma_t)}^2 &\le \|\Delta_B p\|_{L^2(\Gamma_t)}^2 + C\int_{\Gamma_t} |B|^2 |\bar{\nabla} p|^2 dS,\\
\|\bar{\nabla}^3 p\|_{L^2(\Gamma_t)}^2 &\le \|\bar{\nabla}\Delta_B p\|_{L^2(\Gamma_t)}^2 + C\|p\|_{H^2(\Gamma_t)}^2,
\end{aligned}$$

considering the commutation of the tangential derivatives, that are proved by using Simon's identity (see, e.g., [FJM20]). Furthermore, we bound the spatial derivatives of p and $\mathcal{D}_t p$ on the free boundary by their tangential derivatives, $\|\Delta p\|_{H^2(\Omega_t)}$, together with $\|\Delta \mathcal{D}_t p\|_{H^1(\Omega_t)}$. For example,

$$\begin{aligned}
\|\nabla^2 \mathcal{D}_t p\|_{L^2(\Gamma_t)}^2 &\le C\left(\|\Delta \mathcal{D}_t p\|_{H^1(\Omega_t)}^2 + \|\mathcal{D}_t p\|_{H^2(\Gamma_t)}^2\right),\\
\|\nabla^3 p\|_{L^2(\Gamma_t)}^2 &\le C\left(\|\Delta p\|_{H^2(\Omega_t)}^2 + \|p\|_{H^3(\Gamma_t)}^2\right).
\end{aligned}$$

Also, we use the fact that $\|v_n\|_{H^4(\Gamma_t)} \le C$ from the a priori assumptions.

Finally, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\Gamma_t} \bar{\nabla} p \cdot \bar{\nabla}(\nabla v\nu \cdot \nu) dS \le -\frac{1}{2}\|\bar{\nabla}^2 p\|_{L^2(\Gamma_t)}^2 + C,\\
&\frac{d}{dt}\left(\int_{\Gamma_t} \bar{\nabla}^2 p : \bar{\nabla}^2(\nabla v\nu \cdot \nu) dS + \varepsilon \int_{\Gamma_t} |\bar{\nabla}^2 p|^2 dS\right)\\
&\quad \le -\frac{1}{4}\|\bar{\nabla}^3 p\|_{L^2(\Gamma_t)}^2 + C(\|\bar{\nabla}^2 p\|_{L^2(\Gamma_t)}^2 + 1),
\end{aligned}$$

and deduce

$$\sup_{t\in[0,T]} \|\bar{\nabla}^2 p\|_{L^2(\Gamma_t)}^2 + \int_0^T \|\bar{\nabla}^3 p\|_{L^2(\Gamma_t)}^2 dt \le C.$$

Combined with the estimates in the domain Ω_t, (1.11) follows. The initial quantities $\|\bar{\nabla}^2 p\|_{L^2(\Gamma_0)}$ and $\|\nabla p\|_{H^2(\Omega_0)}$ in (1.11) can be easily controlled by the initial data. In fact, we control the initial energy and pressure through the initial velocity, magnetic field, and mean curvature:

$$\bar{E}(0) + \sum_{k=0}^{3} \|\mathcal{D}_t^{3-k} p\|^2_{H^{\frac{3}{2}k+1}(\Omega_0)} \leq C\left(\mathcal{M}_0, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}, \|\mathcal{A}\|_{H^5(\Gamma_0)}\right),$$

using (2.9) and (2.10), where $\mathcal{M}_0 := \mathcal{R} - \|h_0\|_{L^\infty(\Gamma)}$. In the same way, it holds

$$\bar{E}(t) + \sum_{k=0}^{3} \|\mathcal{D}_t^{3-k} p\|^2_{H^{\frac{3}{2}k+1}(\Omega_t)} \leq C,$$

where the constant C depends on $\mathcal{R} - \|h(\cdot,t)\|_{L^\infty(\Gamma)}$, $\|v\|_{H^6(\Omega_t)}$, $\|H\|_{H^6(\Omega_t)}$, and $\|\mathcal{A}\|_{H^5(\Gamma_t)}$, provided $\mathcal{R} - \|h(\cdot,t)\|_{L^\infty(\Gamma)} > 0$.

1.4.4. *Curvature, Kato-Ponce and div-curl estimates.* Thanks to the pressure estimates, we are able to bound the second fundamental form. As a result, we can extend the function to the entire space, enabling us to apply the Kato-Ponce estimates to half-integer Sobolev spaces and control the constants by the curvature bound. Indeed, the estimates in [JLM22] for the Euler equations

$$\|f_1 \dots f_j\|_{H^{\frac{1}{2}}(\Omega_t)} \leq C \sum_{i=1}^{j} \|f_i\|_{L^\infty(\Omega_t)} \prod_{k\neq i} \|f_k\|_{H^{\frac{1}{2}}(\Omega_t)}$$

are not suitable for our case due to the existence of the magnetic field, and we shall apply (A.7) to estimate the error terms, e.g.,

$$\begin{aligned} \|\nabla v \star \nabla \mathcal{D}_t^l v\|_{H^{\frac{1}{2}}(\Omega_t)} \leq C\Big(&\|\nabla v\|_{L^\infty(\Omega_t)} \|\nabla \mathcal{D}_t^l v\|_{H^{\frac{1}{2}}(\Omega_t)} \\ &+ \|\nabla v\|_{W^{\frac{1}{2},6}(\Omega_t)} \|\nabla \mathcal{D}_t^l v\|_{L^3(\Omega_t)}\Big). \end{aligned}$$

Nevertheless, controlling the error associated with the magnetic tension $H\cdot\nabla H$ (denoted as $R^0_{\nabla^2 H,H}$) turns out to be challenging. We shall control it by the curl of the magnetic field, i.e.,

$$\|R^0_{\nabla^2 H,H}\|^2_{H^{\frac{3}{2}k-1}(\Omega_t)} \leq C\|\operatorname{curl} H\|^2_{H^{\lfloor \frac{3}{2}l+\frac{1}{2}\rfloor}(\Omega_t)},$$

which is distinct from all other error terms.

To close the energy estimates, we define another energy functional containing $\operatorname{curl} v$ and $\operatorname{curl} H$ as

$$\begin{aligned} \tilde{e}(t) &:= \frac{1}{2}\sum_{k=1}^{3}\left(\|\mathcal{D}_t^{k+1} v\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t^{k+1} H\|^2_{L^2(\Omega_t)} + \|\bar{\nabla}(\mathcal{D}_t^k v\cdot\nu)\|^2_{L^2(\Gamma_t)}\right) \\ &\quad + \frac{1}{2}\left(\|\operatorname{curl} v\|^2_{H^5(\Omega_t)} + \|\operatorname{curl} H\|^2_{H^5(\Omega_t)}\right) + 1, \\ \tilde{e}_l(t) &:= \frac{1}{2}\left(\|\mathcal{D}_t^{l+1} v\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t^{l+1} H\|^2_{L^2(\Omega_t)} + \|\bar{\nabla}(\mathcal{D}_t^l v\cdot\nu)\|^2_{L^2(\Gamma_t)}\right) \\ &\quad + \frac{1}{2}\left(\|\operatorname{curl} v\|^2_{H^{\lfloor\frac{3l+1}{2}\rfloor}(\Omega_t)} + \|\operatorname{curl} H\|^2_{H^{\lfloor\frac{3l+1}{2}\rfloor}(\Omega_t)}\right) + 1, \quad l \geq 4, \end{aligned}$$

and apply the div-curl estimates to obtain

$$\bar{E}(t) \leq C\tilde{e}(t) \leq C\left(1+\bar{e}(t)\right), \text{ and } E_l(t) \leq C\tilde{e}_l(t) \leq C\left(1+e_l(t)\right), \ l \geq 4.$$

Finally, we deduce the desired energy estimates

$$\frac{d}{dt}\bar{e}(t) \leq C\left(1+\bar{e}(t)\right), \text{ and } \frac{d}{dt}e_l(t) \leq C\left(1+e_l(t)\right), \ l \geq 4.$$

1.4.5. *Justification for the validity of the a priori assumptions.* To show that the a priori assumptions (1.8) hold for some time $T_0 \geq c_0 > 0$, where the constant c_0 depends on the initial data, we define

$$I(t) := \|B\|^2_{H^3(\Gamma_t)} + \|p\|^2_{H^3(\Omega_t)} + \|v\|^2_{H^4(\Omega_t)} + \|H\|^2_{H^4(\Omega_t)} + 1,$$

for $t \geq 0$. The curvature bound can recover the regularity of the free boundary (see, e.g., [SZ08a, LX23a]), and as a result, we can apply Proposition 6.1. It turns out that

$$\frac{d}{dt} I(t) \leq C\bar{E}(t) I(t) \leq C\bar{E}(0) I(t),$$

in a short period, and we can obtain a lower bound of T_0. To ensure that the height function is well-defined, we use the fundamental theorem of calculus and the fact that $\partial_t h = v_n$.

1.4.6. *Organization of the paper.* The rest of this paper is organized as follows.

In Section 2, we calculate the commutators (e.g., Lemmas 2.3 and 2.4, and (2.14)), the error terms (e.g., Lemmas 2.6 and 2.8), and additional terms associated with the div-curl estimates (e.g., Lemmas 2.7 and 2.10) in order to establish the energy estimates. These calculations reveal that as the number of terms multiplied in the formula of a commutator increases, the total number of derivatives for all these terms decreases. This is extremely crucial in closing the energy estimates.

In Section 3, we compute the time derivative of the energy functional by canceling out the leading terms. Under the a priori assumptions, we control $\frac{d}{dt}\bar{e}(t)$ using $\left(1 + \|\nabla^2 p\|^2_{L^2(\Omega_t)}\right) \bar{E}(t)$ in conjunction with some error terms. We further assume that $\sup\limits_{0 \leq t < T} E_{l-1}(t) \leq C$ for $l \geq 4$ to demonstrate that $\frac{d}{dt} e_l(t)$ can be bounded by $E_l(t)$ along with some higher-order error terms using the induction arguments.

In Section 4, we will show that $\|p\|_{H^3(\Omega_t)}$ can be uniformly bounded within the time interval of existence by the time $T > 0$, the a priori assumptions $\mathcal{N}_T, \mathcal{M}_T$, and the initial data $\|v_0\|_{H^6(\Omega_0)}$, $\|H_0\|_{H^6(\Omega_0)}$, $\|\mathcal{A}\|_{H^5(\Gamma_0)}$. Furthermore, the initial data can additionally control the initial quantities $\bar{E}(0)$ and $\|\mathcal{D}_t^{3-k} p\|^2_{H^{\frac{3}{2}k+1}(\Omega_0)}$ for $0 \leq k \leq 3$.

In Section 5, we utilize the Kato-Ponce inequalities to estimate the error terms appeared in Section 3. Due to the shared order of derivatives in the error formulas for the product terms, these error terms can be bounded by the product

$$\left(1 + \|\nabla p\|^2_{H^2(\Omega_t)}\right) \bar{E}(t),$$

and the energy $E_l(t)$ for $l \geq 4$, respectively.

In Section 6, we apply the div-curl estimates to close the energy estimates by incorporating another energy functional that includes the curl of the velocity and magnetic fields.

Finally, we prove our main theorem in Section 7.

2. Formulas for the Energy Estimates

All notation will be defined as it is introduced, and a list of symbols will also be provided in Appendix B for a quick reference. Throughout the paper, we will use the Einstein summation convention and the notation $S \star T$ from [Ham82, Man02] to denote a tensor formed by contracting certain indices of tensors S and T with constant coefficients. In particular, for $k, l \in \mathbb{N}$, we denote $\nabla^k f \star \nabla^l g$ a contraction of certain indices of tensors $\nabla^i f$ and $\nabla^j g$ for $0 \leq i \leq k$ and $0 \leq j \leq l$ with constant coefficients. Here, we denote $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the positive integers and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ for the non-negative integers. Note that f and g can be vector fields, and we include the lower-order derivatives along with the function (or vector field) itself. However, we exclude the case of a single term $\nabla^i f$.

Let $u : \Gamma \to \mathbb{R}$ and $F : \Gamma \to \mathbb{R}^3$ be any sufficiently regular function and vector field, respectively. Since the reference hypersurface Γ is embedded in $\mathbb{R}^3$, it has a natural metric g induced by the Euclidean metric. Then, (Γ, g) is a Riemannian manifold, and we denote the Riemannian connection on Γ by $\hat{\nabla}$. For a function $u \in C^\infty(\Gamma)$ and a vector field F, it holds $\hat{\nabla}_F u = Fu$.

We denote the normal part of F on Γ by $F_n := F \cdot \nu_\Gamma$, and the tangential part by $F_\sigma := F - F_n \nu_\Gamma$, where "$\cdot$" denotes the inner product of two vectors. If Γ is smooth, we can extend both u and F to

$\mathbb{R}^3$. Then, we define the tangential differential of u by $\bar{\nabla}u := (\nabla u)_\sigma = \nabla u - (\nabla u \cdot \nu)\nu$, the tangential gradient of F by $\bar{\nabla}F := \nabla F - (\nabla F\nu) \otimes \nu$, i.e., $(\bar{\nabla}F)_{ij} = \partial_j F^i - \partial_l F^i \nu^l \nu_j$, and the tangential divergence by $\operatorname{div}_\sigma F := \operatorname{Tr}(\bar{\nabla}F)$, where Tr is the trace of a square matrix. It is easy to verify that $\operatorname{div}_\sigma F = \partial_j F^j - \partial_l F^j \nu^l \nu_j$. We remark that the tangential gradient and covariant gradient of u are equivalent in the following sense: for any vector field $\tilde{F} : \Gamma \to \mathbb{R}^3$, $\tilde{F}\cdot\nu = 0$, we have $\hat{\nabla}_{\tilde{F}} u = \bar{\nabla}u \cdot \tilde{F}$. Additionally, the second fundamental form and the mean curvature can be written as

$$B = \bar{\nabla}\nu, \text{ and } \mathcal{A} = \operatorname{div}_\sigma \nu. \tag{2.1}$$

We also recall the divergence theorem

$$\int_\Gamma \operatorname{div}_\sigma F dS = \int_\Gamma \mathcal{A}_\Gamma (F \cdot \nu_\Gamma) dS.$$

The Beltrami-Laplacian of u is defined by $\Delta_B u := \operatorname{div}_\sigma(\bar{\nabla}u)$, and it holds

$$\Delta_B u = \Delta u - \left(\nabla^2 u\nu \cdot \nu\right) - \mathcal{A}\partial_\nu u, \tag{2.2}$$

where ∂_ν denotes the outer normal derivative.

We will fix our reference surface Γ, which is a boundary of a smooth, compact set Ω satisfying the interior and exterior ball condition with radius $\mathcal{R}$. We denote the tubular neighborhood of Γ by $U(\mathcal{R},\Gamma)$, given by $U(\mathcal{R},\Gamma) = \{x \in \mathbb{R}^3 : \operatorname{dist}(x,\Gamma) < \mathcal{R}\}$. Then the map $\Psi : \Gamma \times (-\mathcal{R},\mathcal{R}) \to U(\mathcal{R},\Gamma)$ defined as $\Psi(x,s) = x + s\nu_\Gamma(x)$ is a diffeomorphism. We say that a hypersurface $\Gamma_t = \partial\Omega_t$ (or Ω_t) is $H^s(\Gamma)$-regular, if it can be written as $\Gamma_t = \{x + h(x,t)\nu_\Gamma(x) : x \in \Gamma\}$, for a $H^s(\Gamma)$-regular function $h(\cdot,t) : \Gamma \to \mathbb{R}$ with $\|h(\cdot,t)\|_{L^\infty(\Gamma)} < \mathcal{R}$. We say that Γ_t is uniformly $H^s(\Gamma)$-regular if the height-function satisfies $\|h(\cdot,t)\|_{H^s(\Gamma)} \le C$ and $\|h(\cdot,t)\|_{L^\infty(\Gamma)} \le c\mathcal{R}$ for constants C and $c < 1$ (see [JLM22] for similar definitions).

We can express the unit outer normal and the second fundamental form by the tangential derivative of the height function (cf. [Man11])

$$\nu_{\Gamma_t} = a_1\left(h(\cdot,t), \bar{\nabla}h(\cdot,t)\right), \tag{2.3}$$
$$B_{\Gamma_t} = a_2\left(h(\cdot,t), \bar{\nabla}h(\cdot,t)\right)\bar{\nabla}^2 h(\cdot,t), \tag{2.4}$$

where a_1 and a_2 are smooth functions.

Let us next fix the notation for the function spaces. We define the Sobolev space $W^{l,p}(\Omega)$ ($W^{l,p}(\Gamma)$) in a standard way for $p \in [1,\infty]$ by ∇ ($\hat{\nabla}$ or $\bar{\nabla}$), and denote the Hilbert space $H^l = W^{l,2}$. We define the space $H^{\frac{1}{2}}(\Gamma)$ via the harmonic extension: let $u \in L^2(\Gamma)$,

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)} := \|u\|_{L^2(\Gamma)} + \inf\left\{\|\nabla w\|_{L^2(\Omega)} : w \in H^1(\Omega) \text{ and } w = u \text{ on } \Gamma\right\}.$$

The spaces H^{-1} and $H^{-\frac{1}{2}}$ are defined by duality. For any index vector $\alpha = (\alpha)_{i=1}^k \in \mathbb{N}_0^k$, we define its norm by $|\alpha| = \sum_{i=1}^k \alpha_i$.

We extend the unit outer normal ν to Ω using harmonic extension and denote it as $\tilde{\nu}$. With a slight abuse of notation, we sometimes still denote the extended one as ν. From (1.8) and (2.3), we see that $\|\tilde{\nu}\|_{H^{\frac{5}{2}+\delta}(\Omega_t)} \le C$.

Now, we recall the following results including some basic commutator formulas. As usual, we use the Lie bracket to represent the commutators, i.e., $[\mathcal{L}_1, \mathcal{L}_2] := \mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1$.

Lemma 2.1. *For a smooth function f, it holds*

$$[\mathcal{D}_t, \nabla]f = -(\nabla v)^\top \nabla f, \quad [\mathcal{D}_t, \partial_i]f = -\partial_i v^k \partial_k f, \quad [\mathcal{D}_t, \bar{\nabla}]f = -(\bar{\nabla}v)^\top \bar{\nabla} f,$$
$$[\mathcal{D}_t, \bar{\nabla}^2]f = \bar{\nabla}^2 v \star \bar{\nabla}f + \bar{\nabla}v \star \bar{\nabla}^2 f, \quad \mathcal{D}_t\nu = -(\bar{\nabla}v)^\top\nu = -\bar{\nabla}v_n + Bv_\sigma,$$
$$\bar{\nabla}v_n = \bar{\nabla}v^\top\nu + B_\Gamma v_\sigma, \quad [\partial_\nu, \partial_k]u = -\nabla u \cdot \partial_k\nu, \quad [\bar{\nabla}, \nabla]f = \nabla f \star \nabla\nu \star \nu,$$
$$[\mathcal{D}_t, \Delta_B]f = \bar{\nabla}^2 f \star \nabla v - \bar{\nabla}f \cdot \Delta_B v + B \star \nabla v \star \bar{\nabla}f,$$
$$\mathcal{D}_t B = -\bar{\nabla}^2 v \star \nu - \bar{\nabla}v \star B.$$

Proof. Most of the above formulas can be found in [SZ08b, Section 3.1] and the others follows from direct calculations. □

From the definition of curl, namely, $\operatorname{curl} F = \nabla F - (\nabla F)^\top$, a straightforward calculation yields the following lemma.

Lemma 2.2. *Let F and G be smooth vector fields. Then we have*

$$\operatorname{curl}(F\cdot\nabla G) = \nabla G\nabla F - (\nabla F)^\top(\nabla G)^\top + (F\cdot\nabla)(\operatorname{curl} G),$$
$$[\mathcal{D}_t, \operatorname{curl}]F = (\nabla v)^\top(\nabla F)^\top - \nabla F\nabla v.$$

Let $l, k \in \mathbb{N}$ and let f be a smooth function. Then we have

$$[\mathcal{D}_t^{l+1}, \nabla^k]f = \mathcal{D}_t[\mathcal{D}_t^l, \nabla^k]f + [\mathcal{D}_t, \nabla^k]\mathcal{D}_t^l f,$$
$$[\mathcal{D}_t^l, \nabla^{k+1}]f = [\mathcal{D}_t^l, \nabla]\nabla^k f + \nabla[\mathcal{D}_t^l, \nabla^k]f.$$

To derive a general formula for the commutators, we need to apply the following results. It is easy to verify that

$$\mathcal{D}_t a(\nu) = b(\nu)\bar\nabla v, \quad \mathcal{D}_t\nabla\mathcal{D}_t^k v = \nabla\mathcal{D}_t^{k+1}v + \nabla v\star\nabla\mathcal{D}_t^k v,$$
$$\mathcal{D}_t\bar\nabla\mathcal{D}_t^k v = \bar\nabla\mathcal{D}_t^{k+1}v + \bar\nabla v\star\bar\nabla\mathcal{D}_t^k v,$$

for $k\in\mathbb{N}$, where $a(\nu)$ and $b(\nu)$ denote the finite $\star$ product of ν.

As we shall see below, the commutator formulas involve the product of functions, and the total order of derivatives decreases as more terms are multiplied. Thanks to this observation, we can control the errors in Section 5 and establish the energy estimates in Section 6.

Lemma 2.3. *Let $l, k\in\mathbb{N}$ such that $l\geq 2$ and $k\geq 3$. It holds*

$$[\mathcal{D}_t, \nabla^2]f = \nabla v\star\nabla^2 f + \nabla^2 v\star\nabla f,$$
$$[\mathcal{D}_t, \nabla^k]f = \sum_{|\alpha|\leq k-1}\nabla^{1+\alpha_1}v\star\nabla^{1+\alpha_2}f,$$
$$[\mathcal{D}_t^l, \nabla]f = \sum_{2\leq m\leq l+1}\ \sum_{|\beta|\leq l+1-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}f,$$

and

$$[\mathcal{D}_t^l, \nabla^2]f = \sum_{2\leq m\leq l+1}\ \sum_{\substack{|\alpha|\leq 1\\ |\beta|\leq l+1-m}}\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_{m-1}}\mathcal{D}_t^{\beta_{m-1}}v \star\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}f.$$

Roughly speaking, the leading term is $\nabla^k\mathcal{D}_t^{l-1}$ in the commutator $[\mathcal{D}_t^l, \nabla^k]$.

Proof. A direct calculation yields the first claim and the second claim can be found in [JLM22, Lemma 4.1].

We prove the third one by induction, and it is easy to verify the case of $l=2$. For the case of $l\geq 3$, from Lemma 2.2 and the above formulas, it follows that

$$\begin{aligned}
&[\mathcal{D}_t^l, \nabla]f\\
=&\mathcal{D}_t[\mathcal{D}_t^{l-1}, \nabla]f + \nabla v\star\nabla\mathcal{D}_t^{l-1}f\\
=&\mathcal{D}_t\Big(\sum_{2\leq m\leq l}\ \sum_{|\beta|\leq l-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}f\Big) + \nabla v\star\nabla\mathcal{D}_t^{l-1}f\\
=&\nabla v\star\nabla\mathcal{D}_t^{l-1}f + \sum_{2\leq m\leq l}\ \sum_{|\beta|\leq l-m}\nabla\mathcal{D}_t\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}f\\
&+\cdots+\sum_{2\leq m\leq l}\ \sum_{|\beta|\leq l-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t\mathcal{D}_t^{\beta_m}f
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{2\le m\le l}\sum_{|\beta|\le l-m}\nabla v\star\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}f\\
&+\cdots+\sum_{2\le m\le l}\sum_{|\beta|\le l-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla v\star\nabla\mathcal{D}_t^{\beta_m}f\\
=&\sum_{2\le m\le l+1}\sum_{|\beta|\le l+1-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}f.
\end{aligned}
$$

Finally, we prove the last claim. Again by induction, for $l\ge 3$, one has

$$
\begin{aligned}
&[\mathcal{D}_t^l,\nabla^2]f\\
=&\mathcal{D}_t[\mathcal{D}_t^{l-1},\nabla^2]f+[\mathcal{D}_t,\nabla^2]\mathcal{D}_t^{l-1}f\\
=&\mathcal{D}_t\Big(\sum_{2\le m\le l}\sum_{\substack{|\alpha|\le 1\\|\beta|\le l-m}}\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_{m-1}}\mathcal{D}_t^{\beta_{m-1}}v\star\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}f\Big)\\
&+\nabla v\star\nabla^2\mathcal{D}_t^{l-1}f+\nabla^2v\star\nabla\mathcal{D}_t^{l-1}f\\
=&\sum_{2\le m\le l+1}\sum_{\substack{|\alpha|\le 1\\|\beta|\le l+1-m}}\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_{m-1}}\mathcal{D}_t^{\beta_{m-1}}v\star\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}f.
\end{aligned}
$$

□

Below, let $a_\beta(\nu)$ and $a_{\alpha,\beta}(\nu,B)$ denote the finite $\star$ product of the tensors. The following lemma is critical for our estimates, in which we provide a more precise formulation of the quantities than those in [JLM22, Lemma 4.2].

Lemma 2.4. *Let $l\ge 1$, we have*

$$
\begin{aligned}
[\mathcal{D}_t^l,\bar\nabla]f&=\sum_{2\le m\le l+1}\sum_{|\beta|\le l+1-m}\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla\mathcal{D}_t^{\beta_m}f,\\
\mathcal{D}_t^l\nu&=\sum_{1\le m\le l}\sum_{|\beta|\le l-m}a_\beta(\nu)\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_m}v,\\
\mathcal{D}_t^lB&=\sum_{1\le m\le l}\sum_{\substack{|\beta|\le l-m\\|\alpha|\le 1}}a_{\alpha,\beta}(\nu,B)\bar\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v,
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
[\mathcal{D}_t^l,\bar\nabla^2]f=\sum_{2\le m\le l+1}\sum_{\substack{|\beta|\le l+1-m\\|\alpha|\le 1}}&a_{\alpha,\beta}(\nu,B)\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\\
&\star\nabla^{1+\alpha_{m-1}}\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}f.
\end{aligned}
$$

Proof. To prove the first claim, we recall $[\mathcal{D}_t,\bar\nabla]f=-(\bar\nabla v)^\top\bar\nabla f$ in Lemma 2.1. For the case of $l\ge 2$, we have by induction that

$$
\begin{aligned}
[\mathcal{D}_t^l,\bar\nabla]f&=\mathcal{D}_t[\mathcal{D}_t^{l-1},\bar\nabla]f+[\mathcal{D}_t,\bar\nabla]\mathcal{D}_t^{l-1}f\\
&=\mathcal{D}_t\Big(\sum_{2\le m\le l}\sum_{|\beta|\le l-m}\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla\mathcal{D}_t^{\beta_m}f\Big)\\
&\quad+\bar\nabla v\star\bar\nabla\mathcal{D}_t^{l-1}f\\
&=\sum_{2\le m\le l+1}\sum_{|\beta|\le l+1-m}\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla\mathcal{D}_t^{\beta_m}f.
\end{aligned}
$$

Similarly, we can obtain the last claim.

For the second claim, we recall $\mathcal{D}_t\nu = \bar\nabla v \star \nu$, and for $l \geq 2$, it holds by induction that

$$\begin{aligned}\mathcal{D}_t^l\nu =& \mathcal{D}_t\Big(\sum_{1\leq m\leq l-1}\sum_{|\beta|\leq l-1-m} a_\beta(\nu)\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_m}v\Big)\\ =& \sum_{1\leq m\leq l}\sum_{|\beta|\leq l-m} a_\beta(\nu)\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_m}v.\end{aligned}$$

As for the third claim, we have for $l \geq 1$ that

$$\begin{aligned}\mathcal{D}_t^l B =& [\mathcal{D}_t^l, \bar\nabla]\nu + \bar\nabla\mathcal{D}_t^l\nu\\ =& \bar\nabla\Big(\sum_{1\leq m\leq l}\sum_{|\beta|\leq l-m} a_\beta(\nu)\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_m}v\Big)\\ &+\sum_{2\leq m\leq l+1}\sum_{|\beta|\leq l+1-m}\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla\mathcal{D}_t^{\beta_m}\nu\\ =:& I_1 + I_2.\end{aligned}$$

It is clear that

$$I_1 = \sum_{1\leq m\leq l}\sum_{\substack{|\beta|\leq l-m\\ |\alpha|\leq 1}} a_{\alpha,\beta}(\nu, B)\bar\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v.$$

For I_2, it follows that

$$\begin{aligned}I_2 =& \sum_{2\leq m\leq l+1}\sum_{|\beta|\leq l+1-m}\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\\ &\star\Big(\sum_{1\leq n\leq\beta_m}\sum_{\substack{|\lambda|\leq\beta_m-n\\ |\gamma|\leq 1}}\bar\nabla^{1+\gamma_1}\mathcal{D}_t^{\lambda_1}v\star\cdots\star\bar\nabla^{1+\gamma_n}\mathcal{D}_t^{\lambda_n}v\Big)\\ =& \sum_{\substack{2\leq m\leq l+1\\ |\beta|\leq l+1-m}}\sum_{1\leq n\leq\beta_m}\sum_{\substack{|\lambda|\leq\beta_m-n\\ |\gamma|\leq 1}} a_{\beta,\lambda,\gamma}(\nu,B)\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\\ &\qquad\star\bar\nabla^{1+\gamma_1}\mathcal{D}_t^{\lambda_1}v\star\cdots\star\bar\nabla^{1+\gamma_n}\mathcal{D}_t^{\lambda_n}v,\end{aligned}$$

which is also contained in

$$\sum_{1\leq m\leq l}\sum_{|\alpha|\leq 1, |\beta|\leq l-m} a_{\alpha,\beta}(\nu, B)\bar\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v.$$

□

To estimate energy, it is necessary to determine the order of the material and spatial derivatives that appear in the time derivative of the energy functional. Additionally, we will consider the magnetic field. We denote the divergence of a matrix $A = (A_{ij})$ as $(\operatorname{div} A)_i := \sum_j \partial_j A_{ij}$ and recall that the curl of a vector field F is defined by $\operatorname{curl} F = \nabla F - (\nabla F)^\top$.

We begin with the following basic results. By the divergence-free condition, it is clear that $\operatorname{div}\mathcal{D}_t v = \partial_i v^j\partial_j v^i$ and

$$-\Delta p = \partial_i v^j\partial_j v^i - \partial_i H^j\partial_j H^i. \tag{2.6}$$

The operators $\operatorname{curl}\mathcal{D}_t$ and $\mathcal{D}_t\operatorname{curl}$ can be represented in terms of spatial derivatives of lower orders, suggesting that the curl operator should not be regarded merely as the gradient when considering the velocity and magnetic fields. In fact, a direct calculation produces the following identities.

Lemma 2.5. *For the velocity and magnetic fields, we have*

$$\begin{aligned}\operatorname{curl}\mathcal{D}_t v =& (\nabla H)^\top\operatorname{curl} H + \operatorname{curl} H\nabla H + (H\cdot\nabla)(\operatorname{curl} H),\\ \mathcal{D}_t\operatorname{curl} v =& -(\nabla v)^\top\operatorname{curl} v - \operatorname{curl} v\nabla v + (\nabla H)^\top\operatorname{curl} H + \operatorname{curl} H\nabla H\\ &+ (H\cdot\nabla)(\operatorname{curl} H),\\ [\mathcal{D}_t, \operatorname{curl}]v =& -(\nabla v)^\top\operatorname{curl} v - \operatorname{curl} v\nabla v,\end{aligned}$$

$$
\begin{aligned}
\operatorname{curl} \mathcal{D}_t H &= \nabla v \nabla H - (\nabla H)^\top (\nabla v)^\top + (H \cdot \nabla)(\operatorname{curl} v), \\
\mathcal{D}_t \operatorname{curl} H &= (\nabla v)^\top (\nabla H)^\top - \nabla H \nabla v + \nabla v \nabla H - (\nabla H)^\top (\nabla v)^\top \\
&\quad + (H \cdot \nabla)(\operatorname{curl} v), \\
[\mathcal{D}_t, \operatorname{curl}] H &= (\nabla v)^\top (\nabla H)^\top - \nabla H \nabla v.
\end{aligned}
$$

Next, due to the presence of the magnetic tension $H \cdot \nabla H$ in contrast to the Euler equations, we introduce some error terms associated with the magnetic field. These will be necessary for the subsequent computation of the quantities $\operatorname{curl} \mathcal{D}_t^{k+1} v$, $[\mathcal{D}_t^{k+1}, \nabla] p$, and $-\Delta \mathcal{D}_t^{k+1} p$.

Denote $R^0_{\nabla H, H} := 0$, $R^0_{\nabla H, \nabla H} := \nabla H \star \nabla H$, and we define

$$
\begin{aligned}
R^k_{\nabla H, H} := \sum_{3 \le m \le k+2} \sum_{\substack{|\alpha| \le 1 \\ |\beta| \le k+2-m}} & a_{\alpha,\beta}(\nabla v) \nabla^{1+\alpha_1} \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_{m-2}} \mathcal{D}_t^{\beta_{m-2}} v \\
& \star \nabla^{\alpha_{m-1}} H \star H, \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
R^k_{\nabla H, \nabla H} := \sum_{3 \le m \le k+2} \sum_{\substack{|\alpha| \le 2, \alpha_i \le 1 \\ |\beta| \le k+2-m}} & \nabla^{1+\alpha_1} \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_{m-2}} \mathcal{D}_t^{\beta_{m-2}} v \\
& \star \nabla^{\alpha_{m-1}} H \star \nabla^{\alpha_m} H, \qquad (2.8)
\end{aligned}
$$

for $k \ge 1$, where $a_{\alpha,\beta}(\nabla v) = \nabla v \star \nabla v \star \cdots \star \nabla v$ denotes the finite $\star$ product. In the case of $\beta_j = 0$, $\nabla \mathcal{D}_t^{\beta_j}$ can be absorbed into $a_{\alpha,\beta}(\nabla v)$.

A direct calculation shows $\mathcal{D}_t(\nabla H \star \nabla H) = \nabla^2 v \star H \star \nabla H + \nabla v \star \nabla H \star \nabla H$ and $\mathcal{D}_t(\nabla H \star H) = \nabla^2 v \star H \star H + \nabla v \star \nabla H \star H$, and the following is some results for higher-order material derivatives.

Lemma 2.6. *Let $k \in \mathbb{N}$. It follows that*

$$
\mathcal{D}_t^k (\nabla H \star \nabla H) = R^k_{\nabla H, \nabla H}, \quad \mathcal{D}_t^k (\nabla H \star H) = R^k_{\nabla H, H}.
$$

Proof. It is sufficient to consider the case of $k \ge 2$. We claim that given any $k \ge 2$, one has

$$
\begin{aligned}
\mathcal{D}_t^k (\nabla H \star \nabla H) = \sum_{2 \le m \le k+2} \sum_{|\beta| \le k+2-m} & \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-2}} v \star \nabla \mathcal{D}_t^{\beta_{m-1}} H \\
& \star \nabla \mathcal{D}_t^{\beta_m} H, \\
\mathcal{D}_t^k (\nabla H \star H) = \sum_{2 \le m \le k+2} \sum_{|\beta| \le k+2-m} & \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-2}} v \star \nabla \mathcal{D}_t^{\beta_{m-1}} H \\
& \star \mathcal{D}_t^{\beta_m} H.
\end{aligned}
$$

In fact, from Lemma 2.3, we see that

$$
\begin{aligned}
& \mathcal{D}_t^k (\nabla H \star \nabla H) \\
&= \nabla \mathcal{D}_t^k H \star \nabla H + [\mathcal{D}_t^k, \nabla] H \star \nabla H + \sum_{|\gamma| = k, \gamma_1, \gamma_2 \ge 1} [\mathcal{D}_t^{\gamma_1}, \nabla] H \star [\mathcal{D}_t^{\gamma_2}, \nabla] H \\
&\quad + \nabla \mathcal{D}_t^{\gamma_1} H \star [\mathcal{D}_t^{\gamma_2}, \nabla] H + \nabla \mathcal{D}_t^{\gamma_1} H \star \nabla \mathcal{D}_t^{\gamma_2} H \\
&= \nabla \mathcal{D}_t^k H \star \nabla H \\
&\quad + \sum_{2 \le m \le k+1} \sum_{|\beta| \le k+1-m} \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-1}} v \star \nabla \mathcal{D}_t^{\beta_m} H \star \nabla H \\
&\quad + \sum_{2 \le m \le k+2} \sum_{|\beta| \le k+2-m} \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-2}} v \star \nabla \mathcal{D}_t^{\beta_{m-1}} H \star \nabla \mathcal{D}_t^{\beta_m} H \\
&= \sum_{2 \le m \le k+2} \sum_{|\beta| \le k+2-m} \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-2}} v \star \nabla \mathcal{D}_t^{\beta_{m-1}} H \star \nabla \mathcal{D}_t^{\beta_m} H,
\end{aligned}
$$

and

$$
\mathcal{D}_t^k (\nabla H \star H)
$$

$$
\begin{aligned}
&= \nabla\mathcal{D}_t^k H \star H + [\mathcal{D}_t^k, \nabla] H \star H + \mathcal{D}_t^k H \star \nabla H + \sum_{|\gamma|=k, \gamma_i\geq 1} [\mathcal{D}_t^{\gamma_1}, \nabla] H \star \mathcal{D}_t^{\gamma_2} H \\
&\quad + \nabla\mathcal{D}_t^{\gamma_1} H \star \mathcal{D}_t^{\gamma_2} H \\
&= \nabla\mathcal{D}_t^k H \star H + \mathcal{D}_t^k H \star \nabla H \\
&\quad + \sum_{2\leq m\leq k+1} \sum_{|\beta|\leq k+1-m} \nabla\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla\mathcal{D}_t^{\beta_{m-1}} v \star \nabla\mathcal{D}_t^{\beta_m} H \star H \\
&\quad + \sum_{2\leq m\leq k+2} \sum_{|\beta|\leq k+2-m} \nabla\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla\mathcal{D}_t^{\beta_{m-2}} v \star \nabla\mathcal{D}_t^{\beta_{m-1}} H \star \mathcal{D}_t^{\beta_m} H \\
&= \sum_{2\leq m\leq k+2} \sum_{|\beta|\leq k+2-m} \nabla\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla\mathcal{D}_t^{\beta_{m-2}} v \star \nabla\mathcal{D}_t^{\beta_{m-1}} H \star \mathcal{D}_t^{\beta_m} H.
\end{aligned}
$$

By substituting $\mathcal{D}_t H = H \cdot \nabla v$, we remove the material derivatives of the magnetic field. By induction, it is readily verified that

$$
\mathcal{D}_t^j H = \sum_{1\leq m\leq j} \sum_{|\beta|\leq j-m} \nabla\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla\mathcal{D}_t^{\beta_m} v \star H, \tag{2.9}
$$

$$
\begin{aligned}
\nabla^i\mathcal{D}_t^j H = \sum_{1\leq m\leq j} \sum_{\substack{|\alpha|\leq i \\ |\beta|\leq j-m}} & \nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m} v \\
& \star \nabla^{\alpha_{m+1}} H,
\end{aligned} \tag{2.10}
$$

where $i, j \in \mathbb{N}$. These conclude the proof of the lemma. □

The above lemma shows that $\mathcal{D}_t^k(H\cdot\nabla H) = R^k_{\nabla H, H}$. Due to the divergence-free condition, it can be shown that taking the divergence does not increase the order of derivatives. This observation is crucial for establishing the validity of Proposition 3.1 when dealing with $-\Delta\mathcal{D}_t^{k+1} p$.

Lemma 2.7. *We have* $\operatorname{div}\mathcal{D}_t(H\cdot\nabla H) = \nabla^2 v \star \nabla H \star H + \nabla v \star \nabla H \star \nabla H + \nabla^2 H \star \nabla v \star H$, *and it holds*

$$
\operatorname{div}\mathcal{D}_t^k(H\cdot\nabla H) = \partial_j\partial_l\mathcal{D}_t^{k-1} v^i H^l \partial_i H^j + \nabla^3\mathcal{D}_t^{k-2} v \star \nabla v \star H \star H + \mathrm{L.\,O.\,T.},
$$

for any integer $k \geq 2$, *where* L. O. T. *stands for lower-order terms.*

Proof. By Lemma 2.1, a direct calculation gives

$$
\begin{aligned}
\operatorname{div}\mathcal{D}_t(H\cdot\nabla H) &= \partial_j(\mathcal{D}_t\partial_i H^j H^i + \partial_i H^j \mathcal{D}_t H^i) \\
&= \partial_j([\mathcal{D}_t, \partial_i] H^j H^i + \partial_i\mathcal{D}_t H^j H^i + \partial_i H^j \partial_k v^i H^k) \\
&= \nabla^2 v \star \nabla H \star H + \nabla v \star \nabla H \star \nabla H + \nabla^2 H \star \nabla v \star H.
\end{aligned}
$$

For $k \geq 2$, the condition $\operatorname{div} H = 0$ implies that $\partial_j\mathcal{D}_t^\gamma\partial_i H^j = [\partial_j, \mathcal{D}_t^\gamma]\partial_i H^j$, and therefore

$$
\begin{aligned}
&\operatorname{div}\mathcal{D}_t^k(H\cdot\nabla H) \\
&= \partial_j(\mathcal{D}_t^k\partial_i H^j H^i) + \partial_j(\partial_i H^j \mathcal{D}_t^k H^i) + \partial_j\Big(\sum_{|\gamma|=k, \gamma_i<k} \mathcal{D}_t^{\gamma_1}\partial_i H^j \mathcal{D}_t^{\gamma_2} H^i\Big) \\
&= \partial_j\mathcal{D}_t^k\partial_i H^j H^i + \mathcal{D}_t^k\partial_i H^j \partial_j H^i + \partial_i H^j \partial_j\mathcal{D}_t^k H^i \\
&\quad + \mathcal{D}_t^{\gamma_1}\nabla H \star \nabla\mathcal{D}_t^{\gamma_2} H \\
&= \partial_j\mathcal{D}_t^k H^i \partial_i H^j + [\partial_j, \mathcal{D}_t^k]\partial_i H^j H^i + [\mathcal{D}_t^k, \nabla] H \star \nabla H + \nabla\mathcal{D}_t^{\gamma_1} H \star \nabla\mathcal{D}_t^{\gamma_2} H \\
&\quad + [\mathcal{D}_t^{\gamma_1}, \nabla] H \star \nabla\mathcal{D}_t^{\gamma_2} H + \sum_{|\gamma|=k, \gamma_i<k} [\partial_j, \mathcal{D}_t^{\gamma_1}]\partial_i H^j \mathcal{D}_t^{\gamma_2} H^i.
\end{aligned}
$$

In the above, it suffices to consider the most challenging term, i.e., $\partial_j\mathcal{D}_t^k H^i \partial_i H^j$. Note that

$$
\partial_j\mathcal{D}_t^k H^i = \partial_j\mathcal{D}_t^{k-1}(\partial_l v^i H^l)
$$

$$= \partial_j \partial_l \mathcal{D}_t^{k-1} v^i H^l + \sum_{|\gamma|=k-1, \gamma_1<k-1} \partial_j \partial_l \mathcal{D}_t^{\gamma_1} v^i \mathcal{D}_t^{\gamma_2} H^l + \sum_{|\gamma|=k-1} \partial_j [\mathcal{D}_t^{\gamma_1}, \partial_l] v^i \mathcal{D}_t^{\gamma_2} H^l,$$

and we find that

$$\begin{aligned}
&\operatorname{div} \mathcal{D}_t^k (H \cdot \nabla H) \\
&= \partial_j \partial_l \mathcal{D}_t^{k-1} v^i H^l \partial_i H^j + [\nabla, \mathcal{D}_t^k] \nabla H \star H + [\mathcal{D}_t^k, \nabla] H \star \nabla H \\
&\quad + \sum_{\substack{|\gamma|=k \\ \gamma_i<k}} [\nabla, \mathcal{D}_t^{\gamma_1}] \nabla H \star \mathcal{D}_t^{\gamma_2} H + \nabla \mathcal{D}_t^{\gamma_1} H \star \nabla \mathcal{D}_t^{\gamma_2} H + [\mathcal{D}_t^{\gamma_1}, \nabla] H \star \nabla \mathcal{D}_t^{\gamma_2} H \\
&\quad + \sum_{\substack{|\gamma|=k-1 \\ \gamma_1<k-1}} \nabla^2 \mathcal{D}_t^{\gamma_1} v \star \mathcal{D}_t^{\gamma_2} H \star \nabla H + \sum_{|\gamma|=k-1} \nabla [\mathcal{D}_t^{\gamma_1}, \nabla] v \star \mathcal{D}_t^{\gamma_2} H \star \nabla H \\
&=: \partial_j \partial_l \mathcal{D}_t^{k-1} v^i H^l \partial_i H^j + R.
\end{aligned}$$

Here, the highest-order term in R is $\nabla^2 \mathcal{D}_t^{k-1} H \star \nabla v \star H$, resulting from $[\nabla, \mathcal{D}_t^k]$ $\nabla H \star H$. To complete the proof, we replace the material derivative with the spatial derivative, resulting in $\nabla^3 \mathcal{D}_t^{k-2} v \star \nabla v \star H \star H$, along with lower-order terms as shown in (2.10). □

To derive the energy estimates, we need to apply the div-curl estimates. Accordingly, it is inevitable to compute $\operatorname{div} \mathcal{D}_t^l v, \operatorname{div} \mathcal{D}_t^l H, \operatorname{curl} \mathcal{D}_t^l v$, and $\operatorname{curl} \mathcal{D}_t^l v$. The following lemma is crucial for computing $\operatorname{curl} \mathcal{D}_t^l v$ (see Lemma 2.10). Additionally, it indicates that we should reserve the curl operator (not simply ∇) for the highest-order term, and seek opportunities to utilize Lemma 2.5.

Lemma 2.8. *It holds*

$$\mathcal{D}_t((H \cdot \nabla)(\operatorname{curl} H)) = \nabla^2 \operatorname{curl} v \star H \star H + \nabla^2 H \star \nabla v \star H + \nabla^2 v \star \nabla H \star H,$$

and

$$\begin{aligned}
&\mathcal{D}_t^k ((H \cdot \nabla) \operatorname{curl} H) \\
&= \nabla^{k+1} \operatorname{curl} H \star \underbrace{H \star \cdots \star H}_{k \text{ times}} + \sum_{\substack{|\alpha| \le k+2, \alpha_i \le k+1 \\ m \le k+2, F_j = v, H}} \nabla^{\alpha_1} F_1 \star \cdots \star \nabla^{\alpha_m} F_m \\
&\quad + \sum_{\substack{|\alpha|+|\beta| \le k+2 \\ \alpha_i+\beta_i \le k+1, \beta_i \le k-1 \\ m \le k+1, F_j = v, H}} \nabla^{\alpha_1} \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{\alpha_{k-1}} \mathcal{D}_t^{\beta_{k-1}} v \star \nabla^{\alpha_k} F_k \star \cdots \star \nabla^{\alpha_m} F_m,
\end{aligned}$$

if the integer $k \ge 2$ is even. For an odd integer $k \ge 3$, we replace $\nabla^{k+1} \operatorname{curl} H \star \underbrace{H \star \cdots \star H}_{k \text{ times}}$ by $\nabla^{k+1} \operatorname{curl} v \star \underbrace{H \star \cdots \star H}_{k \text{ times}}$ in the above identity.

Proof. First, we apply Lemma 2.5 to obtain

$$\begin{aligned}
\mathcal{D}_t[(H \cdot \nabla)(\operatorname{curl} H)] &= \mathcal{D}_t \partial_i \operatorname{curl} H H^i + \partial_i \operatorname{curl} H \mathcal{D}_t H^i \\
&= [\mathcal{D}_t, \partial_i] \operatorname{curl} H H^i + \partial_i \mathcal{D}_t \operatorname{curl} H H^i + \partial_i \operatorname{curl} H \partial_j v^i H^j \\
&= \nabla^2 \operatorname{curl} v \star H \star H + \nabla^2 H \star \nabla v \star H + \nabla^2 v \star \nabla H \star H.
\end{aligned}$$

In the case of $k = 2$, one has

$$\begin{aligned}
\mathcal{D}_t^2((H \cdot \nabla)(\operatorname{curl} H)) &= \mathcal{D}_t^2 \partial_i \operatorname{curl} H H^i + \partial_i \operatorname{curl} H \mathcal{D}_t^2 H^i + \mathcal{D}_t \partial_i \operatorname{curl} H \mathcal{D}_t H^i \\
&= \partial_i \mathcal{D}_t^2 \operatorname{curl} H H^i + [\mathcal{D}_t^2, \partial_i] \operatorname{curl} H H^i + \nabla^2 H \star \mathcal{D}_t^2 H \\
&\quad + \mathcal{D}_t \nabla^2 H \star \nabla v \star H \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We denote

$$I_1 = \nabla \mathcal{D}_t^2 \operatorname{curl} H \star H = (\nabla \operatorname{curl} \mathcal{D}_t(H\cdot\nabla v)) \star H + \nabla([\mathcal{D}_t^2, \nabla]H) \star H =: I_{11} + I_{12}.$$

Again from Lemma 2.5, it holds

$$\begin{aligned}\operatorname{curl}(H\cdot\nabla\mathcal{D}_t v) &= \nabla H \star \nabla\mathcal{D}_t v + (H\cdot\nabla)\operatorname{curl}\mathcal{D}_t v\\ &= \nabla H \star \nabla\mathcal{D}_t v + (H\cdot\nabla)(\nabla H\star\nabla H)\\ &\quad + (H\cdot\nabla)(H\cdot\nabla)\operatorname{curl} H\\ &= \nabla\mathcal{D}_t v\star\nabla H + \nabla^2\operatorname{curl} H\star H\star H + \nabla^2 H\star\nabla H\star H,\end{aligned}$$

and using Lemma 2.1, it follows that

$$\begin{aligned}I_{11} &= \nabla(\operatorname{curl}(\mathcal{D}_t H\cdot\nabla v))\star H + \nabla(\operatorname{curl}(H\cdot\mathcal{D}_t\nabla v))\star H\\ &= \nabla(\operatorname{curl}(H\cdot\nabla\mathcal{D}_t v))\star H + \nabla^2(H\star\nabla v\star\nabla v)\star H\\ &= \nabla^3\operatorname{curl} H\star H\star H + \nabla^2\mathcal{D}_t v\star\nabla H + \nabla\mathcal{D}_t v\star\nabla^2 H\\ &\quad + \sum_{\substack{|\alpha|\le 4,\alpha_i\le 3, m\le 4\\ F_1,\dots,F_m = v,H}} \nabla^{\alpha_1}F_1\star\cdots\star\nabla^{\alpha_m}F_m.\end{aligned}$$

Applying Lemma 2.3, we have

$$\begin{aligned}I_{12} &= \nabla^2\mathcal{D}_t v\star\nabla H\star H + \nabla\mathcal{D}_t v\star\nabla^2 H\star H + \nabla^2 H\star\nabla v\star\nabla v\star H\\ &\quad + \nabla^2 v\star\nabla H\star\nabla v\star H + \nabla^2 H\star\nabla v\star H + \nabla^2 v\star\nabla H\star H,\end{aligned}$$

and

$$\begin{aligned}I_2 &= \nabla\mathcal{D}_t v\star\nabla^2 H\star H + \nabla^3 v\star\nabla v\star H\star H + \nabla^2 H\star\nabla v\star\nabla v\star H\\ &\quad + \nabla^2 v\star\nabla H\star\nabla v\star H + \nabla^2 H\star\nabla v\star H.\end{aligned}$$

To control the last two terms, (2.9) implies that

$$I_3 = \nabla\mathcal{D}_t v\star\nabla^2 H\star H + \nabla^2 H\star\nabla v\star\nabla v\star H + \nabla^2 H\star\nabla v\star H,$$

and Lemma 2.3 together with (1.1) yields

$$\begin{aligned}I_4 &= [\mathcal{D}_t, \nabla^2]H\star\nabla v\star H + \nabla^2\mathcal{D}_t H\star\nabla v\star H\\ &= \nabla^3 v\star\nabla v\star H\star H + \nabla^2 H\star\nabla v\star\nabla v\star H + \nabla^2 v\star\nabla H\star\nabla v\star H.\end{aligned}$$

We arrive at the following

$$\begin{aligned}\mathcal{D}_t^2((H\cdot\nabla)\operatorname{curl} H) &= \nabla^3\operatorname{curl} H\star H\star H + \sum_{\substack{|\alpha|\le 4,\alpha_i\le 3\\ m\le 4, F_j = v,H}} \nabla^{\alpha_1}F_1\star\cdots\star\nabla^{\alpha_m}F_m\\ &\quad + \sum_{\substack{|\alpha|+|\beta|\le 4,\alpha_i+\beta_i\le 3\\ \beta_i\le 1, m\le 3, F_j = v,H}} \nabla^{\alpha_1}\mathcal{D}_t^{\beta_1}v\star\nabla^{\alpha_2}F_2\star\cdots\star\nabla^{\alpha_m}F_m\\ &=: J_1 + J_2 + J_3.\end{aligned}$$

As for $k = 3$, to calculate $\mathcal{D}_t J_1$, we only focus on the most difficult term. Actually, it holds

$$\begin{aligned}\mathcal{D}_t\nabla^3\operatorname{curl} H &= \nabla^3\mathcal{D}_t\operatorname{curl} H + [\mathcal{D}_t, \nabla^3]\operatorname{curl} H\\ &= \nabla^3(\nabla v\star\nabla H + (H\cdot\nabla)\operatorname{curl} v) + \sum_{|\alpha|\le 5,\alpha_i\le 4}\nabla^{\alpha_1}H\star\nabla^{\alpha_2}v\\ &= \nabla^4\operatorname{curl} v\star H + \sum_{|\alpha|\le 5,\alpha_i\le 4}\nabla^{\alpha_1}H\star\nabla^{\alpha_2}v,\end{aligned}$$

from Lemmas 2.3 and 2.5. With the help of Lemma 2.3, $\mathcal{D}_t J_2$ and $\mathcal{D}_t J_3$ can be treated in the same fashion. Therefore, we obtain

$$\mathcal{D}_t^3((H\cdot\nabla)\operatorname{curl} H)$$

$$= \nabla^4 \operatorname{curl} v \star H \star H \star H + \sum_{\substack{|\alpha|\le 5, \alpha_i \le 4\\ m\le 5, F_j = v, H}} \nabla^{\alpha_1} F_1 \star \cdots \star \nabla^{\alpha_m} F_m$$

$$+ \sum_{\substack{|\alpha|+|\beta|\le 5, \beta_i\le 2, \alpha_i+\beta_i\le 4\\ m\le 4, F_j=v,H}} \nabla^{\alpha_1}\mathcal{D}_t^{\beta_1} v \star \nabla^{\alpha_2}\mathcal{D}_t^{\beta_2} v \star \nabla^{\alpha_3} F_3 \star \cdots \star \nabla^{\alpha_m} F_m.$$

The other cases can be shown in the same way. □

From now on, we denote

$$R^0_{\nabla^2 H,H} := (H\cdot\nabla)\operatorname{curl} H, \quad R^k_{\nabla^2 H,H} := \mathcal{D}_t^k((H\cdot\nabla)\operatorname{curl} H), \quad k \ge 1. \tag{2.11}$$

We proceed to introduce another two types of error terms that are related to $\operatorname{div}\mathcal{D}_t^l v$, $\operatorname{curl}\mathcal{D}_t^l v$ and $[\mathcal{D}_t^{l+1}, \nabla]p$. The first one is written as the form

$$R^0_I = \nabla v \star \nabla v, \quad R^l_I = \sum_{2\le m\le l+1}\ \sum_{|\beta|\le l+2-m} \nabla\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla\mathcal{D}_t^{\beta_{m-1}} v \star \nabla\mathcal{D}_t^{\beta_m} v, \tag{2.12}$$

for any $l \ge 1$. Recall that $\operatorname{div}\mathcal{D}_t v = \partial_i v^j \partial_j v^i$ and $\operatorname{curl}\mathcal{D}_t v = \nabla H \star \nabla H + (H\cdot\nabla)(\operatorname{curl} H)$. We will show that it holds

$$\operatorname{div}\mathcal{D}_t^{l+1} v = R^l_I, \quad \operatorname{curl}\mathcal{D}_t^{l+1} v = R^l_I + R^l_{\nabla H,\nabla H} + R^l_{\nabla^2 H,H},$$

for $l \in \mathbb{N}_0$. The second error term is denoted by

$$\begin{aligned} R^0_{II} &= \nabla v \star \mathcal{D}_t v + \nabla v \star \nabla v \star v,\\ R^l_{II} &= \sum_{2\le m\le l+1}\ \sum_{|\beta|\le l, |\alpha|\le 1} a_{\alpha,\beta}(\nabla v)\nabla\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla\mathcal{D}_t^{\beta_{m-1}} v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+\beta_m} v, \end{aligned} \tag{2.13}$$

where $l \ge 1$ and $a_{\alpha,\beta}(\nabla v)$ denotes the finite $\star$ product as before.

By applying Lemma 2.3 and (1.1), we arrive at the following result.

Lemma 2.9. *For $l \in \mathbb{N}_0$, we have*

$$[\mathcal{D}_t^{l+1}, \nabla]p = \sum_{\beta_1\le l} \nabla\mathcal{D}_t^{\beta_1} v \star \nabla H \star H + R^l_{II} + R^l_{\nabla H,H}. \tag{2.14}$$

Proof. We work by induction on $l \in \mathbb{N}_0$. The case of $l = 0$ can be obtained via a straightforward calculation. As for $l \ge 1$, by Lemmas 2.2 and 2.3, it holds

$$[\mathcal{D}_t^{l+1}, \nabla]p = \mathcal{D}_t([\mathcal{D}_t^l, \nabla]p) + [\mathcal{D}_t, \nabla]\mathcal{D}_t^l p = \mathcal{D}_t([\mathcal{D}_t^l, \nabla]p) - (\nabla v)^\top \nabla\mathcal{D}_t^l p,$$

where

$$-\nabla\mathcal{D}_t^l p = [\mathcal{D}_t^l, \nabla]p + \mathcal{D}_t^l(\mathcal{D}_t v - H\cdot\nabla H) = [\mathcal{D}_t^l, \nabla]p + \mathcal{D}_t^{l+1} v - \mathcal{D}_t^l(H\cdot\nabla H).$$

A direct computation also shows that $\mathcal{D}_t R^{l-1}_{II} = R^l_{II}$ and $\mathcal{D}_t R^{l-1}_{\nabla H,H} = R^l_{\nabla H,H}$. These, combined with $[\mathcal{D}_t^l, \nabla]p = \sum_{\beta_1\le l-1} \nabla\mathcal{D}_t^{\beta_1} v \star \nabla H \star H + R^{l-1}_{II} + R^{l-1}_{\nabla H,H}$ from the induction argument, yield that

$$\begin{aligned} [\mathcal{D}_t^{l+1}, \nabla]p &= \mathcal{D}_t\Big(\sum_{\beta_1\le l-1} \nabla\mathcal{D}_t^{\beta_1} v \star \nabla H \star H + R^{l-1}_{II} + R^{l-1}_{\nabla H,H}\Big) + R^l_{II} + R^l_{\nabla H,H}\\ &= \mathcal{D}_t\Big(\sum_{\beta_1\le l-1} \nabla\mathcal{D}_t^{\beta_1} v \star \nabla H \star H\Big) + R^l_{II} + R^l_{\nabla H,H}\\ &= \sum_{\beta_1\le l} \nabla\mathcal{D}_t^{\beta_1} v \star \nabla H \star H + R^l_{II} + R^l_{\nabla H,H}, \end{aligned}$$

where in the last step, the lower-order terms have been absorbed into R^l_{II} and $R^l_{\nabla H,H}$. □

The following lemma will also be used to prove Propositions 3.1, 6.1 and 6.2.

Lemma 2.10. *Let $l \in \mathbb{N}$. We have*

$$\begin{aligned}\mathcal{D}_t\nabla^l \operatorname{curl} v =& (H\cdot\nabla)\nabla^l \operatorname{curl} H + \nabla v \star \nabla^l \operatorname{curl} v + \nabla^{l+1} v \star \operatorname{curl} v\\ &+ \sum_{|\beta|=l} \nabla^{1+\beta_1} H \star \nabla^{\beta_2} \operatorname{curl} H\\ &+ \sum_{|\alpha|\le l-1, \alpha_2\le l-2} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} \operatorname{curl} v,\\ \mathcal{D}_t\nabla^l \operatorname{curl} H =& (H\cdot\nabla)\nabla^l(\operatorname{curl} v) + \nabla v \star \nabla^l \operatorname{curl} H\\ &+ \sum_{|\beta|=l} \nabla^{1+\beta_1} v \star \nabla^{1+\beta_2} H\\ &+ \sum_{|\alpha|\le l-1, \alpha_2\le l-2} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} \operatorname{curl} H.\end{aligned}$$

Moreover, $\operatorname{div} \mathcal{D}_t^l v$ *and* $\operatorname{curl} \mathcal{D}_t^l v$ *can be written as the form*

$$\operatorname{div} \mathcal{D}_t^l v = R_I^{l-1}, \quad \operatorname{curl} \mathcal{D}_t^l v = R_I^{l-1} + R_{\nabla H,\nabla H}^{l-1} + R_{\nabla^2 H, H}^{l-1}.$$

We may also write $\operatorname{div} \mathcal{D}_t^{l+1} v = \operatorname{div}\operatorname{div}(v\otimes \mathcal{D}_t^l v) + \operatorname{div} R_{II}^{l-1}$.

Proof. The first claim is an immediate consequence of Lemmas 2.3 and 2.5. Indeed, one has

$$\begin{aligned}&\mathcal{D}_t\nabla^l \operatorname{curl} v\\ =& [\mathcal{D}_t, \nabla^l] \operatorname{curl} v + \nabla^l \mathcal{D}_t \operatorname{curl} v\\ =& \nabla^l[-(\nabla v)^\top \operatorname{curl} v - \operatorname{curl} v \nabla v + (\nabla H)^\top \operatorname{curl} H + \operatorname{curl} H \nabla H\\ &+ (H\cdot\nabla)(\operatorname{curl} H)] + \sum_{|\alpha|\le l-1} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} \operatorname{curl} v\\ =& (H\cdot\nabla)\nabla^l \operatorname{curl} H + \nabla v \star \nabla^l \operatorname{curl} v + \nabla^{l+1} v \star \operatorname{curl} v\\ &+ \sum_{|\beta|=l} \nabla^{1+\beta_1} H \star \nabla^{\beta_2} \operatorname{curl} H + \sum_{|\alpha|\le l-1, \alpha_2\le l-2} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} \operatorname{curl} v,\end{aligned}$$

and $\mathcal{D}_t\nabla^l \operatorname{curl} H$ can be computed in the same way.

Regarding $\operatorname{curl} \mathcal{D}_t^l v$ and $\operatorname{div} \mathcal{D}_t^l v$ for $l \ge 2$. Noting that $(\mathcal{D}_t^l \nabla u)^\top = \mathcal{D}_t^l[(\nabla u)^\top]$ and applying Lemmas 2.3 and 2.8, together with Lemma 2.5, we obtain

$$\begin{aligned}\operatorname{curl} \mathcal{D}_t^l v =& \nabla \mathcal{D}_t^{l-1}(\mathcal{D}_t v) - [\nabla \mathcal{D}_t^{l-1}(\mathcal{D}_t v)]^\top\\ =& [\nabla, \mathcal{D}_t^{l-1}](\mathcal{D}_t v) - ([\nabla, \mathcal{D}_t^{l-1}](\mathcal{D}_t v))^\top + \mathcal{D}_t^{l-1} \operatorname{curl} \mathcal{D}_t v\\ =& \sum_{2\le m\le l} \sum_{|\beta|\le l-m} \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-1}} v \star \nabla \mathcal{D}_t^{\beta_m+1} v\\ &+ \mathcal{D}_t^{l-1}(\nabla H \star \nabla H) + \mathcal{D}_t^{l-1}((H\cdot\nabla)(\operatorname{curl} H))\\ =& R_I^{l-1} + R_{\nabla H,\nabla H}^{l-1} + R_{\nabla^2 H, H}^{l-1}.\end{aligned}$$

Similarly, it follows that $\operatorname{div} \mathcal{D}_t^l v = R_I^{l-1}$ thanks to $\operatorname{div} v = 0$.

For the last statement, we need to apply

$$[\mathcal{D}_t, \operatorname{div}]F = -\operatorname{div}(\nabla v F), \text{ and } \operatorname{div}\operatorname{div}(v\otimes \mathcal{D}_t^l v) = \operatorname{div}(\nabla \mathcal{D}_t^l v v), \text{ for } l \ge 1,$$

both of which can be easily computed. Then, we have

$$\begin{aligned}\operatorname{div} \mathcal{D}_t^2 v =& \mathcal{D}_t \operatorname{div} \mathcal{D}_t v - [\mathcal{D}_t, \operatorname{div}]\mathcal{D}_t v\\ =& \mathcal{D}_t \operatorname{div}(\nabla v v) + \operatorname{div}(\nabla v \mathcal{D}_t v)\\ =& \operatorname{div} \mathcal{D}_t(\nabla v v) - \operatorname{div}(\nabla v \nabla v v) + \operatorname{div} R_{II}^0\\ =& \operatorname{div}(\mathcal{D}_t(\nabla v) v) - \operatorname{div}(\nabla v \nabla v v) + \operatorname{div} R_{II}^0\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{div}(\nabla\mathcal{D}_t v v) + \operatorname{div}([\mathcal{D}_t,\nabla]vv) - \operatorname{div}(\nabla v\nabla v v) + \operatorname{div} R_{II}^0\\
&= \operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t v) + \operatorname{div} R_{II}^0.
\end{aligned}
$$

As for $l\geq 2$, we argue by induction, i.e.,

$$
\begin{aligned}
\operatorname{div}\mathcal{D}_t^{l+1}v &= \mathcal{D}_t\operatorname{div}\mathcal{D}_t^l v - [\mathcal{D}_t,\operatorname{div}]\mathcal{D}_t^l v\\
&= \mathcal{D}_t(\operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^{l-1}v) + \operatorname{div} R_{II}^{l-2}) + \operatorname{div}(\nabla v\mathcal{D}_t^l v)\\
&= \mathcal{D}_t\operatorname{div}(\nabla\mathcal{D}_t^{l-1}vv) + \mathcal{D}_t\operatorname{div} R_{II}^{l-2} + \operatorname{div}(\nabla v\mathcal{D}_t^l v).
\end{aligned}
$$

A straightforward calculation yields $\mathcal{D}_t\operatorname{div} R_{II}^{l-2} = \operatorname{div} R_{II}^{l-1}$, $\operatorname{div}(\nabla v\mathcal{D}_t^l v) = \operatorname{div} R_{II}^{l-1}$, and therefore

$$
\begin{aligned}
&\mathcal{D}_t\operatorname{div}(\nabla\mathcal{D}_t^{l-1}vv)\\
=\ &\operatorname{div}\mathcal{D}_t(\nabla\mathcal{D}_t^{l-1}vv) + [\mathcal{D}_t,\operatorname{div}](\nabla\mathcal{D}_t^{l-1}vv)\\
=\ &\operatorname{div}(\mathcal{D}_t(\nabla\mathcal{D}_t^{l-1}v)v) + \operatorname{div}(\nabla\mathcal{D}_t^{l-1}v(\mathcal{D}_t v)) + \operatorname{div}(v\star\nabla v\star\nabla\mathcal{D}_t^{l-1}v)\\
=\ &\operatorname{div}(([\mathcal{D}_t,\nabla]\mathcal{D}_t^{l-1}v + \nabla\mathcal{D}_t^l v)v) + \operatorname{div}(\mathcal{D}_t v\star\nabla\mathcal{D}_t^{l-1}v + v\star\nabla v\star\nabla\mathcal{D}_t^{l-1}v)\\
=\ &\operatorname{div}(\nabla v\star\nabla\mathcal{D}_t^{l-1}v\star v) + \operatorname{div}(\nabla\mathcal{D}_t^l vv + \mathcal{D}_t v\star\nabla\mathcal{D}_t^{l-1}v + v\star\nabla v\star\nabla\mathcal{D}_t^{l-1}v)\\
=\ &\operatorname{div}(\nabla\mathcal{D}_t^l vv) + \operatorname{div} R_{II}^{l-1}\\
=\ &\operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^l v) + \operatorname{div} R_{II}^{l-1}.
\end{aligned}
$$

□

The following lemma will be used to establish Proposition 3.1 and the pressure estimates in Section 4.

Lemma 2.11. *We have*

$$
\begin{aligned}
-\Delta\mathcal{D}_t p =\ &\operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t v) + \operatorname{div}(R_{II}^0 + \nabla v\star H\star\nabla H + H\cdot\nabla(H\cdot\nabla v))\\
=\ &-\operatorname{div}\operatorname{div}(v\otimes\nabla p) + \operatorname{div} R_{II}^0 + \nabla^2 v\star\nabla H\star H + \nabla^2 H\star\nabla v\star H\\
&+ \nabla^2 H\star\nabla H\star v + \nabla v\star\nabla H\star\nabla H,
\end{aligned}
$$

and for $l\geq 1$, it holds

$$
\begin{aligned}
-\Delta\mathcal{D}_t^{l+1}p =\ &\operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^{l+1}v) - \operatorname{div} R_{\nabla^2 H,H}^{l+1}\\
&+ \operatorname{div}\Big(\sum_{\beta_1\leq l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H + R_{II}^l + R_{\nabla H,H}^l\Big).
\end{aligned}
$$

Proof. From the divergence-free condition, Lemma 2.10 and (2.14), $-\Delta\mathcal{D}_t p$ can be written as

$$
\begin{aligned}
&-\Delta\mathcal{D}_t p\\
=\ &-\operatorname{div}\mathcal{D}_t\nabla p + \operatorname{div}[\mathcal{D}_t,\nabla]p\\
=\ &-\operatorname{div}\mathcal{D}_t(H\cdot\nabla H) + \operatorname{div}\mathcal{D}_t^2 v - \operatorname{div}(\nabla v^\top(H\cdot\nabla H)) + \operatorname{div}(\nabla v^\top\mathcal{D}_t v)\\
=\ &\operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t v) + \operatorname{div} R_{II}^0 - \operatorname{div}\mathcal{D}_t(H\cdot\nabla H) - \operatorname{div}(\nabla v^\top(H\cdot\nabla H))\\
=\ &-\operatorname{div}\operatorname{div}(v\otimes\nabla p) + \operatorname{div} R_{II}^0 + \nabla^2 v\star\nabla H\star H + \nabla v\star\nabla H\star\nabla H\\
&+ \nabla^2 H\star\nabla v\star H + v\star\nabla^2 H\star\nabla H.
\end{aligned}
$$

The second claim follows by applying (2.14) that

$$
\begin{aligned}
-\Delta\mathcal{D}_t^{l+1}p =\ &-\operatorname{div}\mathcal{D}_t^{l+1}\nabla p + \operatorname{div}[\mathcal{D}_t^{l+1},\nabla]p\\
=\ &\operatorname{div}\mathcal{D}_t^{l+2}v - \operatorname{div}\mathcal{D}_t^{l+1}(H\cdot\nabla H) + \operatorname{div} R_{II}^l\\
&+ \operatorname{div}\Big(\sum_{\beta_1\leq l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H + R_{\nabla H,H}^l\Big)\\
=\ &\operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^{l+1}v) - \operatorname{div} R_{\nabla^2 H,H}^{l+1}
\end{aligned}
$$

$$+\operatorname{div}\Big(\sum_{\beta_1\le l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H+R_{II}^l+R_{\nabla H,H}^l\Big).$$

□

Remark. In the above formula, the term $\operatorname{div} R^{l+1}_{\nabla^2 H,H}$ appears to be more challenging than the other error terms (e.g., $\operatorname{div} R^l_{II}$). However, taking the divergence ($\operatorname{div} R^{l+1}_{\nabla^2 H,H}$) does not increase the order of derivatives, as indicated in Lemma 2.7 due to the divergence-free condition. The key observation will enable us to conclude the proof of Proposition 3.1.

To establish the energy estimates, it is necessary to derive the formula for $\mathcal{D}_t^l p$ on the free boundary. It is important to note that the solution is well-defined on Γ_t due to our assumption of the local existence result. Furthermore, it is important to mention that the following formulas do not include the magnetic field.

Lemma 2.12. *On the free-boundary Γ_t, it holds*

$$\mathcal{D}_t p=-\Delta_B v\cdot\nu-2B:\bar\nabla v=-\Delta_B v_n-|B|^2v_n+\bar\nabla p\cdot v. \tag{2.15}$$

Proof. We recall that $p=\mathcal{A}$ and from the identities (e.g., [SZ08b, Section 3.1])

$$\mathcal{D}_t\mathcal{A}=-\Delta_B v_n-|B|^2v_n+\bar\nabla\mathcal{A}\cdot v,\quad \Delta_B\nu=-|B|^2\nu+\bar\nabla\mathcal{A}, \tag{2.16}$$

it is clear that

$$\begin{aligned}\mathcal{D}_t p&=-\Delta_B v_n-|B|^2v_n+(\Delta_B\nu+|B|^2\nu)\cdot v\\&=-\Delta_B v_n+\Delta_B\nu\cdot\nu\\&=-\Delta_B v\cdot v-2B:\bar\nabla v.\end{aligned}$$

□

Finally, we introduce the error term R^l_p as described in [JLM22]. We define

$$\begin{aligned}R_p^1=&-|B|^2\mathcal{D}_t v\cdot\nu+\bar\nabla p\cdot\mathcal{D}_t v+a_1(\nu,\nabla v)\star\nabla^2 v+a_2(\nu,\nabla v)\star B,\\ R_p^2=&-|B|^2\mathcal{D}_t^2 v\cdot\nu+\bar\nabla p\cdot\mathcal{D}_t^2 v+a_3(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v\\&+a_4(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star\nabla^2 v+a_5(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star B\\&+a_6(\nu,\nabla v)\star\nabla^2 v+a_7(\nu,\nabla v)\star B,\end{aligned}$$

and

$$\begin{aligned}R_p^3=&-|B|^2\mathcal{D}_t^3 v\cdot\nu+\bar\nabla p\cdot\mathcal{D}_t^3 v\\&+a_8(\nu,\nabla v)\star\nabla^2\mathcal{D}_t^2 v+a_9(\nu,\nabla v)\star\nabla\mathcal{D}_t^2 v\star\nabla^2 v\\&+a_{10}(\nu,\nabla v)\star\nabla\mathcal{D}_t^2 v\star B+a_{11}(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v\star\nabla\mathcal{D}_t v\\&+a_{12}(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v\star B+a_{13}(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star\nabla\mathcal{D}_t v\star\nabla^2 v\\&+a_{14}(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star\nabla\mathcal{D}_t v\star B+\mathrm{L.O.T.},\end{aligned}$$

where $a_i(\nu,\nabla v)$ denotes the finite $\star$ product of ν and ∇v. For $l\ge 4$, we define

$$\begin{aligned}R_p^l=-|B|^2\mathcal{D}_t^l v\cdot\nu+\bar\nabla p\cdot\mathcal{D}_t^l v+\sum_{|\alpha|\le 1,|\beta|\le l-1}&a_{\alpha,\beta}(\nu,B)\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\dots\\&\star\nabla^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}}v,\end{aligned}$$

where $a_{\alpha,\beta}(\nu,B)$ also denotes the finite $\star$ product.

Lemma 2.13. *On the free-boundary Γ_t, we have*

$$\mathcal{D}_t^{l+1}p=-\Delta_B(\mathcal{D}_t^l v\cdot\nu)+R_p^l,$$

for $l\in\mathbb{N}$.

Proof. For $l=1$, we differentiate (2.15) to obtain $\mathcal{D}_t^2 p = -\mathcal{D}_t\Delta_B v\cdot\nu - \Delta_B v\cdot\mathcal{D}_t\nu - 2\mathcal{D}_t B:\bar\nabla v - 2B:\mathcal{D}_t\bar\nabla v$. Recalling the formulas for $[\mathcal{D}_t,\Delta_B]$, $\mathcal{D}_t\nu$ and $\mathcal{D}_t B$ in Lemma 2.1, it holds

$$\mathcal{D}_t^2 p = -\Delta_B\mathcal{D}_t v\cdot\nu - 2B:\bar\nabla\mathcal{D}_t v + a_1(\nu,\nabla v)\star\nabla^2 v + a_2(\nu,\nabla v)\star B.$$

where a_1 and a_2 are finite $\star$ product of ν and ∇v.

As $l=2$, we differentiate $\mathcal{D}_t^2 p$ and calculate $[\mathcal{D}_t,\Delta_B]\mathcal{D}_t v = \bar\nabla^2\mathcal{D}_t v\star\nabla v - \bar\nabla\mathcal{D}_t v\cdot\Delta_B v + B_\Gamma\star\nabla v\star\bar\nabla\mathcal{D}_t v$, $\mathcal{D}_t B = a_1(\nu,\nabla v)\star B + a_2(\nu,\nabla v)\star\nabla^2 v$, $\mathcal{D}_t\bar\nabla\mathcal{D}_t v = \bar\nabla\mathcal{D}_t^2 v + \bar\nabla v\star\bar\nabla\mathcal{D}_t v$, $\mathcal{D}_t a(\nu,\nabla v) = b(\nu,\nabla v)\star\nabla\mathcal{D}_t v$, $\mathcal{D}_t\nabla^2 v = \nabla^2 v\star\nabla v + \nabla^2\mathcal{D}_t v$ to obtain

$$\begin{aligned}\mathcal{D}_t^3 p = &-\Delta_B\mathcal{D}_t^2 v\cdot\nu - 2B:\bar\nabla\mathcal{D}_t^2 v + a_3(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v + a_4(\nu,\nabla v)\star\nabla\mathcal{D}_t v\\ &\star\nabla^2 v + a_5(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star B + a_6(\nu,\nabla v)\star\nabla^2 v + a_7(\nu,\nabla v)\star B.\end{aligned}$$

Similarly, it holds

$$\begin{aligned}\mathcal{D}_t^4 p = &-\Delta_B\mathcal{D}_t^3 v\cdot\nu - 2B:\bar\nabla\mathcal{D}_t^3 v\\ &+ a_8(\nu,\nabla v)\star\nabla^2\mathcal{D}_t^2 v + a_9(\nu,\nabla v)\star\nabla\mathcal{D}_t^2 v\star\nabla^2 v\\ &+ a_{10}(\nu,\nabla v)\star\nabla\mathcal{D}_t^2 v\star B + a_{11}(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v\star\nabla\mathcal{D}_t v\\ &+ a_{12}(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v\star B + a_{13}(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star\nabla\mathcal{D}_t v\star\nabla^2 v\\ &+ a_{14}(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star\nabla\mathcal{D}_t v\star B + \mathrm{L.O.T.},\end{aligned}$$

for $l=3$. As in [JLM22, Lemma 4.7], we can show that

$$\begin{aligned}\mathcal{D}_t^l p = -\Delta_B\mathcal{D}_t^{l-1} v\cdot\nu - 2B:\bar\nabla\mathcal{D}_t^{l-1} v + \sum_{\substack{|\alpha|\le 1\\ |\beta|\le l-1}} a_{\alpha,\beta}(\nu,B)\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1} v\star\dots\\ \star\nabla^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}} v,\end{aligned}$$

for $l\ge 5$. Combined with (2.2) and (2.16), the remaining proof is similar to [JLM22, Lemma 4.7]. □

3. Time Derivatives of the Energy Functionals

In this section, we compute the time derivative of the energy functional $e_l(t)$ by applying Lemmas A.2 and A.3. The main result in this section is the following proposition.

Proposition 3.1. *Assume that the a priori assumptions (1.8) hold for some $T>0$. Then, we have*

$$\begin{aligned}\frac{d}{dt}\bar e(t) \le C\sum_{l=1}^{3}\Big(&\|R_I^l\|^2_{H^{\frac12}(\Omega_t)} + \|R_{II}^l\|^2_{L^2(\Omega_t)} + \|R^l_{\nabla H,H}\|^2_{L^2(\Omega_t)}\\ &+ \|R_p^l\|^2_{H^{\frac12}(\Gamma_t)}\Big) + C\left(1+\|\nabla^2 p\|^2_{L^2(\Omega_t)}\right)\bar E(t),\end{aligned}$$

where the constant C depends on $T,\mathcal{N}_T$, and $\mathcal{M}_T$.

Moreover, we further assume that $\sup_{0\le t<T} E_{l-1}(t)\le C$ for $l\ge 4$. Then, it holds

$$\begin{aligned}\frac{d}{dt}e_l(t) \le C\Big(&E_l(t) + \|R_I^l\|^2_{H^{\frac12}(\Omega_t)} + \|R_{II}^l\|^2_{L^2(\Omega_t)} + \|R^l_{\nabla H,H}\|^2_{L^2(\Omega_t)}\\ &+ \|R_p^l\|^2_{H^{\frac12}(\Gamma_t)}\Big),\end{aligned}$$

for $l\ge 4$, where the constant C depends on $T,\mathcal{N}_T,\mathcal{M}_T$, and $\sup_{0\le t<T} E_{l-1}(t)$.

When computing the time derivative of $e_l(t)$, we denote

$$\begin{aligned}e_l(t) = &\frac12\int_{\Omega_t}|\mathcal{D}_t^{l+1} v|^2 dx + \frac12\int_{\Omega_t}|\mathcal{D}_t^{l+1} H|^2 dx + \frac12\int_{\Gamma_t}|\bar\nabla(\mathcal{D}_t^l v\cdot\nu)|^2 dS\\ &+ \frac12\int_{\Omega_t}|\nabla^{\lfloor\frac{3l+1}{2}\rfloor}\operatorname{curl} v|^2 dx + \frac12\int_{\Omega_t}|\nabla^{\lfloor\frac{3l+1}{2}\rfloor}\operatorname{curl} H|^2 dx =: \sum_{i=1}^{5} I_i^l(t),\end{aligned}$$

and we will apply Lemmas A.2 and A.3 several times.

We begin by considering the material derivative of the velocity field. From equation (1.1) and applying the divergence theorem, we obtain

$$
\begin{aligned}
&\frac{d}{dt}I_1^l(t)\\
=&\int_{\Omega_t}\mathcal{D}_t^{l+2}v\cdot\mathcal{D}_t^{l+1}vdx\\
=&-\int_{\Omega_t}\mathcal{D}_t^{l+1}\nabla p\cdot\mathcal{D}_t^{l+1}vdx+\int_{\Omega_t}\mathcal{D}_t^{l+1}(H\cdot\nabla H)\cdot\mathcal{D}_t^{l+1}vdx\\
=&-\int_{\Omega_t}\nabla\mathcal{D}_t^{l+1}p\cdot\mathcal{D}_t^{l+1}vdx-\int_{\Omega_t}[\mathcal{D}_t^{l+1},\nabla]p\cdot\mathcal{D}_t^{l+1}vdx\\
&+\int_{\Omega_t}\mathcal{D}_t^{l+1}(H^j\partial_jH_i)\mathcal{D}_t^{l+1}v^idx\\
=&-\int_{\Omega_t}\operatorname{div}(\mathcal{D}_t^{l+1}p\mathcal{D}_t^{l+1}v)dx+\int_{\Omega_t}\mathcal{D}_t^{l+1}p\operatorname{div}\mathcal{D}_t^{l+1}vdx\\
&-\int_{\Omega_t}[\mathcal{D}_t^{l+1},\nabla]p\cdot\mathcal{D}_t^{l+1}vdx+\int_{\Omega_t}\mathcal{D}_t^{l+1}(H^j\partial_jH_i)\mathcal{D}_t^{l+1}v^idx\\
\le&\underbrace{\int_{\Omega_t}H^j\partial_j(\mathcal{D}_t^{l+1}H_i)\mathcal{D}_t^{l+1}v^idx}_{=:J_1^l(t)}\underbrace{-\int_{\Gamma_t}\mathcal{D}_t^{l+1}p(\mathcal{D}_t^{l+1}v\cdot\nu)dS}_{=:K_1^l(t)}+\|\mathcal{D}_t^{l+1}v\|_{L^2(\Omega_t)}^2\\
&+\underbrace{\int_{\Omega_t}\mathcal{D}_t^{l+1}p\operatorname{div}\mathcal{D}_t^{l+1}vdx}_{=:I_{11}^l(t)}+\underbrace{\|[\mathcal{D}_t^{l+1},\nabla]p\|_{L^2(\Omega_t)}^2}_{=:I_{12}^l(t)}\\
&+\underbrace{\sum_{k=0}^{l}\int_{\Omega_t}\mathcal{D}_t^kH^j[\mathcal{D}_t^{l+1-k},\partial_j]H_i\mathcal{D}_t^{l+1}v^idx}_{=:I_{13}^l(t)}\\
&+\underbrace{\sum_{k=1}^{l+1}\int_{\Omega_t}\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{l+1-k}H_i\mathcal{D}_t^{l+1}v^idx}_{=:I_{14}^l(t)},
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
&\mathcal{D}_t^{l+1}(H^j\partial_jH_i)\mathcal{D}_t^{l+1}v^i\\
=&H^j\partial_j(\mathcal{D}_t^{l+1}H_i)\mathcal{D}_t^{l+1}v^i+\sum_{k=0}^{l}\mathcal{D}_t^kH^j[\mathcal{D}_t^{l+1-k},\partial_j]H_i\mathcal{D}_t^{l+1}v^i\\
&+\sum_{k=1}^{l+1}\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{l+1-k}H_i\mathcal{D}_t^{l+1}v^i.
\end{aligned}
$$

Similarly, for the magnetic field, it follows that

$$
\begin{aligned}
&\frac{d}{dt}I_2^l(t)\\
=&\int_{\Omega_t}\mathcal{D}_t^{l+2}H\cdot\mathcal{D}_t^{l+1}Hdx\\
=&\int_{\Omega_t}\mathcal{D}_t^{l+1}(H^j\partial_jv^i)\mathcal{D}_t^{l+1}H_idx
\end{aligned}
$$

$$= \underbrace{\int_{\Omega_t} H^j \partial_j (\mathcal{D}_t^{l+1} v^i) \mathcal{D}_t^{l+1} H_i dx}_{=:J_2^l(t)} + \underbrace{\sum_{k=0}^{l} \int_{\Omega_t} \mathcal{D}_t^k H^j [\mathcal{D}_t^{l+1-k}, \partial_j] v^i \mathcal{D}_t^{l+1} H_i dx}_{=:I_{21}^l(t)}$$
$$+ \underbrace{\sum_{k=1}^{l+1} \int_{\Omega_t} \mathcal{D}_t^k H^j \partial_j \mathcal{D}_t^{l+1-k} v^i \mathcal{D}_t^{l+1} H_i dx}_{=:I_{22}^l(t)} .$$

Recalling that $\operatorname{div} H = 0$ in Ω_t and $H \cdot \nu = 0$ on Γ_t, it is clear that

$$J_1^l(t) + J_2^l(t) = 0,$$

and we obtain

$$\frac{d}{dt}\left(I_1^l(t) + I_2^l(t)\right) \le K_1^l(t) + \sum_{i=1}^{4} I_{1i}^l(t) + I_{21}^l(t) + I_{22}^l(t) + \|\mathcal{D}_t^{l+1} v\|_{L^2(\Omega_t)}^2 .$$

To control the third term, we apply Lemma 2.1 to deduce

$$\begin{aligned}
&\frac{d}{dt} I_3^l(t) \\
&= \int_{\Gamma_t} [\mathcal{D}_t, \bar\nabla](\mathcal{D}_t^l v \cdot \nu) \cdot \bar\nabla(\mathcal{D}_t^l v \cdot \nu) dS + \frac{1}{2}\int_{\Gamma_t} |\bar\nabla(\mathcal{D}_t^l v \cdot \nu)|^2 \operatorname{div}_\sigma v dS \\
&\quad + \int_{\Gamma_t} \bar\nabla \mathcal{D}_t(\mathcal{D}_t^l v \cdot \nu) \cdot \bar\nabla(\mathcal{D}_t^l v \cdot \nu) dS \\
&= \int_{\Gamma_t} -(\bar\nabla v)^\top \bar\nabla(\mathcal{D}_t^l v \cdot \nu) \cdot \bar\nabla(\mathcal{D}_t^l v \cdot \nu) dS + \frac{1}{2}\int_{\Gamma_t} |\bar\nabla(\mathcal{D}_t^l v \cdot \nu)|^2 \operatorname{div}_\sigma v dS \\
&\quad + \int_{\Gamma_t} \bar\nabla(\mathcal{D}_t^{l+1} v \cdot \nu) \cdot \bar\nabla(\mathcal{D}_t^l v \cdot \nu) dS + \int_{\Gamma_t} \bar\nabla(\mathcal{D}_t^l v \cdot \mathcal{D}_t \nu) \cdot \bar\nabla(\mathcal{D}_t^l v \cdot \nu) dS \\
&\le \underbrace{-\int_{\Gamma_t} (\mathcal{D}_t^{l+1} v \cdot \nu) \cdot \Delta_B(\mathcal{D}_t^l v \cdot \nu) dS}_{=:K_3^l(t)} + \underbrace{\|\bar\nabla(\mathcal{D}_t^l v \cdot \mathcal{D}_t \nu)\|_{L^2(\Gamma_t)}^2}_{=:I_{31}^l(t)} \\
&\quad + C(\|\bar\nabla v\|_{L^\infty(\Gamma_t)} + 1)\|\bar\nabla(\mathcal{D}_t^l v \cdot \nu)\|_{L^2(\Gamma_t)}^2 .
\end{aligned}$$

Finally, to compute the last two terms involving the curl, we denote $\mu_l := \lfloor \frac{1}{2}(3l+1) \rfloor$. We then utilize the divergence-free condition and the fact that $H \cdot \nu = 0$ on Γ_t to obtain

$$\int_{\Omega_t} \sum_{|\alpha|=l} (H \cdot \nabla)\nabla^\alpha \operatorname{curl} H : \nabla^\alpha \operatorname{curl} v + \sum_{|\alpha|=l} (H \cdot \nabla)\nabla^\alpha \operatorname{curl} v : \nabla^\alpha \operatorname{curl} H dx = 0.$$

Therefore, from Lemma 2.10, it follows that

$$\begin{aligned}
&\frac{d}{dt} I_4^l(t) - \int_{\Omega_t} \sum_{|\alpha|=l} (H \cdot \nabla)\nabla^\alpha \operatorname{curl} H : \nabla^\alpha \operatorname{curl} v dx \\
&= \int_{\Omega_t} \nabla v \star \nabla^{\mu_l} \operatorname{curl} v \star \nabla^{\mu_l} \operatorname{curl} v + \nabla^{\mu_l+1} v \star \operatorname{curl} v \star \nabla^{\mu_l} \operatorname{curl} v \\
&\quad + \sum_{|\beta|=\mu_l} \nabla^{1+\beta_1} H \star \nabla^{\beta_2} \operatorname{curl} H \star \nabla^{\mu_l} \operatorname{curl} v \\
&\quad + \sum_{|\alpha|\le \mu_l-1, \alpha_2 \le \mu_l-2} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} \operatorname{curl} v \star \nabla^{\mu_l} \operatorname{curl} v dx \\
&\le C(\|\nabla v\|_{L^\infty(\Omega_t)} + 1)\|\nabla^{\mu_l+1} v\|_{L^2(\Omega_t)}^2 + \|\nabla H\|_{L^\infty(\Omega_t)}^2 \|\operatorname{curl} H\|_{H^{\mu_l}(\Omega_t)}^2 \\
&\quad + \|\operatorname{curl} H\|_{L^\infty(\Omega_t)}^2 \|\nabla H\|_{H^{\mu_l}(\Omega_t)}^2 + \|\nabla v\|_{L^\infty(\Omega_t)}^2 \|\nabla v\|_{H^{\mu_l}(\Omega_t)}^2 ,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}I_5^l(t) - \int_{\Omega_t} \sum_{|\alpha|=l} (H\cdot\nabla)\nabla^\alpha \operatorname{curl} v : \nabla^\alpha \operatorname{curl} H dx \\
&\le C(\|\nabla v\|_{L^\infty(\Omega_t)}+1)\|\nabla^{\mu_l}\operatorname{curl} H\|^2_{L^2(\Omega_t)} + \|\nabla v\|^2_{H^{\mu_l}(\Omega_t)}\|\nabla H\|^2_{L^\infty(\Omega_t)} \\
&\quad + \|\nabla H\|^2_{H^{\mu_l}(\Omega_t)}\|\nabla v\|^2_{L^\infty(\Omega_t)}.
\end{aligned}$$

Proof of Proposition 3.1. Since we assume the local existence of smooth solution, by (1.8), one has $\|\bar\nabla v\|_{L^\infty(\Gamma_t)} \le C\|\nabla v\|_{L^\infty(\Omega_t)} \le C$. This, combined with the above calculations, and applying Lemma 2.13, $\|\nabla H\|_{L^\infty(\Omega_t)} \le C$ by (1.8), together with the definition (1.5), we obtain

$$K_1^l(t) + K_3^l(t) = -\int_{\Gamma_t} R_p^l(\mathcal{D}_t^{l+1} v\cdot\nu) dS,$$

and

$$\begin{aligned}
\frac{d}{dt}\bar e(t) \le C\bar E + C\sum_{l=1}^3 \Bigg(&-\int_{\Gamma_t} R_p^l(\mathcal{D}_t^{l+1}v\cdot\nu)dS + \sum_{i=1}^4 I_{1i}^l(t) + I_{31}^l(t) \\
&+ I_{21}^l(t) + I_{22}^l(t)\Bigg).
\end{aligned}$$

As for $l\ge 4$, it follows that

$$\begin{aligned}
\frac{d}{dt}e_l(t) \le CE_l(t) + C\Bigg(&-\int_{\Gamma_t} R_p^l(\mathcal{D}_t^{l+1}v\cdot\nu)dS + \sum_{i=1}^4 I_{1i}^l(t) + I_{31}^l(t) \\
&+ I_{21}^l(t) + I_{22}^l(t)\Bigg).
\end{aligned}$$

We divide the remaining proof into six steps.

Step 1. We control $I_{14}^l(t)$ and $I_{22}^l(t)$. Let $l=1$ and assume $F=v, G=H$ or $F=H, G=v$ respectively. From (1.8), it holds

$$\begin{aligned}
&\sum_{k=1}^2 \int_{\Omega_t} \mathcal{D}_t^k H^j \partial_j \mathcal{D}_t^{2-k} F_i \mathcal{D}_t^2 G^i dx \\
&\le C\sum_{k=1}^2 \|\mathcal{D}_t^k H^j\partial_j\mathcal{D}_t^{2-k}F\|^2_{L^2(\Omega_t)} + C\|\mathcal{D}_t^2 G\|^2_{L^2(\Omega_t)} \\
&\le C(E_1(t) + \|\mathcal{D}_t H\|^2_{L^2(\Omega_t)}\|\nabla\mathcal{D}_t F\|^2_{L^\infty(\Omega_t)} + \|\mathcal{D}_t^2 H\|^2_{L^2(\Omega_t)}\|\nabla F\|^2_{L^\infty(\Omega_t)}) \\
&\le C\bar E(t).
\end{aligned}$$

In the case of $l=2$, from the fact that

$$\|\nabla\mathcal{D}_t H\|^2_{L^2(\Omega_t)} \le \|\nabla(H\cdot\nabla v)\|^2_{L^2(\Omega_t)} \le C, \tag{3.1}$$

$$\begin{aligned}
\|\nabla\mathcal{D}_t v\|^2_{L^2(\Omega_t)} &\le \|\nabla(H\cdot\nabla H)\|^2_{L^2(\Omega_t)} + \|\nabla^2 p\|^2_{L^2(\Omega_t)} \\
&\le C(1+\|\nabla^2 p\|^2_{L^2(\Omega_t)}),
\end{aligned} \tag{3.2}$$

it follows that

$$\begin{aligned}
&\sum_{k=1}^3 \int_{\Omega_t} \mathcal{D}_t^k H^j \partial_j \mathcal{D}_t^{3-k} F_i \mathcal{D}_t^3 G^i dx \\
&\le C\sum_{k=1}^3 \|\mathcal{D}_t^k H^j\partial_j\mathcal{D}_t^{3-k}F\|^2_{L^2(\Omega_t)} + C\|\mathcal{D}_t^3 G\|^2_{L^2(\Omega_t)} \\
&\le C(E_2(t) + \|H\cdot\nabla v\|^2_{L^2(\Omega_t)}\|\mathcal{D}_t^2 F\|^2_{H^3(\Omega_t)} + \|\mathcal{D}_t^2 H\|^2_{H^2(\Omega_t)}\|\nabla\mathcal{D}_t F\|^2_{L^2(\Omega_t)}
\end{aligned}$$

$$
\begin{aligned}
&+\|\mathcal{D}_t^3H\|_{L^2(\Omega_t)}^2\|\nabla F\|_{L^\infty(\Omega_t)}^2)\\
&\le C(1+\|\nabla^2 p\|_{L^2(\Omega_t)}^2)\bar{E}(t).
\end{aligned}
$$

As for $l=3$, again by (3.1) and (3.2), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{4}\int_{\Omega_t}\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{4-k}F_i\mathcal{D}_t^4G^i dx\\
&\le C\sum_{k=1}^{4}\|\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{4-k}F\|_{L^2(\Omega_t)}^2+C\|\mathcal{D}_t^4G\|_{L^2(\Omega_t)}^2\\
&\le C(E_3(t)+\|H\cdot\nabla v\|_{L^6(\Omega_t)}^2\|\mathcal{D}_t^3F\|_{H^{\frac{3}{2}}(\Omega_t)}^2+\|\mathcal{D}_t^2H\|_{L^2(\Omega_t)}^2\|\mathcal{D}_t^2F\|_{H^3(\Omega_t)}^2\\
&\qquad+\|\mathcal{D}_t^2(H\cdot\nabla v)\|_{L^\infty(\Omega_t)}^2\|\nabla\mathcal{D}_tF\|_{L^2(\Omega_t)}^2+\|\mathcal{D}_t^4H\|_{L^2(\Omega_t)}^2\|\nabla F\|_{L^\infty(\Omega_t)}^2)\\
&\le C(1+\|\nabla^2 p\|_{L^2(\Omega_t)}^2)\bar{E}(t),
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\|\mathcal{D}_t^2H\|_{L^2(\Omega_t)}^2&\le C\|\mathcal{D}_tH\star\nabla v\|_{L^2(\Omega_t)}^2+C\|H\star\mathcal{D}_t\nabla v\|_{L^2(\Omega_t)}^2\\
&\le C(1+\|\nabla^2 p\|_{L^2(\Omega_t)}^2),
\end{aligned}\tag{3.3}
$$

and

$$
\begin{aligned}
&\|\mathcal{D}_t^3H\|_{L^\infty(\Omega_t)}^2\\
&\le\|\mathcal{D}_t^2(H\cdot\nabla v)\|_{L^\infty(\Omega_t)}^2\\
&\le\|\mathcal{D}_t^2H\star\nabla v\|_{L^\infty(\Omega_t)}^2+\|\mathcal{D}_tH\star\mathcal{D}_t\nabla v\|_{L^\infty(\Omega_t)}^2+\|H\star\mathcal{D}_t^2\nabla v\|_{L^\infty(\Omega_t)}^2\\
&\le C(\|\mathcal{D}_t^2H\|_{L^\infty(\Omega_t)}^2+\|[\mathcal{D}_t,\nabla]v\|_{L^\infty(\Omega_t)}^2+\|\nabla\mathcal{D}_tv\|_{L^\infty(\Omega_t)}^2\\
&\qquad+\|[\mathcal{D}_t^2,\nabla]v\|_{L^\infty(\Omega_t)}^2+\|\nabla\mathcal{D}_t^2v\|_{L^\infty(\Omega_t)}^2)\\
&\le C\bar{E}(t),
\end{aligned}
$$

by utilizing (1.8), Lemmas 2.1 and 2.3. Additionally, one order material derivative has been substituted with the spatial derivative of the velocity field.

As $l\ge 4$, we use the hypotheses $E_{l-1}(t)\le C$ to obtain

$$
\begin{aligned}
&\sum_{k=1}^{l+1}\int_{\Omega_t}\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{l+1-k}F_i\mathcal{D}_t^{l+1}G^i dx\\
&\le C\sum_{k=1}^{l+1}\|\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{l+1-k}F\|_{L^2(\Omega_t)}^2+C\|\mathcal{D}_t^{l+1}G\|_{L^2(\Omega_t)}^2\\
&\le C(\sum_{k=2}^{l}\|\mathcal{D}_t^kH\|_{H^1(\Omega_t)}^2\|\mathcal{D}_t^{l+1-k}F\|_{H^{\frac{3}{2}}(\Omega_t)}^2+\|\mathcal{D}_tH^j\partial_j\mathcal{D}_t^lF\|_{L^2(\Omega_t)}^2)+CE_l(t)\\
&\le CE_l(t)E_{l-1}(t)+CE_l(t)+C\|\mathcal{D}_tH\|_{L^6(\Omega_t)}^2\|\nabla\mathcal{D}_t^lF\|_{L^3(\Omega_t)}^2\le CE_l(t).
\end{aligned}
$$

Step 2. We control $I_{13}^l(t)$ and $I_{21}^l(t)$. As before, we assume $F=v,G=H$ or $F=H,G=v$. We only consider the cases in which $l=2$ and $l=3$, since the case for $l=1$ is simpler. In fact, from the commutator formula of $[\mathcal{D}_t^j,\nabla]$ in Lemma 2.3, (3.1), (3.2) and (3.3), it holds

$$
\begin{aligned}
&\sum_{k=0}^{2}\int_{\Omega_t}\mathcal{D}_t^kH^j[\mathcal{D}_t^{3-k},\partial_j]F_i\mathcal{D}_t^3G^i dx\\
&\le C\sum_{k=0}^{2}\|\mathcal{D}_t^kH^j[\mathcal{D}_t^{3-k},\partial_j]F\|_{L^2(\Omega_t)}^2+CE_2(t)\\
&\le C(\|\mathcal{D}_t^2H^j[\mathcal{D}_t,\partial_j]F\|_{L^2(\Omega_t)}^2+\|\mathcal{D}_tH^j[\mathcal{D}_t^2,\partial_j]F\|_{L^2(\Omega_t)}^2+\|H^j[\mathcal{D}_t^3,\partial_j]F\|_{L^2(\Omega_t)}^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + E_2(t)) \\
&\le C(\bar{E}(t) + \|\mathcal{D}_t^2 H^j \partial_j v^k \partial_k F\|_{L^2(\Omega_t)}^2 \\
&\quad + \|\mathcal{D}_t H \star (\nabla v \star \nabla F + \nabla \mathcal{D}_t v \star \nabla F + \nabla v \star \nabla \mathcal{D}_t F + \nabla v \star \nabla v \star \nabla F)\|_{L^2(\Omega_t)}^2 \\
&\quad + \|H \star (\nabla \mathcal{D}_t^2 v \star \nabla F + \nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t F + \nabla v \star \nabla \mathcal{D}_t^2 F + \nabla \mathcal{D}_t v \star \nabla v \star \nabla F \\
&\qquad + \nabla v \star \nabla v \star \nabla \mathcal{D}_t F + \mathrm{L.O.T.})\|_{L^2(\Omega_t)}^2) \\
&\le C(1 + \|\nabla^2 p\|_{L^2(\Omega_t)}^2)\bar{E}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{3} \int_{\Omega_t} \mathcal{D}_t^k H^j [\mathcal{D}_t^{4-k}, \partial_j] F_i \mathcal{D}_t^4 G^i dx \\
&\le C(E_3(t) + \|\mathcal{D}_t^3 H^j \partial_j v^k \partial_k F\|_{L^2(\Omega_t)}^2 \\
&\quad + \|\mathcal{D}_t^2 H \star (\nabla v \star \nabla F + \nabla \mathcal{D}_t v \star \nabla F + \nabla v \star \nabla \mathcal{D}_t F + \nabla v \star \nabla v \star \nabla F)\|_{L^2(\Omega_t)}^2 \\
&\quad + \|\mathcal{D}_t H \star (\nabla \mathcal{D}_t^2 v \star \nabla F + \nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t F + \nabla v \star \nabla \mathcal{D}_t^2 F + \nabla \mathcal{D}_t v \star \nabla v \\
&\qquad \star \nabla F + \nabla v \star \nabla v \star \nabla \mathcal{D}_t F + \mathrm{L.O.T.})\|_{L^2(\Omega_t)}^2 \\
&\quad + \|H \star (\nabla \mathcal{D}_t^3 v \star \nabla F + \nabla \mathcal{D}_t^2 v \star \nabla \mathcal{D}_t F + \nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t^2 F + \nabla v \star \nabla \mathcal{D}_t^3 F \\
&\qquad + \mathrm{L.O.T.})\|_{L^2(\Omega_t)}^2) \\
&\le C(1 + \|\nabla^2 p\|_{L^2(\Omega_t)}^2)\bar{E}(t).
\end{aligned}
$$

For $l \ge 4$, from Lemma 2.3 and the assumption $E_{l-1}(t) \le C$, we deduce

$$
\begin{aligned}
&\sum_{k=0}^{l} \int_{\Omega_t} \mathcal{D}_t^k H^j [\mathcal{D}_t^{l+1-k}, \partial_j] F_i \mathcal{D}_t^{l+1} G^i dx \\
&\le C(\|\mathcal{D}_t^l H^j [\mathcal{D}_t, \partial_j] F\|_{L^2(\Omega_t)}^2 + \sum_{k=0}^{l-1} \|\mathcal{D}_t^k H^j [\mathcal{D}_t^{l+1-k}, \partial_j] F\|_{L^2(\Omega_t)}^2 + E_l(t)) \\
&\le C(E_l(t) + \|\mathcal{D}_t^l H^j \partial_j v^k \partial_k F\|_{L^2(\Omega_t)}^2 \\
&\quad + \sum_{k=0}^{l-1} \|\mathcal{D}_t^k H \star \sum_{2 \le m \le l+2-k} \sum_{|\beta| \le l+2-k-m} \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-1}} v \\
&\qquad \star \nabla \mathcal{D}_t^{\beta_m} F\|_{L^2(\Omega_t)}^2) \\
&\le C E_{l-1}(t) E_l(t) + C E_l(t) \le C E_l(t).
\end{aligned}
$$

Step 3. To estimate $\int_{\Gamma_t} R_p^l (\mathcal{D}_t^{l+1} v \cdot \nu) dS$, we apply Lemmas 2.10 and A.12 to obtain

$$
\|\mathcal{D}_t^{l+1} v \cdot \nu\|_{H^{-\frac{1}{2}}(\Gamma_t)} \le C(\|\mathcal{D}_t^{l+1} v\|_{L^2(\Omega_t)} + \|\operatorname{div} \mathcal{D}_t^{l+1} v\|_{H^{-1}(\Omega_t)}).
$$

Therefore, it follows that

$$
\begin{aligned}
&|\int_{\Gamma_t} R_p^l (\mathcal{D}_t^{l+1} v \cdot \nu) dS| \le C(\bar{E}(t) + \|R_I^l\|_{L^2(\Omega_t)}^2 + \|R_p^l\|_{H^{\frac{1}{2}}(\Gamma_t)}^2), \quad l \le 3, \\
&|\int_{\Gamma_t} R_p^l (\mathcal{D}_t^{l+1} v \cdot \nu) dS| \le C(E_l(t) + \|R_I^l\|_{L^2(\Omega_t)}^2 + \|R_p^l\|_{H^{\frac{1}{2}}(\Gamma_t)}^2), \quad l \ge 4.
\end{aligned}
$$

It should be noted that later on, we must estimate $\|R_p^l\|_{H^{\frac{1}{2}}(\Gamma_t)}^2$.

Step 4. We estimate $I_{31}^l(t)$. We only present estimates for $l = 3$, and the cases of $l \le 2$ are similar or easier. Actually, by the a priori assumption (1.8) and the trace theorem, one has

$$
\|\bar{\nabla}(\mathcal{D}_t^3 v \cdot \mathcal{D}_t \nu)\|_{L^2(\Gamma_t)}^2
$$

$$\begin{aligned}&\le \|\bar\nabla \mathcal{D}_t^3 v \star \mathcal{D}_t\nu\|^2_{L^2(\Gamma_t)} + \|\mathcal{D}_t^3 v \star \bar\nabla\mathcal{D}_t\nu\|^2_{L^2(\Gamma_t)}\\ &\le C(\|\mathcal{D}_t\nu\|^2_{L^\infty(\Gamma_t)}\|\mathcal{D}_t^3 v\|^2_{H^{\frac{3}{2}}(\Omega_t)} + \|\mathcal{D}_t^3 v\star\bar\nabla^2 v\star\nu\|^2_{L^2(\Gamma_t)}\\ &\quad + \underbrace{\|\mathcal{D}_t^3 v\star\bar\nabla v\star\bar\nabla\nu\|^2_{L^2(\Gamma_t)}}_{=:L^3_{31}(t)})\\ &\le C\bar E(t).\end{aligned}$$

Above, we have applied the Sobolev embedding, i.e., for $p^{-1}+q^{-1}=2^{-1}, p=2\delta^{-1}$ with $\delta>0$ small enough, it holds

$$L^3_{31}(t)\le \|\mathcal{D}_t^3 v\|^2_{L^p(\Gamma_t)}\|\bar\nabla^2 v\|^2_{L^q(\Gamma_t)}\le \|\mathcal{D}_t^3 v\|^2_{H^{1-\delta}(\Gamma_t)}\|\bar\nabla^2 v\|^2_{H^\delta(\Gamma_t)},$$

and $\|\mathcal{D}_t^3 v\|^2_{H^{1-\delta}(\Gamma_t)}\|\bar\nabla^2 v\|^2_{H^\delta(\Gamma_t)} \le \|\mathcal{D}_t^3 v\|^2_{H^{\frac{3}{2}-\delta}(\Omega_t)}\|v\|^2_{H^{\frac{5}{2}+\delta}(\Omega_t)} \le C\bar E(t)$, by using the trace theorem.

As for $l\ge 4$, it follows that

$$\begin{aligned}&\|\bar\nabla(\mathcal{D}_t^l v\cdot\mathcal{D}_t\nu)\|^2_{L^2(\Gamma_t)}\\ &\le C(\|\mathcal{D}_t\nu\|^2_{L^\infty(\Gamma_t)}\|\mathcal{D}_t^l v\|^2_{H^1(\Gamma_t)} + \|\mathcal{D}_t\nu\|^2_{W^{1,4}(\Gamma_t)}\|\mathcal{D}_t^l v\|^2_{L^4(\Gamma_t)})\\ &\le C(\|\mathcal{D}_t^l v\|^2_{H^{\frac{3}{2}}(\Omega_t)} + E_{l-1}(t)\|\mathcal{D}_t^l v\|^2_{H^1(\Omega_t)})\le CE_l(t),\end{aligned}$$

where we have used the fact that $\mathcal{D}_t\nu=\bar\nabla v\star\nu$ from Lemma 2.1 and $\|\nu\|_{H^{2+\delta}(\Gamma_t)}\le C$ by (1.8) together with (2.3).

Step 5. For $I^l_{12}(t)$, we recall that it holds $[\mathcal{D}_t^{l+1},\nabla]p=\sum_{\beta_1\le l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H + R^l_{II}+R^l_{\nabla H,H}$ by (2.14). Clearly, we have $\|\sum_{\beta_1\le l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H\|^2_{L^2(\Omega_t)}\le C\bar E(t)$ for $l\le 3$, and $\|\sum_{\beta_1\le l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H\|^2_{L^2(\Omega_t)}\le CE_l(t)$ as $l\ge 4$. We leave the estimates for $\|R^l_{II}\|^2_{L^2(\Omega_t)}$ and $\|R^l_{\nabla H,H}\|^2_{L^2(\Omega_t)}$ to Section 5 (cf. Lemmas 5.3 and 5.4).

Step 6. Finally, controlling $I^l_{11}(t)$ is trickier and necessitates the application of Lemmas 2.7 and 2.10. Let u be a solution to

$$\begin{cases}-\Delta u=\operatorname{div}\mathcal{D}_t^{l+1}v, & \text{in } \Omega_t,\\ u=0, & \text{on } \Gamma_t,\end{cases}$$

where $l\ge 1$, and we integrate by parts to obtain

$$I^l_{11}(t)=-\int_{\Omega_t}\Delta\mathcal{D}_t^{l+1}pu\,dx-\int_{\Gamma_t}\mathcal{D}_t^{l+1}p\partial_\nu u\,dS=:I^l_{111}(t)+I^l_{112}(t).$$

Again by integration by parts, Lemma 2.11 and the divergence theorem, it follows that

$$\begin{aligned}&I^l_{111}(t)\\ &=\int_{\Omega_t}(\operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^{l+1}v)+\operatorname{div}(R^l_{II}+\sum_{\beta_1\le l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H+R^l_{\nabla H,H}))u\,dx\\ &\quad-\int_{\Omega_t}\operatorname{div}\mathcal{D}_t^{l+1}(H\cdot\nabla H)u\,dx\\ &=\int_{\Omega_t}(v\otimes\mathcal{D}_t^{l+1}v):\nabla^2u-(R^l_{II}+R^l_{\nabla H,H}+\sum_{\beta_1\le l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H)\cdot\nabla u\,dx\\ &\quad-\int_{\Omega_t}\operatorname{div}\mathcal{D}_t^{l+1}(H\cdot\nabla H)u\,dx-\int_{\Gamma_t}v^i\mathcal{D}_t^{l+1}v^j\partial_i u\nu_j\,dS\\ &\le C(\|u\|^2_{H^2(\Omega_t)}+E_l(t)+\|R^l_{II}\|^2_{L^2(\Omega_t)}+\|R^l_{\nabla H,H}\|^2_{L^2(\Omega_t)})\\ &\quad+\underbrace{\int_{\Omega_t}\operatorname{div}(v^i\mathcal{D}_t^{l+1}v\partial_i u)\,dx}_{=:L^l_{1111}(t)}-\underbrace{\int_{\Omega_t}\operatorname{div}\mathcal{D}_t^{l+1}(H\cdot\nabla H)u\,dx}_{=:L^l_{1112}(t)}.\end{aligned}$$

We estimate the first term by using Lemma 2.10. Indeed, it holds

$$
\begin{aligned}
|L_{1111}^l(t)| = &| \int_{\Omega_t} \nabla v \star \mathcal{D}_t^{l+1} v \star \nabla u + v \star \operatorname{div} \mathcal{D}_t^{l+1} v \star \nabla u \\
&+ v \star \mathcal{D}_t^{l+1} v \star \nabla^2 u dx| \\
\leq & C(\|u\|_{H^2(\Omega_t)}^2 + E_l(t) + \|R_I^l\|_{L^2(\Omega_t)}^2).
\end{aligned}
$$

To control $L_{1112}^l(t)$, it is important to note that the integration by parts method used previously is not applicable. However, as indicated in Lemmas 2.6 and 2.7, one-order material derivative can be substituted for one-order spatial derivative due to the divergence-free condition. Roughly speaking, we reduce the spatial derivative of $\frac{1}{2}$-order , which enables us to close the energy estimates.

In fact, we have from Lemma 2.7 that

$$
\operatorname{div} \mathcal{D}_t^{l+1}(H \cdot \nabla H) = \partial_i \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i + \nabla^3 \mathcal{D}_t^{l-1} v \star H \star H + \text{L. O. T.},
$$

and

$$
\begin{aligned}
|L_{1112}^l(t)| \leq &| \int_{\Omega_t} \partial_i \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i u dx| + C\|u\|_{L^2(\Omega_t)}^2 \\
&+ C\|\nabla^3 \mathcal{D}_t^{l-1} v \star H \star H\|_{L^2(\Omega_t)}^2 + M_{1112}^l(t) \\
\leq & C\|u\|_{H^1(\Omega_t)}^2 + CE_l(t) + M_{1112}^l(t),
\end{aligned}
$$

where we have used $H \cdot \nu = 0$, and

$$
\begin{aligned}
&| \int_{\Omega_t} \partial_i \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i u dx| \\
= &| - \int_{\Omega_t} \partial_m \mathcal{D}_t^l v^j \partial_i (\partial_j H^m H^i u) dx| + | \int_{\Gamma_t} \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i u \nu_i dS| \\
= &| \int_{\Omega_t} \partial_m \mathcal{D}_t^l v^j \partial_i \partial_j H^m H^i u + \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i \partial_i u dx| \\
\leq & CE_l(t) + C\|u\|_{H^1(\Omega_t)}^2
\end{aligned}
$$

by integration by parts. Also, $M_{1112}^l(t)$ contains lower-order terms (at most $\nabla^2 \mathcal{D}_t^{l-1}$) which can be controlled in the same fashion as before. These, together with

$$
\|u\|_{H^2(\Omega_t)}^2 \leq \| \operatorname{div} \mathcal{D}_t^{l+1} v\|_{L^2(\Omega_t)}^2 \leq C\|R_I^l\|_{L^2(\Omega_t)}^2,
$$

it follows that

$$
I_{111}^l(t) \leq C(\bar{E}(t) + \|R_I^l\|_{L^2(\Omega_t)}^2 + \|R_{II}^l\|_{L^2(\Omega_t)}^2 + \|R_{\nabla H,H}^l\|_{L^2(\Omega_t)}^2),
$$

for $l \leq 3$, and

$$
I_{111}^l(t) \leq C(E_l(t) + \|R_I^l\|_{L^2(\Omega_t)}^2 + \|R_{II}^l\|_{L^2(\Omega_t)}^2 + \|R_{\nabla H,H}^l\|_{L^2(\Omega_t)}^2),
$$

for $l \geq 4$.

We are left with $I_{112}^l(t)$. Applying Lemma 2.13 and by integration by parts, one has

$$
\begin{aligned}
\int_{\Gamma_t} \mathcal{D}_t^{l+1} p \partial_\nu dS u = &\int_{\Gamma_t} (-\Delta_B (\mathcal{D}_t^l v \cdot \nu) + R_p^l) \partial_\nu u dS \\
= &\int_{\Gamma_t} \bar{\nabla}(\mathcal{D}_t^l v \cdot \nu) \cdot \bar{\nabla} \partial_\nu u dS + \int_{\Gamma_t} R_p^l \partial_\nu u dS.
\end{aligned}
$$

Then, we use Lemmas A.9 and A.17 to deduce

$$
\begin{aligned}
|I_{112}^l(t)| \leq & C(\|\bar{\nabla}(\mathcal{D}_t^l v \cdot \nu)\|_{L^2(\Gamma_t)}^2 + \|\bar{\nabla} \partial_\nu u\|_{L^2(\Gamma_t)}^2 + \|R_p^l\|_{L^2(\Gamma_t)}^2) \\
\leq & C(\bar{E}(t) + \| \operatorname{div} \mathcal{D}_t^{l+1} v\|_{H^{\frac{1}{2}}(\Omega_t)}^2 + \|R_p^l\|_{L^2(\Gamma_t)}^2) \\
\leq & C(\bar{E}(t) + \|R_I^l\|_{H^{\frac{1}{2}}(\Omega_t)}^2 + \|R_p^l\|_{L^2(\Gamma_t)}^2),
\end{aligned}
$$

for $l \le 3$. Similarly, it holds $|I_{112}^l(t)| \le C(E_l(t) + \|R_I^l\|^2_{H^{\frac{1}{2}}(\Omega_t)} + \|R_p^l\|^2_{L^2(\Gamma_t)})$ for $l \ge 4$. This completes the proof of the proposition. □

4. Estimates for the Pressure

In this section, we treat the pressure and will show that

$$\sup_{t\in[0,T]} \|p\|_{H^3(\Omega_t)} \le C, \tag{4.1}$$

where the constant C depends on the time $T > 0$, the a priori assumptions $\mathcal{N}_T, \mathcal{M}_T$, and the initial data $\|v_0\|_{H^6(\Omega_0)}$, $\|H_0\|_{H^6(\Omega_0)}$ and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$.

For this purpose, we assume the a priori assumptions (1.8) for some $T > 0$. As a result, it follows that $\sup_{0\le t<T} \|h\|_{H^{3+\delta}(\Gamma)} \le C$ and $\sup_{0\le t<T} \|B\|_{H^{1+\delta}(\Gamma_t)} \le C$. In particular, we have $\|p\|_{H^{1+\delta}(\Gamma_t)} \le C$ and

$$\int_0^T \|p\|^2_{H^1(\Gamma_t)} dt \le C(\mathcal{N}_T, \mathcal{M}_T) T. \tag{4.2}$$

Recalling we define $H^{\frac{1}{2}}(\Gamma_t)$ via the harmonic extension. From Lemma A.15 and (A.4), we obtain

$$\begin{aligned}
\|\partial_\nu p\|^2_{L^2(\Gamma_t)} \le& C(\|\bar\nabla p\|^2_{L^2(\Gamma_t)} + \|\nabla p\|^2_{L^2(\Omega_t)} + \|\Delta p\|^2_{L^2(\Omega_t)}) \\
\le& C(\|\bar\nabla p\|^2_{L^2(\Gamma_t)} + \|p\|^2_{H^{\frac{1}{2}}(\Gamma_t)} + \|\Delta p\|^2_{L^2(\Omega_t)}) \\
\le& C(\|p\|^2_{H^1(\Gamma_t)} + \|\Delta p\|^2_{L^2(\Omega_t)}) \\
\le& C(\mathcal{N}_T, \mathcal{M}_T)(1+T).
\end{aligned} \tag{4.3}$$

For higher-order derivatives, we have the following results.

Proposition 4.1. *Assume that Γ_t is uniformly $H^{3+\delta}(\Gamma)$-regular for $\delta > 0$ sufficiently small. For smooth function f, it holds*

$$\|\nabla^2 f\|^2_{L^2(\Gamma_t)} \le C\left(\|\Delta f\|^2_{H^1(\Omega_t)} + \|f\|^2_{H^2(\Gamma_t)}\right), \tag{4.4}$$

$$\|\nabla^3 f\|^2_{L^2(\Gamma_t)} \le C\left(\|\Delta f\|^2_{H^2(\Omega_t)} + \|f\|^2_{H^3(\Gamma_t)}\right). \tag{4.5}$$

In particular, we have

$$\|\nabla^2 p\|^2_{L^2(\Gamma_t)} \le C\left(\|\Delta p\|^2_{H^1(\Omega_t)} + \|p\|^2_{H^2(\Gamma_t)}\right), \tag{4.6}$$

$$\|\nabla^2 \mathcal{D}_t p\|^2_{L^2(\Gamma_t)} \le C\left(\|\Delta \mathcal{D}_t p\|^2_{H^1(\Omega_t)} + \|\mathcal{D}_t p\|^2_{H^2(\Gamma_t)}\right), \tag{4.7}$$

$$\|\nabla^3 p\|^2_{L^2(\Gamma_t)} \le C\left(\|\Delta p\|^2_{H^2(\Omega_t)} + \|p\|^2_{H^3(\Gamma_t)}\right). \tag{4.8}$$

Proof. For any $k \in \{1,2,3\}$, it follows that

$$\|\nabla \partial_k f\|^2_{L^2(\Gamma_t)} \le C(\|\bar\nabla \partial_k f\|^2_{L^2(\Gamma_t)} + \|\nabla^2 f\|^2_{L^2(\Omega_t)} + \|\nabla \Delta f\|^2_{L^2(\Omega_t)}),$$

by applying Lemma A.15. Recall that we extend the unit outer normal ν to Ω_t by the harmonic extension and $\|\tilde\nu\|_{H^{\frac{5}{2}+\delta}(\Omega_t)} \le C$. This, combined with Lemmas 2.1 and A.15 implies that

$$\begin{aligned}
\|\bar\nabla \partial_k f\|^2_{L^2(\Gamma_t)} &\le C(\|\nabla \bar\nabla f\|^2_{L^2(\Gamma_t)} + \|\nabla f \star \nabla \tilde\nu \star \tilde\nu\|^2_{L^2(\Gamma_t)}) \\
&\le C(\|\bar\nabla^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla \bar\nabla f\|^2_{L^2(\Omega_t)} + \|\Delta \bar\nabla f\|^2_{L^2(\Omega_t)} \\
&\qquad + \|\nabla f\|^2_{H^1(\Omega_t)}) \\
&\le C(\|\bar\nabla^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla \Delta f\|^2_{L^2(\Omega_t)} + \|\nabla f\|^2_{H^1(\Omega_t)} \\
&\qquad + \|\nabla f \star \nabla \tilde\nu \star \nabla \tilde\nu\|^2_{L^2(\Omega_t)} + \|\nabla^2 f \star \nabla \tilde\nu\|^2_{L^2(\Omega_t)}) \\
&\le C(\|\bar\nabla^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla \Delta f\|^2_{L^2(\Omega_t)} + \|\nabla f\|^2_{H^1(\Omega_t)}),
\end{aligned}$$

and therefore,

$$\|\nabla\partial_k f\|^2_{L^2(\Gamma_t)} \le C(\|\bar\nabla^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla\Delta f\|^2_{L^2(\Omega_t)} + \|\nabla f\|^2_{H^1(\Omega_t)}).$$

Next, we apply (A.4) and Lemma A.16 to find that

$$\begin{aligned}\|\nabla f\|^2_{H^1(\Omega_t)} &\le C(\|\partial_\nu f\|^2_{H^{\frac12}(\Gamma_t)} + \|\nabla f\|^2_{L^2(\Omega_t)} + \|\Delta f\|^2_{L^2(\Omega_t)})\\ &\le C(\|\partial_\nu f\|^2_{H^{\frac12}(\Gamma_t)} + \|f\|^2_{H^{\frac12}(\Gamma_t)} + \|\Delta f\|^2_{L^2(\Omega_t)}).\end{aligned}$$

To control $\|\partial_\nu f\|^2_{H^{\frac12}(\Gamma_t)}$, using Lemma A.15 and by interpolation, one has

$$\begin{aligned}\|\partial_\nu f\|^2_{H^{\frac12}(\Gamma_t)} &\le \varepsilon\|\bar\nabla\partial_\nu f\|^2_{L^2(\Gamma_t)} + C_\varepsilon\|\partial_\nu f\|^2_{L^2(\Gamma_t)}\\ &\le \varepsilon(\|\nabla^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla f\|^2_{H^1(\Omega_t)})\\ &\quad + C_\varepsilon(\|\bar\nabla f\|^2_{L^2(\Gamma_t)} + \|f\|^2_{H^{\frac12}(\Gamma_t)} + \|\Delta f\|^2_{L^2(\Omega_t)}),\end{aligned}$$

where $\varepsilon > 0$ is sufficiently small. We conclude that

$$\|\nabla f\|^2_{H^1(\Omega_t)} \le \varepsilon\|\nabla^2 f\|^2_{L^2(\Gamma_t)} + C(\|f\|^2_{H^1(\Gamma_t)} + \|\Delta f\|^2_{L^2(\Omega_t)}), \tag{4.9}$$

and then (4.4) follows.

To prove the second claim, by Lemma A.15 again with $k \in \{1,2,3\}$, it holds

$$\|\nabla\partial_k\partial_l f\|^2_{L^2(\Gamma_t)} \le C(\|\bar\nabla\partial_k\partial_l f\|^2_{L^2(\Gamma_t)} + \|\nabla^3 f\|^2_{L^2(\Omega_t)} + \|\nabla^2\Delta f\|^2_{L^2(\Omega_t)}).$$

To estimate $\|\nabla^3 f\|^2_{L^2(\Omega_t)}$, from Lemma A.16, we obtain

$$\|\partial_i f\|^2_{H^2(\Omega_t)} \le C(\|\partial_\nu\partial_i f\|^2_{H^{\frac12}(\Gamma_t)} + \|\nabla f\|^2_{L^2(\Omega_t)} + \|\nabla\Delta f\|^2_{L^2(\Omega_t)})$$

for $i \in \{1,2,3\}$, and by interpolation, we see that

$$\|\partial_\nu\partial_i f\|^2_{H^{\frac12}(\Gamma_t)} \le \varepsilon\|\bar\nabla\partial_\nu\partial_i f\|^2_{L^2(\Gamma_t)} + C_\varepsilon\|\partial_\nu\partial_i f\|^2_{L^2(\Gamma_t)},$$

where $\varepsilon > 0$ is small enough. These, combined with (4.4), (A.4) and the fact that $\|\tilde\nu\|_{H^{\frac52+\delta}(\Omega_t)} \le C$, yield

$$\begin{aligned}\|\nabla f\|^2_{H^2(\Omega_t)} &\le \varepsilon(\|\nabla^3 f\|^2_{L^2(\Gamma_t)} + \|\nabla^2 f\star\nabla\tilde\nu\|^2_{L^2(\Gamma_t)}) + \|f\|^2_{H^2(\Gamma_t)}\\ &\quad + \|\Delta f\|^2_{H^1(\Omega_t)}\\ &\le \varepsilon\|\nabla^3 f\|^2_{L^2(\Gamma_t)} + \|f\|^2_{H^2(\Gamma_t)} + \|\Delta f\|^2_{H^1(\Omega_t)}.\end{aligned} \tag{4.10}$$

Then, we control $\|\bar\nabla\partial_k\partial_l f\|^2_{L^2(\Gamma_t)}$ by Lemma A.15 and the fact that $\Delta\tilde\nu = 0$, i.e.,

$$\begin{aligned}&\|\bar\nabla\partial_k\partial_l f\|^2_{L^2(\Gamma_t)}\\ &\le C\|\partial_k\bar\nabla\partial_l f\|^2_{L^2(\Gamma_t)} + C\|\nabla^2 f\star\nabla\tilde\nu\star\tilde\nu\|^2_{L^2(\Gamma_t)}\\ &\le C(\|\bar\nabla^2\partial_l f\|^2_{L^2(\Gamma_t)} + \|\nabla\bar\nabla\nabla f\|^2_{L^2(\Omega_t)} + \|\Delta\bar\nabla\nabla f\|^2_{L^2(\Omega_t)} + \|\nabla^2 f\|^2_{L^2(\Gamma_t)})\\ &\le C(\|\bar\nabla^2\partial_l f\|^2_{L^2(\Gamma_t)} + \|\nabla^2 f\|^2_{H^1(\Omega_t)} + \|\Delta f\|^2_{H^2(\Omega_t)} + \|f\|^2_{H^2(\Gamma_t)}).\end{aligned}$$

Again by (4.4) and Lemma A.15, we obtain

$$\begin{aligned}&\|\bar\nabla^2\partial_l f\|^2_{L^2(\Gamma_t)}\\ &\le \|\partial_l\bar\nabla^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla^2 f\star\nabla\tilde\nu\|^2_{L^2(\Gamma_t)} + \|\nabla f\star\bar\nabla\nabla\tilde\nu\|^2_{L^2(\Gamma_t)}\\ &\quad + \|\nabla f\star\nabla\tilde\nu\star\nabla\tilde\nu\|^2_{L^2(\Gamma_t)}\\ &\le \|\bar\nabla^3 f\|^2_{L^2(\Gamma_t)} + \|\nabla^3 f\|^2_{L^2(\Omega_t)} + \|\nabla^3 f\star\nabla\tilde\nu\|^2_{L^2(\Omega_t)} + \|\nabla^2 f\star\nabla\tilde\nu\|^2_{L^2(\Omega_t)}\\ &\quad + \|\nabla^2\Delta f\|^2_{L^2(\Omega_t)} + \|\nabla f\star\nabla^2\tilde\nu\|^2_{L^2(\Omega_t)} + \|\nabla^2 f\|^2_{L^2(\Gamma_t)}\\ &\quad + \|\nabla^2 f\star\nabla^2\tilde\nu\|^2_{L^2(\Omega_t)} + \|\nabla f\|^2_{H^{\frac32+\delta}(\Omega_t)}\end{aligned}$$

$$\le C(\|f\|^2_{H^3(\Gamma_t)} + \|\nabla f\|^2_{H^2(\Omega_t)} + \|\Delta f\|^2_{H^2(\Omega_t)}).$$

Recalling (4.10), we conclude that

$$\|\nabla^3 f\|^2_{L^2(\Gamma_t)} \le \varepsilon \|\nabla^3 f\|^2_{L^2(\Gamma_t)} + C(\|f\|^2_{H^3(\Gamma_t)} + \|\Delta f\|^2_{H^2(\Omega_t)}).$$

This completes the proof. □

We will proceed with the estimates for the pressure.

Lemma 4.2. *Assume that* (1.8) *holds for some* $T > 0$. *Then, we have*

$$\sup_{t\in[0,T]} \|\nabla p\|^2_{L^2(\Omega_t)} \le e^{C(\mathcal{N}_T,\mathcal{M}_T)(1+T)} \left(1 + \|\nabla p\|^2_{L^2(\Omega_0)}\right).$$

Proof. From Lemmas 2.1 and A.2, and the divergence-free condition, we differentiate as follows

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\Omega_t} |\nabla p|^2 dx &= \int_{\Omega_t} \mathcal{D}_t \nabla p \cdot \nabla p dx \\
&= \int_{\Omega_t} \nabla \mathcal{D}_t p \cdot \nabla p dx + \int_{\Omega_t} \nabla v \star \nabla p \star \nabla p dx \\
&=: I_1(t) + I_2(t).
\end{aligned}$$

Clearly, the a priori assumptions (1.8) imply that $|I_2(t)| \le C\|\nabla p\|^2_{L^2(\Omega_t)}$. For the first term, by (2.15), (4.3), and the divergence theorem, we have

$$\begin{aligned}
I_1(t) &\le \int_{\Omega_t} \operatorname{div}(\mathcal{D}_t p \nabla p) - \mathcal{D}_t p \Delta p dx \\
&\le \int_{\Gamma_t} \mathcal{D}_t p \partial_\nu p dS - \int_{\Omega_t} \mathcal{D}_t p \Delta p dx \\
&\le C(1 + \|p\|^2_{H^1(\Gamma_t)}) - \int_{\Omega_t} \mathcal{D}_t p \Delta p dx.
\end{aligned}$$

To control $\int_{\Omega_t} \mathcal{D}_t p \Delta p dx$, we consider the following elliptic equation

$$\begin{cases} -\Delta u = \Delta p, & \text{in } \Omega_t, \\ u = 0, & \text{on } \Gamma_t. \end{cases}$$

Then, we see that

$$-\int_{\Omega_t} \mathcal{D}_t p \Delta p dx = \int_{\Omega_t} \Delta \mathcal{D}_t p u dx + \int_{\Gamma_t} \mathcal{D}_t p \partial_\nu u dS =: I_{11}(t) + I_{12}(t).$$

Note that (2.6) implies $|\Delta p| \le C$, and we have $\|u\|_{H^1(\Omega_t)} \le C$. Also, from Lemma A.15, we get $\|\nabla u\|^2_{L^2(\Gamma_t)} \le C$ and

$$|I_{12}(t)| \le \|\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + \|\partial_\nu u\|^2_{L^2(\Gamma_t)} \le C(1 + \|p\|^2_{H^1(\Gamma_t)}).$$

We are left with $I_{11}(t)$, for which one can repeat the argument in [JLM22, Propsition 6.3] to deduce $\|u\|^2_{H^2(\Omega_t)} \le C(1 + \|p\|^2_{H^1(\Gamma_t)})$. Then, by (1.1), (1.8), Lemma 2.11, (2.13) and (4.3), we integrate by parts to obtain

$$I_{11}(t) \le C(1 + \|p\|^2_{H^1(\Gamma_t)} + \|\nabla p\|^2_{L^2(\Omega_t)}).$$

Combining the above calculations, it follows that

$$I_1(t) + I_2(t) \le C(1 + \|p\|^2_{H^1(\Gamma_t)} + \|\nabla p\|^2_{L^2(\Omega_t)}).$$

With the help of estimate (4.2), the proof is complete since we have

$$\frac{d}{dt} \ln(1 + \|\nabla p\|^2_{L^2(\Omega_t)}) \le C(1 + \|p\|^2_{H^1(\Gamma_t)}).$$

□

Lemma 4.3. *Assume that* (1.8) *holds for some* $T>0$. *Then, we have*

$$\int_0^T \|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)}dt \le C(\mathcal{N}_T,\mathcal{M}_T)(1+T).$$

Proof. We define

$$I(t) := \int_{\Gamma_t} \bar\nabla p\cdot\bar\nabla(\nabla v\nu\cdot\nu)dS,$$

and from the hypothesis (1.8) and (4.2), we see that

$$|I(t)| \le C\|\bar\nabla p\|^2_{L^2(\Gamma_t)} + C\|\nabla^2 v\|^2_{L^2(\Gamma_t)} + C\|\nabla v\star B\|^2_{L^2(\Gamma_t)} \le C.$$

Again by (1.8) and applying the divergence theorem, Lemmas 2.1 and A.3, we deduce for sufficiently small $\varepsilon>0$ that

$$\begin{aligned}\frac{d}{dt}I(t) &\le C|I(t)| + \int_{\Gamma_t} \mathcal{D}_t\bar\nabla p\cdot\bar\nabla(\nabla v\nu\cdot\nu) + \bar\nabla p\cdot\mathcal{D}_t\bar\nabla(\nabla v\nu\cdot\nu)dS\\ &\le C_\varepsilon + \varepsilon\|\bar\nabla\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + \int_{\Gamma_t}\bar\nabla p\cdot\bar\nabla\mathcal{D}_t(\nabla v\nu\cdot\nu)dS\\ &\le C_\varepsilon + \varepsilon\underbrace{\|\bar\nabla\mathcal{D}_t p\|^2_{L^2(\Gamma_t)}}_{=:I_1(t)} \underbrace{- \int_{\Gamma_t}\Delta_B p\mathcal{D}_t(\nabla v\nu\cdot\nu)dS}_{=:I_2(t)}.\end{aligned}$$

By (1.8), (2.15) and (4.2), it holds

$$\begin{aligned}|I_1(t)| &\le C(1+\|v_n\|^2_{H^3(\Gamma_t)} + \|\bar\nabla B\star B\star v_n\|^2_{L^2(\Gamma_t)} + \|\bar\nabla p\|^2_{H^1(\Gamma_t)})\\ &\le C(1+\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)}).\end{aligned}$$

For the second term, from (1.8), Lemma (2.1) and the divergence theorem, we have

$$\begin{aligned}|I_2(t)| \le& -\int_{\Gamma_t}\Delta_B p(\nabla\mathcal{D}_t v\nu\cdot\nu)dS + C\|\bar\nabla p\|_{L^1(\Gamma_t)}\\ =& -\int_{\Gamma_t}\Delta_B p(\nabla(-\nabla p + H\cdot\nabla H)\nu\cdot\nu)dS + \varepsilon\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon\\ \le& \int_{\Gamma_t}\Delta_B p(\nabla^2 p\nu\cdot\nu)dS - \int_{\Gamma_t}\Delta_B p\star\nabla^2 H\star\nabla H\star\nu\star\nu dS\\ &+\varepsilon\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon\\ \le& \int_{\Gamma_t}\Delta_B p(\nabla^2 p\nu\cdot\nu)dS + \varepsilon\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon.\end{aligned}$$

Recalling $|\Delta p|\le C$ and by (2.2), (4.3), the divergence theorem, for $\varepsilon>0$ small enough, we deduce

$$\begin{aligned}\int_{\Gamma_t}\Delta_B p(\nabla^2 p\nu\cdot\nu)dS =& \int_{\Gamma_t}\Delta_B p\Delta p - \Delta_B p\Delta_B p - \Delta_B p\mathcal{A}\partial_\nu p dS\\ &\le C + \frac{\varepsilon}{2}\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon\|p\|^2_{H^1(\Gamma_t)} - \int_{\Gamma_t}|\bar\nabla^2 p|^2 dS\\ &\quad + \|\Delta_B p\|_{L^2(\Gamma_t)}\|\partial_\nu p\|_{L^2(\Gamma_t)}\|p\|_{L^\infty(\Gamma_t)}\\ &\le -\frac{3}{4}\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon.\end{aligned}$$

Above, we have applied the results in [FJM20, Remark 2.4] that

$$\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} \le \|\Delta_B p\|^2_{L^2(\Gamma_t)} + C\int_{\Gamma_t}|B|^2|\bar\nabla p|^2 dS.$$

Combining the above calculations, it follows that

$$\frac{d}{dt}I(t) \le -\frac{1}{2}\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C.$$

Integrating over $[0,T]$, we obtain

$$\int_0^T \|\bar{\nabla}^2 p\|_{L^2(\Gamma_t)}^2 dt \le C(1+T+I(0)) \le C(\mathcal{N}_T,\mathcal{M}_T)(1+T).$$

□

Lemma 4.4. *Assume that (1.8) holds for some $T>0$. Then, we have*

$$\sup_{t\in[0,T]} \|\nabla^2 p\|_{L^2(\Omega_t)}^2 \le e^{C(\mathcal{N}_T,\mathcal{M}_T)(1+T)}(1+\|\nabla^2 p\|_{L^2(\Omega_0)}^2).$$

Proof. We differentiate and apply Lemma 2.3 to obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\Omega_t}|\nabla^2 p|^2 &= \frac{1}{2}\int_{\Omega_t}|\nabla^2 p|^2 \overbrace{\operatorname{div} v}^{=0}\, dx + \int_{\Omega_t}\mathcal{D}_t\nabla^2 p : \nabla^2 p dx\\ &= \int_{\Omega_t}\nabla^2\mathcal{D}_t p : \nabla^2 p dx\\ &\quad + \int_{\Omega_t}\nabla^2 v\star\nabla p\star\nabla^2 p + \nabla v\star\nabla^2 p\star\nabla^2 p dx\\ &=: I_1(t)+I_2(t).\end{aligned}$$

From (1.8), (2.6) and using Lemma 4.2, we have

$$\begin{aligned}I_1(t) &\le \int_{\Omega_t}\sum_{i,j}\partial_i(\partial_j\mathcal{D}_t p\partial_i\partial_j p)dx - \int_{\Omega_t}\nabla\mathcal{D}_t p\cdot\nabla\Delta p dx\\ &\le \int_{\Gamma_t}\sum_j \partial_j\mathcal{D}_t p\partial_\nu\partial_j p dS + \int_{\Omega_t}\Delta\mathcal{D}_t p\Delta p dx - \int_{\Gamma_t}\partial_\nu\mathcal{D}_t p\Delta p dS\\ &\le C\sum_j\|\partial_\nu\partial_j p\|_{L^2(\Gamma_t)}^2 + C\|\partial_\nu\mathcal{D}_t p\|_{L^2(\Gamma_t)}^2 + C\|\Delta\mathcal{D}_t p\|_{L^2(\Omega_t)}^2\\ &=: I_{11}(t)+I_{12}(t)+I_{13}(t),\\ I_2(t) &\le C(\|v\|_{H^{\frac{7}{2}}(\Omega_t)}^2\|\nabla p\|_{L^6(\Omega_t)}^2 + \|\nabla^2 p\|_{L^2(\Omega_t)}^2)\\ &\le C(1+\|\nabla^2 p\|_{L^2(\Omega_t)}^2).\end{aligned}$$

We apply Lemmas 2.11 and 4.2, and (4.6) to obtain $|I_{13}(t)| \le C(1+\|\nabla^2 p\|_{L^2(\Omega_t)}^2)$ and $|I_{11}(t)| \le C(1+\|p\|_{H^2(\Gamma_t)}^2)$. Finally, (1.8), Lemmas 2.11 and A.15, and (A.4) imply that

$$\begin{aligned}|I_{12}(t)| &\le C(\|\bar{\nabla}\mathcal{D}_t p\|_{L^2(\Gamma_t)}^2 + \|\nabla\mathcal{D}_t p\|_{L^2(\Omega_t)}^2 + \|\Delta\mathcal{D}_t p\|_{L^2(\Omega_t)}^2)\\ &\le C(\|\bar{\nabla}\mathcal{D}_t p\|_{L^2(\Gamma_t)}^2 + \|\mathcal{D}_t p\|_{H^{\frac{1}{2}}(\Gamma_t)}^2 + \|\Delta\mathcal{D}_t p\|_{L^2(\Omega_t)}^2)\\ &\le C(1+\|p\|_{H^2(\Gamma_t)}^2 + \|\nabla^2 p\|_{L^2(\Omega_t)}^2).\end{aligned}$$

Combined with (4.2) and Lemma 4.3, the proof is complete, since

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\Omega_t}|\nabla^2 p|^2 dx &\le \sum_{i=1}^3 I_{1i}(t)+I_2(t)\\ &\le C(1+\|\nabla^2 p\|_{L^2(\Omega_t)}^2 + \|p\|_{H^2(\Gamma_t)}^2).\end{aligned}$$

□

We move on to higher-order pressure estimates.

Lemma 4.5. *Assume that (1.8) holds for some $T>0$. Then, we have*

$$\begin{aligned}&\sup_{t\in[0,T]}\|\bar{\nabla}^2 p\|_{L^2(\Gamma_t)}^2 + \int_0^T \|\bar{\nabla}^3 p\|_{L^2(\Gamma_t)}^2 dt\\ &\le C\left(T,\mathcal{N}_T,\mathcal{M}_T,\|\bar{\nabla}^2 p\|_{L^2(\Gamma_0)},\|\nabla p\|_{H^1(\Omega_0)}\right).\end{aligned}$$

Proof. We define

$$I(t) := \int_{\Gamma_t} \bar\nabla^2 p : \bar\nabla^2(\nabla v\nu\cdot\nu)dS + \varepsilon\int_{\Gamma_t}|\bar\nabla^2 p|^2 dS =: I_1(t)+\varepsilon I_2(t),$$

where $\varepsilon>0$ will be chosen later. From (1.8), (4.2), Lemmas 4.3 and A.11, we have

$$\begin{aligned}|I_1(t)| \le& \frac{\varepsilon}{2}\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon(\|\nabla^3 v\|^2_{L^2(\Gamma_t)} + \|\nabla^2 v\star B\|^2_{L^2(\Gamma_t)}\\ &+\|\nabla v\star\bar\nabla B\|^2_{L^2(\Gamma_t)})\\ \le& \frac{\varepsilon}{2}\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon,\end{aligned}$$

and $I(t)\ge -C_\varepsilon + \frac{\varepsilon}{2}\|\bar\nabla^2 p(\cdot,t)\|^2_{L^2(\Gamma_t)}$. We differentiate and use (1.8), (4.2), the divergence theorem, Lemmas 2.1 and A.11 to obtain

$$\begin{aligned}\frac{d}{dt}I_1(t) \le& C|I_1(t)| + \int_{\Gamma_t}\mathcal{D}_t\bar\nabla^2 p : \bar\nabla^2(\nabla v\nu\cdot\nu) + \bar\nabla^2 p : \mathcal{D}_t\bar\nabla^2(\nabla v\nu\cdot\nu)dS\\ \le& C_\varepsilon + \varepsilon(\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + \|\bar\nabla^2\mathcal{D}_t p\|^2_{L^2(\Gamma_t)})\\ &+\int_{\Gamma_t}\bar\nabla^2 p : \bar\nabla^2\mathcal{D}_t(\nabla v\nu\cdot\nu)dS\\ \le& \varepsilon\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon\\ &+\varepsilon\underbrace{\|\bar\nabla^2\mathcal{D}_t p\|^2_{L^2(\Gamma_t)}}_{=:I_{11}(t)}\underbrace{-\int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla\mathcal{D}_t(\nabla v\nu\cdot\nu)dS}_{=:I_{12}(t)}.\end{aligned}$$

The first term can be controlled by (1.8), (2.15), (4.2) and Lemma A.11, i.e.,

$$\begin{aligned}|I_{11}(t)| \le& C(\|v_n\|^2_{H^4(\Gamma_t)} + \|p\|^2_{H^3(\Gamma_t)} + \|B\|^2_{H^2(\Gamma_t)})\\ \le& C(1+\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)} + \|\bar\nabla^3 p\|^2_{L^2(\Gamma_t)}).\end{aligned}$$

As for $I_{12}(t)$, applying (1.8), Lemma (2.1) and the divergence theorem, it follows that

$$\begin{aligned}I_{12}(t) \le& -\int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla(\nabla\mathcal{D}_t v\nu\cdot\nu)dS + C(\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)}+1)\\ =& -\int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla(\nabla(-\nabla p + H\cdot\nabla H)\nu\cdot\nu)dS + C(\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)}+1)\\ \le& \int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla(\nabla^2 p\nu\cdot\nu)dS - \int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla(\nabla^2 H\star H\star\nu\star\nu)dS\\ &+C(\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)}+1)\\ \le& \int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla(\nabla^2 p\nu\cdot\nu)dS + \varepsilon\|\bar\nabla^3 p\|^2_{L^2(\Gamma_t)} + C(\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)}+1).\end{aligned}$$

To estimate $\int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla(\nabla^2 p\nu\cdot\nu)dS$, by (1.8), (2.2), (2.6), (4.3), Lemma A.16 and the divergence theorem, it holds

$$\begin{aligned}&\int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla(\nabla^2 p\nu\cdot\nu)dS\\ =& \int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla\Delta p dS - \int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla\Delta_B p dS - \int_{\Gamma_t}\bar\nabla\Delta_B p\cdot\bar\nabla(\mathcal{A}\partial_\nu p)dS\\ \le& C_\varepsilon\|\Delta p\|^2_{H^{\frac32}(\Omega_t)} + \varepsilon\|\bar\nabla^3 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon\|p\|^2_{H^2(\Gamma_t)} - \frac78\int_{\Gamma_t}|\bar\nabla^3 p|^2 dS\\ &+\|\bar\nabla^3 p\|_{L^2(\Gamma_t)}(\|\bar\nabla\partial_\nu p\|_{L^2(\Gamma_t)}\|p\|_{L^\infty(\Gamma_t)} + \|\partial_\nu p\|_{L^4(\Gamma_t)}\|\bar\nabla p\|_{L^4(\Gamma_t)})\\ \le& C_\varepsilon - \frac34\int_{\Gamma_t}|\bar\nabla^3 p|^2 dS + C_\varepsilon\|\nabla\partial_\nu p\|^2_{L^2(\Gamma_t)} + \|\bar\nabla^3 p\|_{L^2(\Gamma_t)}\|\nabla p\|^2_{H^1(\Omega_t)}\end{aligned}$$

$$\leq C_\varepsilon - \frac{1}{2}\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)} + C\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)}.$$

Above, we have used Lemma 2.1 and (4.6) to deduce

$$\begin{aligned}\|\nabla\partial_\nu p\|^2_{L^2(\Gamma_t)} &\leq C(\|\partial_\nu \nabla p\|^2_{L^2(\Gamma_t)} + \|\nabla p \star \nabla\tilde{\nu}\|^2_{L^2(\Gamma_t)})\\ &\leq C(1 + \|p\|^2_{H^2(\Gamma_t)} + \|\Delta p\|^2_{H^1(\Omega_t)}),\end{aligned}$$

and the following result in [FJM20, Lemma 2.3]

$$\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)} \leq \|\bar{\nabla}\Delta_B p\|^2_{L^2(\Gamma_t)} + C\|p\|^2_{H^2(\Gamma_t)}.$$

Similarly, we can obtain

$$\frac{d}{dt} I_2(t) \leq C(1 + \|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + \|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)}).$$

Combined the above calculations and by choosing suitable $\varepsilon > 0$, it follows that

$$\frac{d}{dt} I(t) \leq -\frac{1}{4}\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)} + C\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C.$$

Integrating the above over $[0, t]$ with $0 < t \leq T$ and recalling (4.2) together with $I(t) \geq -C_\varepsilon + \frac{\varepsilon}{2}\|\bar{\nabla}^2 p(\cdot, t)\|^2_{L^2(\Gamma_t)}$, it follows that

$$\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + \int_0^t \|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_s)} ds \leq C\left(T, \mathcal{N}_T, \mathcal{M}_T, \|\bar{\nabla}^2 p\|_{L^2(\Gamma_0)}, \|\nabla p\|_{H^1(\Omega_0)}\right).$$

□

Lemma 4.6. *Assume that (1.8) holds for some $T > 0$. Then, we have*

$$\sup_{t\in[0,T]} \|\nabla^3 p\|^2_{L^2(\Omega_t)} \leq C\left(\mathcal{N}_T, \mathcal{M}_T, \|\bar{\nabla}^2 p\|_{L^2(\Gamma_0)}, \|\nabla p\|_{H^2(\Omega_0)}, T\right).$$

Moreover, we have

$$\sup_{t\in[0,T]} \|p\|^2_{H^3(\Omega_t)} \leq C\left(\mathcal{N}_T, \mathcal{M}_T, \|\bar{\nabla}^2 p\|_{L^2(\Gamma_0)}, \|\nabla p\|_{H^2(\Omega_0)}, T\right).$$

Proof. We differentiate and apply Lemma 2.3 to obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\Omega_t} |\nabla^3 p|^2 dx &= \frac{1}{2}\int_{\Omega_t} \sum_{ijk} \mathcal{D}_t \partial_{ijk} p \partial_{ijk} p dx\\ &= \int_{\Omega_t} \sum_{ijk} \partial_{ijk}\mathcal{D}_t p \partial_{ijk} p dx + \int_{\Omega_t} \nabla^3 v \star \nabla p \star \nabla^3 p\\ &\quad + \nabla^2 v \star \nabla^2 p \star \nabla^3 p + \nabla v \star \nabla^3 p \star \nabla^3 p dx\\ &=: I_1(t) + I_2(t).\end{aligned}$$

From (1.8), (2.6) and Lemma 2.11, we have

$$\begin{aligned}I_1(t) &\leq \int_{\Omega_t} \sum_{ijk} \partial_i(\partial_{jk}\mathcal{D}_t p \partial_{ijk} p) dx - \int_{\Omega_t} \sum_{jk} \partial_{jk}\mathcal{D}_t p \partial_{jk}\Delta p dx\\ &\leq \int_{\Gamma_t} \sum_{jk} \partial_{jk}\mathcal{D}_t p \partial_\nu \partial_{jk} p dS + \int_{\Omega_t} \sum_k \partial_k \Delta\mathcal{D}_t p \partial_k \Delta p dx\\ &\quad - \int_{\Gamma_t} \sum_k \partial_\nu \partial_k \mathcal{D}_t p \partial_k \Delta p dS\\ &\leq C\sum_{jk} \|\partial_\nu \partial_{jk} p\|^2_{L^2(\Gamma_t)} + C\sum_{jk} \|\partial_{jk}\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + C\|\nabla\Delta\mathcal{D}_t p\|_{L^1(\Omega_t)}\end{aligned}$$

$$\leq C\underbrace{\sum_{jk}\|\partial_\nu\partial_{jk}p\|^2_{L^2(\Gamma_t)}}_{=:I_{11}(t)}+C\underbrace{\sum_{jk}\|\partial_{jk}\mathcal{D}_tp\|^2_{L^2(\Gamma_t)}}_{=:I_{12}(t)}+C(1+\|\nabla p\|^2_{H^2(\Omega_t)}),$$

and

$$\begin{aligned}|I_2(t)|&\leq C(\|\nabla^3v\|^2_{L^3(\Omega_t)}\|\nabla p\|^2_{L^6(\Omega_t)}+\|\nabla^2v\|^2_{L^\infty(\Omega_t)}\|\nabla^2p\|^2_{L^2(\Omega_t)}\\&\quad+\|\nabla^3p\|^2_{L^2(\Omega_t)})\\&\leq C(1+\|\nabla p\|^2_{H^2(\Omega_t)}).\end{aligned}$$

Applying (4.7) and (4.8), we obtain

$$\begin{aligned}\sum_{i=1}^2 I_{1i}(t)&\leq C(\|\Delta p\|^2_{H^2(\Omega_t)}+\|\Delta\mathcal{D}_tp\|^2_{H^1(\Omega_t)}+\|\mathcal{D}_tp\|^2_{H^2(\Gamma_t)}+\|p\|^2_{H^3(\Gamma_t)})\\&\leq C(\|\Delta p\|^2_{H^2(\Omega_t)}+\|\nabla p\|^2_{H^2(\Omega_t)}+\|\mathcal{D}_tp\|^2_{H^2(\Gamma_t)}+\|p\|^2_{H^3(\Gamma_t)})\\&\leq C(1+\|\nabla p\|^2_{H^2(\Omega_t)}+\|p\|^2_{H^3(\Gamma_t)}).\end{aligned}$$

The first claim follows from (4.2), Lemmas 4.2, 4.3, 4.4 and 4.5, since

$$\frac{d}{dt}\frac{1}{2}\int_{\Omega_t}|\nabla^3p|^2\leq C(1+\|\nabla p\|^2_{H^2(\Omega_t)}+\|p\|^2_{H^3(\Gamma_t)}).$$

This, together with the previous pressure estimates and Lemma A.16, yields the second claim. □

We conclude this section by stating the following result: the initial quantities $\bar{E}(0)$ and $\sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|^2_{H^{\frac{3}{2}k+1}(\Omega_0)}$ can be controlled by the initial velocity, magnetic field and mean curvature.

Proposition 4.7. *Assume that Ω_0 is a smooth domain such that $\|h_0\|_{L^\infty(\Gamma)}<\mathcal{R}$. Then, we have*

$$\bar{E}(0)+\sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|^2_{H^{\frac{3}{2}k+1}(\Omega_0)}\leq C,$$

where the constant C depends on $\mathcal{M}_0:=\mathcal{R}-\|h_0\|_{L^\infty(\Gamma)}$, $\|v_0\|_{H^6(\Omega_0)}$, $\|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}\|_{H^5(\Gamma_0)}$.

Remark. It should be noted that the result remains valid for any $t\in(0,T)$, provided $\|h(\cdot,t)\|_{L^\infty(\Gamma)}\leq\mathcal{R}$, i.e.,

$$\begin{aligned}&\bar{E}(t)+\sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|^2_{H^{\frac{3}{2}k+1}(\Omega_t)}\\&\leq C\left(\mathcal{R}-\|h(\cdot,t)\|_{L^\infty(\Gamma)},\|v\|_{H^6(\Omega_t)},\|H\|_{H^6(\Omega_t)},\|\mathcal{A}\|_{H^5(\Gamma_t)}\right).\end{aligned}\tag{4.11}$$

Proof of Proposition 4.7. We divide the proof into three steps.

Step 1. We control $\|\mathcal{D}_t^{4-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_0)}$ by the lower-order velocity terms using (2.9) and (2.10). For $k=0$, we apply (2.9) to obtain

$$\begin{aligned}&\|\mathcal{D}_t^4H\|^2_{L^2(\Omega_0)}\\&\leq C\|\sum_{1\leq m\leq 4}\sum_{|\beta|\leq 4-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_m}v\star H\|^2_{L^2(\Omega_0)}\\&\leq C\|H\|^2_{L^\infty(\Omega_0)}(\sum_{|\beta|\leq 3}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{L^2(\Omega_0)}+\sum_{|\beta|\leq 2}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{L^3(\Omega_0)}\|\nabla\mathcal{D}_t^{\beta_2}v\|^2_{L^6(\Omega_0)}\\&\quad+\sum_{|\beta|\leq 1}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{L^6(\Omega_0)}\|\nabla\mathcal{D}_t^{\beta_2}v\|^2_{L^6(\Omega_0)}\|\nabla\mathcal{D}_t^{\beta_3}v\|^2_{L^6(\Omega_0)}+\|v\|^8_{H^3(\Omega_0)})\\&\leq C\|\mathcal{D}_t^3v\|^2_{H^1(\Omega_0)}+C(1+\|\mathcal{D}_t^2v\|^2_{H^2(\Omega_0)})(1+\|\mathcal{D}_tv\|^2_{H^2(\Omega_0)}).\end{aligned}$$

We claim that

$$\begin{aligned}
&\sum_{k=1}^{3}\|\mathcal{D}_t^{4-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_0)}\\
\le{}& C(\|v\|_{H^4(\Omega_0)},\|H\|_{H^4(\Omega_0)})\\
&\cdot(1+\sum_{k=1}^{3}\|\mathcal{D}_t^{4-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_0)}+\|v\|_{H^6(\Omega_0)}+\|H\|_{H^6(\Omega_0)}).
\end{aligned}\tag{4.12}$$

Indeed, by (2.9), it follows that

$$\begin{aligned}
&\|\mathcal{D}_t^3H\|^2_{H^{\frac{3}{2}}(\Omega_0)}\\
\le{}& C\|\sum_{1\le m\le 3}\sum_{|\beta|\le 3-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_m}v\star H\|^2_{H^{\frac{3}{2}}(\Omega_0)}\\
\le{}& C\sum_{|\beta|\le 2}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{H^{\frac{3}{2}}(\Omega_0)}\|H\|^2_{H^2(\Omega_0)}+C\sum_{|\beta|\le 1}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{H^2(\Omega_0)}\\
&\cdot\|\nabla\mathcal{D}_t^{\beta_m}v\|^2_{H^2(\Omega_0)}\|H\|^2_{H^2(\Omega_0)}+C\|v\|^6_{H^3(\Omega_0)}\|H\|^2_{H^2(\Omega_0)}\\
\le{}& C(\|v\|_{H^4(\Omega_0)},\|H\|_{H^4(\Omega_0)})(1+\|\mathcal{D}_t^2v\|^2_{H^{\frac{5}{2}}(\Omega_0)}+\|\mathcal{D}_tv\|^2_{H^{\frac{5}{2}}(\Omega_0)}).
\end{aligned}$$

As for $\|\mathcal{D}_t^2H\|^2_{H^3(\Omega_0)}$, again from (2.9), we see that

$$\begin{aligned}
\|\mathcal{D}_t^2H\|^2_{H^3(\Omega_0)}\le{}& C\|\sum_{1\le m\le 2}\sum_{|\beta|\le 2-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_m}v\star H\|^2_{H^3(\Omega_0)}\\
\le{}& C\|\nabla\mathcal{D}_tv\|^2_{H^3(\Omega_0)}\|H\|^2_{H^3(\Omega_0)}+C\|\nabla v\|^4_{H^3(\Omega_0)}\|H\|^2_{H^3(\Omega_0)}\\
\le{}& C(\|v\|_{H^4(\Omega_0)},\|H\|_{H^4(\Omega_0)})(1+\|\mathcal{D}_tv\|^2_{H^4(\Omega_0)}),
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{D}_tH\|^2_{H^{\frac{9}{2}}(\Omega_0)}\le{}& C(\|H\|^2_{L^\infty(\Omega_0)}\|v\|^2_{H^{\frac{11}{2}}(\Omega_0)}+\|H\|^2_{H^{\frac{9}{2}}(\Omega_0)}\|v\|^2_{L^\infty(\Omega_0)})\\
\le{}& C(\|v\|_{H^4(\Omega_0)},\|H\|_{H^4(\Omega_0)})(\|v\|^2_{H^{\frac{11}{2}}(\Omega_0)}+\|H\|^2_{H^{\frac{9}{2}}(\Omega_0)}),
\end{aligned}$$

by using Lemma A.9.

Step 2. We control $\|\mathcal{D}_t^{4-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_0)}$ by the pressure terms, i.e., $\|p\|^2_{H^{\frac{11}{2}}(\Omega_0)}$, $\|\nabla\mathcal{D}_tp\|^2_{H^3(\Omega_0)}$, $\|\nabla\mathcal{D}_t^2p\|^2_{H^{\frac{3}{2}}(\Omega_0)}$, and $\|\nabla\mathcal{D}_t^3p\|^2_{L^2(\Omega_0)}$. Note that

$$\begin{aligned}
\|\mathcal{D}_tv\|^2_{H^{\frac{9}{2}}(\Omega_0)}\le{}& C\|p\|^2_{H^{\frac{11}{2}}(\Omega_0)}+C\|H\|^2_{H^{\frac{11}{2}}(\Omega_0)}\|H\|^2_{H^{\frac{9}{2}}(\Omega_0)}\\
\le{}& C\|p\|^2_{H^{\frac{11}{2}}(\Omega_0)}+C,
\end{aligned}$$

and by Lemma 2.8, we have

$$\begin{aligned}
\|\mathcal{D}_t^2v\|^2_{H^3(\Omega_0)}\le{}&\|\mathcal{D}_t\nabla p\|^2_{H^3(\Omega_0)}+\|\mathcal{D}_t(H\cdot\nabla H)\|^2_{H^3(\Omega_0)}\\
\le{}&\|\nabla\mathcal{D}_tp\|^2_{H^3(\Omega_0)}+\|[\mathcal{D}_t,\nabla]p\|^2_{H^3(\Omega_0)}+C\\
\le{}& C\|\nabla\mathcal{D}_tp\|^2_{H^3(\Omega_0)}+C\|p\|^2_{H^4(\Omega_0)}+C.
\end{aligned}$$

Similarly, applying Lemma 2.8 and (2.14), we obtain

$$\|\mathcal{D}_t^3v\|^2_{H^{\frac{3}{2}}(\Omega_0)}\le C(\|\nabla\mathcal{D}_t^2p\|^2_{H^{\frac{3}{2}}(\Omega_0)}+\|\nabla\mathcal{D}_tp\|^2_{H^{\frac{3}{2}}(\Omega_0)}+\|p\|^2_{H^{\frac{9}{2}}(\Omega_0)}+1),$$

and

$$\begin{aligned}
\|\mathcal{D}_t^4v\|^2_{L^2(\Omega_0)}\le{}& C(\|\nabla\mathcal{D}_t^3p\|^2_{L^2(\Omega_0)}+\|\nabla\mathcal{D}_t^2p\|^2_{L^2(\Omega_0)}+\|\nabla\mathcal{D}_tp\|^2_{H^2(\Omega_0)}\\
&+\|p\|^2_{H^{\frac{9}{2}}(\Omega_0)}).
\end{aligned}$$

Step 3. We show that $\sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|^2_{H^{\frac{3}{2}k+1}(\Omega_0)}\le C$. Consider the following elliptic equation

$$\begin{cases} -\Delta p=\partial_i v^j\partial_j v^i-\partial_i H^j\partial_j H^i, & \text{in } \Omega_0,\\ p=\mathcal{A}_{\Gamma_0}, & \text{on } \Gamma_0. \end{cases}$$

From the standard elliptic estimates, we find that

$$\|p\|_{H^{\frac{11}{2}}(\Omega_0)}\le C(\|\partial_i v^j\partial_j v^i-\partial_i H^j\partial_j H^i\|_{H^{\frac{7}{2}}(\Omega_0)}+\|\mathcal{A}\|_{H^5(\Gamma_0)})\le C.$$

Again by the elliptic estimates and by Lemma A.17, it holds

$$\|\mathcal{D}_t p\|_{H^4(\Omega_0)}\le C(\|\Delta\mathcal{D}_t p\|_{H^2(\Omega_0)}+\|\mathcal{D}_t p\|_{H^{\frac{7}{2}}(\Gamma_0)}),$$

and

$$\|\mathcal{D}_t^2 p\|_{H^{\frac{5}{2}}(\Omega_0)}\le C(\|\Delta\mathcal{D}_t^2 p\|_{H^{\frac{1}{2}}(\Omega_0)}+\|\mathcal{D}_t^2 p\|_{H^2(\Gamma_0)}).$$

Also, by (A.5), we obtain

$$\|\mathcal{D}_t^3 p\|_{H^1(\Omega_0)}\le C(\|\Delta\mathcal{D}_t^3 p\|_{L^2(\Omega_0)}+\|\mathcal{D}_t^3 p\|_{H^{\frac{1}{2}}(\Gamma_0)}).$$

The calculations of the remaining terms on the right-hand side are straightforward applications of Lemmas 2.11 and 2.13, and (2.15), since we have $\|p\|_{H^{\frac{11}{2}}(\Omega_0)}\le C$.

Finally, for $1\le j\le 3$, $\|\bar\nabla(\mathcal{D}_t^j v\cdot\nu)\|^2_{L^2(\Gamma_0)}$ can be estimated by the trace theorem due to the regularity of the boundary. In fact, using the mean curvature bound, we apply Lemma A.11 to obtain $\|B\|_{H^2(\Gamma_0)}\le C$ and therefore

$$\begin{aligned}\|\bar\nabla(\mathcal{D}_t^j v\cdot\nu)\|^2_{L^2(\Gamma_0)}&\le C(\|\bar\nabla\mathcal{D}_t^j v\star\nu\|^2_{L^2(\Gamma_0)}+\|\mathcal{D}_t^j v\star B\|^2_{L^2(\Gamma_0)})\\&\le C\|\mathcal{D}_t^j v\|^2_{H^{\frac{3}{2}}(\Omega_0)}\le C.\end{aligned}$$

This concludes the proof of the proposition. □

5. Estimates for the Error Terms

In this section, we estimate the error terms by the energy functional and the pressure. We will assume the a priori assumptions hold for some $T>0$, and $\sup_{0\le t<T}E_{l-1}(t)\le C$ for $l\ge 4$.

Lemma 5.1. *Assume that* (1.8) *holds for* $T>0$. *Then, we have* $\|B\|_{H^{\frac{5}{2}}(\Gamma_t)}\le C$, *and* $\|B\|_{H^k(\Gamma_t)}\le C\left(1+\|p\|_{H^k(\Gamma_t)}\right)$ *for* $k\in\frac{\mathbb{N}}{2}, k\le\frac{9}{2}$.

Assume further that $\sup_{0\le t<T}E_{l-1}(t)\le C$ *for* $l\ge 4$. *Then, it holds* $\|B\|_{H^{\frac{3}{2}l-1}(\Gamma_t)}\le C$, *and* $\|B\|_{H^k(\Gamma_t)}\le C\left(1+\|p\|_{H^k(\Gamma_t)}\right)$ *for* $k\in\frac{\mathbb{N}}{2}, k\le\frac{3}{2}l+1$.

Proof. We recall (4.1) that $\|p\|_{H^3(\Omega_t)}\le C$ by the results in Section 4. Since Γ_t is uniformly $H^{3+\delta}(\Gamma)$-regular, it holds $\|B\|_{L^\infty(\Gamma_t)}+\|B\|_{H^1(\Gamma_t)}\le C$. Applying Lemma A.11, for $k\in\frac{\mathbb{N}}{2}, k\le 3$, we see that

$$\|B\|_{H^k(\Gamma_t)}\le C(1+\|\mathcal{A}\|_{H^k(\Gamma_t)})\le C(1+\|p\|_{H^k(\Gamma_t)}),$$

and $\|B\|_{H^{\frac{5}{2}}(\Gamma_t)}\le C$. Again by Lemma A.11, the first claim follows. As for $l\ge 4$, the assumption implies that

$$\begin{aligned}\|p\|^2_{H^{\frac{3l}{2}-1}(\Gamma_t)}&\le C(1+\|\nabla p\|^2_{H^{\frac{3(l-1)}{2}}(\Omega_t)})\\&\le C(1+\|\mathcal{D}_t v\|^2_{H^{\frac{3(l-1)}{2}}(\Omega_t)}+\|H\cdot\nabla H\|^2_{H^{\lfloor\frac{3l}{2}-1\rfloor}(\Omega_t)})\le C.\end{aligned}$$

For $l = 4$, we have $\|p\|_{H^5(\Gamma_t)} \le C$ and $\|B\|_{H^{\frac{9}{2}}(\Gamma_t)} \le C$ by the first claim. Moreover, by Lemma A.11, it implies $\|B\|_{H^5(\Gamma_t)} \le C(1+\|p\|_{H^5(\Gamma_t)}) \le C$, i.e., $\|B\|_{H^{\frac{3l}{2}-1}(\Gamma_t)} \le C$ in this case. Therefore, it holds

$$\|B\|_{H^k(\Gamma_t)} \le C(1+\|\mathcal{A}\|_{H^k(\Gamma_t)}) \le (1+\|p\|_{H^k(\Gamma_t)}), \quad k \in \frac{\mathbb{N}}{2}, k \le \frac{3}{2}l+1.$$

Using a similar argument, the second claim follows for $l \ge 5$. □

Now we begin estimating the error terms.

Lemma 5.2. *Assume that (1.8) holds for $T > 0$. For $l \le 3$, we have*

$$\|R_I^l\|^2_{H^{\frac{1}{2}}(\Omega_t)} \le C\left(1+\|\nabla^2 p\|^2_{H^{\frac{1}{2}}(\Omega_t)}\right)\bar{E}(t).$$

Assume further that $\sup_{0\le t<T} E_{l-1}(t) \le C$ for $l \ge 4$, then we have

$$\|R_I^l\|^2_{H^{\frac{1}{2}}(\Omega_t)} \le CE_l(t), \tag{5.1}$$

and for $k \in \mathbb{N}, 1 \le k \le l$, it holds

$$\|R_I^{l-k}\|^2_{H^{\frac{3}{2}k-1}(\Omega_t)} \le \varepsilon E_l(t) + C_\varepsilon, \tag{5.2}$$

with some $\varepsilon > 0$ small enough.

Proof. Thanks to Lemmas 5.1 and A.10, it is feasible to extend functions in $H^2(\Omega_t)$ to the entire space $\mathbb{R}^3$ and then apply (A.7). To simplify the notation, we will not distinguish between the original function and its extension. It suffices to estimate R_I^3 defined in (2.12) since R_I^1 and R_I^2 are easier to handle. In fact, we need to control

$$\|R_I^3\|^2_{H^{\frac{1}{2}}(\Omega_t)} = \|\sum_{2\le m\le 4}\sum_{|\beta|\le 5-m} \nabla\mathcal{D}_t^{\beta_1}v \star\cdots\star \nabla\mathcal{D}_t^{\beta_{m-1}}v \star \nabla\mathcal{D}_t^{\beta_m}v\|^2_{H^{\frac{1}{2}}(\Omega_t)}.$$

We deal with the case of $m = 2$, i.e., $\sum_{|\beta|\le 3}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla\mathcal{D}_t^{\beta_2}v$ and we only show the estimates when $|\beta| = 3$. From (1.8) and (A.7), we see that

$$\begin{aligned}
&\|\nabla v\star\nabla\mathcal{D}_t^3 v\|_{H^{\frac{1}{2}}(\Omega_t)}\\
&\le C(\|\nabla v\|_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^3 v\|_{H^{\frac{1}{2}}(\Omega_t)} + \|\nabla v\|_{W^{\frac{1}{2},6}(\Omega_t)}\|\nabla\mathcal{D}_t^3 v\|_{L^3(\Omega_t)})\\
&\le C(\|\nabla v\|_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^3 v\|_{H^{\frac{1}{2}}(\Omega_t)} + \|v\|_{H^{\frac{5}{2}}(\Omega_t)}\|\mathcal{D}_t^3 v\|_{H^{\frac{3}{2}}(\Omega_t)}) \le C\bar{E}(t)^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla\mathcal{D}_t^2 v\star\nabla\mathcal{D}_t v\|_{H^{\frac{1}{2}}(\Omega_t)}\\
&\le C(\|\nabla\mathcal{D}_t v\|_{H^{\frac{1}{2}}(\Omega_t)}\|\nabla\mathcal{D}_t^2 v\|_{L^\infty(\Omega_t)} + \|\nabla\mathcal{D}_t v\|_{L^3(\Omega_t)}\|\nabla\mathcal{D}_t^2 v\|_{W^{\frac{1}{2},6}(\Omega_t)})\\
&\le C(1+\|\nabla^2 p\|_{H^{\frac{1}{2}}(\Omega_t)})\bar{E}(t)^{\frac{1}{2}}.
\end{aligned}$$

If $l \ge 4$, the assumption $E_{l-1}(t) \le C$ also ensures that the functions in $H^{\frac{3l}{2}+1}(\Omega_t)$ can be extended by Lemma 5.1 and the extension Theorem A.10. Then, it follows that

$$\begin{aligned}
&\|\nabla v\star\nabla\mathcal{D}_t^l v\|_{H^{\frac{1}{2}}(\Omega_t)}\\
&\le C(\|\nabla v\|_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^l v\|_{H^{\frac{1}{2}}(\Omega_t)} + \|v\|_{H^{\frac{5}{2}}(\Omega_t)}\|\mathcal{D}_t^l v\|_{H^{\frac{3}{2}}(\Omega_t)}) \le CE_l(t)^{\frac{1}{2}}.
\end{aligned}$$

For $1 \le j \le l-j \le l-1$, we have $j \le \lfloor\frac{l}{2}\rfloor \le l-2$ due to $l \ge 4$, and obtain

$$\begin{aligned}
&\|\nabla\mathcal{D}_t^j v\star\nabla\mathcal{D}_t^{l-j}v\|_{H^{\frac{1}{2}}(\Omega_t)}\\
&\le C(\|\nabla\mathcal{D}_t^j v\|_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^{l-j}v\|_{H^{\frac{1}{2}}(\Omega_t)} + \|\mathcal{D}_t^j v\|_{H^{\frac{5}{2}}(\Omega_t)}\|\nabla\mathcal{D}_t^{l-j}v\|_{H^{\frac{3}{2}}(\Omega_t)})
\end{aligned}$$

$$\leq CE_l(t)^{\frac{1}{2}},$$

where we have used the fact that $\|\mathcal{D}_t^j v\|_{H^{\frac{5}{2}+\varepsilon}(\Omega_t)} \leq E_{l-1}(t) \leq C$. Again from the hypothesis $E_{l-1}(t) \leq C$, the terms involving the product of more than three items can be controlled since we will have fewer material derivatives in this case. For example, if $1 \leq j \leq l-1-j \leq l-2$, we see that $j \leq \lfloor \frac{l-1}{2} \rfloor \leq l-3$ and

$$\begin{aligned}
&\|\nabla v \star \nabla \mathcal{D}_t^j v \star \nabla \mathcal{D}_t^{l-1-j} v\|_{H^{\frac{1}{2}}(\Omega_t)} \\
\leq& C(\|\nabla \mathcal{D}_t^j v\|_{L^\infty(\Omega_t)} \|\nabla \mathcal{D}_t^l v\|_{H^{\frac{1}{2}}(\Omega_t)} + \|\nabla v \star \nabla \mathcal{D}_t^j v\|_{H^{\frac{5}{2}}(\Omega_t)} \|\mathcal{D}_t^l v\|_{H^{\frac{3}{2}}(\Omega_t)}) \\
\leq& C(\|\nabla \mathcal{D}_t^j v\|_{L^\infty(\Omega_t)} + \|\nabla \mathcal{D}_t^j v\|_{H^{\frac{5}{2}}(\Omega_t)}) \|\mathcal{D}_t^l v\|_{H^{\frac{3}{2}}(\Omega_t)} \\
\leq& CE_{l-1}(t)^{\frac{1}{2}} E_l(t)^{\frac{1}{2}} \\
\leq& CE_l(t)^{\frac{1}{2}}.
\end{aligned}$$

To prove (5.2), we first consider the estimate of $\|R_I^0\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)}$. The definition (2.12) yields

$$\|R_I^0\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)} \leq C\|\nabla v\|^2_{L^\infty(\Omega_t)} \|\nabla v\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)} \leq C\|\nabla v\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)}.$$

By interpolation, it holds

$$\|R_I^0\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)} \leq \varepsilon E_l(t) + C_\varepsilon, \quad l = 5, 7, 9, \cdots,$$

and

$$\|R_I^0\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)} \leq CE_{l-1}(t) \leq C, \quad l = 4, 6, 8, \cdots.$$

Then we control the case of $k = 1$. When $l \geq 5$, applying the previous estimates, it follows that $\|R_I^{l-1}\|^2_{H^{\frac{1}{2}}(\Omega_t)} \leq CE_{l-1}(t) \leq C$. If $l = 4$, we have by the definition of $E_3(t)$ that

$$\|R_I^{l-1}\|^2_{H^{\frac{1}{2}}(\Omega_t)} \leq C(1 + \|\nabla^2 p\|^2_{H^{\frac{1}{2}}(\Omega_t)}) E_{l-1}(t) \leq C,$$

where we use the fact that $\|\nabla^2 p\|^2_{H^{\frac{1}{2}}(\Omega_t)} \leq \|\nabla(H \cdot \nabla H - \mathcal{D}_t v)\|^2_{H^{\frac{1}{2}}(\Omega_t)} \leq C$.

We are left with the case of $2 \leq k \leq l-1$. Note that (2.12) gives

$$R_I^{l-k} = \sum_{2 \leq m \leq l-k+1} \sum_{|\beta| \leq l-k+2-m} \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-1}} v \star \nabla \mathcal{D}_t^{\beta_m} v.$$

We only estimate the case of $k = m = 2$, i.e., $\nabla \mathcal{D}_t^{l-2-j} v \star \nabla \mathcal{D}_t^j v$ and the others are similar or easier. As before, we assume that $0 \leq j \leq l-2-j \leq l-2$ and it holds

$$j \leq \lfloor \frac{l-2}{2} \rfloor \leq l-2,\ l = 4, \text{ and } j \leq \lfloor \frac{l-2}{2} \rfloor \leq l-3,\ l \geq 5.$$

We deal with the first case, i.e., $\|\nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t v\|^2_{H^2(\Omega_t)} + \|\nabla \mathcal{D}_t^2 v \star \nabla v\|^2_{H^2(\Omega_t)}$, since the same arguments work for $l \geq 5$ ($j \leq l-3$ in this case). We deduce that

$$\begin{aligned}
&\|\nabla v \star \nabla \mathcal{D}_t^2 v\|^2_{H^2(\Omega_t)} \\
\leq& C(\|\nabla v\|^2_{L^\infty(\Omega_t)} \|\nabla \mathcal{D}_t^2 v\|^2_{H^2(\Omega_t)} + \|\nabla v\|^2_{H^3(\Omega_t)} \|\nabla \mathcal{D}_t^2 v\|^2_{H^{\frac{5}{2}}(\Omega_t)}) \\
\leq& C\|\nabla \mathcal{D}_t^2 v\|^2_{H^{\frac{5}{2}}(\Omega_t)} \leq \varepsilon \|\nabla \mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)} + C\|\nabla \mathcal{D}_t^2 v\|^2_{H^2(\Omega_t)} \leq \varepsilon E_l(t) + C_\varepsilon,
\end{aligned}$$

and

$$\|\nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t v\|^2_{H^2(\Omega_t)} \leq C\|\nabla \mathcal{D}_t v\|^2_{L^\infty(\Omega_t)} \|\nabla \mathcal{D}_t v\|^2_{H^2(\Omega_t)} \leq CE_{l-1}(t) \leq C.$$

□

We proceed to bound $R^l_{\nabla H, H}$, $R^l_{\nabla H, \nabla H}$ and $R^l_{\nabla^2 H, H}$.

Lemma 5.3. *Assume that (1.8) holds for $T>0$. For $l\le 3$, we have*

$$\begin{aligned}&\|R^l_{\nabla H,H}\|^2_{H^{\frac12}(\Omega_t)}+\|R^l_{\nabla H,\nabla H}\|^2_{H^{\frac12}(\Omega_t)}+\|R^l_{\nabla^2 H,H}\|^2_{H^{\frac12}(\Omega_t)}\\ &\le C\left(1+\|\nabla^2 p\|^2_{H^{\frac12}(\Omega_t)}\right)\bar{E}(t).\end{aligned}$$

Assume further that $\sup_{0\le t<T} E_{l-1}(t)\le C$ *for* $l\ge 4$, *then we have*

$$\|R^l_{\nabla H,H}\|^2_{H^{\frac12}(\Omega_t)}+\|R^l_{\nabla H,\nabla H}\|^2_{H^{\frac12}(\Omega_t)}+\|R^l_{\nabla^2 H,H}\|^2_{H^{\frac12}(\Omega_t)}\le CE_l(t), \tag{5.3}$$

$$\|R^0_{\nabla H,H}\|^2_{H^{\frac32 l-1}(\Omega_t)}+\|R^0_{\nabla H,\nabla H}\|^2_{H^{\frac32 l-1}(\Omega_t)}\le \varepsilon E_l(t)+C_\varepsilon, \tag{5.4}$$

$$\|R^0_{\nabla^2 H,H}\|^2_{H^{\frac32 k-1}(\Omega_t)}\le C\|\operatorname{curl} H\|^2_{H^{\lfloor\frac32 l+\frac12\rfloor}(\Omega_t)}, \tag{5.5}$$

and

$$\begin{aligned}&\|R^{l-k}_{\nabla H,H}\|^2_{H^{\frac32 k-1}(\Omega_t)}+\|R^{l-k}_{\nabla H,\nabla H}\|^2_{H^{\frac32 k-1}(\Omega_t)}+\|R^{l-k}_{\nabla^2 H,H}\|^2_{H^{\frac32 k-1}(\Omega_t)}\\ &\le \varepsilon E_l(t)+C_\varepsilon,\end{aligned} \tag{5.6}$$

for $k\in\mathbb{N}, 1\le k<l$. In the above, $\varepsilon>0$ is a constant small enough.

Proof. We note that $R^l_{\nabla^2 H,H}$ contains all the highest-order terms in $R^l_{\nabla H,H}$ and $R^l_{\nabla H,\nabla H}$, since $\|v\|_{H^4(\Omega_t)}+\|H\|_{H^4(\Omega_t)}\le C$. We focus on the estimate for $R^l_{\nabla^2 H,H}$.

To control $R^3_{\nabla^2 H,H}$ in the case of $l\le 3$, we recall that

$$\begin{aligned}&R^3_{\nabla^2 H,H}\\ &=\nabla^4\operatorname{curl} H\star H\star H\star H+\sum_{\substack{|\alpha|\le 5,\alpha_i\le 4\\ m\le 5,F_j=v,H}}\nabla^{\alpha_1}F_1\star\cdots\star\nabla^{\alpha_m}F_m\\ &\quad+\sum_{\substack{|\alpha|+|\beta|\le 5,\alpha_i+\beta_i\le 4\\ \beta_i\le 2,m\le 4,F_j=v,H}}\nabla^{\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{\alpha_{l-1}}\mathcal{D}_t^{\beta_{l-1}}v\star\nabla^{\alpha_l}F_l\star\cdots\star\nabla^{\alpha_m}F_m.\end{aligned}$$

From (1.8), we have

$$\|\nabla^4\operatorname{curl} H\star H\star\cdots\star H\|^2_{H^{\frac12}(\Omega_t)}\le C\|H\|_{H^6(\Omega_t)}\le C\bar{E}(t),$$

and

$$\|\nabla^2\mathcal{D}_t^2 v\star\nabla F_2\star F_3\|^2_{H^{\frac12}(\Omega_t)}\le C\bar{E}(t),$$

as in Lemma 5.2. The leading terms in $R^3_{\nabla^2 H,H}$ have been controlled, and the estimates of the lower-order terms follow from the same arguments as in Lemma 5.2.

As for $l\ge 4$, to prove (5.3), it is sufficient to bound $\nabla^{l+1}\operatorname{curl} v\star\underbrace{H\star\cdots\star H}_{l\text{ times}}$ and $\nabla^{l+1}\operatorname{curl} H\star\underbrace{H\star\cdots\star H}_{l\text{ times}}$ since the other terms are either simpler or have already been estimated in Lemma 5.2. From the assumption $E_{l-1}(t)\le C$, we have $\|v\|_{H^{\lfloor\frac32 l\rfloor}(\Omega_t)}+\|H\|_{H^{\lfloor\frac32 l\rfloor}(\Omega_t)}\le C$. As before, we extend the functions and estimate as in Lemma 5.2 to obtain

$$\begin{aligned}&\|\nabla^{l+1}\operatorname{curl} v\star\underbrace{H\star\cdots\star H}_{l\text{ times}}\|_{H^{\frac12}(\Omega_t)}\\ &\le C(\|\underbrace{H\star\cdots\star H}_{l\text{ times}}\|_{L^\infty(\Omega_t)}\|\nabla^{l+1}\operatorname{curl} v\|_{H^{\frac12}(\Omega_t)}\\ &\quad+\|\underbrace{H\star\cdots\star H}_{l\text{ times}}\|_{W^{\frac12,6}(\Omega_t)}\|\nabla^{l+1}\operatorname{curl} v\|_{L^3(\Omega_t)})\\ &\le C\|v\|_{H^{l+\frac52}(\Omega_t)}\le CE_l(t)^{\frac12}.\end{aligned}$$

In the last step, the condition $l \geq 4$ implies $l + \frac{5}{2} \leq \lfloor \frac{3}{2}(l+1) \rfloor$ and therefore, it holds $\|v\|_{H^{l+\frac{5}{2}}(\Omega_t)} \leq \|v\|_{H^{\lfloor \frac{3}{2}(l+1) \rfloor}(\Omega_t)}$.

Next, to verify (5.6) for $k \in \mathbb{N}$ with $1 \leq k < l$, we show how to control $\|R^{l-k}_{\nabla^2 H,H}\|^2_{H^{\frac{3}{2}k-1}(\Omega_t)}$. For this purpose, we concentrate on the estimate for the product $\|\nabla^{l-k+1} \operatorname{curl} v \star \underbrace{H \star \cdots \star H}_{l-k \text{ times}}\|^2_{H^{\frac{3}{2}k-1}(\Omega_t)}$. This time we obtain for $1 \leq k < l$ that

$$\|\nabla^{l-k+1} \operatorname{curl} v \star \underbrace{H \star \cdots \star H}_{l-k \text{ times}}\|^2_{H^{\frac{3}{2}k-1}(\Omega_t)} \leq C\|v\|_{H^{l+1+\frac{k}{2}}(\Omega_t)} \leq C\|v\|_{H^{\frac{3}{2}l+\frac{1}{2}}(\Omega_t)}.$$

By interpolation, it holds

$$\|v\|^2_{H^{\frac{3}{2}l+\frac{1}{2}}(\Omega_t)} \leq \varepsilon\|v\|^2_{H^{\lfloor \frac{3}{2}(l+1) \rfloor}(\Omega_t)} + C_\varepsilon\|v\|^2_{H^{\lfloor \frac{3}{2}l \rfloor}(\Omega_t)} \leq \varepsilon E_l(t) + C_\varepsilon.$$

Finally, to obtain (5.4) and (5.5), we need to bound the most difficult term, i.e., $R^0_{\nabla^2 H,H} = (H \cdot \nabla) \operatorname{curl} H$. Since $l \geq 4$, we have

$$\begin{aligned}
\|R^0_{\nabla^2 H,H}\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)} &= \|(H \cdot \nabla) \operatorname{curl} H\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)} \\
&\leq C\|H\|^2_{H^{\frac{3}{2}l-1}(\Omega_t)}\|\operatorname{curl} H\|^2_{H^{\frac{3}{2}l}(\Omega_t)} \\
&\leq C\|H\|^2_{H^{\lfloor \frac{3}{2}l \rfloor}(\Omega_t)}\|\operatorname{curl} H\|^2_{H^{\lfloor \frac{3}{2}l+\frac{1}{2} \rfloor}(\Omega_t)} \\
&\leq C\|\operatorname{curl} H\|^2_{H^{\lfloor \frac{3}{2}l+\frac{1}{2} \rfloor}(\Omega_t)},
\end{aligned}$$

and the proof is complete. □

For the error R^l_{II}, we have the following results.

Lemma 5.4. *Assume that* (1.8) *holds for* $T > 0$. *For* $l \leq 3$, *we have*

$$\|R^l_{II}\|^2_{L^2(\Omega_t)} \leq C\left(1 + \|\nabla p\|^2_{H^{\frac{3}{2}}(\Omega_t)}\right)\bar{E}(t).$$

Assume further that $\sup_{0 \leq t < T} E_{l-1}(t) \leq C$ *for* $l \geq 4$, *then we have*

$$\|R^l_{II}\|^2_{L^2(\Omega_t)} \leq CE_l(t),$$

and for $k \in \mathbb{N}, 1 \leq k \leq l-1$, *it holds*

$$\|R^{l-k}_{II}\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \leq C.$$

Proof. To prove the first claim, we show the estimate of

$$R^3_{II} = \sum_{1 \leq m \leq 4} \sum_{\substack{|\beta| \leq 3, |\alpha| \leq 1 \\ \beta_1, \ldots, \beta_{m-1} \geq 1}} a_{\alpha,\beta}(\nabla v)\nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-1}} v \star \nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2+\beta_m} v.$$

If $m = 1$, we consider the case of $|\beta| = \beta_1 = 3$ and $|\alpha| = 1$. We should control $a(\nabla v)\mathcal{D}_t^4 v + b(\nabla v)\nabla \mathcal{D}_t^3 v$. From the hypothesis (1.8), it is clear that

$$\|a(\nabla v)\mathcal{D}_t^4 v\|^2_{L^2(\Omega_t)} + \|b(\nabla v)\nabla \mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} \leq C\bar{E}(t).$$

For $m = 2, |\beta| = 3$ and $|\alpha| = 1$, we show the estimates of $a(\nabla v)\nabla \mathcal{D}_t v \star \mathcal{D}_t^3 v$ and $b(\nabla v)\nabla \mathcal{D}_t^2 v \star \mathcal{D}_t^2 v$. Choosing $1/p + 1/q = 1/2, p = 3/\delta$ with $\delta > 0$ small enough, we see that $\|\nabla^2 H\|^2_{L^q(\Omega_t)} \leq C\|H\|^2_{H^{\frac{5}{2}+\delta}(\Omega_t)}$,

$$\begin{aligned}
&\|a(\nabla v)\nabla \mathcal{D}_t v \star \mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} \\
&\leq C\|\nabla \mathcal{D}_t v\|^2_{L^q(\Omega_t)}\|\mathcal{D}_t^3 v\|^2_{L^p(\Omega_t)} \\
&\leq C\|\nabla^2 p + \nabla H \star \nabla H + H \star \nabla^2 H\|^2_{L^q(\Omega_t)}\|\mathcal{D}_t^3 v\|^2_{H^{\frac{3}{2}}(\Omega_t)}
\end{aligned}$$

$$\leq C(1+\|\nabla^2 p\|^2_{L^q(\Omega_t)})E_3(t) \leq C(1+\|\nabla^2 p\|^2_{H^{\frac{1}{2}}(\Omega_t)})\bar{E}(t),$$

and

$$\begin{aligned}&\|a(\nabla v)\nabla\mathcal{D}_t^2 v \star \mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)}\\ &\leq C\|\nabla\mathcal{D}_t^2 v\|^2_{L^\infty(\Omega_t)}\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)} \leq C\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)}\bar{E}(t).\end{aligned}$$

To control $\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)}$, from $\|\Delta p\|_{H^1(\Omega_t)} \leq C$ and using (2.6), (2.11), (2.13), together with (A.4), we obtain

$$\begin{aligned}&\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)}\\
&\leq \|\mathcal{D}_t\nabla p\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t(H\cdot\nabla H)\|^2_{L^2(\Omega_t)}\\
&\leq \|\nabla\mathcal{D}_t p\|^2_{L^2(\Omega_t)} + \|[\mathcal{D}_t,\nabla]p\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t H\star\nabla H + H\star\mathcal{D}_t\nabla H\|^2_{L^2(\Omega_t)}\\
&\leq \|\Delta\mathcal{D}_t p\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t p\|^2_{H^{\frac{1}{2}}(\Gamma_t)} + \|\nabla v\star\nabla p\|^2_{L^2(\Omega_t)}\\
&\quad + \|H\star\nabla v\star\nabla H + H\star\nabla v\star\nabla H + H\star\nabla^2 v\star H\|^2_{L^2(\Omega_t)}\\
&\leq \|\operatorname{div}\operatorname{div}(v\otimes\nabla p)\|^2_{L^2(\Omega_t)} + \|\nabla p\|^2_{L^2(\Omega_t)} + C\\
&\quad + \|\operatorname{div} R^0_{II} + \nabla^2 v\star\nabla H\star H + \nabla v\star\nabla H\star\nabla H\\
&\qquad + \nabla^2 H\star\nabla v\star H + v\star\nabla^2 H\star\nabla H\|^2_{L^2(\Omega_t)}\\
&\leq \|\partial_j\partial_i(v^i\partial_j p)\|^2_{L^2(\Omega_t)} + \|\nabla p\|^2_{H^1(\Omega_t)} + C\\
&\leq C(1+\|\nabla p\|^2_{H^1(\Omega_t)}).\end{aligned}$$

In the case of $m=3$ and $m=4$, we estimate in the same fashion, and obtain

$$\|R^l_{II}\|^2_{L^2(\Omega_t)} \leq C(1+\|\nabla p\|^2_{H^{\frac{3}{2}}(\Omega_t)})\bar{E}(t),$$

as desired.

To control R^l_{II} for $l \geq 4$, we still focus on the case of $|\beta| = l$ and $|\alpha| = 1$. If $m = 1$, it holds $\|a(\nabla v)\mathcal{D}_t^{l+1}v + b(\nabla v)\nabla\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)} \leq CE_{l-1}(t) \leq C$.

Next, we handle the product of functions as follows. We simply assume $\alpha_2 = 1$ since the material derivative $\mathcal{D}_t$ is $\frac{1}{2}$-higher than the spatial derivative. If $1 \leq j \leq l+1-j \leq l$, it follows that $1 \leq j \leq \lfloor\frac{l+1}{2}\rfloor \leq l-2$, and we have

$$\begin{aligned}\|a(\nabla v)\nabla\mathcal{D}_t^j v\star\mathcal{D}_t^{l+1-j}v\|^2_{L^2(\Omega_t)} &\leq C\|\nabla\mathcal{D}_t^j v\|^2_{L^\infty(\Omega_t)}\|\mathcal{D}_t^{l+1-j}v\|^2_{L^2(\Omega_t)}\\
&\leq C\|\mathcal{D}_t^j v\|^2_{H^{\frac{5}{2}+\varepsilon}(\Omega_t)}\|\mathcal{D}_t^{l+1-j}v\|^2_{L^2(\Omega_t)}\\
&\leq CE_{l-1}(t)E_l(t)\\
&\leq CE_l(t).\end{aligned}$$

If $1 \leq l+1-j < j \leq l$, we find that $\lfloor\frac{l+1}{2}\rfloor + 1 \leq j$ and $1 \leq l+1-j \leq l-2$. Then, we obtain

$$\begin{aligned}&\|a(\nabla v)\nabla\mathcal{D}_t^j v\star\mathcal{D}_t^{l+1-j}v\|^2_{L^2(\Omega_t)}\\
&\leq C\|\nabla\mathcal{D}_t^j v\|^2_{L^2(\Omega_t)}\|\mathcal{D}_t^{l+1-j}v\|^2_{L^\infty(\Omega_t)}\\
&\leq C\|\mathcal{D}_t^j v\|^2_{H^1(\Omega_t)}\|\mathcal{D}_t^{l+1-j}v\|^2_{H^{\frac{3}{2}+\varepsilon}(\Omega_t)} \leq CE_l(t).\end{aligned}$$

The others can be estimated in the same way.

We are left with the last claim. For $k = 1$, it follows by applying the above estimates with $l-1$ if $l \geq 5$. As $k = 1$ and $l = 4$, it follows from the hypothesis that $E_3(t) \leq C$. Therefore, $\|\nabla p\|^2_{H^1(\Omega_t)} \leq C\|H\cdot\nabla H - \mathcal{D}_t v\|^2_{H^1(\Omega_t)} \leq C$. This concludes the proof for $k = 1$.

Assume that $2 \le k \le l-1$ and we shall control $\|R_{II}^{l-k}\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)}$ defined in (2.13):

$$\sum_{1\le m\le l-k+1} \sum_{\substack{|\beta|\le l-k, |\alpha|\le 1\\ \beta_1,\dots,\beta_{m-1}\ge 1}} a_{\alpha,\beta}(\nabla v)\nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-1}} v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+\beta_m} v.$$

If $m=1, |\beta| = l-k$ and $|\alpha|=1$, it is clear that

$$\|a(\nabla v)\mathcal{D}_t^{l+1-k} v + b(\nabla v)\nabla \mathcal{D}_t^{l-k} v\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \le CE_{l-1}(t) \le C.$$

To bound the product of functions, e.g., $m=2, |\beta| = l-k, |\alpha| = 1$ and $1\le j\le l-k-j\le l-k-1$, we note that $1 \le j \le \lfloor \frac{l-k}{2} \rfloor$ and

$$\|a(\nabla v)\|^2_{H^{\frac{3}{2}k-\frac{1}{2}}(\Omega_t)} \le C\|\nabla v\|^2_{L^\infty(\Omega_t)}\cdots\|\nabla v\|^2_{L^\infty(\Omega_t)}\|v\|^2_{\lfloor\frac{3l}{2}\rfloor(\Omega_t)} \le C.$$

This, combined with the Sobolev embedding and (A.7), we deduce that

$$\begin{aligned}
&\|a(\nabla v)\nabla \mathcal{D}_t^j v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1} v\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)}\\
\le{}& C\|a(\nabla v)\|^2_{W^{\frac{3}{2}(k-1),6}(\Omega_t)}\|\nabla \mathcal{D}_t^j v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1} v\|^2_{L^3(\Omega_t)}\\
&+ C\|a(\nabla v)\|^2_{L^\infty(\Omega_t)}\|\nabla \mathcal{D}_t^j v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1} v\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)}\\
\le{}& C\|\nabla \mathcal{D}_t^j v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1} v\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)}\\
\le{}& C\|\nabla \mathcal{D}_t^j v\|^2_{H^{\frac{3}{2}k-\frac{1}{2}}(\Omega_t)}\|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1} v\|^2_{L^3(\Omega_t)}\\
&+ C\|\nabla \mathcal{D}_t^j v\|^2_{L^\infty(\Omega_t)}\|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1} v\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \le C,
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&\|\nabla \mathcal{D}_t^j v\|^2_{H^{\frac{3}{2}k-\frac{1}{2}}(\Omega_t)} + \|\nabla \mathcal{D}_t^j v\|^2_{L^\infty(\Omega_t)}\\
\le{}& C(\|\mathcal{D}_t^j v\|^2_{H^{\frac{3}{2}k+\frac{1}{2}}(\Omega_t)} + \|\mathcal{D}_t^j v\|^2_{H^{\frac{5}{2}+\varepsilon}(\Omega_t)}) \le C,
\end{aligned}$$

for $\varepsilon > 0$ small enough. Thus, the proof is complete since the other terms can be estimated by using the similar arguments. □

For the error term R_p^l on the free boundary, we shall control it using $\|\nabla p\|_{H^2(\Omega_t)}$ for $l \le 3$.

Lemma 5.5. *Assume that (1.8) holds for $T>0$. For $l\le 3$, we have*

$$\|R_p^l\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \le C\left(1 + \|\nabla p\|^2_{H^2(\Omega_t)}\right)\bar{E}(t).$$

Assume further that $\sup\limits_{0\le t<T} E_{l-1}(t) \le C$ *for $l \ge 4$, then we have*

$$\|R_p^l\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \le CE_l(t),$$

and for $k\in\mathbb{N}, 1\le k\le l-1$, it holds

$$\|R_p^{l-k}\|^2_{H^{\frac{3}{2}k-1}(\Gamma_t)} \le \varepsilon E_l(t) + C_\varepsilon,$$

for some $\varepsilon > 0$ small enough.

Proof. It is sufficient to show the estimate for $l=3$, since the other cases are easier. Recall the definition of R_p^3, and we denote

$$\begin{aligned}
R_p^3 ={}& \underbrace{-|B|^2\mathcal{D}_t^3 v\cdot\nu}_{=:I_1} + \underbrace{\bar\nabla p\cdot \mathcal{D}_t^3 v}_{=:I_2} + \underbrace{a_8(\nu,\nabla v)\star\nabla^2\mathcal{D}_t^2 v}_{=:I_3} + \underbrace{a_9(\nu,\nabla v)\star\nabla \mathcal{D}_t^2 v\star B}_{=:I_4}\\
&+ \underbrace{a_{10}(\nu,\nabla v)\star\nabla\mathcal{D}_t^2 v\star\nabla^2 v}_{=:I_5} + \underbrace{a_{11}(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v\star\nabla\mathcal{D}_t v}_{=:I_6}
\end{aligned}$$

$$+ a_{12}(\nu, \nabla v) \star \nabla^2 \mathcal{D}_t v \star B + a_{13}(\nu, \nabla v) \star \nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t v \star \nabla^2 v$$
$$+ a_{14}(\nu, \nabla v) \star \nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t v \star B + \mathrm{L.\,O.\,T.}.$$

To control the second term, we estimate as follows

$$\begin{aligned}
\|I_2\|^2_{H^{\frac{1}{2}}(\Gamma_t)} &\le C\|\bar{\nabla} p\|^2_{W^{\frac{1}{2},4}(\Gamma_t)} \|\mathcal{D}_t^3 v\|^2_{L^4(\Gamma_t)} + C\|\bar{\nabla} p\|^2_{L^4(\Gamma_t)} \|\mathcal{D}_t^3 v\|^2_{W^{\frac{1}{2},4}(\Gamma_t)} \\
&\le C\|\bar{\nabla} p\|^2_{H^1(\Gamma_t)} E_3(t) \\
&\le C\|\nabla p\|^2_{H^{\frac{3}{2}}(\Omega_t)} \bar{E}(t),
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} &\le \|\bar{\nabla}(\nabla p - \nabla p \cdot \nu\nu)\|^2_{L^2(\Gamma_t)} \\
&\le C(\|\nabla p\|^2_{H^1(\Gamma_t)} + \|\nabla p \star B\|^2_{L^2(\Gamma_t)}) \\
&\le C\|\nabla p\|^2_{H^1(\Gamma_t)},
\end{aligned}$$

and the trace theorem. Similarly, for I_1, we obtain by (1.8) that

$$\|\mathcal{D}_t^3 v \cdot \nu\|^2_{L^2(\Gamma_t)} \le C\|\mathcal{D}_t^3 v\|^2_{H^1(\Omega_t)} \le C\bar{E}(t),$$

and

$$\|I_1\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \le C\||B|^2\|^2_{H^1(\Gamma_t)} \|\mathcal{D}_t^3 v \cdot \nu\|^2_{H^1(\Gamma_t)} \le C(1 + \|\nabla p\|^2_{H^1(\Omega_t)})\bar{E}(t).$$

Again from (1.8), it follows that

$$\begin{aligned}
\|I_3\|^2_{H^{\frac{1}{2}}(\Gamma_t)} &\le C\|\nabla^2 \mathcal{D}_t^2 v\|^2_{H^{\frac{1}{2}}(\Gamma_t)} + C\|a_8(\nu, \nabla v)\|^2_{W^{\frac{1}{2},4}(\Gamma_t)} \|\nabla^2 \mathcal{D}_t^2 v\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \\
&\le C\bar{E}(t), \\
\|I_4\|^2_{H^{\frac{1}{2}}(\Gamma_t)} &\le C\|\nabla \mathcal{D}_t^2 v \star B\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \\
&\le C\|\nabla \mathcal{D}_t^2 v\|^2_{W^{\frac{1}{2},4}(\Gamma_t)} \|B\|^2_{L^4(\Gamma_t)} + C\|B\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \|\nabla \mathcal{D}_t^2 v\|^2_{L^\infty(\Gamma_t)} \\
&\le C\bar{E}(t), \\
\|I_5\|^2_{H^{\frac{1}{2}}(\Gamma_t)} &\le C\|\nabla \mathcal{D}_t^2 v \star \nabla^2 v\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \\
&\le C\|\nabla^2 v\|^2_{H^{\frac{1}{2}}(\Gamma_t)} (\|\nabla \mathcal{D}_t^2 v\|^2_{W^{\frac{1}{2},4}(\Gamma_t)} + \|\nabla \mathcal{D}_t^2 v\|^2_{L^\infty(\Gamma_t)}) \\
&\le C\bar{E}(t), \\
\|I_6\|^2_{H^{\frac{1}{2}}(\Gamma_t)} &\le C\|\nabla(-\nabla p + H \cdot \nabla H)\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \\
&\qquad \cdot (\|\nabla \mathcal{D}_t^2 v\|^2_{W^{\frac{1}{2},4}(\Gamma_t)} + \|\nabla \mathcal{D}_t^2 v\|^2_{L^\infty(\Gamma_t)}) \\
&\le C(1 + \|\nabla^2 p\|^2_{H^1(\Omega_t)})\bar{E}(t),
\end{aligned}$$

and the other terms can be estimated in the same way. For $l \ge 4$, the proof is similar to [JLM22, Lemma 5.8], so we omit the details. □

Applying the above error estimates and recalling Proposition 3.1 as well as (4.1), we conclude this section by presenting the following improved version of Proposition 3.1.

Proposition 5.6. *Assume that* (1.8) *holds for* $T > 0$. *Then, we have*

$$\frac{d}{dt}\bar{e}(t) \le C\bar{E}(t),$$

where C *depends on* $T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$, *and* $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$.

For $l \ge 4$, *assume further that* $\sup\limits_{0 \le t < T} E_{l-1}(t) \le C$, *then we have*

$$\frac{d}{dt}e_l(t) \le CE_l(t),$$

where the constant C depends on $T, \mathcal{N}_T, \mathcal{M}_T$, and $\sup_{0\le t<T} E_{l-1}(t)$.

6. Closing the Energy Estimates

In this section, we will close the energy estimates, and we introduce the following energy functional

$$\begin{aligned}\tilde{e}(t) &:= \frac{1}{2}\sum_{k=1}^{3}\Big(\|\mathcal{D}_t^{k+1}v\|_{L^2(\Omega_t)}^2 + \|\mathcal{D}_t^{k+1}H\|_{L^2(\Omega_t)}^2 + \|\bar{\nabla}(\mathcal{D}_t^k v\cdot\nu)\|_{L^2(\Gamma_t)}^2\Big)\\ &\quad + \frac{1}{2}\Big(\|\operatorname{curl} v\|_{H^5(\Omega_t)}^2 + \|\operatorname{curl} H\|_{H^5(\Omega_t)}^2\Big) + 1,\\ \tilde{e}_l(t) &:= \frac{1}{2}\Big(\|\mathcal{D}_t^{l+1}v\|_{L^2(\Omega_t)}^2 + \|\mathcal{D}_t^{l+1}H\|_{L^2(\Omega_t)}^2 + \|\bar{\nabla}(\mathcal{D}_t^l v\cdot\nu)\|_{L^2(\Gamma_t)}^2\Big)\\ &\quad + \frac{1}{2}\Big(\|\operatorname{curl} v\|_{H^{\lfloor\frac{3l+1}{2}\rfloor}(\Omega_t)}^2 + \|\operatorname{curl} H\|_{H^{\lfloor\frac{3l+1}{2}\rfloor}(\Omega_t)}^2\Big) + 1, \quad l\ge 4.\end{aligned}$$

Note that from the a priori assumptions (1.8), it holds

$$\|\operatorname{curl} v\|_{L^2(\Omega_t)}^2 + \|\operatorname{curl} H\|_{L^2(\Omega_t)}^2 \le C.$$

By interpolation, we have $\tilde{e}(t)\le C(\bar{e}+1)$ and $\tilde{e}_l(t)\le C(e_l(t)+1)$ for $l\ge 4$.

We first control the energy functional $\bar{E}(t)$ by $\bar{e}(t)$ under a slightly different hypothesis compared with the a priori assumptions (1.8). In fact, we have the following result.

Proposition 6.1. *Assume that $\Gamma_t\in H^{3+\delta}(\Gamma)$ with $\delta>0$ small enough. Assume that the pressure, velocity and magnetic field satisfy*

$$\|p\|_{H^3(\Omega_t)} + \|v\|_{H^4(\Omega_t)} + \|H\|_{H^4(\Omega_t)} \le C_0.$$

Then we have

$$\bar{E}(t)\le C(1+\bar{e}(t)), \tag{6.1}$$

and

$$\|B\|_{H^{\frac{9}{2}}(\Gamma_t)}^2 \le C(1+\bar{e}(t)), \tag{6.2}$$

where the constant C depends on $\mathcal{M}_t$, $\|h(\cdot,t)\|_{H^{3+\delta}(\Gamma)}$, $\|p\|_{H^3(\Omega_t)}$, $\|v\|_{H^4(\Omega_t)}$, and $\|H\|_{H^4(\Omega_t)}$.

Proof. We shall show that $\bar{E}(t)\le C\tilde{e}(t)$. For this purpose, we need to control $\|\mathcal{D}_t^{4-k}v\|_{H^{\frac{3}{2}k}(\Omega_t)}^2$, $\|\mathcal{D}_t^{4-k}H\|_{H^{\frac{3}{2}k}(\Omega_t)}^2$ with $1\le k\le 3$, $\|v\|_{H^6(\Omega_t)}^2$ and $\|H\|_{H^6(\Omega_t)}^2$. Recall that we have already deduced the estimates for $\|\mathcal{D}_t^{4-k}H\|_{H^{\frac{3}{2}k}(\Omega_t)}^2$ in (4.12), thanks to (2.9) and (2.10). Then, it is sufficient to control

$$\|\mathcal{D}_t^3 v\|_{H^{\frac{3}{2}}(\Omega_t)}^2,\ \|\mathcal{D}_t^2 v\|_{H^3(\Omega_t)}^2,\ \|\mathcal{D}_t v\|_{H^{\frac{9}{2}}(\Omega_t)}^2,\ \|v\|_{H^6(\Omega_t)}^2, \text{ and } \|H\|_{H^6(\Omega_t)}^2.$$

We divide the proof into three steps.

Step 1. We control $\|\mathcal{D}_t^3 v\|_{H^{\frac{3}{2}}(\Omega_t)}^2$. Recalling that $\|\tilde{\nu}\|_{H^{\frac{5}{2}+\delta}(\Omega_t)}\le C$ and by the definition of $\tilde{e}(t)$, we have

$$\begin{aligned}\|\mathcal{D}_t^3 v\cdot\nu\|_{L^2(\Gamma_t)}^2 &= \int_{\Gamma_t}[(\mathcal{D}_t^3 v\cdot\nu)\mathcal{D}_t^3 v]\cdot\nu dS\\ &\le \Big|\int_{\Omega_t}(\mathcal{D}_t^3 v\cdot\nu)\operatorname{div}\mathcal{D}_t^3 v dx\Big| + \Big|\int_{\Omega_t}\nabla\mathcal{D}_t^3 v\star\mathcal{D}_t^3 v dx\Big|\\ &\quad + \Big|\int_{\Omega_t}\mathcal{D}_t^3 v\star\nabla\nu\star\mathcal{D}_t^3 v dx\Big|\\ &\le C(\|\mathcal{D}_t^3 v\|_{L^2(\Omega_t)}^2 + \|\operatorname{div}\mathcal{D}_t^3 v\|_{L^2(\Omega_t)}^2\\ &\qquad + \|\nabla\mathcal{D}_t^3 v\|_{L^2(\Omega_t)}\|\mathcal{D}_t^3 v\|_{L^2(\Omega_t)})\\ &\le \varepsilon\|\nabla\mathcal{D}_t^3 v\|_{L^2(\Omega_t)}^2 + C_\varepsilon\tilde{e}(t) + C\|\operatorname{div}\mathcal{D}_t^3 v\|_{L^2(\Omega_t)}^2.\end{aligned}$$

This, combined with Lemmas 2.10, 5.2 and A.13, it follows that

$$\|\mathcal{D}_t^3 v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \le C(\|\mathcal{D}_t^3 v\cdot\nu\|^2_{H^1(\Gamma_t)} + \|\mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\mathcal{D}_t^3 v\|^2_{H^{\frac{1}{2}}(\Omega_t)} + \|\operatorname{curl}\mathcal{D}_t^3 v\|^2_{H^{\frac{1}{2}}(\Omega_t)}),$$

and therefore,

$$\|\mathcal{D}_t^3 v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \le C(\tilde e(t) + \underbrace{\|R_I^2\|^2_{H^{\frac{1}{2}}(\Omega_t)}}_{=:I_1(t)} + \underbrace{\|R^2_{\nabla H,\nabla H}\|^2_{H^{\frac{1}{2}}(\Omega_t)}}_{=:I_2(t)} + \underbrace{\|R^2_{\nabla^2 H,H}\|^2_{H^{\frac{1}{2}}(\Omega_t)}}_{=:I_3(t)}).$$

To control the second term, using (2.8), we estimate as follows. Indeed, by the assumption, applying Young's inequality and Lemma A.8, we obtain

$$\begin{aligned}
&\|\nabla^2\mathcal{D}_t v\star\nabla H\star H\|^2_{H^{\frac{1}{2}}(\Omega_t)} + \|\nabla\mathcal{D}_t v\star\nabla^2 H\star H\|^2_{H^{\frac{1}{2}}(\Omega_t)}\\
&\le C\|\mathcal{D}_t v\|^2_{H^3(\Omega_t)}\|H\|^4_{H^3(\Omega_t)}\\
&\le \varepsilon\|\mathcal{D}_t v\|^2_{H^{\frac{9}{2}}(\Omega_t)} + C_\varepsilon\|\mathcal{D}_t v\|^2_{L^2(\Omega_t)}\\
&\le \varepsilon\bar E(t) + C_\varepsilon\|p\|^2_{H^1(\Omega_t)} + C_\varepsilon\|H\cdot\nabla H\|^2_{L^2(\Omega_t)}\\
&\le \varepsilon\bar E(t) + C_\varepsilon.
\end{aligned}$$

As for $I_3(t)$, we recall Lemma (2.11), and we handle the most difficult term, i.e.,

$$\|\nabla^3\operatorname{curl}H\star H\star H\star H\|^2_{H^{\frac{1}{2}}(\Omega_t)} \le C\|\operatorname{curl}H\|^2_{H^4(\Omega_t)} \le C\tilde e(t).$$

The other terms can be estimated using the same argument. Again by the Young's inequality and Lemma A.8, we can control $I_1(t)$. In fact, we have

$$\begin{aligned}
&\|\nabla\mathcal{D}_t^2 v\star\nabla v\|^2_{H^{\frac{1}{2}}(\Omega_t)} + \|\nabla\mathcal{D}_t v\star\nabla\mathcal{D}_t v\|^2_{H^{\frac{1}{2}}(\Omega_t)}\\
&\le C\|v\|^2_{H^2(\Omega_t)}\|\mathcal{D}_t^2 v\|^2_{H^2(\Omega_t)} + C\|\nabla\mathcal{D}_t v\|^2_{L^3(\Omega_t)}\|\nabla\mathcal{D}_t v\|^2_{H^{\frac{3}{2}}(\Omega_t)}\\
&\le \varepsilon\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)} + C_\varepsilon\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)}\\
&\quad + (\varepsilon\|\mathcal{D}_t v\|^2_{H^{\frac{9}{2}}(\Omega_t)} + C_\varepsilon\|\mathcal{D}_t v\|^2_{L^2(\Omega_t)})(\|\nabla^2 p\|^2_{L^3(\Omega_t)} + \|\nabla(H\cdot\nabla H)\|^2_{L^3(\Omega_t)})\\
&\le C_\varepsilon\tilde e(t) + \varepsilon\bar E(t).
\end{aligned}$$

Combining the above estimates, it follows that

$$\|\mathcal{D}_t^3 v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \le \varepsilon\bar E(t) + C_\varepsilon\tilde e(t).$$

Step 2. We estimate $\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)}$ and $\|\mathcal{D}_t v\|^2_{H^{\frac{9}{2}}(\Omega_t)}$. Applying Lemmas 2.10 and A.14, it holds

$$\begin{aligned}
\|\mathcal{D}_t v\|^2_{H^{\frac{9}{2}}(\Omega_t)} &\le C(\|\Delta_B(\mathcal{D}_t v\cdot\nu)\|^2_{H^2(\Gamma_t)} + \|\mathcal{D}_t v\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\mathcal{D}_t v\|^2_{H^{\frac{7}{2}}(\Omega_t)}\\
&\qquad + \|\operatorname{curl}\mathcal{D}_t v\|^2_{H^{\frac{7}{2}}(\Omega_t)})\\
&\le C\tilde e(t) + C(\|\Delta_B(\mathcal{D}_t v\cdot\nu)\|^2_{H^2(\Gamma_t)} + \|\nabla v\star\nabla v\|^2_{H^{\frac{7}{2}}(\Omega_t)}\\
&\qquad + \|\nabla H\star\nabla H\|^2_{H^{\frac{7}{2}}(\Omega_t)} + \|H\star\nabla\operatorname{curl}H\|^2_{H^{\frac{7}{2}}(\Omega_t)})\\
&\le C\tilde e(t) + C\|\Delta_B(\mathcal{D}_t v\cdot\nu)\|^2_{H^2(\Gamma_t)},
\end{aligned}$$

and

$$\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)} \le C(\|\Delta_B(\mathcal{D}_t^2 v\cdot\nu)\|^2_{H^{\frac{1}{2}}(\Gamma_t)} + \|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\mathcal{D}_t^2 v\|^2_{H^2(\Omega_t)}$$

$$
\begin{aligned}
&+\|\operatorname{curl}\mathcal{D}_t^2 v\|_{H^2(\Omega_t)}^2)\\
&\le C\tilde{e}(t)+C(\|\Delta_B(\mathcal{D}_t^2 v\cdot\nu)\|_{H^{\frac12}(\Gamma_t)}^2\\
&\quad+\underbrace{\|R_I^1\|_{H^2(\Omega_t)}^2}_{=:I_4(t)}+\underbrace{\|R^1_{\nabla H,\nabla H}\|_{H^2(\Omega_t)}^2}_{=:I_5(t)}+\underbrace{\|R^1_{\nabla^2 H,H}\|_{H^2(\Omega_t)}^2}_{=:I_6(t)}).
\end{aligned}
$$

We control $I_4(t)$ by the bilinear inequality, i.e.,

$$
\|\nabla\mathcal{D}_t v\star\nabla v\|_{H^2(\Omega_t)}^2\le C\|\mathcal{D}_t v\|_{H^3(\Omega_t)}^2\|v\|_{H^3(\Omega_t)}^2\le\varepsilon\bar{E}(t)+C_\varepsilon\tilde{e}(t).
$$

For $I_5(t)$, it holds that

$$
\|\nabla^2 v\star\nabla H\star H\|_{H^2(\Omega_t)}^2+\|\nabla v\star\nabla H\star\nabla H\|_{H^2(\Omega_t)}^2\le C\|v\|_{H^4(\Omega_t)}^2\|H\|_{H^3(\Omega_t)}^4\le C,
$$

from the assumption. The estimate for $I_6(t)$ follows since

$$
\begin{aligned}
&\|\nabla^2\operatorname{curl} v\star H\star H\|_{H^2(\Omega_t)}^2+\|\nabla^2 H\star\nabla v\star H\|_{H^2(\Omega_t)}^2\\
&+\|\nabla^2 v\star\nabla H\star H\|_{H^2(\Omega_t)}^2\\
\le\,& C(\|\operatorname{curl} v\|_{H^4(\Omega_t)}^2\|H\|_{H^2(\Omega_t)}^4+\|H\|_{H^4(\Omega_t)}^2\|v\|_{H^3(\Omega_t)}^2\|H\|_{H^3(\Omega_t)}^2\\
&+\|v\|_{H^4(\Omega_t)}^2\|H\|_{H^3(\Omega_t)}^4)\le C\tilde{e}(t).
\end{aligned}
$$

We are left with $\|\Delta_B(\mathcal{D}_t^2 v\cdot\nu)\|_{H^{\frac12}(\Gamma_t)}^2$ and $\|\Delta_B(\mathcal{D}_t v\cdot\nu)\|_{H^2(\Gamma_t)}^2$. We focus on the estimate of $\|\Delta_B(\mathcal{D}_t^2 v\cdot\nu)\|_{H^{\frac12}(\Gamma_t)}^2$ since the other one is similar. Recalling that from Lemma 2.13, we have $\mathcal{D}_t^3 p=-\Delta_B(\mathcal{D}_t^2 v\cdot\nu)+R_p^2$. Since $\|R_p^2\|_{H^{\frac12}(\Gamma_t)}^2$ is easier to control than $\|\mathcal{D}_t^3 p\|_{H^{\frac12}(\Gamma_t)}^2$, we only bound $\|\mathcal{D}_t^3 p\|_{H^{\frac12}(\Gamma_t)}^2$. By (A.3), it holds

$$
\|\mathcal{D}_t^3 p\|_{H^{\frac12}(\Gamma_t)}^2\le C\|\mathcal{D}_t^3 p\|_{L^2(\Gamma_t)}^2+C\|\nabla\mathcal{D}_t^3 p\|_{L^2(\Omega_t)}^2=:I_7(t)+I_8(t).
$$

Applying (2.5), for the first term, we have

$$
\begin{aligned}
I_7(t)&\le C\|\mathcal{D}_t^3 B\|_{L^2(\Gamma_t)}^2\\
&\le C\|\sum_{1\le m\le 3}\sum_{|\beta|\le 3-m,|\alpha|\le 1}a_{\alpha,\beta}(\nu,B)\bar{\nabla}^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar{\nabla}^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v\|_{L^2(\Gamma_t)}^2.
\end{aligned}
$$

In the above, if $m=1$, from $\|B\|_{L^\infty(\Gamma_t)}\le C$, we control $a(\nu,B)\bar{\nabla}^2\mathcal{D}_t^2 v$ by the trace theorem and by interpolation:

$$
\|a(\nu,B)\bar{\nabla}^2\mathcal{D}_t^2 v\|_{L^2(\Gamma_t)}^2\le C\|\mathcal{D}_t^2 v\|_{H^{\frac52}(\Omega_t)}^2\le\varepsilon\bar{E}(t)+C_\varepsilon\tilde{e}(t).
$$

The other cases are either simpler or similar. As for $I_8(t)$, it follows that

$$
\begin{aligned}
I_8(t)&\le C\|\mathcal{D}_t^3\nabla p\|_{L^2(\Omega_t)}^2+C\|[\nabla,\mathcal{D}_t^3]p\|_{L^2(\Omega_t)}^2\\
&\le C\tilde{e}(t)+C\|\mathcal{D}_t^3(H\cdot\nabla H)\|_{L^2(\Omega_t)}^2+C\|[\nabla,\mathcal{D}_t^3]p\|_{L^2(\Omega_t)}^2.
\end{aligned}
$$

To control $\|\mathcal{D}_t^3(H\cdot\nabla H)\|_{L^2(\Omega_t)}^2$, again by interpolation, we see that

$$
\begin{aligned}
&\|\nabla^2\mathcal{D}_t^2 v\star H\star H\|_{L^2(\Omega_t)}^2+\|\nabla^2\mathcal{D}_t v\star H\star H\|_{L^2(\Omega_t)}^2\\
\le\,& C\|\mathcal{D}_t^2 v\|_{H^2(\Omega_t)}^2+C\|\mathcal{D}_t v\|_{H^2(\Omega_t)}^2\le\varepsilon\bar{E}(t)+C_\varepsilon\tilde{e}(t),
\end{aligned}
$$

and we estimate $\|[\nabla,\mathcal{D}_t^3]p\|_{L^2(\Omega_t)}^2$ as follows

$$
\begin{aligned}
&\|\nabla\mathcal{D}_t^2 v\star\nabla p\|_{L^2(\Omega_t)}^2+\|\nabla\mathcal{D}_t v\star\nabla\mathcal{D}_t p\|_{L^2(\Omega_t)}^2+\|\nabla v\star\nabla\mathcal{D}_t^2 p\|_{L^2(\Omega_t)}^2\\
\le\,& C(\|\mathcal{D}_t^2 v\|_{H^2(\Omega_t)}^2\|p\|_{H^{\frac32}(\Omega_t)}^2+\|\nabla(H\cdot\nabla H)\star\nabla\mathcal{D}_t p\|_{L^2(\Omega_t)}^2\\
&+\|\nabla^2 p\star\nabla\mathcal{D}_t p\|_{L^2(\Omega_t)}^2+\|\nabla\mathcal{D}_t^2 p\|_{L^2(\Omega_t)}^2)\\
\le\,& C\|\mathcal{D}_t^2 v\|_{H^2(\Omega_t)}^2+C\|\nabla\mathcal{D}_t p\|_{L^3(\Omega_t)}^2+C\|\nabla\mathcal{D}_t^2 p\|_{L^2(\Omega_t)}^2.
\end{aligned}
$$

We note that $\|\nabla\mathcal{D}_t^2 p\|^2_{L^2(\Omega_t)}$ and $\|\nabla\mathcal{D}_t p\|^2_{L^3(\Omega_t)}$ have fewer material derivatives than $\|\nabla\mathcal{D}_t^3 p\|^2_{L^2(\Omega_t)}$. Therefore, it can be estimated as $I_8(t)$ in the same fashion, and we can obtain

$$\|\mathcal{D}_t^3 p\|^2_{H^{\frac{1}{2}}(\Gamma_t)} \le C\tilde{e}(t) + \varepsilon\bar{E}(t).$$

Similarly, it holds

$$\|\mathcal{D}_t^2 p\|^2_{H^2(\Gamma_t)} \le C\tilde{e}(t) + \varepsilon\bar{E}(t).$$

Combining the above estimates, we conclude that

$$\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)} + \|\mathcal{D}_t v\|^2_{H^{\frac{9}{2}}(\Omega_t)} \le C\tilde{e}(t) + \varepsilon\bar{E}(t).$$

Step 3. Finally, we bound $\|v\|^2_{H^6(\Omega_t)}$ and $\|H\|^2_{H^6(\Omega_t)}$. From Lemma A.14, we see that

$$\begin{aligned}
\|v\|^2_{H^6(\Omega_t)} &\le C(\tilde{e}(t) + \|\Delta_B v_n\|^2_{H^{\frac{7}{2}}(\Gamma_t)} + \|B\|^2_{H^{\frac{9}{2}}(\Gamma_t)}),\\
\|H\|^2_{H^6(\Omega_t)} &\le C(\tilde{e}(t) + \|B\|^2_{H^{\frac{9}{2}}(\Gamma_t)}).
\end{aligned}$$

Recalling Lemma 5.1 and by the trace theorem, it follows that

$$\begin{aligned}
\|B\|^2_{H^{\frac{9}{2}}(\Gamma_t)} &\le C(1 + \|p\|^2_{H^{\frac{9}{2}}(\Gamma_t)})\\
&\le C(1 + \|H\cdot\nabla H - \mathcal{D}_t v\|^2_{H^4(\Omega_t)})\\
&\le C + \|H\|^2_{H^5(\Omega_t)} + \varepsilon\|\mathcal{D}_t v\|^2_{H^{\frac{9}{2}}(\Omega_t)} + C_\varepsilon\|\mathcal{D}_t v\|^2_{L^2(\Omega_t)}\\
&\le \varepsilon\bar{E}(t) + \|H\|^2_{H^5(\Omega_t)} + C_\varepsilon.
\end{aligned}$$

Again by Lemma A.14, we can estimate in $H^5(\Omega_t)$ and deduce $\|H\|^2_{H^5(\Omega_t)} \le C\tilde{e}(t) + \|B\|^2_{H^{\frac{7}{2}}(\Gamma_t)}$. Similarly, it holds

$$\|B\|^2_{H^{\frac{7}{2}}(\Gamma_t)} \le \varepsilon\bar{E}(t) + \|H\|^2_{H^4(\Omega_t)} + C_\varepsilon \le \varepsilon\bar{E}(t) + C_\varepsilon.$$

Thus, we see that

$$\|B\|^2_{H^{\frac{9}{2}}(\Gamma_t)} \le \varepsilon\bar{E}(t) + C_\varepsilon, \quad \|p\|^2_{H^{\frac{9}{2}}(\Gamma_t)} \le \varepsilon\bar{E}(t) + C_\varepsilon,$$

and $\|H\|^2_{H^6(\Omega_t)} \le \varepsilon\bar{E}(t) + C_\varepsilon$. To obtain the desired estimate, we are left with $\|\Delta_B v_n\|^2_{H^{\frac{7}{2}}(\Gamma_t)}$.

From (2.15) and by the above calculations, it follows that

$$\begin{aligned}
\|\Delta_B v_n\|^2_{H^{\frac{7}{2}}(\Gamma_t)} &\le C\|\mathcal{D}_t p\|^2_{H^{\frac{7}{2}}(\Gamma_t)} + C\||B|^2 v_n\|^2_{H^{\frac{7}{2}}(\Gamma_t)} + C\|\bar{\nabla} p\cdot v\|^2_{H^{\frac{7}{2}}(\Gamma_t)}\\
&\le C\|v\|^2_{H^4(\Omega_t)}\|B\|^2_{L^\infty(\Gamma_t)}\|B\|^2_{H^{\frac{7}{2}}(\Gamma_t)} + \varepsilon\bar{E}(t) + C_\varepsilon\\
&\le \varepsilon\bar{E}(t) + C_\varepsilon,
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\|\mathcal{D}_t p\|^2_{H^{\frac{7}{2}}(\Gamma_t)} &\le C(\|\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + \|\nabla\mathcal{D}_t p\|^2_{H^3(\Omega_t)})\\
&\le C(1 + \|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)} + \|\mathcal{D}_t(H\cdot\nabla H)\|^2_{H^3(\Omega_t)}\\
&\qquad + \|\nabla v\star(H\cdot\nabla H - \mathcal{D}_t v)\|^2_{H^3(\Omega_t)})\\
&\le C\tilde{e}(t) + \frac{\varepsilon}{2}\bar{E}(t) + C\|\nabla\mathcal{D}_t H\|^2_{H^3(\Omega_t)}\|H\|^2_{H^3(\Omega_t)}\\
&\le C\tilde{e}(t) + \varepsilon\bar{E}(t),
\end{aligned}$$

since $\|\mathcal{D}_t H\|^2_{H^4(\Omega_t)}$ and $\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)}$ have already been controlled. This completes the proof. □

Now we prove the higher-order energy estimate.

Proposition 6.2. *Let $l \geq 4$. Assume that (1.8) holds for some $T > 0$ and $\sup_{0\leq t<T} E_{l-1}(t) \leq C$. Then we have*

$$E_l(t) \leq C(1 + e_l(t)),$$

where the constant C depends on $l, T, \mathcal{N}_T, \mathcal{M}_T$ and $\sup_{0\leq t<T} E_{l-1}(t)$.

Proof. We will show that $E_l(t) \leq C\tilde{e}_l(t)$ and we divide the proof into three steps.

Step 1. We claim that it is sufficient to bound $\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)}$ with $k \in \{1, 2, \dots, l\}$, $\|v\|^2_{H^{\lfloor \frac{3l+3}{2} \rfloor}(\Omega_t)}$ and $\|H\|^2_{H^{\lfloor \frac{3l+3}{2} \rfloor}(\Omega_t)}$. Indeed, for $k \in \{1, 2, \dots, l\}$, we can control $\|\mathcal{D}_t^{l+1-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_t)}$ by the sum of

$$\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)},\ \|v\|^2_{H^{\lfloor \frac{3l+3}{2} \rfloor}(\Omega_t)},\ \text{and}\ \|H\|^2_{H^{\lfloor \frac{3l+3}{2} \rfloor}(\Omega_t)}.$$

Starting with the case of $2 \leq k \leq l-1$, from the hypothesis $E_{l-1}(t) \leq C$, (2.9) and (2.10), we have

$$\begin{aligned}
&\|\mathcal{D}_t^{l+1-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \\
\leq& C\Big\| \sum_{1\leq m\leq l+1-k} \sum_{|\beta|\leq l+1-k-m} \nabla\mathcal{D}_t^{\beta_1}v \star \cdots \star \nabla\mathcal{D}_t^{\beta_m}v \star H \Big\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \\
\leq& C \sum_{\substack{1\leq m\leq l+1-k \\ |\beta|\leq l+1-k-m}} \|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \cdots \|\nabla\mathcal{D}_t^{\beta_m}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \|H\|^2_{H^{\frac{3}{2}k}(\Omega_t)}.
\end{aligned}$$

If $m = 1$, we see that

$$\begin{aligned}
\|\mathcal{D}_t^{l+1-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_t)} &\leq C(\|\nabla\mathcal{D}_t^{l-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + E_{l-1}(t)) \\
&\leq C(\|\mathcal{D}_t^{l+1-(k+1)}v\|^2_{H^{\frac{3}{2}(k+1)}(\Omega_t)} + 1),
\end{aligned}$$

since $\|H\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \leq CE_{l-1}(t) \leq C$.

For $m \geq 2$, it holds

$$\|\mathcal{D}_t^{l+1-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \leq C \underbrace{E_{l-1}(t) \dots E_{l-1}(t)}_{\text{finite product}} \leq C.$$

Next, we deal with the case of $k = 1$, and it follows that

$$\begin{aligned}
\|\mathcal{D}_t^{l}H\|^2_{H^{\frac{3}{2}}(\Omega_t)} &\leq C\Big\| \sum_{1\leq m\leq l} \sum_{|\beta|\leq l-m} \nabla\mathcal{D}_t^{\beta_1}v \star \cdots \star \nabla\mathcal{D}_t^{\beta_m}v \star H \Big\|^2_{H^{\frac{3}{2}}(\Omega_t)} \\
&\leq C \sum_{\beta_1\leq l-1} \|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \|H\|^2_{H^2(\Omega_t)} \\
&\quad + C \sum_{\substack{2\leq m\leq l \\ |\beta|\leq l-m}} \|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{H^2(\Omega_t)} \cdots \|\nabla\mathcal{D}_t^{\beta_m}v\|^2_{H^2(\Omega_t)} \|H\|^2_{H^2(\Omega_t)} \\
&\leq C(\|\nabla\mathcal{D}_t^{l-1}v\|^2_{H^{\frac{3}{2}}(\Omega_t)} + 1) \leq C(\|\mathcal{D}_t^{l+1-2}v\|^2_{H^3(\Omega_t)} + 1).
\end{aligned}$$

Finally, for even integer $k = l$, from $E_{l-1}(t) \leq C$, one has

$$\begin{aligned}
\|\mathcal{D}_tH\|^2_{H^{\frac{3}{2}l}(\Omega_t)} &\leq C\|H\|^2_{H^{\frac{3}{2}l}(\Omega_t)} \|v\|^2_{H^{\frac{3}{2}l+1}(\Omega_t)} \\
&\leq C\|H\|^2_{H^{\lfloor \frac{3}{2}l \rfloor}(\Omega_t)} \|v\|^2_{H^{\lfloor \frac{3}{2}l+1 \rfloor}(\Omega_t)} \\
&\leq C\|v\|^2_{H^{\lfloor \frac{3l+3}{2} \rfloor}(\Omega_t)},
\end{aligned}$$

and if $k = l$ is odd, we have by Lemma A.9 that

$$\|\mathcal{D}_tH\|^2_{H^{\frac{3}{2}l}(\Omega_t)} \leq C(\|H\|^2_{L^\infty(\Omega_t)} \|v\|^2_{H^{\frac{3}{2}l+1}(\Omega_t)} + \|H\|^2_{H^{\frac{3}{2}l}(\Omega_t)} \|v\|^2_{L^\infty(\Omega_t)})$$

$$\leq C\|v\|^2_{H^{\lfloor\frac{3l+3}{2}\rfloor}(\Omega_t)} + C\|H\|^2_{H^{\lfloor\frac{3l+3}{2}\rfloor}(\Omega_t)}.$$

Step 2. We claim that $\|\mathcal{D}_t^l v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \leq \varepsilon E_l(t) + C\tilde{e}_l(t)$. Note that from $\|\nu\|_{H^{\frac{5}{2}+\delta}(\Omega_t)} \leq C$ and the assumption $E_{l-1}(t) \leq C$, we have

$$\begin{aligned}
\|\mathcal{D}_t^l v\cdot\nu\|^2_{L^2(\Gamma_t)} &= \int_{\Gamma_t} [(\mathcal{D}_t^l v\cdot\nu)\mathcal{D}_t^l v]\cdot\nu dS\\
&\leq |\int_{\Omega_t}(\mathcal{D}_t^l v\cdot\nu)\operatorname{div}\mathcal{D}_t^l v dx| + |\int_{\Omega_t}\nabla\mathcal{D}_t^l v\star\mathcal{D}_t^l v dx|\\
&\quad + |\int_{\Omega_t}\mathcal{D}_t^l v\star\nabla\nu\star\mathcal{D}_t^l v dx|\\
&\leq C(\|\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)}\\
&\qquad + \|\nabla\mathcal{D}_t^l v\|_{L^2(\Omega_t)}\|\mathcal{D}_t^l v\|_{L^2(\Omega_t)})\\
&\leq C(\|\operatorname{div}\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)} + \varepsilon\|\nabla\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)} + C_\varepsilon E_{l-1}(t))\\
&\leq \varepsilon\|\nabla\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)} + C(1+\|\operatorname{div}\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)}).
\end{aligned}$$

This, combined with Lemma A.13, we see that

$$\begin{aligned}
\|\mathcal{D}_t^l v\|^2_{H^{\frac{3}{2}}(\Omega_t)} &\leq C(\|\mathcal{D}_t^l v\cdot\nu\|^2_{H^1(\Gamma_t)} + \|\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\mathcal{D}_t^l v\|^2_{H^{\frac{1}{2}}(\Omega_t)}\\
&\quad + \|\operatorname{curl}\mathcal{D}_t^l v\|^2_{H^{\frac{1}{2}}(\Omega_t)})\\
&\leq C(\varepsilon\|\mathcal{D}_t^l v\|^2_{H^1(\Omega_t)} + 1 + E_{l-1}(t) + \|\bar{\nabla}(\mathcal{D}_t^l v\cdot\nu)\|^2_{L^2(\Gamma_t)}\\
&\quad + \|\operatorname{div}\mathcal{D}_t^l v\|^2_{H^{\frac{1}{2}}(\Omega_t)} + \|\operatorname{curl}\mathcal{D}_t^l v\|^2_{H^{\frac{1}{2}}(\Omega_t)}).
\end{aligned}$$

Then, it follows that

$$\|\mathcal{D}_t^l v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \leq C(\tilde{e}_l(t) + \underbrace{\|\operatorname{div}\mathcal{D}_t^l v\|^2_{H^{\frac{1}{2}}(\Omega_t)}}_{=:I_1(t)} + \underbrace{\|\operatorname{curl}\mathcal{D}_t^l v\|^2_{H^{\frac{1}{2}}(\Omega_t)}}_{=:I_2(t)}).$$

Applying Lemmas 2.10, 5.2 and 5.3, we arrive at

$$\begin{aligned}
I_1(t)+I_2(t) &\leq C(\|R_I^{l-1}\|^2_{H^{\frac{1}{2}}(\Omega_t)} + \|R^{l-1}_{\nabla H,\nabla H}\|^2_{H^{\frac{1}{2}}(\Omega_t)} + \|R^{l-1}_{\nabla^2 H,H}\|^2_{H^{\frac{1}{2}}(\Omega_t)})\\
&\leq \varepsilon E_l(t) + C_\varepsilon,
\end{aligned}$$

where $\varepsilon > 0$ is sufficiently small. This concludes the claim.

Step 3. We claim that for $2 \leq k \leq l$, it holds

$$\|\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \leq C\|\mathcal{D}_t^{l+3-k} v\|^2_{H^{\frac{3}{2}k-3}(\Omega_t)} + \varepsilon E_l(t) + C_\varepsilon\tilde{e}_l(t). \tag{6.3}$$

Once we have these estimates, it follows that $\|\mathcal{D}_t^{l-1}v\|^2_{H^3(\Omega_t)} \leq \varepsilon E_l(t) + C_\varepsilon\tilde{e}_l$. This, combined with Step 2, will control $\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)}$ for any $3 \leq k \leq l$.

To prove (6.3), from Lemmas 2.10, 5.2, 5.3 and A.14, it holds

$$\begin{aligned}
&\|\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)}\\
&\leq C(\|\Delta_B(\mathcal{D}_t^{l+1-k} v\cdot\nu)\|^2_{H^{\frac{3k-5}{2}}(\Gamma_t)} + \|\mathcal{D}_t^{l+1-k} v\|^2_{L^2(\Omega_t)}\\
&\quad + \|\operatorname{div}\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3k-2}{2}}(\Omega_t)} + \|\operatorname{curl}\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3k-2}{2}}(\Omega_t)})\\
&\leq C(\|\Delta_B(\mathcal{D}_t^{l+1-k} v\cdot\nu)\|^2_{H^{\frac{3k-5}{2}}(\Gamma_t)} + \|R_I^{l-k}\|^2_{H^{\frac{3k-2}{2}}(\Omega_t)}\\
&\quad + \|R^{l-k}_{\nabla H,\nabla H}\|^2_{H^{\frac{3k-2}{2}}(\Omega_t)} + \|R^{l-k}_{\nabla^2 H,H}\|^2_{H^{\frac{3k-2}{2}}(\Omega_t)} + E_{l-1}(t))\\
&\leq C\|\Delta_B(\mathcal{D}_t^{l+1-k} v\cdot\nu)\|^2_{H^{\frac{3k-5}{2}}(\Gamma_t)} + \varepsilon E_l(t) + C_\varepsilon.
\end{aligned}$$

Lemmas 2.13 and 5.5 imply that $\mathcal{D}_t^{l+2-k}p = -\Delta_B(\mathcal{D}_t^{l+1-k}v\cdot\nu)+R_p^{l+1-k}$ as well as $\|R_p^{l+1-k}\|^2_{H^{\frac{3k-5}{2}}(\Gamma_t)} \le \|R_p^{l-(k-1)}\|^2_{H^{\frac{3}{2}(k-1)-1}(\Gamma_t)} \le \varepsilon E_l(t) + C_\varepsilon$. Then, we obtain

$$\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \le C\|\mathcal{D}_t^{l+2-k}p\|^2_{H^{\frac{3k-5}{2}}(\Gamma_t)} + \varepsilon E_l(t) + C_\varepsilon.$$

By (A.4), we see that

$$\begin{aligned}\|\mathcal{D}_t^{l+2-k}p\|^2_{H^{\frac{3k-5}{2}}(\Gamma_t)} &\le \|\mathcal{D}_t^{l+2-k}p\|^2_{H^{\frac{3k-6}{2}}(\Gamma_t)} + \|\nabla\mathcal{D}_t^{l+2-k}p\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)}\\ &=: I_3(t) + I_4(t).\end{aligned}$$

The first term can be controlled by using Lemma 2.4 as in Proposition 6.1, and we have $I_3(t) \le \varepsilon E_l(t) + C_\varepsilon$. For the second term, by (1.1), Lemmas 5.3 and 5.4, it holds

$$\begin{aligned}I_4(t) \le{}& \|[\nabla, \mathcal{D}_t^{l+2-k}]p\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)} + \|\mathcal{D}_t^{l+3-k}v\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)}\\ &+ \|\mathcal{D}_t^{l+2-k}(H\cdot\nabla H)\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)}\\ \le{}& \|\mathcal{D}_t^{l+3-k}v\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)} + \Big\|\sum_{\beta\le l+1-k}\nabla\mathcal{D}_t^{\beta}v\star\nabla H\star H\Big\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)}\\ &+ \|R_{II}^{l+1-k}\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)} + \|R_{\nabla H,H}^{l+2-k}\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)}\\ \le{}& \|\mathcal{D}_t^{l+3-k}v\|^2_{H^{\frac{3k-6}{2}}(\Omega_t)} + \varepsilon E_l(t) + C_\varepsilon.\end{aligned}$$

Combining the above estimates, (6.3) follows.

Finally, it remains to verify that $\|v\|^2_{H^{\lfloor\frac{3l+3}{2}\rfloor}(\Omega_t)} + \|H\|^2_{H^{\lfloor\frac{3l+3}{2}\rfloor}(\Omega_t)} \le \varepsilon E_l(t) + C_\varepsilon\tilde{e}_l(t)$. Note that from Lemma 5.1 with $l \ge 4$, it follows that $\|B\|_{H^{\frac{3}{2}l-1}(\Gamma_t)} \le C$ and $\|B\|_{H^k(\Gamma_t)} \le C(1+\|p\|_{H^k(\Gamma_t)})$ for $k \in \frac{\mathbb{N}}{2}, k \le \frac{3}{2}l$. Then, we can apply the same argument as in Proposition 6.1. This completes the proof. □

7. Proof of the Main Theorem

We are ready to prove the main theorem.

Proof of Theorem 1.1. We divide the proof into three parts.

Step 1. Assume that the quantities $\mathcal{N}_T$ and $\mathcal{M}_T$, defined in (1.7) and (1.6) respectively, satisfy the a priori assumptions (1.8) for some $T > 0$. We claim that

$$\begin{aligned}&\sup_{0\le t<T}\left(\bar{E}(t) + \sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|_{H^{\frac{3}{2}k+1}(\Omega_t)} + \|B_{\Gamma_t}\|_{H^5(\Gamma_t)}\right)\\ &\le \bar{C}\left(T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}, \|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}\right),\end{aligned}\tag{7.1}$$

and

$$\sup_{0\le t<T} E_l(t) \le C_l\left(T, \mathcal{N}_T, \mathcal{M}_T, E_l(0)\right), \quad l \ge 4. \tag{7.2}$$

These estimates quantify the regularity of the flow, provided that the a priori assumptions are bounded.

Recalling the estimates in Section 4 that

$$\bar{E}(0) + \sup_{0\le t<T}\|p\|^2_{H^3(\Omega_t)} \le C,$$

where C depends on $T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$. Then, the assumptions of Proposition 6.1 hold for any $0 \le t < T$, and Propositions 5.6 and 6.1 allow us to obtain

$$\frac{d}{dt}\bar{e}(t) \le C\bar{E}(t) \le C(1+\bar{e}(t)) \tag{7.3}$$

for $0 \le t < T$. Integrating over $(0,t)$, we have

$$\sup_{0\le t<T} \bar{e}(t) \le C(1+\bar{e}(0))e^{CT}.$$

Again by Proposition 6.1, we see that

$$\sup_{0\le t<T} \bar{E}(t) \le C + C(1+\bar{e}(0))e^{CT} \le C + C\bar{E}(0)e^{CT} \le \bar{C}_0, \tag{7.4}$$

where $\bar{C}_0$ depends on $T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$.

With $\sup_{0\le t<T} (\bar{E}(t) + \|p\|^2_{H^3(\Omega_t)}) \le \bar{C}_0$, applying Lemma 5.1 and the trace theorem, it follows that

$$\|B\|^2_{H^{\frac{9}{2}}(\Gamma_t)} \le C(1+\|p\|^2_{H^{\frac{9}{2}}(\Gamma_t)}) \le C(1+\|H\cdot\nabla H - \mathcal{D}_t v\|^2_{H^4(\Omega_t)}) \le C(\bar{C}_0),$$

which means $\|B\|^2_{H^{\frac{9}{2}}(\Gamma_t)} + \|p\|^2_{H^5(\Omega_t)} \le C(\bar{C}_0)$. We proceed to find that

$$\|p\|^2_{H^{\frac{11}{2}}(\Omega_t)} \le C(1+\|\nabla p\|^2_{H^{\frac{9}{2}}(\Omega_t)}) \le C(1+\|H\cdot\nabla H - \mathcal{D}_t v\|^2_{H^{\frac{9}{2}}(\Omega_t)}) \le C(\bar{C}_0),$$

and utilize Lemma A.11 to obtain

$$\|B\|^2_{H^5(\Gamma_t)} \le C(1+\|p\|^2_{H^5(\Gamma_t)}) \le C(\bar{C}_0).$$

In particular, it follows that $\|\mathcal{A}\|^2_{H^5(\Gamma_t)} \le C(\bar{C}_0)$, and (4.11) yields

$$\sum_{k=0}^{3} \|\mathcal{D}_t^{3-k} p\|^2_{H^{\frac{3}{2}k+1}(\Omega_t)} \le C,$$

where C depends on $\mathcal{R} - \|h(\cdot,t)\|_{L^\infty(\Gamma)}, \|v\|_{H^6(\Omega_t)}, \|H\|_{H^6(\Omega_t)}$, and $\|\mathcal{A}\|_{H^5(\Gamma_t)}$. Combining the above estimates, we conclude that

$$\sup_{0\le t<T} \left(\bar{E}(t) + \sum_{k=0}^{3} \|\mathcal{D}_t^{3-k} p\|_{H^{\frac{3}{2}k+1}(\Omega_t)} + \|B_{\Gamma_t}\|_{H^5(\Gamma_t)} \right) \le \bar{C},$$

where $\bar{C}$ depends on $T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$.

To verify the second claim, for $l \ge 4$, we apply Propositions 5.6 and 6.2 by induction: if $\sup_{0\le t<T} E_{l-1}(t) \le C$, then it follows that

$$\frac{d}{dt} e_l(t) \le CE_l(t) \le C(1+e_l(t)).$$

Similarly, we integrate over $(0,t)$ and use Proposition 6.2 again to obtain

$$\sup_{0\le t<T} e_l(t) \le C(1+e_l(0))e^{CT},$$

and

$$\sup_{0\le t<T} E_l(t) \le C + C(1+e_l(0))e^{CT} \le C_l, \tag{7.5}$$

where the constant C_l depends on $l, T, \mathcal{N}_T, \mathcal{M}_T, \sup_{0\le t<T} E_{l-1}(t)$, and $e_l(0)$. However, the induction argument implies that (7.5) holds for all l and the constant C_l which depends on $l, T, \mathcal{N}_T, \mathcal{M}_T, e_l(0)$ and $\bar{e}(0)$ from (7.4). Note that $\bar{e}(0) + e_l(0) \le CE_l(0)$, and the constant C_l in fact depends on $l, T, \mathcal{N}_T, \mathcal{M}_T$, and $E_l(0)$. This completes the proof of our claim.

Step 2. We prove the last statement in Theorem 1.1, i.e., the a priori assumptions (1.8) hold for some time $T_0 \ge c_0 > 0$, where the constant c_0 depends on $\mathcal{M}_0, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$ and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$. To this aim, we define

$$I(t) := \|B\|^2_{H^3(\Gamma_t)} + \|p\|^2_{H^3(\Omega_t)} + \|v\|^2_{H^4(\Omega_t)} + \|H\|^2_{H^4(\Omega_t)} + 1, \quad t \ge 0.$$

Suppose that it holds $I(t) \le 2I(0)$ and $\mathcal{M}_t \ge \mathcal{M}_0/2$ for some time $t > 0$, where $\mathcal{M}_0 = \mathcal{R} - \|h_0\|_{L^\infty(\Gamma)}$. Then we have $\|\mathcal{A}_{\Gamma_t}\|^2_{H^3(\Gamma_t)} \le C(I(0))$. Therefore, applying Lemma A.4, it follows that $\|h(\cdot,t)\|_{H^{3+\delta}(\Gamma)} \le C$, for $\delta > 0$ small enough, where the constant C depends on $\|\mathcal{A}_{\Gamma_t}\|_{H^{1+\delta}(\Gamma_t)}$, and hence on $I(0)$. An application of Proposition 6.1 allows us to obtain that there exists a constant C, which depends on $I(0)$ and $\mathcal{M}_0$ such that

$$\bar{E}(t) \le C(1+\bar{e}(t)). \tag{7.6}$$

From the above argument, we define $T_0 \in (0,1]$ to be the largest number such that

$$[0,T_0] \subset \left\{ t\in[0,1] : I(t) \ge \frac{I(0)}{2}, \mathcal{M}_t \ge \frac{\mathcal{M}_0}{2}, \text{ and } \bar{e}(t) \le 1+\bar{e}(0) \right\}. \tag{7.7}$$

Here, we make the assumption that $T_0 < 1$, since the claim would be trivial otherwise. We note that the last condition together with (7.6) implies that

$$\sup_{0\le t\le T_0} \bar{E}(t) \le C(1+\bar{e}(t)) \le C(2+\bar{e}(0)) \le C\bar{E}(0). \tag{7.8}$$

Also, we observe that the $\mathcal{N}_T$ defined in (1.7) satisfies

$$\mathcal{N}_{T_0}^2 \le C \sup_{0\le t<T_0} \bar{E}(t),$$

thanks to the curvature bound $\|B\|_{H^3(\Gamma_t)} \le 2I(0)$. Indeed, from $\bar{\nabla} v_n = \bar{\nabla} v\cdot \nu - v\star B$, we can bound $\|v_n\|_{H^4(\Gamma_t)}$ by using $\|v\|_{H^4(\Gamma_t)}$ and $\|B\|_{H^3(\Gamma_t)}$.

The estimate (7.8) ensures that the a priori assumptions (1.8) hold for time $T = T_0$, and the claim follows once we show that the time T_0 specified in (7.7) has a lower bound $c_0 > 0$, depending only on the initial data.

According to the definition of T_0, at least one of the three conditions has equality. Assume that $I(T_0) = 2I(0)$. Then, it holds $\bar{E}(t) \le C\bar{E}(0)$, for all $t \le T_0$ by (7.8). We will show that

$$\frac{d}{dt} I(t) \le C\bar{E}(t)I(t) \le C\bar{E}(0)I(t). \tag{7.9}$$

We focus on the computation of the highest-order terms. In fact, Lemma 2.3 yields

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla^4 v\|_{L^2(\Omega_t)}^2 + \|\nabla^4 H\|_{L^2(\Omega_t)}^2\right) \\
=& \int_{\Omega_t} \mathcal{D}_t \nabla^4 v \star \nabla^4 v + \mathcal{D}_t \nabla^4 H \star \nabla^4 H dx \\
\le& \int_{\Omega_t} \nabla^4 \mathcal{D}_t v \star \nabla^4 v dx + \int_{\Omega_t} \sum_{|\alpha|\le 3} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} v \star \nabla^4 v dx \\
&+ \int_{\Omega_t} \nabla^4 \mathcal{D}_t H \star \nabla^4 H dx + \int_{\Omega_t} \sum_{|\alpha|\le 3} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} H \star \nabla^4 H dx \\
\le& \|\nabla^4 \mathcal{D}_t v\|_{L^3(\Omega_t)} \|\nabla^4 v\|_{L^6(\Omega_t)} + \|\nabla^4 \mathcal{D}_t H\|_{L^3(\Omega_t)} \|\nabla^4 H\|_{L^6(\Omega_t)} \\
&+ \|v\|_{H^5(\Omega_t)}^2 \|v\|_{H^4(\Omega_t)} + \|H\|_{H^5(\Omega_t)}^2 \|H\|_{H^4(\Omega_t)} \\
\le& C\bar{E}(t)I(t).
\end{aligned}$$

Applying Lemmas 2.3 and 2.6, we see that

$$\begin{aligned}
&\frac{d}{dt}\|\nabla^3 p\|_{L^2(\Omega_t)}^2 \\
=& \int_{\Omega_t} \mathcal{D}_t \nabla^3 p \star \nabla^3 p dx \\
=& \int_{\Omega_t} \nabla^2 \mathcal{D}_t \nabla p \star \nabla^3 p dx + \int_{\Omega_t} \nabla^2 v \star \nabla^2 p \star \nabla^3 p + \nabla v \star \nabla^3 p \star \nabla^3 p \\
&\qquad + \nabla v \star \nabla^2 p \star \nabla^3 p dx \\
\le& \left| \int_{\Omega_t} (\nabla^2 \mathcal{D}_t^2 v + \nabla^2(\nabla^2 v \star H \star H) + \nabla^2(\nabla v \star \nabla H) \star H) \star \nabla^3 p dx \right| \\
&+ C\|v\|_{H^3(\Omega_t)} \|p\|_{H^3(\Omega_t)}^2
\end{aligned}$$

$$\leq C\bar{E}(t)I(t).$$

Similarly, we obtain by Lemma 2.1 that

$$\begin{aligned}
&\frac{d}{dt}\|\bar{\nabla}^3 B\|^2_{L^2(\Gamma_t)} \\
&= \int_{\Gamma_t} |\bar{\nabla}^3 B|^2 \operatorname{div}_\sigma v dS + \int_{\Gamma_t} \mathcal{D}_t \bar{\nabla}^3 B \star \bar{\nabla}^3 B dS \\
&\leq \|B\|^2_{H^3(\Gamma_t)} \|\nabla v\|_{L^\infty(\Gamma_t)} + \int_{\Gamma_t} \bar{\nabla}^3(\bar{\nabla}^2 v \star \nu + \bar{\nabla} v \star B) \star \bar{\nabla}^3 B dS \\
&\quad + \int_{\Gamma_t} \sum_{|\alpha|\leq 2} \bar{\nabla}^{1+\alpha_1} v \star \bar{\nabla}^{1+\alpha_2} B \star \bar{\nabla}^3 B dS \\
&\leq C\bar{E}(t)I(t).
\end{aligned}$$

By integrating (7.9) over $(0, T_0)$ and using $I(T_0) = 2I(0)$, we obtain

$$\ln 2 = \ln I(T_0) - \ln I(0) \leq CT_0\bar{E}(0).$$

Then we have

$$T_0 \geq \frac{C}{\bar{E}(0)} = c_0,$$

where the constant c_0 depends on $I(0), \mathcal{M}_0$, and $\bar{E}(0)$. Moreover, by Lemma 5.1 and Proposition 4.7, the constant c_0 depends only on $\mathcal{M}_0$, $\|v_0\|_{H^6(\Omega_0)}$, $\|H_0\|_{H^6(\Omega_0)}$ and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$.

A similar argument applies if we have an equality in the third condition, i.e., $\bar{e}(T_0) = 1 + \bar{e}(0)$. In fact, it follows that

$$\frac{d}{dt}\bar{e}(t) \leq C\bar{E}(t) \leq C\bar{E}(0),$$

by (7.3) and (7.8), and we integrate the above over $(0, T_0)$ to obtain

$$1 = \bar{e}(T_0) - \bar{e}(0) \leq C\bar{E}(0)T_0.$$

This results in $T_0 \geq c_0 > 0$ again, with the constant c_0 depending on the same initial data.

Finally, we assume that $\mathcal{M}_{T_0} = \mathcal{M}_0/2$. Recalling that

$$\mathcal{M}_T = \mathcal{R} - \sup_{0\leq t<T} \|h(\cdot,t)\|_{L^\infty(\Gamma)},$$

and $\mathcal{M}_0 > 0$, we define $0 < T_1 \leq T_0$ by

$$\mathcal{M}_{T_0} = \mathcal{R} - \|h(\cdot, T_1)\|_{L^\infty(\Gamma)}.$$

It is clear that $\|v_n\|^2_{L^\infty(\Omega_t)} \leq C\bar{E}(t) \leq C\bar{E}(0)$ by using (7.8). Recalling the fact that $\frac{d}{dt}h = v_n$, we have by the fundamental Theorem of calculus that

$$\begin{aligned}
\mathcal{M}_{T_0} &= \mathcal{R} - \|h(\cdot, T_1)\|_{L^\infty(\Gamma)} \\
&\geq \mathcal{R} - \|h_0\|_{L^\infty(\Gamma)} - \int_0^{T_1} \|v_n\|_{L^\infty(\Omega_t)}\, dt \\
&\geq \mathcal{M}_0 - C\bar{E}(0)^{\frac{1}{2}} T_1,
\end{aligned}$$

which means $T_0 \geq T_1 \geq C\mathcal{M}_0/\bar{E}(0)^{\frac{1}{2}} > 0$. This concludes the claim.

Step 3. We prove the first three statements of Theorem 1.1. According to the a priori assumptions, the estimates (7.1) and (7.2) hold. In particular, we conclude by Lemmas 5.1 and A.4 that the regularity of the curvature implies the regularity of the free boundary, i.e., $\Gamma_T \in C^\infty$. Additionally, the quantitative regularity estimates show that $v(\cdot, T), H(\cdot, T) \in C^\infty(\Omega_T)$.

Then, we apply the results in Step 2 to the domain Ω_T and conclude that system (1.1) is well-defined and the a priori assumptions hold for some time $\tau > 0$. Moreover, by (7.7) and (7.8), it follows that

$$\sup_{T\leq t<T+\tau} \bar{E}(t) \leq C\bar{E}(T), \text{ and } \mathcal{M}_{T+\tau} \geq \frac{\mathcal{M}_T}{2}.$$

Therefore, applying the same argument as in Step 1 yields

$$\sup_{0\le t<T+\tau}\left(\bar{E}(t)+\sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|_{H^{\frac{3}{2}k+1}(\Omega_t)}+\|B_{\Gamma_t}\|_{H^5(\Gamma_t)}\right)\le\bar{C},$$

where $\bar{C}$ depends on $T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$. Again by induction as in (7.2), we obtain

$$\sup_{0\le t<T+\tau}E_l(t)\le C_l,$$

where the constant C_l depends on $l, T, \mathcal{N}_T, \mathcal{M}_T$, and $E_l(0)$. This completes the proof of the theorem. □

Appendix A. Some Estimates and Formulas

Lemma A.1 ([BM18]). *Let Ω be a standard domain, i.e., Ω is either $\mathbb{R}^n$ or a half-space or a Lipschitz bounded domain in $\mathbb{R}^n$. For real numbers $s_1, s_2, s \ge 0, \theta \in (0,1)$ and $1 \le p_1, p_2, p \le \infty$, satisfy the relations*

$$s=\theta s_1+(1-\theta)s_2,\quad \frac{1}{p}=\frac{\theta}{p_1}+\frac{1-\theta}{p_2}.$$

(1) *If $s_1 < s < s_2$,*

$$\|f\|_{W^{s,p}(\Omega)}\le C\|f\|^{\theta}_{W^{s_1,p_1}(\Omega)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}\tag{A.1}$$

holds, if and only if

$$s_2\in\mathbb{N}^+,p_2=1,s_1-\frac{1}{p_1}\ge s_2-\frac{1}{p_2}\tag{A.2}$$

fails. More precisely, if (A.2) fails then, for every $\theta \in (0,1)$, there exists a constant C depending on $s_1, s_2, p_1, p_2, \theta$ and Ω such that

$$\|f\|_{W^{s,p}(\Omega)}\le C\|f\|^{\theta}_{W^{s_1,p_1}(\Omega)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}.$$

If (A.2) holds, there exists some $f \in W^{s_1,p_1}(\Omega)\cap W^{s_2,p_2}(\Omega)$ such that $f \notin W^{s,p}(\Omega), \forall\theta \in (0,1)$.

(2) *If $s_1 = s_2$, it is simply Hölder's inequality.*

Lemma A.2 (Reynolds transport theorem). *For all smooth function $f(\cdot,t) : \Omega_t \to \mathbb{R}$, it holds*

$$\frac{d}{dt}\int_{\Omega_t}f\,dx=\int_{\Omega_t}\mathcal{D}_tf\,dx.$$

Lemma A.3 ([SZ08b]). *For all smooth function $f(\cdot,t) : \Gamma_t \to \mathbb{R}$, it holds*

$$\frac{d}{dt}\int_{\Gamma_t}f\,dS=\int_{\Gamma_t}\mathcal{D}_tf+f\,\mathrm{div}_\sigma\, v\,dS.$$

Lemma A.4 ([SZ08b, Proposition A.2]). *Let $\Omega \subset \mathbb{R}^3$ be a domain such that $\partial\Omega \in H^{s_0}$, $s_0 > 2$. Suppose $\|\mathcal{A}\|_{H^{s-2}(\Gamma_t)} \le C$ with $s \ge s_0$, then $\partial\Omega \in H^s$.*

Definition A.5. Let $\Gamma = \partial\Omega$ and Ω be a smooth domain. Let $u \in L^2(\Gamma)$. We say $u \in H^{\frac{1}{2}}(\Gamma)$ if

$$\begin{aligned}\|u\|_{H^{\frac{1}{2}}(\Gamma)}&:=\|u\|_{L^2(\Gamma)}+\inf\{\|\nabla w\|_{L^2(\Omega)}: w\in H^1(\Omega)\text{ and }w=u\text{ on }\Gamma\}\\&=\|u\|_{L^2(\Gamma)}+\|\nabla v\|_{L^2(\Omega)},\end{aligned}$$

where $v \in H^1(\Omega)$ such that $v|_\Gamma = u$ in the trace sense and $\Delta v = 0$ in the weak sense.

We note that for $u \in H^1(\Omega)$, it holds

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)}\le\|u\|_{L^2(\Gamma)}+\|\nabla u\|_{L^2(\Omega)}.\tag{A.3}$$

Moreover, since we define the space $H^{\frac{1}{2}}(\Gamma)$ via the harmonic extension, for $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ such that $u|_\Gamma$ is the trace of v on Γ, we have

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega)}\le&\|\nabla(u-v)\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\\\le&\|(u-v)\Delta u\|_{L^1(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&\leq \varepsilon\|u-v\|_{L^2(\Omega)}^2 + C_\varepsilon\|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \\
&\leq \varepsilon\|\nabla(u-v)\|_{L^2(\Omega)}^2 + C_\varepsilon\|\Delta u\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \\
&\leq \varepsilon\|\nabla u\|_{L^2(\Omega)}^2 + C_\varepsilon\|\Delta u\|_{L^2(\Omega)}^2 + C\|\nabla v\|_{L^2(\Omega)}^2 \\
&\leq \varepsilon\|\nabla u\|_{L^2(\Omega)}^2 + C(\|\Delta u\|_{L^2(\Omega)}^2 + \|u\|_{H^{\frac{1}{2}}(\Gamma)}^2),
\end{aligned}
$$

where we have used the fact that $v-u\in H_0^1(\Omega)$ and Poincaré's inequality. Therefore, we see that

$$
\|\nabla u\|_{L^2(\Omega)} \leq C\left(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^{\frac{1}{2}}(\Gamma)}\right). \tag{A.4}
$$

Furthermore, if v is the harmonic extension of $u|_\Gamma$, it follows that $\|v\|_{H^1(\Omega)}\leq C\|u\|_{H^{\frac{1}{2}}(\Gamma)}$ and we can obtain

$$
\|u\|_{H^1(\Omega)} \leq C\left(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^{\frac{1}{2}}(\Gamma)}\right). \tag{A.5}
$$

Lemma A.6 ([Man02, Proposition 6.5]). *Assume $\Gamma\subset\mathbb{R}^3$ is a compact hypersurface which is $C^{1,\alpha}$-regular and $\|B\|_{L^4(\Gamma)}\leq C$. Then for $k,l\in\mathbb{N}_0, 0\leq k<l, p,r\in[1,\infty)$, and $q\in[1,\infty]$, we have for all tensor fields T that*

$$
\|\bar\nabla^k T\|_{L^p(\Gamma)} \leq C\|T\|_{W^{l,r}(\Gamma)}^\theta \|T\|_{L^q(\Gamma)}^{(1-\theta)},
$$

where $p,\theta\in[0,1]$ are given by $1/p = k/2+\theta(1/r-l/2)+(1-\theta)/q$.

In particular, for $k,l\in\mathbb{N}_0, 0\leq k<l, q\in[1,\infty]$, we have

$$
\|\bar\nabla^k u\|_{L^2(\Gamma)} \leq C\|u\|_{H^l(\Gamma)}^\theta \|u\|_{L^q(\Gamma)}^{(1-\theta)},
$$

where $\theta\in[0,1]$ are given by $1=k+\theta\,(1-l)+(2-2\theta)/q$.

Lemma A.7 ([JLM22, Corollary 2.9]). *Let $m\in\mathbb{N}_0$ and $\Gamma\subset\mathbb{R}^3$ be a compact 2-dimensional hypersurface which is $C^{1,\alpha}$-regular such that $\Gamma=\partial\Omega$ and satisfies the condition (H_m), i.e.,*

$$
\|B\|_{L^4(\Gamma)}\leq C, \text{ if } m=2, \quad \|B\|_{L^\infty(\Gamma)}+\|B\|_{H^{m-2}(\Gamma)}\leq C, \text{ if } m>2. \tag{A.6}
$$

Then for all $k,l\in\frac{\mathbb{N}}{2}$ with $k<l\leq m$ and for $q\in[1,\infty]$, it holds

$$
\|u\|_{H^k(\Gamma)} \leq C\|u\|_{H^l(\Gamma)}^\theta \|u\|_{L^q(\Gamma)}^{1-\theta},
$$

where $\theta\in[0,1]$ is given by $1=k-\theta(l-1)+(2-2\theta)/q$, and

$$
\|u\|_{H^k(\Omega)} \leq C\|u\|_{H^l(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta},
$$

where $\theta\in[0,1]$ is given by $1/2=k/3+\theta(1/2-l/3)+(1-\theta)/q$.

Moreover, for $k,l\in\mathbb{N}_0$ with $k<l\leq m$ and for $p\in[1,\infty), q\in[1,\infty]$, it holds

$$
\|\nabla^k u\|_{L^p(\Omega)} \leq C\|u\|_{H^l(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta},
$$

where $\theta\in[0,1]$ is given by $1/p=k/3+\theta(1/2-l/3)+(q-\theta)/q$.

Lemma A.8 ([CS17, JLM22]). *For $f,g\in C_0^\infty(\mathbb{R}^n)$ and numbers $2\leq p_1,q_2<\infty, 2\leq p_2,q_1\leq\infty$ with $1/p_1+1/q_1=1/p_2+1/q_2=1/2$, then we have for all $k\in\frac{\mathbb{N}}{2}$,*

$$
\|fg\|_{H^k(\mathbb{R}^n)} \leq C\|f\|_{W^{k,p_1}(\mathbb{R}^n)}\|g\|_{L^{q_1}(\mathbb{R}^n)} + C\|g\|_{W^{k,q_2}(\mathbb{R}^n)}\|f\|_{L^{p_2}(\mathbb{R}^n)}. \tag{A.7}
$$

Lemma A.9 ([JLM22, Proposition 2.10]). *Let $m\in\mathbb{N}$ and assume $\Gamma=\partial\Omega$ is $C^{1,\alpha}$-regular and satisfies the condition (H_m) defined in (A.6). Then for all $k\in\frac{\mathbb{N}}{2}, k\leq m$, it holds*

$$
\|fg\|_{H^k(\Gamma)} \leq C\|f\|_{H^k(\Gamma)}\|g\|_{L^\infty(\Gamma)} + C\|f\|_{L^\infty(\Gamma)}\|g\|_{H^k(\Gamma)},
$$

and

$$
\|fg\|_{H^k(\Omega)} \leq C\|f\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)} + C\|f\|_{L^\infty(\Omega)}\|g\|_{H^k(\Omega)}.
$$

Moreover, assume that $\|B\|_{L^4}\leq C$ and let $k\in\mathbb{N}_0$. Then for $p_1,p_2,q_1,q_2\in[2,\infty]$ with $p_1,q_2<\infty$ satisfying $1/p_1+1/q_1=1/p_2+1/q_2=1/2$, we have

$$
\|fg\|_{H^k(\Gamma)} \leq C\|f\|_{W^{k,p_1}(\Gamma)}\|g\|_{L^{q_1}(\Gamma)} + C\|f\|_{L^{p_2}(\Gamma)}\|g\|_{W^{k,q_2}(\Gamma)}.
$$

Lemma A.10 ([JLM22, Proposition 2.1]). *Let $m \in \mathbb{N}, m \geq 2$, and let Ω be a smooth domain, which is uniformly $C^{1,\alpha}$-regular and satisfies the condition* (H_m) *defined in* (A.6)*. Then, there is an extension operator $T : H^m(\Omega) \to H_0^m(\mathbb{R}^3)$ such that*

$$\|T(u)\|_{H^m(\mathbb{R}^3)} \leq C_m \|u\|_{H^m(\Omega)}.$$

Lemma A.11 ([JLM22, Proposition 2.12]). *Assume that Γ is $C^{1,\alpha}$-regular. Then for every $p \in (1, \infty)$ it holds*

$$\|B_\Gamma\|_{L^p(\Gamma)} \leq C\left(1 + \|\mathcal{A}_\Gamma\|_{L^p(\Gamma)}\right).$$

If in addition $\|B_\Gamma\|_{L^4(\Gamma)} \leq C$, then for $k = \frac{1}{2}, 1, 2$, it holds

$$\|B_\Gamma\|_{H^k(\Gamma)} \leq C\left(1 + \|\mathcal{A}_\Gamma\|_{H^k(\Gamma)}\right).$$

Finally, let $m \in \frac{\mathbb{N}}{2}, m \geq 3$, and assume that Γ satisfies additionally $\|B\|_{L^\infty(\Gamma)} + \|B\|_{H^{m-2}(\Gamma)} \leq C$. Then the above estimate holds for all half-integers $k \in \frac{\mathbb{N}}{2}$ with $k \leq m$.

Lemma A.12 ([CCS08, Theorem 3.1],[CLS10, Lemma 5.1]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain such that $\partial\Omega \in H^3$ or $\partial\Omega \in C^2$, then*

$$\|u_\sigma\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\operatorname{curl} u\|_{H^{-1}(\Omega)}\right),$$
$$\|u \cdot \nu\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{H^{-1}(\Omega)}\right),$$

for some constant C independent of u. In particular, we have

$$\|u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{H^{-1}(\Omega)} + \|\operatorname{curl} u\|_{H^{-1}(\Omega)}\right).$$

Lemma A.13 ([JLM22, Theorem 3.1]). *Let $l \geq 2$ be an integer and let Ω be a smooth domain with $\Gamma = \partial\Omega$, such that $\|B_\Gamma\|_{H^{\frac{3}{2}l-1}(\Gamma)} \leq C$. Then for all smooth vector fields $F : \Omega \to \mathbb{R}^3$ and every $k \in \{\frac{3}{2}, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, \cdots, \frac{3}{2}l\}$, it holds*

$$\|F\|_{H^k(\Omega)} \leq C\Big(\|F_n\|_{H^{k-\frac{1}{2}}(\Gamma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}(\Omega)}\Big).$$

Moreover, for $k = \lfloor \frac{3}{2}(l+1) \rfloor$, it holds

$$\|F\|_{H^k(\Omega)} \leq C\Big(\|\bar{\nabla} F_n\|_{H^{k-\frac{3}{2}}(\Gamma)} + (1 + \|B_\Gamma\|_{H^{\frac{3}{2}l}(\Gamma)})\|F\|_{L^\infty(\Omega)} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}(\Omega)}\Big).$$

Lemma A.14 ([JLM22, Proposition 3.2]). *Let l and Ω be as in Lemma A.13. Then for all smooth vector fields $F : \Omega \to \mathbb{R}^3$ and $k \in \{\frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, \cdots, \frac{3}{2}l\}$, it holds*

$$\|F\|_{H^k(\Omega)} \leq C\Big(\|\Delta_\Gamma F_n\|_{H^{k-\frac{5}{2}}(\Gamma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}(\Omega)}\Big).$$

Moreover, for $k = \lfloor \frac{3}{2}(l+1) \rfloor$, it holds

$$\|F\|_{H^k(\Omega)} \leq C\Big(\|\Delta_B F_n\|_{H^{k-\frac{5}{2}}(\Gamma)} + (1 + \|B\|_{H^{\frac{3}{2}l}(\Gamma)})\|F\|_{L^\infty(\Omega)} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}(\Omega)}\Big).$$

Lemma A.15 ([JLM22, Lemma 3.3]). *Let $\Omega \subset \mathbb{R}^3$ with $\Gamma = \partial\Omega$ be C^1-regular. Then for all vector fields $F : \Omega \to \mathbb{R}^3$ such that $\|\nabla F\|_{L^2(\Omega)} + \|F\|_{L^6(\Omega)} < \infty$, it holds*

$$\|F\|_{L^2(\Gamma)}^2 \leq C\left(\|F_n\|_{L^2(\Gamma)}^2 + \|F\|_{L^2(\Omega)}^2 + \|\operatorname{div} F\|_{L^2(\Omega)}^2 + \|\operatorname{curl} F\|_{L^2(\Omega)}^2\right),$$

and

$$\|F\|^2_{L^2(\Gamma)} \le C\left(\|F_\sigma\|^2_{L^2(\Gamma)} + \|F\|^2_{L^2(\Omega)} + \|\operatorname{div} F\|^2_{L^2(\Omega)} + \|\operatorname{curl} F\|^2_{L^2(\Omega)}\right).$$

Note that Ω may be unbounded, but its boundary is compact.

Lemma A.16 ([JLM22, Lemma 3.5]). *Assume that Ω, with $\Gamma = \partial\Omega$, is C^1-regular and $\|B_\Gamma\|_{L^4} \le C$ and $u : \Omega \to \mathbb{R}$ is a smooth function. Then it holds*

$$\|u\|_{H^2(\Omega)} \le C\left(\|\partial_\nu u\|_{H^{\frac12}(\Gamma)} + \|\nabla u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\right),$$
$$\|u\|_{H^2(\Omega)} \le C\left(\|\partial_\nu u\|_{H^{\frac12}(\Gamma)} + \|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\right).$$

Lemma A.17 ([JLM22, Proposition 3.8]). *Assume Ω, with $\Gamma = \partial\Omega$, is C^1-regular and $\|B_\Gamma\|_{H^{\frac12}(\Gamma)} \le C$. Then the solution of the following Dirichlet problem*

$$\begin{cases} \Delta u = f, & x \in \Omega, \\ u = 0, & x \in \Gamma, \end{cases}$$

satisfies

$$\|\partial_\nu u\|_{H^1(\Gamma)} + \|\nabla u\|_{H^{\frac32}(\Omega)} \le C\|f\|_{H^{\frac12}(\Omega)}.$$

Appendix B. Notations

- $\mathbb{N}$: positive integers $\{1, 2, \dots\}$
- $\frac{\mathbb{N}}{2}$: positive half-integers $\{\frac{k}{2} : k \in \mathbb{N}\}$
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$: non-negative integers
- $\lfloor\cdot\rfloor$: integer part of a given number
- $[\cdot,\cdot]$: Lie bracket
- $\alpha = (\alpha)_{i=1}^k \in \mathbb{N}_0^k$: an index vector, $|\alpha| = \sum_{i=1}^k \alpha_i$
- $\Omega \subset \mathbb{R}^3$: reference domain
- $\Gamma = \partial\Omega$: reference surface
- $\mathcal{R}$: the interior and exterior ball radius of Ω (or $\Gamma = \partial\Omega$)
- ν_Γ: unit outer normal to a compact hypersurface $\Gamma \subset \mathbb{R}^3$
- ∂: differentiation with respect to spatial variables
- ∂_ν: outer normal derivative
- ∇: gradient operator
- $\hat\nabla$: Riemannian connection, $\hat\nabla_F u = Fu$ for a vector field F and a function u
- $\mathcal{D}_t = \partial_t + v\cdot\nabla$: material derivative along the particle path
- $\operatorname{div} F = \partial_i F^i$: divergence of a vector field F
- $(\operatorname{div} A)_i = \sum_j \partial_j A_{ij}$: divergence of a matrix $A = (A_{ij})$
- $\operatorname{curl} F = \nabla F - (\nabla F)^\top$: curl of a vector field F
- $\bar\nabla u = (\nabla u)_\sigma$: tangential differential of a function $u : \Gamma \to \mathbb{R}, \bar\nabla_j u = \partial_j u - \partial_l u \nu^l \nu_j$
- $\bar\nabla F = \nabla F - (\nabla F\nu) \otimes \nu$: tangential gradient of a vector field $F : \Gamma \to \mathbb{R}^3$
- Tr: trace of a square matrix
- $\operatorname{div}_\sigma F = \operatorname{Tr}(\bar\nabla F)$: tangential divergence of a vector field $F : \Gamma \to \mathbb{R}^3$
- $\mathcal{A}_\Gamma = \operatorname{div}_\sigma \nu_\Gamma$: mean curvature of Γ
- $\Delta_B = \operatorname{div}_\sigma \bar\nabla$: Beltrami-Laplace operator on Γ
- $v\cdot\nabla$ and $H\cdot\nabla$: directional derivatives
- $h(\cdot,t) : \Gamma \to \mathbb{R}, t \ge 0$: height function of $\Gamma_t, h_0(\cdot) = h(\cdot, 0)$
- $A = (A_{ij})$: a 3×3 matrix (i-row, j-column)
- $A^\top$: transpose of a matrix
- $A : B = \sum_{i,j} A_{ij}B_{ij}$: inner product of two matrices
- $x\cdot y$: inner product of two vectors $x, y \in \mathbb{R}^3$
- $S \star T$: a tensor formed by contraction on some indexes of tensors S and T with constant coefficients
- $a(u_1, \dots, u_m)$: finite $\star$ product of $u_1, \dots, u_m$

- $F_n = F \cdot \nu$: normal part of a vector field F
- $F_\sigma = F - F_n\nu$: tangential part of a vector field F
- $B_\Gamma = \bar{\nabla}\nu_\Gamma$: second fundamental form of Γ
- $\nabla^k u \star \nabla^l v$ ($\bar{\nabla}^k u \star \bar{\nabla}^l v$): contraction on some indexes of tensors $\nabla^i u$ ($\bar{\nabla}^i u$) and $\nabla^j v$ ($\bar{\nabla}^j v$) for any $i \leq k$ and $j \leq l$ (including the lower-order derivatives)
- $W^{l,p}(\Omega), p \in [1, \infty]$: usual Sobolev space, $H^l(\Omega) = W^{l,2}(\Omega)$
- $W^{l,p}(\Gamma), p \in [1, \infty]$: Sobolev space defined on $\Gamma, H^l(\Gamma) = W^{l,2}(\Gamma)$

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References

[BKM84] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.

[BM18] Haïm Brezis and Petru Mironescu. Gagliardo-Nirenberg inequalities and non-inequalities: the full story. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 35(5):1355–1376, 2018.

[CCS08] Ching-Hsiao Arthur Cheng, Daniel Coutand, and Steve Shkoller. On the motion of vortex sheets with surface tension in three-dimensional Euler equations with vorticity. *Comm. Pure Appl. Math.*, 61(12):1715–1752, 2008.

[CL00] Demetrios Christodoulou and Hans Lindblad. On the motion of the free surface of a liquid. *Comm. Pure Appl. Math.*, 53(12):1536–1602, 2000.

[CLS10] Daniel Coutand, Hans Lindblad, and Steve Shkoller. A priori estimates for the free-boundary 3D compressible Euler equations in physical vacuum. *Comm. Math. Phys.*, 296(2):559–587, 2010.

[CS07] Daniel Coutand and Steve Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.*, 20(3):829–930, 2007.

[CS14] Daniel Coutand and Steve Shkoller. On the finite-time splash and splat singularities for the 3-D free-surface Euler equations. *Comm. Math. Phys.*, 325(1):143–183, 2014.

[CS17] C. H. Arthur Cheng and Steve Shkoller. Solvability and regularity for an elliptic system prescribing the curl, divergence, and partial trace of a vector field on Sobolev-class domains. *J. Math. Fluid Mech.*, 19(3):375–422, 2017.

[DE14] Marcelo M. Disconzi and David G. Ebin. On the limit of large surface tension for a fluid motion with free boundary. *Comm. Partial Differential Equations*, 39(4):740–779, 2014.

[DE16] Marcelo M. Disconzi and David G. Ebin. The free boundary Euler equations with large surface tension. *J. Differential Equations*, 261(2):821–889, 2016.

[DKT19] Marcelo M. Disconzi, Igor Kukavica, and Amjad Tuffaha. A Lagrangian interior regularity result for the incompressible free boundary Euler equation with surface tension. *SIAM J. Math. Anal.*, 51(5):3982–4022, 2019.

[FHYZ23] Jie Fu, Chengchun Hao, Siqi Yang, and Wei Zhang. A Beale-Kato-Majda criterion for free boundary incompressible ideal magnetohydrodynamics. *J. Math. Phys.*, 64(9):Paper No. 091505, 16, 2023.

[FJM20] Nicola Fusco, Vesa Julin, and Massimiliano Morini. The surface diffusion flow with elasticity in three dimensions. *Arch. Ration. Mech. Anal.*, 237(3):1325–1382, 2020.

[GLZ22] Xumin Gu, Chenyun Luo, and Junyan Zhang. Zero surface tension limit of the free-boundary problem in incompressible magnetohydrodynamics. *Nonlinearity*, 35(12):6349–6398, 2022.

[GLZ23] Xumin Gu, Chenyun Luo, and Junyan Zhang. Local well-posedness of the free-boundary incompressible magnetohydrodynamics with surface tension. *Journal de Mathématiques Pures et Appliquées*, 2023.

[GMS12] P. Germain, N. Masmoudi, and J. Shatah. Global solutions for the gravity water waves equation in dimension 3. *Ann. of Math. (2)*, 175(2):691–754, 2012.

[GW19] Xumin Gu and Yanjin Wang. On the construction of solutions to the free-surface incompressible ideal magnetohydrodynamic equations. *J. Math. Pures Appl. (9)*, 128:1–41, 2019.

[Ham82] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geometry*, 17(2):255–306, 1982.

[HL14] Chengchun Hao and Tao Luo. A priori estimates for free boundary problem of incompressible inviscid magnetohydrodynamic flows. *Arch. Ration. Mech. Anal.*, 212(3):805–847, 2014.

[HL20] Chengchun Hao and Tao Luo. Ill-posedness of free boundary problem of the incompressible ideal MHD. *Comm. Math. Phys.*, 376(1):259–286, 2020.

[HL21] Chengchun Hao and Tao Luo. Well-posedness for the linearized free boundary problem of incompressible ideal magnetohydrodynamics equations. *J. Differential Equations*, 299:542–601, 2021.

[HLZ23] Guangyi Hong, Tao Luo, and Zhonghao Zhao. On the splash singularity for the free-boundary problem of the viscous and non-resistive incompressible magnetohydrodynamic equations in 3d. *Preprint, arXiv:2309.09747*, 2023.

[HY24] Chengchun Hao and Siqi Yang. Splash singularity for the free boundary incompressible viscous MHD. *J. Differential Equations*, 379:26–103, 2024.

[IP15] Alexandru D. Ionescu and Fabio Pusateri. Global solutions for the gravity water waves system in 2d. *Invent. Math.*, 199(3):653–804, 2015.

[JLM22] Vesa Julin and Domenico Angelo La Manna. A priori estimates for the motion of charged liquid drop: A dynamic approach via free boundary euler equations. *Preprint, arXiv:2111.10158*, 2022.

[Lan05] David Lannes. Well-posedness of the water-waves equations. *J. Amer. Math. Soc.*, 18(3):605–654, 2005.

[Lee17] Donghyun Lee. Uniform estimate of viscous free-boundary magnetohydrodynamics with zero vacuum magnetic field. *SIAM J. Math. Anal.*, 49(4):2710–2789, 2017.

[Lin03] Hans Lindblad. Well-posedness for the linearized motion of an incompressible liquid with free surface boundary. *Comm. Pure Appl. Math.*, 56(2):153–197, 2003.

[Lin05] Hans Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. of Math. (2)*, 162(1):109–194, 2005.

[LX23a] Sicheng Liu and Zhouping Xin. Local well-posedness of the incompressible current-vortex sheet problems. *Preprint, arXiv:2309.03534*, 2023.

[LX23b] Sicheng Liu and Zhouping Xin. On the free boundary problems for the ideal incompressible mhd equations. *Preprint, arXiv:2311.06581*, 2023.

[LZ20] Chenyun Luo and Junyan Zhang. A regularity result for the incompressible magnetohydrodynamics equations with free surface boundary. *Nonlinearity*, 33(4):1499–1527, 2020.

[LZ21] Chenyun Luo and Junyan Zhang. A priori estimates for the incompressible free-boundary magnetohydrodynamics equations with surface tension. *SIAM J. Math. Anal.*, 53(2):2595–2630, 2021.

[LZ23] Hans Lindblad and Junyan Zhang. Anisotropic regularity of the free-boundary problem in compressible ideal magnetohydrodynamics. *Arch. Ration. Mech. Anal.*, 247(5):Paper No. 89, 94, 2023.

[Man02] C. Mantegazza. Smooth geometric evolutions of hypersurfaces. *Geom. Funct. Anal.*, 12(1):138–182, 2002.

[Man11] Carlo Mantegazza. *Lecture notes on mean curvature flow*, volume 290 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011.

[MTT08] Alessandro Morando, Yuri Trakhinin, and Paola Trebeschi. Stability of incompressible current-vortex sheets. *J. Math. Anal. Appl.*, 347(2):502–520, 2008.

[Sch05] Ben Schweizer. On the three-dimensional Euler equations with a free boundary subject to surface tension. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 22(6):753–781, 2005.

[SWZ18] Yongzhong Sun, Wei Wang, and Zhifei Zhang. Nonlinear stability of the current-vortex sheet to the incompressible MHD equations. *Comm. Pure Appl. Math.*, 71(2):356–403, 2018.

[SWZ19] Yongzhong Sun, Wei Wang, and Zhifei Zhang. Well-posedness of the plasma-vacuum interface problem for ideal incompressible MHD. *Arch. Ration. Mech. Anal.*, 234(1):81–113, 2019.

[SZ08a] Jalal Shatah and Chongchun Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. *Comm. Pure Appl. Math.*, 61(5):698–744, 2008.

[SZ08b] Jalal Shatah and Chongchun Zeng. A priori estimates for fluid interface problems. *Comm. Pure Appl. Math.*, 61(6):848–876, 2008.

[SZ11] Jalal Shatah and Chongchun Zeng. Local well-posedness for fluid interface problems. *Arch. Ration. Mech. Anal.*, 199(2):653–705, 2011.

[Tra05] Yuri Trakhinin. Existence of compressible current-vortex sheets: variable coefficients linear analysis. *Arch. Ration. Mech. Anal.*, 177(3):331–366, 2005.

[Tra09] Yuri Trakhinin. The existence of current-vortex sheets in ideal compressible magnetohydrodynamics. *Arch. Ration. Mech. Anal.*, 191(2):245–310, 2009.

[TW21] Yuri Trakhinin and Tao Wang. Well-posedness of free boundary problem in non-relativistic and relativistic ideal compressible magnetohydrodynamics. *Arch. Ration. Mech. Anal.*, 239(2):1131–1176, 2021.

[Wu09] Sijue Wu. Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.*, 177(1):45–135, 2009.

[Wu11] Sijue Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.*, 184(1):125–220, 2011.

[WX21] Yanjin Wang and Zhouping Xin. Global well-posedness of free interface problems for the incompressible inviscid resistive MHD. *Comm. Math. Phys.*, 388(3):1323–1401, 2021.

[ZZ08] Ping Zhang and Zhifei Zhang. On the free boundary problem of three-dimensional incompressible Euler equations. *Comm. Pure Appl. Math.*, 61(7):877–940, 2008.

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