

LARGE TIME BEHAVIOR AND GLOBAL EXISTENCE OF SOLUTION TO THE BIPOLAR DEFOCUSING NONLINEAR SCHRÖDINGER-POISSON SYSTEM

By

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Abstract. In this paper, we study the large time behavior and the existence of globally defined smooth solutions to the Cauchy problem for the bipolar defocusing nonlinear Schrödinger-Poisson system in the space $\mathbb{R}^3$.

1. Introduction. In the present paper, we study the global existence and large time behavior for the bipolar defocusing nonlinear Schrödinger-Poisson (BDNLSP) system

$$i\varepsilon\psi_j = -\frac{\varepsilon^2}{2}\Delta\psi_j + (q_jV + h_j(|\psi_j|^2))\psi_j, \quad j = 1, 2, \quad (1.1)$$

$$-\lambda^2\Delta V = |\psi_1|^2 - |\psi_2|^2, \quad (1.2)$$

with the initial data

$$\psi_j(0, \cdot) = \varphi_j, \quad j = 1, 2, \quad (1.3)$$

where the wave function $\psi_j = \psi_j(t, x) : \mathbb{R}^{1+d} \rightarrow \mathbb{C}$, $j = 1, 2$, $\dot{\psi}_j = \partial\psi_j/\partial t$, Δ is the Laplace operator on $\mathbb{R}^d$, and the electrostatic potential $V = V(t, x)$. The nonlinear self-interacting potential $h_j(s)$ is assumed to be given by

$$h_j(s) = a_j^2 s^{\gamma_j}, \quad \text{for } s \geq 0 \text{ and some } a_j > 0, \quad \frac{2}{d} < \gamma_j < \alpha(d),$$

where $\alpha(d) = \frac{2}{d-2}$ if $d \geq 3$ and $\alpha(d) = \infty$ if $d = 1, 2$. The charges of the particles described by the wave functions ψ_j are defined by $q_1 = 1$, $q_2 = -1$, respectively. ε is the scaled Planck constant and λ is the scaled Debye length.

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We assume that the initial value

$$\varphi_j(x) \in \Sigma(\mathbb{R}^d) := \{u \in H^1(\mathbb{R}^d) : |x|u \in L^2(\mathbb{R}^d)\}, \quad j = 1, 2, \quad (1.4)$$

with the norm

$$\|\psi_j\|_{\Sigma} = \|\psi_j\|_{H^1} + \| |x| \psi_j \|_{L^2}.$$

This system appears in quantum mechanics as well as semi-conductor and plasma physics. A large amount of interesting work has been devoted to the study of the Schrödinger-Poisson systems (see [2], [3], [4], [6], [7] and references therein). In [4], by applying the estimates of a modulated energy functional and the Wigner measure method, Jüngel and Wang discussed the combined semi-classical and quasineutral limit of the (BDNLSP) system with the initial data (1.3) in the whole space where $a_1 = a_2$ and $\gamma_1 = \gamma_2$, provided the solution of (1.1)–(1.3) exists. But they only declared the existence and uniqueness of global small smooth solution under the assumption that the initial data were sufficiently small in H^s where $s > d/2 + 2$. And in [3], Castella proved the global existence and the asymptotic behavior of solutions in the function space L^2 for the mixed-state unipolar Schrödinger-Poisson systems without the defocusing nonlinearity. In [6], with the help of madelung transform and WKB expansion, Li and Lin discussed the following unipolar nonlinear Schrödinger-Poisson system:

$$\begin{aligned} i\varepsilon \psi_t^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - (V^\varepsilon(x, t) + f'(|\psi^\varepsilon|^2)) \psi^\varepsilon - (\arg \psi^\varepsilon) \psi^\varepsilon &= 0, \\ -\Delta V^\varepsilon &= |\psi^\varepsilon|^2 - \mathcal{C}(x), \quad V \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{aligned}$$

subject to the rapidly oscillating (WKB) initial condition

$$\psi^\varepsilon(x, 0) = \psi_0^\varepsilon = A_0^\varepsilon(x) e^{\frac{i}{\varepsilon} S_0(x)}, \quad (1.5)$$

where $f \in C^\infty(\mathbb{R}^+; \mathbb{R})$, $S_0 \in H^s(\mathbb{R}^d)$, $d \geq 1$, for $s \geq d/2 + 2$, A_0^ε was a function, polynomial in ε , with coefficients of Sobolev regularity in x , and the function $\mathcal{C}(x) > 0$ denoted the background ions. They obtained the existence of smooth solution where the wave function was of the form $\psi^\varepsilon(x, t) = A^\varepsilon(x, t) e^{\frac{i}{\varepsilon} S^\varepsilon(x, t)}$, with A^ε and ∇S^ε bounded in $L^\infty([0, T]; H^s(\mathbb{R}^d))$ and the initial data being sufficiently small in $H^s(\mathbb{R}^d)$. However, to our knowledge, there is no previous result on the global existence and the asymptotic behavior of solutions for the (BDNLSP) system with arbitrary initial data in $\Sigma(\mathbb{R}^3)$. In this paper, by using the pseudo-conformal conservation law of the (BDNLSP) system and applying the time-space $L^p - L^{p'}$ estimate method, we shall establish the global existence and uniqueness of the solution to the (BDNLSP) system with initial data in $\Sigma(\mathbb{R}^3)$. As a byproduct, the large time behavior to the solution is also obtained. Although the above results are established for the single bipolar defocusing nonlinear Schrödinger-Poisson system, the results can be extended to the mixed-state bipolar defocusing nonlinear Schrödinger-Poisson system within the same framework.

For convenience, we first introduce some notation. For any $p \in [2, \infty)$, we denote $\frac{1}{\gamma(p)} = \frac{3}{2}(\frac{1}{2} - \frac{1}{p})$. $S(t)$ denotes the unitary group generated by $\frac{\varepsilon}{2} i \Delta$ in $L^2(\mathbb{R}^3)$. For $p \in [1, \infty]$, we denote by p' the conjugate exponent of p , defined by $1/p + 1/p' = 1$. $\bar{z}$ denotes the conjugate of the complex number z .

Now we state the main result of this paper.

THEOREM 1.1 (Existence and uniqueness). Let $\varphi_j \in \Sigma(\mathbb{R}^3)$. Assume that $\rho \in [2, 6)$. Then, there exists a unique solution

$$\psi_j \in C(\mathbb{R}; \Sigma(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}; H^1(\mathbb{R}^3)) \cap L_{loc}^{\gamma(\rho)}(\mathbb{R}; H_\rho^1(\mathbb{R}^3)), \text{ for } j = 1, 2$$

to the (BDNLSP) system with the initial data (1.3).

Moreover, the solution (ψ_1, ψ_2, V) satisfies the L^2 -norm, the energy, and the pseudo-conformal conservation laws (for details, one can see Proposition 2.1 in the case $d = 3$).

THEOREM 1.2 (Large time behavior). Let (ψ_1, ψ_2, V) and ρ be as in Theorem 1.1. Then, there exist constants C depending only on $\|\varphi_j\|_{H^1}$ and $\|x|\varphi_j\|_2$ such that

$$\|\psi_j\|_\rho \leq C|t|^{-\frac{1}{\gamma(\rho)}}, \quad \forall \rho \in [2, 6), \quad \forall |t| \geq 1, \quad (1.6)$$

$$\|\nabla V(t)\|_\rho \leq C|t|^{-(1-\frac{3}{2\rho})}, \quad \forall \rho \in (\frac{3}{2}, \infty), \quad \forall |t| \geq 1, \quad (1.7)$$

$$\|V(t)\|_\rho \leq C|t|^{-\frac{1}{2}(1-\frac{3}{\rho})}, \quad \forall \rho \in (3, \infty), \quad \forall |t| \geq 1. \quad (1.8)$$

In the next section we shall derive three conservation identities including the L^2 -norm, the energy, and the pseudo-conformal conservation laws in the whole space $\mathbb{R}^d$ for any $d \in \mathbb{N}$. In Sec. 3, we will give some basic estimates used in our proofs for $d = 3$. Section 4 is devoted to the proof of the existence and uniqueness of the solutions for the initial data in $\Sigma(\mathbb{R}^3)$. Finally, the large time behavior of the solution is obtained in Sec. 5.

2. Derivation of the conservation laws.

PROPOSITION 2.1. Let $d \in \mathbb{N}$, $\{\psi_j\}$ be a solution of the (BDNLSP) system with the initial value $\varphi_j(x) \in \Sigma(\mathbb{R}^d)$. Then, we have the following conservation laws for all $t \in \mathbb{R}$:

(i) L^2 -norm law:

$$\|\psi_j(t)\|_2 = \|\varphi_j\|_2 \quad \text{for } j = 1, 2; \quad (2.1)$$

(ii) Energy conservation law:

$$\varepsilon^2 \sum_{j=1}^2 \|\nabla \psi_j(t)\|_2^2 + \lambda^2 \|\nabla V\|_2^2 + 2 \sum_{j=1}^2 \frac{a_j^2}{\gamma_j + 1} \|\psi_j(t)\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} = \text{const}; \quad (2.2)$$

(iii) Pseudo-conformal conservation law (cf. [9]):

$$\begin{aligned} & \sum_{j=1}^2 \|x\psi_j + i\varepsilon t \nabla \psi_j\|_2^2 + \lambda^2 t^2 \|\nabla V\|_2^2 + 2t^2 \sum_{j=1}^2 \frac{a_j^2}{\gamma_j + 1} \|\psi_j\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} \\ & \quad + 2 \sum_{j=1}^2 \frac{a_j^2(d\gamma_j - 2)}{\gamma_j + 1} \int_0^t \tau \|\psi_j(\tau)\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} d\tau \\ & = \sum_{j=1}^2 \|x|\varphi_j\|_2^2 + (4-d)\lambda^2 \int_0^t \tau \|\nabla V(\tau)\|_2^2 d\tau. \end{aligned} \quad (2.3)$$

Proof. Denote $eq(\psi_j) =: i\varepsilon\dot{\psi}_j + \frac{\varepsilon^2}{2}\Delta\psi_j - (q_jV + h_j(|\psi_j|^2))\psi_j$.

(i) It is well known that (2.1) holds for $j = 1, 2$. We omit its proof.

(ii) We consider

$$\Re(eq(\psi_j), \dot{\psi}_j) = 0$$

where $(\cdot, \cdot)$ denotes the L^2 -inner product. From the above, we can get

$$\int_{\mathbb{R}^d} \left\{ \frac{\varepsilon}{4} \partial_t |\nabla \psi_j|^2 + \frac{1}{2} (q_j V + h_j(|\psi_j|^2)) \partial_t |\psi_j|^2 \right\} dx = 0.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^d} \left\{ \frac{\varepsilon}{4} \partial_t (|\nabla \psi_1|^2 + |\nabla \psi_2|^2) + \frac{1}{2} V \partial_t (|\psi_1|^2 - |\psi_2|^2) \right. \\ \left. + \frac{1}{2} h_1(|\psi_1|^2) \partial_t |\psi_1|^2 + \frac{1}{2} h_2(|\psi_2|^2) \partial_t |\psi_2|^2 \right\} dx = 0. \end{aligned} \quad (2.3)$$

Integrating (2.4) over $[0, t]$, we obtain the desired identity.

(iii) Considering

$$\Re(eq(\psi_j), \bar{\psi}_j) = 0,$$

we have

$$-\varepsilon \Im \partial_t \psi_j \bar{\psi}_j + \frac{\varepsilon^2}{2} \Re \nabla \cdot (\nabla \psi_j \bar{\psi}_j) - \frac{\varepsilon^2}{2} |\nabla \psi_j|^2 - (q_j V + h_j(|\psi_j|^2)) |\psi_j|^2 = 0. \quad (2.4)$$

Noticing

$$\Re(eq(\psi_j), \bar{\psi}_{j,r} r) = 0 \quad \text{with } r := |x|,$$

we obtain

$$\begin{aligned} -\varepsilon \Im \partial_t \psi_j \bar{\psi}_{j,r} r + \frac{\varepsilon^2}{2} \Re \nabla \cdot (\nabla \psi_j \bar{\psi}_{j,r} r - \frac{x}{2} |\nabla \psi_j|^2) + \frac{\varepsilon^2}{4} (d-2) |\nabla \psi_j|^2 \\ - \frac{1}{2} (q_j V + h_j(|\psi_j|^2)) r \partial_r |\psi_j|^2 = 0. \end{aligned} \quad (2.5)$$

Due to

$$\partial_t (\psi_j \bar{\psi}_{j,r} r) + \nabla \cdot (x \bar{\psi}_j \partial_t \psi_j) = 2 \partial_t \psi_j \bar{\psi}_{j,r} r + 2 \Re \psi_j x \cdot \nabla \partial_t \bar{\psi}_j + d \partial_t \psi_j \bar{\psi}_j,$$

we have, by taking the imaginary part, that

$$\Im \partial_t (\psi_j \bar{\psi}_{j,r} r) + \Im \nabla \cdot (x \bar{\psi}_j \partial_t \psi_j) = d \Im \partial_t \psi_j \bar{\psi}_j + 2 \Im (\partial_t \psi_j \bar{\psi}_{j,r} r).$$

From (2.5) and (2.6), we have

$$\begin{aligned} -\varepsilon \Im \partial_t (\psi_j \bar{\psi}_{j,r} r) - \Im \nabla \cdot (x \bar{\psi}_j \partial_t \psi_j) - \frac{\varepsilon^2}{2} d \Re \nabla \cdot (\nabla \psi_j \bar{\psi}_j) - d (q_j V + h_j(|\psi_j|^2)) |\psi_j|^2 \\ + \varepsilon^2 \Re \nabla \cdot (\nabla \psi_j \bar{\psi}_{j,r} r - \frac{x}{2} |\nabla \psi_j|^2) - (q_j V + h_j(|\psi_j|^2)) r \partial_r |\psi_j|^2 = 0. \end{aligned}$$

We also have

$$\partial_t |x \psi_j + i \varepsilon t \nabla \psi_j|^2 = \partial_t (|\psi_j r|^2 + \varepsilon^2 t^2 |\nabla \psi_j|^2 + 2 \varepsilon t \Im \psi_j x \cdot \nabla \bar{\psi}_j).$$

Since

$$\begin{aligned}
 \frac{\varepsilon}{2} \partial_t |\psi_j r|^2 &= \varepsilon \Re(\partial_t \psi_j \bar{\psi}_j) r^2 = \Im i \varepsilon \partial_t \psi_j \bar{\psi}_j r^2 \\
 &= \Im \left[-\frac{\varepsilon^2}{2} \Delta \psi_j + (q_j V + h_j(|\psi_j|^2)) \psi_j \right] \bar{\psi}_j r^2 \\
 &= -\frac{\varepsilon^2}{2} \Im \Delta \psi_j \bar{\psi}_j r^2, \\
 \Im \nabla \cdot (\nabla \psi_j \bar{\psi}_j r^2) &= \Im \Delta \psi_j \bar{\psi}_j r^2 - 2 \Im (\nabla \bar{\psi}_j \cdot x) \psi_j.
 \end{aligned}$$

we have

$$\begin{aligned}
 \partial_t |x \psi_j + i \varepsilon t \nabla \psi_j|^2 &= 2 \varepsilon^2 t |\nabla \psi_j|^2 + \varepsilon^2 t^2 \partial_t |\nabla \psi_j|^2 - \varepsilon \Im \nabla \cdot (\nabla \psi_j \bar{\psi}_j r^2) \\
 &\quad + 2 \varepsilon t \Im \partial_t (\psi_j x \cdot \nabla \bar{\psi}_j).
 \end{aligned} \tag{2.7}$$

Integrating (2.7) over $\mathbb{R}^d$, we obtain that for $j = 1, 2$

$$\begin{aligned}
 \partial_t \|x \psi_j + i \varepsilon t \nabla \psi_j\|_2^2 &= \varepsilon^2 t^2 \partial_t \|\nabla \psi_j\|_2^2 - 2t \int_{\mathbb{R}^d} \{d(q_j V + h_j(|\psi_j|^2)) |\psi_j|^2 \\
 &\quad + (q_j V + h_j(|\psi_j|^2)) r \partial_r |\psi_j|^2\} dx.
 \end{aligned} \tag{2.8}$$

From the above, we obtain

$$\begin{aligned}
 \partial_t \sum_{j=1}^2 \|x \psi_j + i \varepsilon t \nabla \psi_j\|_2^2 &+ 4t^2 \partial_t \left(\frac{\lambda^2}{4} \|\nabla V\|_2^2 + \sum_{j=1}^2 \frac{a_j^2}{2(\gamma_j + 1)} \|\psi_j\|_{2(\gamma_j + 1)}^{2(\gamma_j + 1)} \right) \\
 &+ 2t \int_{\mathbb{R}^d} \{dV(-\lambda^2 \Delta V) + V \partial_r(-\lambda^2 \Delta V) r + d \sum_{j=1}^2 a_j^2 |\psi_j|^{2(\gamma_j + 1)} \\
 &+ \sum_{j=1}^2 h_j(|\psi_j|^2) r \partial_r |\psi_j|^2\} dx = 0.
 \end{aligned} \tag{2.9}$$

Noticing that

$$h_j(s) r \partial_r s = a_j^2 s^{\gamma_j} r \partial_r s = \frac{a_j^2}{\gamma_j + 1} [\nabla \cdot (x s^{\gamma_j + 1}) - d s^{\gamma_j + 1}] \tag{2.10}$$

and

$$\begin{aligned}
 \int_{\mathbb{R}^d} V r \partial_r (-\Delta V) dx &= \int_{\mathbb{R}^d} \nabla V_r \cdot \nabla (V r) dx = \int_{\mathbb{R}^d} (\nabla V_r \cdot \nabla V) r + (\nabla V_r \cdot \nabla r) V dx \\
 &= \int_{\mathbb{R}^d} \frac{1}{2} r \partial_r |\nabla V|^2 dx + \int_{\mathbb{R}^d} V \Delta V dx \\
 &= \int_{\mathbb{R}^d} \frac{1}{2} [\nabla \cdot (x |\nabla V|^2) - d |\nabla V|^2] dx - \int_{\mathbb{R}^d} \nabla V \cdot \nabla V dx \\
 &= -\left(\frac{d}{2} + 1\right) \|\nabla V\|_2^2.
 \end{aligned} \tag{2.11}$$

we can obtain, in view of (2.4) and (2.8)–(2.11), that

$$\begin{aligned} \partial_t \left[\sum_{j=1}^2 \|x\psi_j + i\varepsilon t \nabla \psi_j\|_2^2 + \lambda^2 t^2 \|\nabla V\|_2^2 + \sum_{j=1}^2 \frac{2a_j^2}{\gamma_j + 1} t^2 \|\psi_j\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} \right] \\ + 2 \sum_{j=1}^2 \frac{a_j^2(d\gamma_j - 2)}{\gamma_j + 1} t \|\psi_j(t)\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} - (4-d)\lambda^2 t \|\nabla V(t)\|_2^2 = 0, \end{aligned}$$

which yields the pseudo-conformal conservation law. $\square$

3. Basic estimates.

LEMMA 3.1 (Estimate I). Let $\frac{1}{\rho'} = \frac{2}{a} + \frac{1}{\rho} + \frac{1}{3} - 1$, $u, v \in L^{\gamma(a)}(0, T; H_a^1)$ and $w \in L^{\gamma(\rho)}(0, T; H_\rho^1)$. Then we have the estimate

$$\begin{aligned} \left\| \left(\frac{1}{r} * uv \right) w \right\|_{L^{\gamma(\rho)'}(0, T; H_{\rho'}^1)} \leq CT^{1/2} \|u\|_{L^{\gamma(a)}(0, T; H_a^1)} \\ \cdot \|v\|_{L^{\gamma(a)}(0, T; H_a^1)} \|w\|_{L^{\gamma(\rho)}(0, T; H_\rho^1)}. \end{aligned} \quad (3.1)$$

Proof. By the known estimate as in [8], we have the following:

$$\begin{aligned} \left\| \left(\frac{1}{r} * uv \right) w \right\|_{H_{\rho'}^1} &\leq C \left\| \frac{1}{r} * uv \right\|_{L^\rho} \|w\|_{H_q^1} + C \left\| \frac{1}{r} * uv \right\|_{H_q^1} \|w\|_{L^\rho} \\ &\leq C \|u\|_{H_{2m}^1} \|v\|_{H_{2m}^1} \|w\|_{H_q^1} \end{aligned}$$

where $\frac{1}{\rho'} = \frac{1}{\rho} + \frac{1}{q}$, $\frac{1}{\rho} = \frac{1}{m} + \frac{1}{3} - 1$. Let $2m = a$, $q = \rho$, i.e., $\frac{1}{\rho'} = \frac{2}{a} + \frac{1}{\rho} + \frac{1}{3} - 1$. By the Sobolev embedding theorem (cf. [1]), we obtain

$$\left\| \left(\frac{1}{r} * uv \right) w \right\|_{H_{\rho'}^1} \leq C \|u\|_{H_a^1} \|v\|_{H_a^1} \|w\|_{H_\rho^1}.$$

Since

$$\frac{1}{\gamma(\rho)'} = 1 - \frac{3}{2} \left(\frac{1}{\rho'} - \frac{1}{2} \right) = \frac{1}{2} + \frac{2}{\gamma(a)} + \frac{1}{\gamma(\rho)},$$

we have the desired result. $\square$

LEMMA 3.2 (Estimate II). Let $\rho \in [2, 6)$; we have

$$\| |u|^p u \|_{L^{\gamma(\rho)'}(0, T; H_{\rho'}^1)} \leq CT^{1-\frac{2}{\gamma(\rho)}} \|u\|_{L^\infty(0, T; H^1)}^p \|u\|_{L^{\gamma(\rho)}(0, T; H_\rho^1)}. \quad (3.2)$$

Proof. From the identity

$$\begin{aligned} \nabla(|u|^p u) &= \nabla(|u|^p)u + |u|^p \nabla u = \frac{p}{2} |u|^{p-2} (\nabla u \bar{u} + u \nabla \bar{u})u + |u|^p \nabla u \\ &= \left(\frac{p}{2} + 1 \right) |u|^p \nabla u + \frac{p}{2} |u|^{p-2} u^2 \nabla \bar{u}, \end{aligned}$$

we have, in view of $\frac{1}{\rho'} = \frac{p}{q} + \frac{1}{\rho}$, that

$$\begin{aligned} \|\nabla(|u|^p u)\|_{L^{\rho'}} &\leq C \| |u|^p \nabla u \|_{L^{\rho'}} + C \| |u|^{p-2} u^2 \nabla \bar{u} \|_{L^{\rho'}} \\ &\leq C \|u\|_{L^q}^p \|\nabla u\|_{L^\rho}. \end{aligned}$$

By the Sobolev embedding theorem, it yields

$$\| |u|^p u \|_{H_{\rho'}^1} \leq C \|u\|_{H^1}^p \|u\|_{H_\rho^1},$$

which implies the desired result in view of the Hölder inequality with respect to the time variable. $\square$

Now we introduce the Galilei-type operator

$$J(t) = x + i\varepsilon t \nabla. \quad (3.3)$$

Let $\mathcal{M}(t) = e^{\frac{i|x|^2}{2\varepsilon t}}$ and $w_j = \mathcal{M}(-t)\psi_j$; we easily see that

$$J(t) = S(t)xS(-t) = \mathcal{M}(t)(i\varepsilon t \nabla)\mathcal{M}(-t), \quad (3.4)$$

$$|w_j(t)| = |\psi_j(t)|, \quad |J(t)\psi_j(t)| = \varepsilon|t||\nabla w_j(t)|. \quad (3.5)$$

LEMMA 3.3 (Estimate III). It holds

$$\|J(|\psi_j|^p \psi_j)\|_{L^{\gamma(\rho)'}(0,T;L^{\rho'})} \leq CT^{1-\frac{2}{\gamma(\rho)}} \|\psi_j\|_{L^\infty(0,T;H^1)}^p \|J\psi_j\|_{L^{\gamma(\rho)}(0,T;L^\rho)}. \quad (3.6)$$

Proof. We have, in view of (3.5), that

$$\|J(|\psi_j|^p \psi_j)\|_{L^{\rho'}} = \varepsilon|t| \|\nabla(|w_j|^p w_j)\|_{L^{\rho'}}. \quad (3.7)$$

Since

$$\nabla(|w_j|^p w_j) = \nabla(|w_j|^p)w_j + |w_j|^p \nabla w_j,$$

we obtain from Hölder's inequality

$$\|\nabla(|w_j|^p w_j)\|_{L^{\rho'}} \leq C \|w_j\|_{H^1}^p \|\nabla w_j\|_{L^\rho}. \quad (3.8)$$

Then, by (3.5), (3.7), and (3.8), we see that

$$\|J(|\psi_j|^p \psi_j)\|_{L^{\rho'}} \leq C \|\psi_j\|_{H^1}^p \|J\psi_j\|_{L^\rho},$$

which implies the desired result (3.6) in view of the Hölder inequality with respect to the time variable t . $\square$

LEMMA 3.4 (Estimate IV). We have the estimate

$$\begin{aligned} \|J(V\psi_j)\|_{L^{\gamma(\rho)'}(0,T;L^{\rho'})} &\leq CT^{1/2} \|(\psi_1, \psi_2)\|_{L^{\gamma(a)}(0,T;L^a)} \\ &\quad \cdot \|(J\psi_1, J\psi_2)\|_{L^{\gamma(\rho)}(0,T;L^\rho)} \end{aligned} \quad (3.9)$$

where $\|(u, v)\|_X := \|u\|_X + \|v\|_X$.

Proof. Noticing that

$$\begin{aligned} \nabla(Vw_j) &= \nabla V w_j + V \nabla w_j, \\ \nabla V &= C \nabla \left(\frac{1}{r} * (|w_1|^2 - |w_2|^2) \right) = \frac{C}{r} * \nabla(|w_1|^2 - |w_2|^2) \\ &= \frac{C}{r} * (w_1 \nabla \bar{w}_1 + \nabla w_1 \bar{w}_1 - w_2 \nabla \bar{w}_2 - \nabla w_2 \bar{w}_2), \end{aligned}$$

we have for $\frac{1}{\rho'} = \frac{1}{\rho} + \frac{2}{a} + \frac{1}{3} - 1$

$$\|\nabla(Vw_j)\|_{L^{\rho'}} \leq C(\|w_1\|_{L^a}^2 + \|w_2\|_{L^a}^2)(\|\nabla w_1\|_{L^\rho} + \|\nabla w_2\|_{L^\rho}).$$

Thus, we can get

$$\begin{aligned} \|J(V\psi_j)\|_{L^{\rho'}} &= \varepsilon|t| \|\nabla(Vw_j)\|_{L^{\rho'}} \leq C\varepsilon|t|(\|w_1\|_{L^a}^2 + \|w_2\|_{L^a}^2)(\|\nabla w_1\|_{L^\rho} + \|\nabla w_2\|_{L^\rho}) \\ &\leq C(\|\psi_1\|_{L^a}^2 + \|\psi_2\|_{L^a}^2)(\|J\psi_1\|_{L^\rho} + \|J\psi_2\|_{L^\rho}), \end{aligned}$$

which implies the desired result. $\square$

LEMMA 3.5 (Estimate V). We have the following estimate

$$\begin{aligned} & \| (V_k \psi_{1k} - V_l \psi_{1l}, V_k \psi_{2k} - V_l \psi_{2l}) \|_{L^{\gamma(\rho)'}(0,T; H_\rho^1)} \\ & \leq CT^{1/2} \| (\psi_{1k}, \psi_{2k}, \psi_{1l}, \psi_{2l}) \|_{L^{\gamma(a)}(0,T; H_a^1)}^2 \\ & \quad \cdot \| (\psi_{1k} - \psi_{1l}, \psi_{2k} - \psi_{2l}) \|_{L^{\gamma(\rho)}(0,T; H_\rho^1)}. \end{aligned} \quad (3.10)$$

Proof. Since

$$\begin{aligned} & C(V(\psi_{1k}, \psi_{2k})\psi_{jk} - V(\psi_{1l}, \psi_{2l})\psi_{jl}) \\ &= \left(\frac{1}{r} * (|\psi_{1k}|^2 - |\psi_{2k}|^2)\right)\psi_{jk} - \left(\frac{1}{r} * (|\psi_{1l}|^2 - |\psi_{2l}|^2)\right)\psi_{jl} \\ &= \left(\frac{1}{r} * (|\psi_{1k}|^2 - |\psi_{2k}|^2)\right)(\psi_{jk} - \psi_{jl}) + \psi_{jl} \left[\frac{1}{r} * ((|\psi_{1k}|^2 - |\psi_{1l}|^2) - (|\psi_{2k}|^2 - |\psi_{2l}|^2))\right] \\ &= \left(\frac{1}{r} * (|\psi_{1k}|^2 - |\psi_{2k}|^2)\right)(\psi_{jk} - \psi_{jl}) + \left[\frac{1}{r} * ((\bar{\psi}_{1k} - \bar{\psi}_{1l})\psi_{1k}) + \frac{1}{r} * ((\psi_{1k} - \psi_{1l})\bar{\psi}_{1l})\right] \\ & \quad + \frac{1}{r} * ((\bar{\psi}_{2k} - \bar{\psi}_{2l})\psi_{2k}) + \frac{1}{r} * ((\psi_{2k} - \psi_{2l})\bar{\psi}_{2l})] \psi_{jl}, \end{aligned}$$

we have the desired result by the Hölder inequality. $\square$

4. The proof of the existence. In this section, we will prove the local existence of the Cauchy problem (BDNLSP) with the initial data (1.3) first. Let $S(t) := e^{\frac{1}{2}i\varepsilon\Delta t}$ and consider the integral equation

$$\psi_j(t) = S(t)\varphi_j - \frac{1}{\varepsilon}i \int_0^t S(t-\tau)(q_j V(\tau) + h_j(|\psi_j(\tau)|^2))\psi_j(\tau) d\tau. \quad (4.1)$$

Define the workspace $(\mathcal{D}, d)$ as

$$\mathcal{D} := \{(\psi_j)_{j=1,2} : \|\psi_j\|_{L^\infty(0,T; H^1) \cap L^{\gamma(a)}(0,T; H_a^1)} \leq M, \text{ for any } a \in [2, 6)\}, \quad (4.2)$$

with the distance

$$d((\psi_{1k}, \psi_{2k}), (\psi_{1l}, \psi_{2l})) = \|(\psi_{1k} - \psi_{1l}, \psi_{2k} - \psi_{2l})\|_{L^{\gamma(\rho)}(0,T; H_\rho^1)} \quad (4.3)$$

where $M \geq 2 \max_{j=1,2} \|\varphi_j\|_{H^1}$ and $\rho \in [2, 6)$. It is clear that $(\mathcal{D}, d)$ is a Banach space. Let us consider the mapping $T = T_1 \otimes T_2 : (\mathcal{D}, d) \rightarrow (\mathcal{D}, d)$ defined by

$$T_j : \psi_j(t) \mapsto S(t)\varphi_j - \frac{1}{\varepsilon}i \int_0^t S(t-\tau)(q_j V(\tau) + h_j(|\psi_j(\tau)|^2))\psi_j(\tau) d\tau, \quad j = 1, 2. \quad (4.4)$$

By Lemmas 3.1–3.5 and the Strichartz estimates (cf. [5]), we have

$$\begin{aligned}
\|\mathcal{T}_j \psi_j\|_{L^{\gamma(\rho)}(0,T;H_\rho^1)} &\leq \|\varphi_j\|_{H^1} + \|V\psi_j\|_{L^{\gamma(\rho)'}(0,T;H_\rho^1)} + \|h_j(|\psi_j|^2)\psi_j\|_{L^{\gamma(\rho)'}(0,T;H_\rho^1)} \\
&\leq \|\varphi_j\|_{H^1} + CT^{1/2} \left(\sum_{j=1}^2 \|\psi_j\|_{L^{\gamma(\rho)}(0,T;H_\rho^1)}^2 \right) \|\psi_j\|_{L^{\gamma(\rho)}(0,T;H_\rho^1)} \\
&\quad + CT^{1-2/\gamma(\rho)} \|\psi_j\|_{L^\infty(0,T;H^1)}^{2\gamma_j} \|\psi_j\|_{L^{\gamma(\rho)}(0,T;H_\rho^1)} \\
&\leq M/2 + (CT^{1/2}M^2 + CT^{1-2/\gamma(\rho)}M^{2\gamma_j})M \\
&\leq M,
\end{aligned} \tag{4.5}$$

where we have taken T so small that $CT^{1/2}M^2 + CT^{1-2/\gamma(\rho)}M^{2\gamma_j} < \frac{1}{2}$. Similar to the above, a straightforward computation shows that it holds

$$\begin{aligned}
&\|\mathcal{T}(\psi_{1k}, \psi_{2k}) - \mathcal{T}(\psi_{1l}, \psi_{2l})\|_{L^{\gamma(\rho)}(0,T;H_\rho^1)} \\
&\leq \frac{1}{2} \|(\psi_{1k} - \psi_{1l}, \psi_{2k} - \psi_{2l})\|_{L^{\gamma(\rho)}(0,T;H_\rho^1)}.
\end{aligned} \tag{4.6}$$

Hence, $\mathcal{T}$ is a contracted mapping from the Banach space $(\mathcal{D}, d)$ to itself. By the Banach contraction mapping principle, we know that there exists a unique solution $(\psi_1, \psi_2) \in L^{\gamma(\rho)}(0, T; H_\rho^1) \times L^{\gamma(\rho)}(0, T; H_\rho^1)$ to the (BDNLSP) system with the initial data (1.3). From (4.1), (3.4) and (3.5), we may easily obtain $J\psi_1, J\psi_2 \in L^{\gamma(\rho)}(0, T; L^\rho)$ with the help of Lemmas 3.3–3.4. Thus, we can use the standard argument (cf. [3]) to extend it to a global one satisfying for any $T > 0$

$$\psi_1(t, x), \psi_2(t, x) \in C(\mathbb{R}; \Sigma(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}; H^1(\mathbb{R}^3)) \cap L^{\gamma(\rho)}(-T, T; H_\rho^1(\mathbb{R}^3)),$$

and prove the uniqueness of the global solution. We omit the details.

5. Large time behavior of the solution. By the pseudo-conformal conservation law, we get for $d = 3$

$$\lambda^2 t^2 \|\nabla V\|_2^2 \leq I + \lambda^2 \int_1^t \tau \|\nabla V(\tau)\|_2^2 d\tau. \tag{5.1}$$

where $I := \sum_{j=1}^2 \|x|\psi_j\|_2^2 + \lambda^2 \int_0^1 \tau \|\nabla V(\tau)\|_2^2 d\tau$.

From the Gronwall inequality, we have

$$\|\nabla V\|_2 \leq \frac{I^{1/2}}{\lambda} |t|^{-1/2}. \tag{5.2}$$

By the energy conservation law and the Sobolev embedding theorem, we obtain

$$\begin{aligned} \lambda^2 \|\nabla V\|_2^2 &\leq \varepsilon^2 \sum_{j=1}^2 \|\nabla \varphi_j\|_2^2 + \|\nabla V(0)\|_2^2 + \sum_{j=1}^2 \frac{2a_j^2}{\gamma_j + 1} \|\varphi_j\|_{2(\gamma_j+1)}^{2(\gamma_j+1)} \\ &\leq C(\varepsilon, \gamma_j, \|\varphi_j\|_{H^1}) + \|\nabla V(0)\|_2^2. \end{aligned} \quad (5.3)$$

$$\begin{aligned} \|\nabla V(0)\|_2 &= C \|\nabla(|x|^{-2} * (|\varphi_1|^2 - |\varphi_2|^2))\|_2 \leq C \|\frac{1}{|x|^2} * (|\varphi_1|^2 - |\varphi_2|^2)\|_2 \\ &\leq C \sum_{j=1}^2 \|\varphi_j\|_{L^{12/5}}^2 \leq C \sum_{j=1}^2 \|\varphi_j\|_{H^1}^2 \\ &\leq C(\varepsilon, \gamma_j, \|\varphi_j\|_{H^1}). \end{aligned} \quad (5.4)$$

Therefore, we have the estimate

$$\|\nabla V\|_2 \leq \frac{C}{\lambda} |t|^{-\frac{1}{2}}. \quad (5.5)$$

By the Sobolev embedding theorem and the pseudo-conformal conservation law, we have

$$\begin{aligned} \|\psi_j\|_\rho &= \|\mathcal{M}(-t)\psi_j\|_\rho \leq C \|\nabla \mathcal{M}(-t)\psi_j\|_2^{2/\gamma(\rho)} \|\mathcal{M}(t)\psi_j\|_2^{1-\gamma(\rho)} \\ &\leq C \|\mathcal{M}(-t)(\frac{x}{i\varepsilon t}\psi_j + \nabla \psi_j)\|_2^{2/\gamma(\rho)} \|\psi_j\|_2^{1-\gamma(\rho)} \\ &\leq C|t|^{-1/\gamma(\rho)}. \end{aligned} \quad (5.6)$$

From the above and the Hardy-Littlewood-Sobolev inequality, we obtain

$$\|\nabla V(t)\|_\rho \leq C|t|^{-(1-\frac{3}{2\rho})}, \quad \forall \rho \in (\frac{3}{2}, \infty), \quad \forall |t| \geq 1, \quad (5.7)$$

$$\|V(t)\|_\rho \leq C|t|^{-\frac{1}{2}(1-\frac{3}{\rho})}, \quad \forall \rho \in (3, \infty), \quad \forall |t| \geq 1. \quad (5.8)$$

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