Journal of Mathematical Physics

Vol. 64, Iss. 9, Sep. 2023

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Available Online: pubs.aip.org/aip/jmp

RESEARCH ARTICLE | SEPTEMBER 21 2023

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J. Math. Phys. 64, 091505 (2023)
https://doi.org/10.1063/5.0167954

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Cite as: J. Math. Phys. 64, 091505 (2023); doi: 10.1063/5.0167954
Submitted: 16 July 2023 • Accepted: 7 September 2023 •
Published Online: 21 September 2023

View Online Export Citation CrossMark

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ABSTRACT

We prove a continuation criterion for the free boundary problem of three-dimensional incompressible ideal magnetohydrodynamic (MHD) equations in a bounded domain, which is analogous to the theorem given in Beale, Kato, and Majda [Commun. Math. Phys. **94**, 61–66 (1984)]. We combine the energy estimates of our previous works [C. Hao and T. Luo, Arch. Ration. Mech. Anal. **212**(3), 805–847 (2014)] on incompressible ideal MHD and some analogous estimates in Ginsberg [SIAM J. Math. Anal. **53**, 3366–3384 (2021); arXiv:1811.06154] to show that the solution can be continued as long as the curls of the magnetic field and velocity, the second fundamental form and injectivity radius of the free boundary and some norms of the pressure remain bounded, provided that the Taylor-type sign condition holds.

I. INTRODUCTION

We consider the following three-dimensional incompressible ideal magnetohydrodynamics (MHD) equations:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = B\cdot\nabla B, & \text{in } \mathscr{D}, \\ B_t + u\cdot\nabla B = B\cdot\nabla u, & \text{in } \mathscr{D}, \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0, & \text{in } \mathscr{D}, \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} \partial_t + u^k\partial_k \text{ is tangent to } \partial\mathscr{D}_t, & \text{on } \partial\mathscr{D}_t, \\ p = 0, & \text{on } \partial\mathscr{D}_t, \\ |B| = \kappa, \quad B\cdot n = 0, & \text{on } \partial\mathscr{D}_t, \end{cases} \tag{1.2}$$

and initial conditions

$$\{x : (0,x) \in \mathscr{D}\} = \mathscr{D}_0, \quad (u,B)|_{t=0} = (u_0(x), B_0(x)) \text{ for } x \in \mathscr{D}_0, \tag{1.3}$$

where $u = (u_1, u_2, u_3)$ denotes the velocity, $B = (B_1, B_2, B_3)$ denotes the magnetic field, p denotes the total pressure, $\mathscr{D} := \cup_{t\in[0,T]}(\{t\}\times\mathscr{D}_t)$ with a bounded domain $\mathscr{D}_t \subset \mathbb{R}^3$ occupied by the conducting fluid whose boundary $\partial\mathscr{D}_t$ moves with the velocity of the fluid, n is the outward unit normal to $\partial\mathscr{D}_t$, κ is a non-negative constant and $\mathscr{D}_0$ is diffeomorphic to the unit ball. In Ref. 1, Hao and Luo have given the *a priori* bounds for (1.1)–(1.3) in some Sobolev spaces with H^4 initial data under the raised Taylor-type sign condition on the total pressure

$$\partial_n p \le -\varepsilon < 0 \quad \text{on } \partial\mathscr{D}_t, \tag{1.4}$$

with a constant $\varepsilon > 0$, which will hold within a period if it holds initially. They also showed in Ref. 2 that the above free boundary problem (1.1)–(1.3) under consideration would be ill-posed at least for the two-dimensional case if condition (1.4) was violated. Thus, it will be much reasonable and necessary to require condition (1.4) in the studies of well-posedness of the considering free boundary problem of incompressible ideal MHD equations.

In the absence of the magnetic field, the problem reduces to the free-boundary incompressible Euler equations. For the irrotational case without surface tension, Wu[3,4] obtained the well-posedness in Sobolev spaces under the Taylor sign condition. The ill-posedness of incompressible Euler equations without the Taylor sign condition was proved by Ebin in Ref. 5. One can refer to Refs. 6–9 for more results of the case without the irrotationality assumption. In the presence of surface tension, the Taylor sign condition is no longer required to establish the local well-posedness or *a priori* estimates, such as Refs. 10 and 11 and so on.

If the magnetic field exists, the problem becomes the incompressible free-boundary MHD equations. With the motivation of contributing to the study of the ideal MHD free surface problem with free closed curved surface with large curvature, the well-posedness of a linearized problem was proved by Hao and Luo in Ref. 12, via geometric approaches motivated by Refs. 6, 8, and 13 for the Euler equations of fluids, and developed in Ref. 1 for MHD equations. The *a priori* estimate for solutions in a bounded domain with small volume with minimal regularity assumptions on the initial data was established by Luo and Zhang in Ref. 14. One can refer to Refs. 15–17 for some results in other domains.

There are many studies on the breakdown or blow-up criterion for free or fixed boundary problems and Cauchy problems. Ginsberg[18] proved an alternative breakdown criterion for the incompressible Euler equations assuming that the Taylor sign condition holds. One can refer to Refs. 19–21 for other results about the breakdown criterion for Euler equations. The blow-up criterion for the Cauchy problem of 3D incompressible ideal MHD was considered by Zhang and Liu.[22] Through the Fourier frequency localization and Bony paraproduct decomposition, Cannone *et al.* established a blow-up criterion of smooth solutions to the ideal MHD equations in Ref. 23. For compressible MHD equations, Xu and Zhang proved a blow-up criterion of strong solutions in Refs. 24 and 25. In Ref. 26, Kim gave a blow-up criterion for the ideal MHD equations with respect to vorticities in three dimensions.

We now describe the energy estimates developed by Hao and Luo in Ref. 1. To define higher order energies, we introduce the second fundamental form of the free surface and tensor products given in Ref. 6. We want to project the system to the tangent space of the boundary. Let α is (r, s) tensor, the orthogonal projection Π to the tangent space of the boundary of α is defined to be the projection of each component along the normal:

$$(\Pi\alpha)^{j_1\cdots j_r}_{i_1\cdots i_s} = \Pi^{k_1}_{i_1}\cdots\Pi^{k_s}_{i_s}\Pi^{j_1}_{l_1}\cdots\Pi^{j_r}_{l_r}\alpha^{l_1\cdots l_r}_{k_1\cdots k_s}, \quad \text{where} \quad \Pi^j_i = \delta^j_i - n_i n^j.$$

Let $\bar\partial_i = \Pi^j_i\partial_j$ be the tangential derivative. If $p = 0$ on $\partial\mathscr{D}_t$, it follows that $\bar\partial_i p = 0$ and

$$\left(\Pi\partial^2 p\right)_{ij} = \theta_{ij}\partial_n p, \tag{1.5}$$

where $\theta_{ij} = \bar\partial_i n_j$ is the second fundamental form of $\partial\mathscr{D}_t$.

Then we define the quadratic form Q of the form:

$$Q(\alpha,\beta) = \langle\Pi\alpha,\Pi\beta\rangle = f^{i_1 j_1}\cdots f^{i_r j_r}\alpha_{i_1\cdots i_r}\beta_{j_1\cdots j_r},$$

where

$$f^{ij} = \delta^{ij} - \eta^2(d)n^i n^j, \quad d(x) = \mathrm{dist}(x,\partial\mathscr{D}_t), \quad n^i = -\delta^{ij}\partial_j d.$$

Here η is a smooth cut-off function satisfying $0 \le \eta(d) \le 1$, $\eta(d) = 1$ if $d < d_0/4$, and $\eta(d) = 0$ if $d > d_0/2$. $d_0 > 0$ is a fixed number that is smaller than the injectivity radius ς_0 of the normal exponential map, defined to be the largest number ς_0 such that the map

$$\partial\mathscr{D}_t \times (-\varsigma_0,\varsigma_0) \to \left\{x \in \mathbb{R}^3 : \mathrm{dist}(x,\partial\mathscr{D}_t) < \varsigma_0\right\},$$

given by $(\bar x,\varsigma) \to x = \bar x + \varsigma n(\bar x)$ is an injection.

Definition 1.1. Let $0 < \varepsilon_1 < 2$ *be a fixed number, and let* $\varsigma_1 = \varsigma_1(\varepsilon_1)$ *the largest number such that* $|n(\bar x_1) - n(\bar x_2)| \le \varsigma_1$ *whenever* $|\bar x_1 - \bar x_2| \le \varsigma_1$, $\bar x_1, \bar x_2 \in \partial\mathscr{D}_t$.

In this paper, we all require $\varsigma_1 \ge 1/K_1$. Since Ref. 1, Lemma 6.4 allows us to pick a K_1 depending only on initial conditions, we can assume that K_1 is just a constant, it won't affect the breakdown of the solution.

The fundamental geometric assumption that we will make is

$$|\theta| + \frac{1}{\varsigma_0} \le K \quad \text{on } \partial\mathscr{D}_t,$$

which ensures that the domain $\mathscr{D}_t$ satisfies the "uniform exterior sphere condition" by Ref. 6.

Next, we define the higher energies for $r \geq 1$ as

$$
\begin{aligned}
E_r(t) = &\int_{\mathscr{D}_t} \delta^{ij}\big(Q(\partial^r u_i, \partial^r u_j) + Q(\partial^r B_i, \partial^r B_j)\big)dx + \int_{\mathscr{D}_t} \left|\partial^{r-1}\operatorname{curl} u\right|^2 + \left|\partial^{r-1}\operatorname{curl} B\right|^2 dx \\
&+ \operatorname{sgn}(r-1)\int_{\partial\mathscr{D}_t} Q(\partial^r p, \partial^r p)\vartheta dS,
\end{aligned}
\tag{1.6}
$$

where $\operatorname{sgn}(\cdot)$ denotes the sign function and $\vartheta = (-\partial_n p)^{-1}$.

A. Reformulation in Lagrangian coordinates

We introduce the Lagrangian coordinates to transform the free boundary problem to a fixed boundary problem. Let Ω be the unit ball in $\mathbb{R}^3$, the connection between the Eulerian coordinates x and the Lagrangian coordinates y is given by $x = x(t, y) = f_t(y)$ and

$$
\frac{dx}{dt} = u(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega, \tag{1.7}
$$

where $f_0 : \Omega \to \mathscr{D}_0$ is a diffeomorphism. The Euclidean metric δ_{ij} in $\mathscr{D}_t$ induces a metric in Ω for each fixed t

$$
g_{ab}(t, y) = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b},
$$

we can get its inverse

$$
g^{cd}(t, y) = \delta^{kl}\frac{\partial y^c}{\partial x^k}\frac{\partial y^d}{\partial x^l}.
$$

We will work with the covariant derivative associated to g, if $\alpha = \alpha_{a_1\cdots a_r} dy^{a_1}\cdots dy^{a_r}$ is a $(0, r)$ tensor, then $\nabla\alpha$ is a $(0, r+1)$ tensor with components:

$$
\nabla_a \alpha_{a_1\cdots a_r} = \partial_a \alpha_{a_1\cdots a_r} - \Gamma^b_{a_1 a}\alpha_{b a_2\cdots a_r} - \cdots - \Gamma^b_{a a_r}\alpha_{a_1\cdots a_{r-1} b},
$$

where the Christoffel symbols Γ^c_{ab} are defined by

$$
\Gamma^c_{ab} = \frac{1}{2}g^{cd}\left(\frac{\partial}{\partial y^a}g_{bd} + \frac{\partial}{\partial y^b}g_{ad} - \frac{\partial}{\partial y^d}g_{ab}\right).
$$

Then, in the y-coordinates, we have

$$
\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}.
$$

Let us introduce the notation for the material derivative

$$
D_t = \frac{\partial}{\partial t}\bigg|_{y=\text{const}} = \frac{\partial}{\partial t}\bigg|_{x=\text{const}} + u^k\frac{\partial}{\partial x^k}.
$$

If $k(t, x)$ is a $(0, r)$ tensor expressed in the x-coordinates, we have

$$
D_t w_{a_1\cdots a_r} = \frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}\left(D_t k_{i_1\cdots i_r} + \frac{\partial u^\ell}{\partial x^{i_1}}k_{\ell\cdots i_r} + \cdots + \frac{\partial u^\ell}{\partial x^{i_r}}k_{i_1\cdots\ell}\right),
$$

where $w(t, y)$ expressed in the y-coordinates is given by

$$
w_{a_1\cdots a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}}\cdots\frac{\partial x^{i_r}}{\partial y^{a_r}}k_{i_1\cdots i_r}(t, x), \quad x = x(t, y).
$$

Let $v(t,y)$, $\beta(t,y)$ and $q(t,y)$ represent the velocity, the magnetic field, and the pressure in the Lagrangian coordinates, respectively. Then from Ref. 1, system (1.1)–(1.3) can be rewritten in the Lagrangian coordinates as

$$\begin{cases} D_t v_a + \nabla_a q = v^c \nabla_a v_c + \beta^d \nabla_d \beta_a, & \text{in } [0,T]\times\Omega, \\ D_t \beta_a = \beta^d \nabla_d v_a + \beta^c \nabla_a v_c, & \text{in } [0,T]\times\Omega, \\ \nabla_a v^a = 0, \quad \nabla_a \beta^a = 0, & \text{in } [0,T]\times\Omega, \\ |\beta| = \kappa, \quad \beta_a N^a = 0, & \text{on } [0,T]\times\partial\Omega, \\ q = 0, & \text{on } [0,T]\times\partial\Omega, \\ |\theta| + \dfrac{1}{\varsigma_0} \le K, \quad |\nabla q| \ge \varepsilon > 0, & \text{on } [0,T]\times\partial\Omega. \end{cases} \tag{1.8}$$

B. Main result

Our results indicate that the breakdown condition can be replaced with a condition on the vorticity of magnetic field and velocity and some norms of u, B and p on the free boundary $\partial\mathscr{D}_t$.

In a seminal paper,[27] Beale, Kato, and Majda showed that if u is a smooth solution of the incompressible Euler equations in $[0,T)\times\mathbb{R}^3$ and satisfies

$$\int_0^T \|\nabla \times u(t)\|_{L^\infty(\mathbb{R}^3)} dt < +\infty,$$

then the solution can be extended after $t = T$. See also Ref. 28 for a generalization to the Beale–Kato–Majda theorem for ideal MHD. The main result of this paper is a Beale–Kato–Majda type breakdown criterion for the free boundary problem of the incompressible MHD. By *a priori* estimates in Ref. 1, it gives the following breakdown criterion for solutions to (1.8) in Lagrangian coordinates.

Theorem 1.1. *Let (v,β) be a solution to (1.8) satisfying*

$$v(t), \beta(t) \in H^4(\Omega), \quad 0 \le t \le T. \tag{1.9}$$

Let $\mathscr{N}$ be the Dirichlet-to-Neumann operator on $\partial\Omega$; for $\psi : \partial\Omega \to \mathbb{R}$, $\psi_{\mathscr{H}} : \Omega \to \mathbb{R}$ denotes the harmonic extension of ψ to Ω and N denotes the outward unit normal vector on $\partial\Omega$, then

$$\mathscr{N}\psi = (N\cdot\nabla\psi_{\mathscr{H}})|_{\partial\Omega}.$$

We also write $V = v|_{\partial\Omega}$. Define

$$\begin{aligned} \mathscr{A}(t) &= \|\nabla\times v\|_{L^\infty(\Omega)} + \|\nabla\times\beta\|_{L^\infty(\Omega)} + \|\nabla v(t)\|_{L^\infty(\partial\Omega)} + \|\nabla\beta(t)\|_{L^\infty(\partial\Omega)} + \|\mathscr{N}V(t)\|_{L^\infty(\partial\Omega)}, \\ \mathscr{B}(t) &= \|\theta(t)\|_{L^\infty(\partial\Omega)} + \frac{1}{\varsigma_0(t)} + \left\|(\nabla_N q(t))^{-1}\right\|_{L^\infty(\partial\Omega)}. \end{aligned} \tag{1.10}$$

Suppose that T^* is the largest time so that v and β can be continued as a solution to (1.8) in the class (1.9). Then either $T^* = \infty$, or $\limsup_{t\nearrow T^*}\mathscr{B}(t) = \infty$, *or*

$$\int_0^{T^*}\left[\mathscr{A}(t)+\mathscr{A}^2(t) + \|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q(t)\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q\|^2_{L^\infty(\partial\Omega)}\right]dt = \infty. \tag{1.11}$$

In particular, if (1.11) occurs, then

$$\limsup_{t\nearrow T^*}\left[\mathscr{A}(t) + \|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q(t)\|_{L^\infty(\partial\Omega)}\right] = \infty.$$

Remark 1. Although there is no local well-posedness of (1.1)–(1.4) with initial domain Ω to be the unit ball so far, it does not hinder the mathematical research on its breakdown criterion. Here, the solution in $H^4(\Omega)$ we considered is the possible lowest regularity of solutions in view of the a priori estimates in Ref. 1. Unlike the Euler equation, low-order terms of magnetic field and pressure will always appear in estimating energy. In this case, we cannot use the Sobolev theorem to deal with them, but we can use $\mathscr{A}(t)$ to control them.

II. ELLIPTIC ESTIMATES

As is well known, to deal with the blow-up criterion, we need a log-type inequality about velocity and magnetic fields whether in $\mathbb{R}^3$ or the fixed domain. This is difficult to handle on the free boundary because we do not know the value of velocity on the moving boundary $\partial\mathscr{D}_t$ or $\partial\Omega$. Therefore, we introduce the Dirichlet–Neumann operator $\mathscr{N}$ to handle the velocity on the boundary, and for the treatment of magnetic fields, we can directly use the conclusions in Ref. 28 due to the boundary condition $\beta\cdot N = 0$.

To carry out the point above, we start by writing $v = v_1 + v_2$:

$$\begin{aligned} &\Delta v_1 = \nabla\times\nabla\times v, \quad \text{in }\Omega, \quad v_1 = 0, \quad \text{on }\partial\Omega,\\ &\Delta v_2 = 0, \qquad\qquad\quad \text{in }\Omega, \quad v_2 = v, \quad \text{on }\partial\Omega. \end{aligned} \tag{2.1}$$

The following estimates are well known. The first and second estimates are from Ref. 18, Proposition 2 and the last estimate is from Ref. 29, Theorem 3.2.

Proposition 2.1. Let $\Omega\subset\mathbb{R}^3$ be diffeomorphic to the unit ball, $|\theta| + \frac{1}{\varsigma_0}\le K$ on $\partial\Omega$ and $1 < p < \infty$, we have:

(i) *For any $q\in H^{2,p}(\Omega)\cap H_0^{1,p}(\Omega)$, it holds*

$$\|q\|_{L^p(\Omega)} + \|\nabla q\|_{L^p(\Omega)} + \left\|\nabla^2 q\right\|_{L^p(\Omega)} \le C(K)\|\Delta q\|_{L^p(\Omega)}.$$

(ii) *If $\Delta f = \nabla\times g + \rho$ for vector fields f, g and $\rho\in H_0^{1,p}(\Omega)$, then*

$$\|\nabla f\|_{L^p(\Omega)} \le C(K)\Big(\|g\|_{L^p(\Omega)} + \|\rho\|_{L^1(\Omega)}\Big).$$

Assume that Ω is a bounded and smooth domain in $\mathbb{R}^3$. Let $1 < p < \infty$, $v\in H^{1,p}(\Omega)$ with $v\cdot N = 0$ on $\partial\Omega$ in the sense of the trace. Then the estimate

$$\|\nabla v\|_{L^p(\Omega)} \le C(K)\big(\|\mathrm{div}\ v\|_{L^p(\Omega)} + \|\nabla\times v\|_{L^p(\Omega)}\big)$$

is true for all v as above if and only if Ω is the unit ball.

Since v_2 is harmonic, we can get $\|\nabla v_2\|_{L^\infty(\Omega)} \le \|\nabla v_2\|_{L^\infty(\partial\Omega)}$ by the maximum principle. Now, using $v = v_1 + v_2$ in (2.1), the Hölder inequality, Proposition 2.1, and the fact that v_2 is harmonic, we can obtain for $1 < p < \infty$

$$\begin{aligned} \|\nabla v\|_{L^p(\Omega)} &\le \|\nabla v_1\|_{L^p(\Omega)} + \|\nabla v_2\|_{L^p(\Omega)}\\ &\le C(K,\mathrm{Vol}(\Omega))\big(\|\nabla\times v\|_{L^\infty(\Omega)} + \|\bar{\nabla} V\|_{L^\infty(\partial\Omega)} + \|\mathscr{N}V\|_{L^\infty(\partial\Omega)}\big)\\ &\le C(K,\mathrm{Vol}(\Omega))\mathscr{A}, \end{aligned} \tag{2.2}$$

and similarly, by Proposition 2.1 and $\beta\cdot N = 0$ on $\partial\Omega$,

$$\|\nabla\beta\|_{L^p(\Omega)} \le C(K,\mathrm{Vol}(\Omega))\|\nabla\times\beta\|_{L^\infty(\Omega)} \le C(K,\mathrm{Vol}(\Omega))\mathscr{A}, \tag{2.3}$$

with $\mathscr{A}$ defined in (1.10).

III. ENERGY ESTIMATES

Let μ_g, μ_γ, h, h_{NN} be defined as in Ref. 1. Define the zero-order energy as

$$E_0(t) = \frac{1}{2}\int_\Omega\big(|v(t,y)|^2 + |\beta(t,y)|^2\big)d\mu_g. \tag{3.1}$$

Obviously, the energy of the system is conserved. Now the r-th order energy for $r\ge 1$ is defined as:

$$\begin{aligned} E_r(t) = &\int_\Omega g^{ab}\big(Q(\nabla^r v_a,\nabla^r v_b) + Q(\nabla^r\beta_a,\nabla^r\beta_b)\big)d\mu_g + \int_\Omega\Big[\big|\nabla^{r-1}\,\mathrm{curl}\,v\big|^2\\ &+ \big|\nabla^{r-1}\,\mathrm{curl}\,\beta\big|^2\Big]d\mu_g + \mathrm{sgn}\,(r-1)\int_{\partial\Omega} Q(\nabla^r q,\nabla^r q)\vartheta d\mu_\gamma \end{aligned} \tag{3.2}$$

where $\vartheta = 1/(-\nabla_N q)$ as before. Then we have the following energy estimates.

Proposition 3.1. Assume

$$|\theta| + \frac{1}{\varsigma_0}\le K, \quad |\nabla q|\ge\varepsilon > 0,$$

on $\partial\Omega$, then for $r = 1, 2, 3$, we have

$$\frac{d}{dt}E_r \le C(K,\varepsilon^{-1})\Big(\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla \beta\|_{L^\infty(\Omega)}\Big) \cdot \Big(1 + \|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big) + \|\nabla q\|_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\sum_{s=0}^{r} E_s(t); \tag{3.3}$$

and for $r = 4$, we can obtain

$$\begin{aligned}\frac{d}{dt}E_4 \le C(K,\varepsilon^{-1})\Big(&\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla \beta\|_{L^\infty(\Omega)}\Big)\\ &\cdot \Big(1 + \|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big)\\ &+ \|\nabla q\|_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\Big(1 + \sum_{s=0}^{3} E_s(t)\Big)(1 + E_4(t)).\end{aligned} \tag{3.4}$$

Proof. For $r = 1$, since Δ is invariant, we have

$$\Delta q = -\nabla_a v^b \nabla_b v^a + \nabla_a \beta^b \nabla_b \beta^a. \tag{3.5}$$

From the Poincaré inequality (Ref. 1, Lemma A.10), we get

$$\|\nabla q\|_{L^2(\Omega)} \le C(\mathrm{Vol}\Omega)^{1/6}\|\Delta q\|_{L^2(\Omega)} \le C(\mathrm{Vol}\Omega)\Big(\|\nabla v \nabla v\|_{L^2(\Omega)} + \|\nabla \beta \nabla \beta\|_{L^2(\Omega)}\Big). \tag{3.6}$$

By some known results in Ref. 1, (3.6), (Ref. 1, Lemma A.2) and the Hölder inequality, we can directly get

$$\begin{aligned}\frac{d}{dt}E_1(t) &\le CK\|\nabla v\|_{L^2(\Omega)}\|\nabla q\|_{L^2(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}\|\beta\|_{L^2(\Omega)}\|\nabla \beta\|_{L^2(\Omega)}\\ &\quad + C\|\nabla v\|_{L^\infty(\Omega)}\Big(\|\nabla v\|^2_{L^2(\Omega)} + \|\nabla \beta\|^2_{L^2(\Omega)} + \|\operatorname{curl} v\|^2_{L^2(\Omega)} + \|\operatorname{curl} \beta\|^2_{L^2(\Omega)}\Big)\\ &\le C(K)\big(\|\nabla v\|_{L^\infty(\Omega)} + \|\nabla \beta\|_{L^\infty(\Omega)}\big)(E_0 + E_1).\end{aligned}$$

For $2 \le r \le 4$, we can set

$$\mathscr{K}_r(t) := \int_\Omega \big|\nabla^{r-1}\operatorname{curl} v\big|^2 d\mu_g + \int_\Omega \big|\nabla^{r-1}\operatorname{curl} \beta\big|^2 d\mu_g$$

and

$$\begin{aligned}\mathscr{E}_r(t) := &\int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a v_b \nabla_F^{r-1}\nabla_f v_d\, d\mu_g + \int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \beta_b \nabla_F^{r-1}\nabla_f \beta_d\, d\mu_g\\ &+ \int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a q \nabla_F^{r-1}\nabla_f q \vartheta\, d\mu_\gamma.\end{aligned}$$

Firstly, the derivative of $\mathscr{E}_r(t)$ with respect to t is

$$\begin{aligned}\frac{d}{dt}\mathscr{E}_r(t) = &\int_\Omega D_t\Big(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a v_b \nabla_F^{r-1}\nabla_f v_d\Big)d\mu_g\\ &+ \int_\Omega D_t\Big(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \beta_b \nabla_F^{r-1}\nabla_f \beta_d\Big)d\mu_g\\ &+ \int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a v_b \nabla_F^{r-1}\nabla_f v_d \operatorname{tr}h\, d\mu_g\\ &+ \int_\Omega g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a \beta_b \nabla_F^{r-1}\nabla_f \beta_d \operatorname{tr}h\, d\mu_g\\ &+ \int_{\partial\Omega} D_t\Big(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a q \nabla_F^{r-1}\nabla_f q\Big)\vartheta\, d\mu_\gamma\\ &+ \int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a q \nabla_F^{r-1}\nabla_f q\Big(\frac{\vartheta_t}{\vartheta} + \operatorname{tr}h - h_{NN}\Big)\vartheta\, d\mu_\gamma.\end{aligned} \tag{3.7}$$

Hence, we can obtain the following estimate by Ref. 1, Lemma A.2 and $\beta\cdot N = 0$ on $\partial\Omega$,

$$\begin{aligned}
&\int_\Omega D_t\Big(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a v_b\nabla_F^{r-1}\nabla_f v_d\Big) + D_t\Big(g^{bd}\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a\beta_b\nabla_F^{r-1}\nabla_f\beta_d\Big)d\mu_g \\
&\quad+\int_{\partial\Omega} D_t\Big(\gamma^{af}\gamma^{AF}\nabla_A^{r-1}\nabla_a q\nabla_F^{r-1}\nabla_f q\Big)\vartheta d\mu_\gamma \\
&\le C\big(\|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\big)E_r(t) + CKE_r^{1/2}(t)\|\nabla^r q\|_{L^2(\Omega)} + CK\|\beta\|_{L^\infty(\Omega)}E_r(t) \\
&\quad+ CE_r^{1/2}(t)\sum_{s=1}^{r-2}\Big(\big\|\nabla^{s+1}\beta\big(\nabla^{r-s}v+\nabla^{r-s}\beta\big)\big\|_{L^2(\Omega)} + \big\|\nabla^{s+1}v\big(\nabla^{r-s}v+\nabla^{r-s}\beta\big)\big\|_{L^2(\Omega)}\Big) \\
&\quad+2\int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla_{Aa}^r q\Big(D_t\nabla_{Ff}^r q - \frac{1}{\vartheta}N_b\nabla_{Ff}^r v^b\Big)\vartheta d\mu_\gamma,
\end{aligned} \tag{3.8}$$

where, for the fourth line of (3.8), we can use the interpolation inequality in Ref. 18, Lemma 9 to obtain

$$\|\nabla^{s+1}f\nabla^{r-s}g\|_{L^2(\Omega)} \le C\|\nabla f\|_{L^\infty(\Omega)}\sum_{s=1}^{r}\|\nabla^s g\|_{L^2(\Omega)} + \|\nabla g\|_{L^\infty(\Omega)}\sum_{s=1}^{r}\|\nabla^s f\|_{L^2(\Omega)}. \tag{3.9}$$

The next ingredients we will need are the following L^2 estimates for $\Delta D_t q$. These are similar to the estimates in Ref. 1, except that we need to ensure that $\|\nabla v\|_{L^\infty(\Omega)}$ and $\|\nabla\beta\|_{L^\infty(\Omega)}$ appear with the same homogeneity as $\nabla^{r-2}\Delta D_t q$. It follows that for $2\le r\le 4$,

$$\begin{aligned}
\nabla^{r-2}\Delta q &= \nabla^{r-2}\Big(-\nabla_a v^b\nabla_b v^a + \nabla_a\beta^b\nabla_b\beta^a\Big) \\
&= -\sum_{s=0}^{r-2}\binom{r-2}{s}\nabla^s\nabla_a v^b\nabla^{r-2-s}\nabla_b v^a \\
&\quad+\sum_{s=0}^{r-2}\binom{r-2}{s}\nabla^s\nabla_a\beta^b\nabla^{r-2-s}\nabla_b\beta^a.
\end{aligned}$$

Proposition 3.2. Suppose that $|\theta| + 1/\varsigma_0 \le K$, *we have, for any* $2\le r\le 4$,

$$\begin{aligned}
&\|(\nabla^r v,\nabla^r\beta)\|_{L^2(\Omega)}^2 \le CE_r, \quad \|\Pi\nabla^r q\|_{L^2(\partial\Omega)}^2 \le \|\nabla q\|_{L^\infty(\partial\Omega)}E_r, \\
&\|\nabla^r q\|_{L^2(\partial\Omega)}^2 + \|\nabla^r q\|_{L^2(\Omega)}^2 \le \|\Pi\nabla^r q\|_{L^2(\partial\Omega)}^2 + C(K,\mathrm{Vol}(\Omega))\sum_{s=0}^{r}\|\nabla^s\Delta q\|_{L^2(\Omega)}^2 \\
&\qquad\le C(K)\big(\|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}^2 + \|\nabla\beta\|_{L^\infty(\Omega)}^2\big)\sum_{s=0}^{r}E_s.
\end{aligned} \tag{3.10}$$

In addition,

$$\begin{aligned}
&\|\theta\|_{L^2(\partial\Omega)}^2 \le C\big(\varepsilon^{-1}\big)\|\nabla q\|_{L^\infty(\partial\Omega)}E_2, \\
&\|\bar\nabla^{r-2}\theta\|_{L^2(\partial\Omega)}^2 \le C\big(K,\varepsilon^{-1},\mathrm{Vol}\Omega\big)\big(\|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}^2 + \|\nabla\beta\|_{L^\infty(\Omega)}^2\big)\sum_{s=0}^{r}E_s.
\end{aligned} \tag{3.11}$$

Proof. The first estimate follows from Ref. 1, Lemma A.2, and the second estimate follows from the definition

$$\|\Pi\nabla^r q\|_{L^2(\partial\Omega)}^2 = \int_{\partial\Omega}\gamma^{ij}\gamma^{IJ}\nabla_I^{\gamma-1}\nabla_i q\nabla_J^{\gamma-1}\nabla_j q\vartheta\cdot(-\nabla_N q)d\mu_\gamma \le \|\nabla q\|_{L^\infty(\partial\Omega)}E_r.$$

By Ref. 1, Lemma A.3, we have

$$\begin{aligned}
\|\nabla^r q\|_{L^2(\partial\Omega)}^2 + \|\nabla^r q\|_{L^2(\Omega)}^2 &\le C\|\Pi\nabla^r q\|_{L^2(\partial\Omega)}^2 + C(K,\mathrm{Vol}(\Omega))\sum_{s=0}^{r-1}\|\nabla^s\Delta q\|_{L^2(\Omega)}^2 \\
&\le C\|\nabla q\|_{L^\infty(\partial\Omega)}E_r + C(K,\mathrm{Vol}(\Omega))\sum_{s=0}^{r-1}\|\nabla^s\Delta q\|_{L^2(\Omega)}^2,
\end{aligned}$$

where

$$\|\nabla^{r-1}\Delta q\|_{L^2(\Omega)} \le C\sum_{s=0}^{r-1}\|\nabla^{1+s}v\nabla^{r-s}v\|_{L^2(\Omega)} + \sum_{s=0}^{r-1}\|\nabla^{1+s}\beta\nabla^{r-s}\beta\|_{L^2(\Omega)},$$

it follows from Ref. 18, Lemma 9 that

$$\|\nabla^{r-1}\Delta q\|_{L^2(\Omega)} \le C(K)\big(\|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\big)\left(\sum_{k=0}^{r}\|\nabla^k v\|_{L^2(\Omega)} + \sum_{k=0}^{r}\|\nabla^k\beta\|_{L^2(\Omega)}\right).$$

Thus, we have (3.10).

From (1.5), we get $\Pi\nabla^2 q = \theta\nabla_N q$ and then $\|\theta\|_{L^2(\partial\Omega)} \le C(\varepsilon^{-1})\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}E_2^{1/2}$, the first estimate in (3.11) follows. For the second one, by Ref. 6, Proposition 5.9 combined with estimate (3.10), we have

$$\begin{aligned}\|\bar{\nabla}^{r-2}\theta\|_{L^2(\partial\Omega)} &\le C\left(\|\Pi\nabla^r q\|_{L^2(\partial\Omega)} + \sum_{k=1}^{r-1}\|\theta\|^k_{L^\infty(\partial\Omega)}\|\nabla^{r-k}q\|_{L^2(\partial\Omega)}\right)\\ &\le C(K,\varepsilon^{-1})\big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\big)\sum_{k=0}^{r}E_k^{1/2}.\end{aligned}\tag{3.12}$$

Therefore, we complete the proof. □

Owing to Proposition 3.2, we have for $2 \le r \le 4$

$$\begin{aligned}E_r^{1/2}(t)\|\nabla^r q\|_{L^2(\Omega)} &\le C(K,\varepsilon^{-1})\big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\big)\sum_{s=0}^{r}E_s^{1/2}E_r^{1/2}\\ &\le C(K,\varepsilon^{-1})\big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\big)\sum_{s=0}^{r}E_s.\end{aligned}$$

Now, we need to estimate the other integral, by the Hölder inequality and the fact $-\vartheta^{-1}N_b = \delta^a_b\nabla_a q - \gamma^a_b\nabla_a q = \nabla_b q$, we can get

$$\begin{aligned}&\left|\int_{\partial\Omega}\gamma^{af}\gamma^{AF}\nabla^r_{Aa}q\left(D_t\nabla^r_{Ff}q - \frac{1}{\vartheta}N_b\nabla^r_{Ff}v^b\right)\vartheta d\mu_\gamma\right|\\ &\le C\|\vartheta\|^{1/2}_{L^\infty(\partial\Omega)}E_r^{1/2}(t)\left\|\Pi\left(D_t(\nabla^r q) - \vartheta^{-1}N_b\nabla^r v^b\right)\right\|_{L^2(\partial\Omega)}\\ &= C\|\vartheta\|^{1/2}_{L^\infty(\partial\Omega)}E_r^{1/2}(t)\left\|\Pi\left(D_t(\nabla^r q) + \nabla^r v\cdot\nabla q\right)\right\|_{L^2(\partial\Omega)}\end{aligned}\tag{3.13}$$

By Ref. 1, Lemma 2.3, it follows that

$$D_t\nabla^r q + \nabla^r v\cdot\nabla q = \operatorname{sgn}(2-r)\sum_{s=1}^{r-2}\begin{pmatrix} r\\ s+1\end{pmatrix}(\nabla^{s+1}v)\cdot\nabla^{r-s}q + \nabla^r D_t q.$$

We have to control $\|\nabla q\|_{L^\infty(\Omega)}$. By the Sobolev embedding (Ref. 1, Lemma A.9) with Lebesgue exponent $p = 4$, (2.2) and (2.3), Proposition 2.1 and (3.5), we obtain

$$\begin{aligned}\|\nabla q\|_{L^\infty(\Omega)} &\le C(K)\big(\|\nabla q\|_{L^4(\Omega)} + \|\nabla^2 q\|_{L^4(\Omega)}\big) \le C(K)\|\Delta q\|_{L^4(\Omega)}\\ &\le C(K)\big(\|\nabla v\|_{L^\infty(\Omega)}\|\nabla v\|_{L^4(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\|\nabla\beta\|_{L^4(\Omega)}\big)\\ &\le C(K,\mathrm{Vol}(\Omega))\big(\|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\big)\mathscr{A}.\end{aligned}\tag{3.14}$$

In addition, we can use the Sobolev inequality with Lebesgue exponent $p = 2$ and Proposition 3.2 to get

$$\begin{aligned}\|\nabla q\|^2_{L^\infty(\Omega)} &\le C(K,\mathrm{Vol}(\Omega))\sum_{k=1}^{3}\|\nabla^k q\|^2_{L^2(\Omega)}\\ &\le C(K,\mathrm{Vol}(\Omega))\big(\|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla v\|^2_{L^\infty(\Omega)} + +\|\nabla\beta\|^2_{L^\infty(\Omega)}\big)\sum_{s=0}^{3}E_s.\end{aligned}\tag{3.15}$$

For $2 \le r \le 4$, by Ref. 1, Lemma 2.3, (3.5), (Ref. 1, Lemma A.1) and (1.8), it yields

$$\begin{aligned}\Delta D_t q = 4g^{ac}\nabla_c v^b \nabla_a \nabla_b q + (\Delta v^e)\nabla_e q + 2\nabla_e v^b \nabla_b v^a \nabla_a v^e \\ - \frac{1}{2}\nabla_b v^a \nabla_a \beta^c \nabla_c \beta^b - \frac{1}{2}\nabla_b v^a \beta^c \nabla_a \nabla_c \beta^b + \frac{1}{2}\nabla_b \beta^a \beta^e \nabla_e \nabla_a v^b.\end{aligned} \tag{3.16}$$

When we estimate $\|\nabla^{r-2}\Delta D_t q\|_{L^2(\Omega)}$, we need to control the following terms:

$$\begin{aligned}&\|(\nabla^{r-2}\Delta v)(\nabla q)\|_{L^2(\Omega)}, \quad r = 2,3,4,\\ &\|(\nabla^{1+s} v)(\nabla^{r-s} q)\|_{L^2(\Omega)}, \quad s = 0,\dots,r-2,\\ &\|(\nabla^{1+r_1} v)(\nabla^{1+r_2} v)(\nabla^{1+r_3} v)\|_{L^2(\Omega)}, \quad r_1+r_2+r_3 = r-2,\\ &\|(\nabla^{1+r_1} v)(\nabla^{1+r_2} \beta)(\nabla^{1+r_3} \beta)\|_{L^2(\Omega)}, \quad r_1+r_2+r_3 = r-2,\\ &\|(\nabla^{2+r_1} v)(\nabla^{r_2} \beta)(\nabla^{1+r_3} \beta)\|_{L^2(\Omega)}, \quad r_1+r_2+r_3 = r-2,\\ &\|(\nabla^{2+r_1} \beta)(\nabla^{r_2} \beta)(\nabla^{1+r_3} v)\|_{L^2(\Omega)}, \quad r_1+r_2+r_3 = r-2.\end{aligned} \tag{3.17}$$

For $r = 2,3$, by (3.14) and Proposition 3.2, we can get

$$\|(\nabla^r v)\cdot\nabla q\|_{L^2(\Omega)} \le C(K,\mathrm{Vol}(\Omega))E_r^{1/2}\big(\|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\big)\mathscr{A}. \tag{3.18}$$

For $r = 4$, by (3.15) and Proposition 3.2, we have

$$\|(\nabla^r v)\cdot\nabla q\|_{L^2(\Omega)} \le C(K,\varepsilon^{-1})E_r^{1/2}(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)})\sum_{s=0}^{3} E_s^{1/2}. \tag{3.19}$$

Next, we consider $\|(\nabla^{1+s}v)(\nabla^{r-s}q)\|_{L^2(\Omega)}$. For $r = 2$, it yields by (2.2) and (2.3), Proposition 2.1 and (3.5),

$$\begin{aligned}\|\nabla v\|_{L^\infty(\Omega)}\|\nabla^2 q\|_{L^2(\Omega)} &\le \|\nabla v\|_{L^\infty(\Omega)}\|\nabla v\|^2_{L^4(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}\|\nabla\beta\|^2_{L^4(\Omega)}\\ &\le C(K,\mathrm{Vol}\Omega)\|\nabla v\|_{L^\infty(\Omega)}\mathscr{A}\Big(\|\nabla v\|_{L^4(\Omega)} + \|\nabla\beta\|_{L^4(\Omega)}\Big)\\ &\le C(K,\mathrm{Vol}\Omega)\|\nabla v\|_{L^\infty(\Omega)}\mathscr{A}\left(\sum_{s=0}^{2} E_s^{1/2}\right).\end{aligned}$$

From Ref. 18, Lemma 9 and Ref. 1, Lemma A.9, it follows that for $f = v$ or β

$$\|\nabla^{s+1} f\|_{L^4(\Omega)} \le C\|\nabla^s f\|^{1/2}_{L^\infty(\Omega)}\left(\sum_{\ell=0}^{2}\|\nabla^{s+\ell} f\|_{L^2(\Omega)}\right)^{1/2} \le C(K)\sum_{\ell=0}^{2} E^{1/2}_{s+\ell}(t). \tag{3.20}$$

By Ref. 1, Lemma A.3 with $\delta = 1$ and Proposition 3.2, we can get

$$\begin{aligned}\|\nabla^3 q\|^2_{L^2(\Omega)} &\le \|\Pi\nabla^3 q\|^2_{L^2(\partial\Omega)} + C(K,\mathrm{Vol}\Omega)\sum_{s\le 1}\|\nabla^s \Delta q\|^2_{L^2(\Omega)}\\ &\le C(K,\mathrm{Vol}\Omega)\|\nabla q\|_{L^\infty(\partial\Omega)}E_3 + \|\nabla v\nabla v + \nabla\beta\nabla\beta\|^2_{L^2(\Omega)}\\ &\quad + \|\nabla(\nabla v\nabla v + \nabla\beta\nabla\beta)\|^2_{L^2(\Omega)}.\end{aligned}$$

From (3.22), (2.2), and (2.3), we have

$$\begin{aligned}\|\nabla v\nabla v + \nabla\beta\nabla\beta\|_{L^2(\Omega)} &\le \|\nabla v\|_{L^4(\Omega)}\|\nabla v\|_{L^4(\Omega)} + \|\nabla\beta\|_{L^4(\Omega)}\|\nabla\beta\|_{L^4(\Omega)}\\ &\le C(K,\mathrm{Vol}\Omega)\mathscr{A}\left(\sum_{s=0}^{2} E_s^{1/2}(t)\right),\end{aligned}$$

and

$$\|\nabla(\nabla v\nabla v + \nabla\beta\nabla\beta)\|_{L^2(\Omega)} \le C(K,\mathrm{Vol}\Omega)\mathscr{A}\left(\sum_{s=0}^{3} E_s^{1/2}(t)\right).$$

Thus, we have

$$\|\nabla^3 q\|_{L^2(\Omega)} \le C(K, \mathrm{Vol}\Omega)(\|\nabla q\|_{L^\infty(\partial\Omega)}^{1/2} + \mathscr{A})\left(\sum_{s=0}^{3} E_s^{1/2}(t)\right). \tag{3.21}$$

For $r = 3$ and $s = 0, 1$, we have by (3.21)

$$\begin{aligned}\|(\nabla v)(\nabla^3 q)\|_{L^2(\Omega)} &\le \|\nabla v\|_{L^\infty(\Omega)}\|\nabla^3 q\|_{L^2(\Omega)}\\ &\le C(K, \varepsilon^{-1})\|\nabla v\|_{L^\infty(\Omega)}(\mathscr{A} + \|\nabla q\|_{L^\infty(\partial\Omega)}^{1/2})\left(\sum_{s=0}^{3} E_s^{1/2}(t)\right)\end{aligned}$$

and by (3.20), Proposition 2.1, (2.2) and (2.3)

$$\begin{aligned}\|(\nabla^2 v)(\nabla^2 q)\|_{L^2(\Omega)} &\le \|\nabla^2 v\|_{L^4(\Omega)}\|\nabla^2 q\|_{L^4(\Omega)}\\ &\le C(K)\left(\sum_{s=0}^{3} E_s^{1/2}(t)\right)\left(\|\nabla v\nabla v\|_{L^4(\Omega)} + \|\nabla\beta\nabla\beta\|_{L^4(\Omega)}\right)\\ &\le C(K)\|\nabla v\|_{L^\infty(\Omega)}\mathscr{A}\left(\sum_{s=0}^{3} E_s^{1/2}(t)\right).\end{aligned}$$

Owing to Ref. 18, Lemma 9, for $r = 4$, we can get

$$\|(\nabla^{1+s} v)(\nabla^{4-s} q)\|_{L^2(\Omega)} \le \|\nabla v\|_{L^\infty(\Omega)}\sum_{k=1}^{4}\|\nabla^k q\|_{L^2(\Omega)} + \|\nabla q\|_{L^\infty(\Omega)}\sum_{k=1}^{4}\|\nabla^k v\|_{L^2(\Omega)},$$

then, by Sobolev's inequality (Ref. 1, Lemma A.10), (3.15), Proposition 3.2 and the Hölder inequality, we obtain

$$\begin{aligned}&\|(\nabla^{1+s} v)(\nabla^{r-s} q)\|_{L^2(\Omega)}\\ &\le C(K, \varepsilon^{-1})\left(\|\nabla q\|_{L^\infty(\partial\Omega)}^{1/2} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\right)\sum_{s=0}^{3} E_s^{1/2}\sum_{k=0}^{4} E_k^{1/2}.\end{aligned} \tag{3.22}$$

Next, we estimate the remaining terms. We just need to estimate

$$(\nabla^{1+r_1} v)(\nabla^{1+r_2} v)(\nabla^{1+r_3} v) \quad \text{and} \quad (\nabla^{2+r_1} v)(\nabla^{r_2}\beta)(\nabla^{1+r_3}\beta),$$

then for other terms, we can obtain similar results by using the same method. By Sobolev's inequality (Ref. 1, Lemma A.9), (2.3) and the fact that $E_0(t)$ is conserved, we can get

$$\begin{aligned}\|\beta\|_{L^\infty(\Omega)} &\le \|\beta\|_{L^4(\Omega)} + \|\nabla\beta\|_{L^4(\Omega)}\\ &\le C(K, \mathrm{Vol}\Omega)\left(\|\beta\|_{L^2(\Omega)} + \|\nabla\beta\|_{L^2(\Omega)} + \|\nabla\beta\|_{L^4(\Omega)}\right)\\ &\le C(K, \mathrm{Vol}\Omega)(1 + \mathscr{A}).\end{aligned} \tag{3.23}$$

For $r = 2$, we get by (2.2) and (3.20)

$$\|\nabla v\nabla v\nabla v\|_{L^2(\Omega)} \le \|\nabla v\|_{L^\infty(\Omega)}\|\nabla v\|_{L^4(\Omega)}^2 \le C(K, \mathrm{Vol}\Omega)\|\nabla v\|_{L^\infty(\Omega)}\mathscr{A}\left(\sum_{s=0}^{2} E_s^{1/2}\right),$$

and by (3.23)

$$\|\nabla^2 v\beta\nabla\beta\|_{L^2(\Omega)} \le \|\beta\|_{L^\infty(\Omega)}\|\nabla\beta\|_{L^\infty(\Omega)}\|\nabla^2 v\|_{L^2(\Omega)} \le C(K, \mathrm{Vol}\Omega)(1 + \mathscr{A})\|\nabla\beta\|_{L^\infty(\Omega)}E_2^{1/2}.$$

For $r = 3$, by (2.2), (3.20), and (3.23), we have

$$\|\nabla v\|_{L^\infty(\Omega)}\|\nabla v\|_{L^4(\Omega)}\|\nabla^2 v\|_{L^4(\Omega)} \le C(K, \mathrm{Vol}\Omega)\|\nabla v\|_{L^\infty(\Omega)}\mathscr{A}\sum_{s=0}^{3} E_s^{1/2},$$

$$\|\nabla^3 v\|_{L^2(\Omega)}\|\beta\|_{L^\infty(\Omega)}\|\nabla\beta\|_{L^\infty(\Omega)} \le C(K, \mathrm{Vol}\Omega)\|\nabla\beta\|_{L^\infty(\Omega)}(1 + \mathscr{A})E_3^{1/2},$$

and

$$\|\nabla^2\beta\beta\nabla^2 v\|_{L^2(\Omega)} \le C(K)\|\beta\|_{L^\infty(\Omega)}\|\nabla^2 v\|_{L^4(\Omega)}\|\nabla^2\beta\|_{L^4(\Omega)}$$
$$\le C(K)\|\beta\|_{L^\infty(\Omega)}\|\nabla\beta\|^{1/2}_{L^\infty(\Omega)}\|\nabla v\|^{1/2}_{L^\infty(\Omega)}\sum_{s=0}^{3}E_s^{1/2}$$
$$\le C(K)(1+\mathscr{A})\big(\|\nabla\beta\|_{L^\infty(\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}\big)\sum_{s=0}^{3}E_s^{1/2}.$$

For $r=4$, we use Sobolev's embedding and (3.20) to bound them by

$$\|\nabla^3 v\nabla v\nabla v\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^\infty(\Omega)}\|\nabla^3 v\|_{L^2(\Omega)}\|\nabla v\|_{L^\infty(\Omega)}$$
$$\le C(K)\|\nabla v\|_{L^\infty(\Omega)}\left(\sum_{s=0}^{3}\|\nabla^s v\|_{L^2(\Omega)}\right)E_3^{1/2},$$

and

$$\|\nabla^2 v\nabla^2 v\nabla v\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^4(\Omega)}\|\nabla^2 v\|_{L^\infty(\Omega)}\|\nabla^2 v\|_{L^4(\Omega)}$$
$$\le C(K)\mathscr{A}\left(\sum_{s=0}^{3}\|\nabla^s v\|_{L^2(\Omega)}\right)\left(\sum_{s=0}^{4}\|\nabla^s v\|_{L^2(\Omega)}\right).$$

For the terms involving magnetic fields, we only consider the highest order term and other terms can be treated similarly. Indeed, we have

$$\|\nabla^4 v\beta\nabla\beta\|_{L^2(\Omega)} \le \|\nabla^4 v\|_{L^2(\Omega)}\|\nabla\beta\|_{L^\infty(\Omega)}\|\beta\|_{L^\infty(\Omega)} \le C(K)(1+\mathscr{A})\|\nabla\beta\|_{L^\infty(\Omega)}E_4^{1/2},$$

and

$$\|\nabla^3\beta\beta\nabla^2 v\|_{L^2(\Omega)} \le \|\beta\|_{L^\infty(\Omega)}\|\nabla^2 v\|_{L^4(\Omega)}\|\nabla^3\beta\|_{L^4(\Omega)} \le C(K)(1+\mathscr{A})\sum_{s=0}^{4}E_s^{1/2}\sum_{s=0}^{3}E_s^{1/2}.$$

Therefore, we have shown that for $r=2,3$

$$\begin{aligned}\left\|\nabla^{r-2}\Delta D_t q\right\|_{L^2(\Omega)} &\le C(K)\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}+\|\nabla\beta\|_{L^\infty(\Omega)}\Big)\\ &\quad\cdot(1+\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}+\mathscr{A})\sum_{k=0}^{r}E_k^{1/2},\end{aligned}\tag{3.24}$$

and for $r=4$

$$\begin{aligned}\left\|\nabla^{r-2}\Delta D_t q\right\|_{L^2(\Omega)} &\le C(K)\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}+\|\nabla\beta\|_{L^\infty(\Omega)}\Big)\\ &\quad\cdot(1+\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}+\mathscr{A})(E_r^{1/2}+1)\left(1+\sum_{k=0}^{r-1}E_k^{1/2}\right).\end{aligned}\tag{3.25}$$

Now, we can estimate $\Pi\nabla^r D_t q$. For $r=2,3$, applying Ref. 1, Lemma A.5 and (3.24), we get

$$\begin{aligned}\|\nabla D_t q\|_{L^2(\partial\Omega)} &\le C(K,\mathrm{Vol}(\Omega))\|\Delta D_t q\|_{L^2(\Omega)}\\ &\le C(K,\mathrm{Vol}(\Omega))\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}+\|\nabla\beta\|_{L^\infty(\Omega)}\Big)\\ &\quad\cdot(1+\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}+\mathscr{A})\sum_{k=0}^{2}E_k^{1/2},\end{aligned}$$

by Ref. 6, Proposition 5.9 and (3.11), we have

$$\begin{aligned}\left\|\Pi\nabla^2 D_t q\right\|_{L^2(\partial\Omega)} &\le \|\theta\|_{L^2(\partial\Omega)}\|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}+\|\theta\|_{L^\infty(\partial\Omega)}\|\nabla D_t q\|_{L^2(\partial\Omega)}\\ &\le C(K,\varepsilon^{-1},\mathrm{Vol}(\Omega))\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}+\|\nabla v\|_{L^\infty(\Omega)}+\|\nabla\beta\|_{L^\infty(\Omega)}\Big)\\ &\quad\cdot\Big(1+\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)}+\mathscr{A}+\|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big)\sum_{k=0}^{2}E_k^{1/2}.\end{aligned}$$

And by Ref. 1, Lemma A.5, Proposition 3.2 and (3.24), it yields

$$\begin{aligned}\|\nabla^2 D_t q\|_{L^2(\partial\Omega)} &\le C(K,\mathrm{Vol}(\Omega))\Big(\|\theta\|_{L^2(\partial\Omega)}\|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \|\Delta D_t q\|_{L^2(\Omega)} + \|\nabla\Delta D_t q\|_{L^2(\Omega)}\Big)\\ &\le C(K,\varepsilon^{-1})\Big(\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\Big)\Big(1+\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A}\\ &\quad + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big)\Big)\sum_{k=0}^{3} E_k^{1/2},\end{aligned}$$

then from Ref. 6, Proposition 5.9 and Proposition 3.2, we obtain

$$\begin{aligned}&\|\Pi\nabla^3 D_t q\|_{L^2(\partial\Omega)}\\ &\le C(K)\Big(\|\bar\nabla\theta\|_{L^2(\partial\Omega)}\|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \|\nabla D_t q\|_{L^2(\partial\Omega)} + \|\nabla^2 D_t q\|_{L^2(\partial\Omega)}\Big)\\ &\le C(K,\varepsilon^{-1})\Big(\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\partial\Omega)} + \|\nabla\beta\|_{L^\infty(\partial\Omega)}\Big)\|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\sum_{s=0}^{3} E_s^{1/2}\\ &+\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\Big)\Big(1+\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big)\\ &\cdot\sum_{s=0}^{3} E_s^{1/2}\Big).\end{aligned}$$

For $r = 4$, by Ref. 6, Proposition 5.9, Ref. 1, Lemma A.5, Proposition 3.2, we similarly have

$$\begin{aligned}&\|\Pi\nabla^4 D_t q\|_{L^2(\partial\Omega)}\\ &\le C(K)\Big(\|\bar\nabla^2\theta\|_{L^2(\partial\Omega)}\|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \sum_{s=1}^{3}\|\nabla^s D_t q\|_{L^2(\partial\Omega)}\Big)\\ &\le C(K,\varepsilon^{-1})\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\partial\Omega)} + \|\nabla\beta\|_{L^\infty(\partial\Omega)}\Big)\|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\sum_{s=0}^{4} E_s^{1/2}\\ &\quad + C(K,\varepsilon^{-1})\Big(\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\Big)\\ &\quad\cdot\Big(1+\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big)\Big)(E_4^{1/2}+1)\Big(1+\sum_{s=0}^{3} E_s^{1/2}\Big)\\ &\le C(K,\varepsilon^{-1})\Big(\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\Big)\\ &\quad\cdot\Big(1+\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big)\Big)(E_4^{1/2}+1)\Big(1+\sum_{s=0}^{3} E_s^{1/2}\Big).\end{aligned}$$

To estimate (3.13), it only remain to estimate

$$\|\Pi(\nabla^{1+s}v)\cdot(\nabla^{r-s}q)\|_{L^2(\partial\Omega)} \quad \text{for} \quad 1\le s\le r-2.$$

For $r = 3$, by Ref. 1, Lemma A.2, (1.5) and (3.5), we get

$$\|\nabla^2 q\|_{L^\infty(\partial\Omega)} \le C\Big(\|\nabla v\|^2_{L^\infty(\partial\Omega)} + \|\nabla\beta\|^2_{L^\infty(\partial\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)}\|\nabla_N q\|_{L^\infty(\partial\Omega)}\Big). \tag{3.26}$$

Hence, we have by Ref. 1, Lemma A.11 and (3.26)

$$\begin{aligned}\|\Pi((\nabla^2 v)\cdot\nabla^2 q)\|_{L^2(\partial\Omega)} &\le \|\nabla^2 v\|_{L^2(\partial\Omega)}\|\nabla^2 q\|_{L^\infty(\partial\Omega)}\\ &\le C(K,\mathrm{Vol}\Omega)\Big(\|\nabla^3 v\|_{L^2(\Omega)} + \|\nabla^2 v\|_{L^2(\Omega)}\Big)\\ &\quad\cdot\Big(\|\nabla v\|^2_{L^\infty(\partial\Omega)} + \|\nabla\beta\|^2_{L^\infty(\partial\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)}\|\nabla_N q\|_{L^\infty(\partial\Omega)}\Big)\\ &\le C(K,\varepsilon^{-1})\Big(\mathscr{A}^2 + \|\nabla q\|_{L^\infty(\partial\Omega)}\Big)\Big(E_3^{1/2}(t) + E_2^{1/2}(t)\Big).\end{aligned} \tag{3.27}$$

For $r = 4$ and $s = 1$, by Ref. 1, (A.6), Ref. 1, Lemma A.11 and (3.10), it holds

$$\begin{aligned}
&\|\Pi((\nabla^2 v)\cdot\nabla^3 q)\|_{L^2(\partial\Omega)} = \|\Pi\nabla^2 v\cdot\Pi\nabla^3 q + \Pi(\nabla^2 v\cdot N)\tilde{\otimes}\Pi(N\cdot\nabla^3 q)\|_{L^2(\partial\Omega)}\\
&\le C\|\Pi\nabla^2 v\|_{L^4(\partial\Omega)}\|\Pi\nabla^3 q\|_{L^4(\partial\Omega)} + C\|\Pi(N^a\nabla^2 v_a)\|_{L^4(\partial\Omega)}\|\Pi(\nabla_N\nabla^2 q)\|_{L^4(\partial\Omega)}\\
&\le C\|\nabla^2 v\|_{L^4(\partial\Omega)}\|\nabla^3 q\|_{L^4(\partial\Omega)}\\
&\le C(K,\mathrm{Vol}\Omega)\left(\|\nabla^3 v\|_{L^2(\Omega)} + \|\nabla^2 v\|_{L^2(\Omega)}\right)\left(\|\nabla^4 q\|_{L^2(\Omega)} + \nabla^3 q\|_{L^2(\Omega)}\right)\\
&\le C(K,\varepsilon^{-1})\left(E_3^{1/2}(t) + E_2^{1/2}(t)\right)\left(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\right)\sum_{s=0}^{4} E_s^{1/2}.
\end{aligned} \tag{3.28}$$

For $s = 2$, we can get

$$\begin{aligned}
\|\Pi((\nabla^3 v)\cdot\nabla^2 q)\|_{L^2(\partial\Omega)} &\le \|\nabla^3 v\|_{L^2(\partial\Omega)}\|\nabla^2 q\|_{L^\infty(\partial\Omega)}\\
&\le C(K,\mathrm{Vol}\Omega)\left(\|\nabla^4 v\|_{L^2(\Omega)} + \|\nabla^3 v\|_{L^2(\Omega)}\right)\\
&\quad\cdot\left(\|\nabla v\|^2_{L^\infty(\partial\Omega)} + \|\nabla\beta\|^2_{L^\infty(\partial\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)}\|\nabla_N q\|_{L^\infty(\partial\Omega)}\right)\\
&\le C(K,\varepsilon^{-1},\mathrm{Vol}\Omega)\left(\mathscr{A}^2 + \|\nabla q\|_{L^\infty(\partial\Omega)}\right)\left(E_4^{1/2}(t) + E_3^{1/2}(t)\right).
\end{aligned} \tag{3.29}$$

Therefore, we have shown that for $r = 2, 3$

$$\begin{aligned}
|(3.13)| \le C(K,\varepsilon^{-1})\Big(&\left(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\right)\\
&\cdot\left(1 + \|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\right)\\
&+ \|\nabla q\|_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\left(\sum_{s=0}^{r} E_s(t)\right),
\end{aligned}$$

and for $r = 4$

$$\begin{aligned}
|(3.13)| \le C(K,\varepsilon^{-1})\Big(&\left(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\right)\\
&\cdot\left(1 + \|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\right)\\
&+ \|\nabla q\|_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\left(1 + \sum_{s=0}^{r-1} E_s(t)\right)(1 + E_r(t)).
\end{aligned} \tag{3.30}$$

Finally, we only need to estimate the remainder terms in (3.7). By Ref. 1, Lemma A.1 and Ref. 1, Lemma 2.3, we get

$$\frac{\vartheta_t}{\vartheta} = -\frac{2h_d^a N^d\nabla_a q}{\nabla_N q} + h_{NN} + \frac{\nabla_N D_t q}{\nabla_N q}.$$

Thus, the remainder integrals can be controlled by the right-hand side of (3.30). Therefore, we obtain for $r = 1, 2, 3$

$$\begin{aligned}
\frac{d}{dt}\mathscr{E}_r(t) \le C(K,\varepsilon^{-1})\Big(&\left(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\right)\\
&\cdot\left(1 + \|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\right)\\
&+ \|\nabla q\|_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\left(\sum_{s=0}^{r} E_s(t)\right),
\end{aligned} \tag{3.31}$$

and for $r = 4$

$$\begin{aligned}
\frac{d}{dt}\mathscr{E}_r(t) \le C(K,\varepsilon^{-1})\Big(&\left(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\right)\\
&\cdot\left(1 + \|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\right)\\
&+ \|\nabla q\|_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\left(1 + \sum_{s=0}^{r-1} E_s(t)\right)(1 + E_r(t)).
\end{aligned} \tag{3.32}$$

We now calculate the material derivatives of $\mathscr{K}_r(t)$. From Ref. 1, by the Hölder inequality and the Gauss formula and the same method as in obtaining (3.31) and (3.32), we can get for $r = 2, 3$

$$\begin{aligned}\frac{d}{dt}\mathscr{K}_r(t) \le C\big(K,\varepsilon^{-1}\big)\Big(\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\Big)\\ \cdot\Big(1 + \|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big)\\ + \|\nabla q\|_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\left(\sum_{s=0}^{r} E_s(t)\right),\end{aligned} \tag{3.33}$$

and for $r = 4$

$$\begin{aligned}\frac{d}{dt}\mathscr{K}_r(t) \le C\big(K,\varepsilon^{-1}\big)\Big(\Big(\|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)}\Big)\\ \cdot\Big(1 + \|\nabla q\|^{1/2}_{L^\infty(\partial\Omega)} + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)}\Big)\\ + \|\nabla q\|_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\left(1 + \sum_{s=0}^{r-1} E_s(t)\right)(1 + E_r(t)).\end{aligned} \tag{3.34}$$

Thus, we complete the Proof of Proposition 3.1. □

IV. PROOF OF THE MAIN THEOREM

Because Ω is the unit ball, we can get the following elliptic-type estimates from Ref. 19, Proposition 1 and Corollary 1.

Lemma 4.1. Let $F = F(t,x)$ be a smooth vector field defined in Ω, satisfying

$$\nabla\cdot F = 0,\ \text{in}\,\Omega, \quad \text{and}\, F\cdot N = 0,\ \text{on}\,\partial\Omega.$$

Then, for any $s \ge 3$

$$\|F(t)\|_{H^{1,\infty}(\Omega)} \le C\Big(\Big(1 + \log^+\|\nabla\times F(t)\|_{H^{s-1}(\Omega)}\Big)\|\nabla\times F(t)\|_{L^\infty(\Omega)} + 1\Big),$$

where $\log^+ f = \max(0, \log f)$.

From Proposition 3.1, we can obtain the following consequence.

Corollary 4.2. Let

$$|\theta| + \frac{1}{\varsigma_0} \le K, \quad |\nabla q| \ge \varepsilon > 0,$$

on ∂Ω, then we have

$$\begin{aligned}\frac{d}{dt}E_4 \le C\big(K,\varepsilon^{-1}\big)\Big(1 + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q\|^2_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\\ \cdot(1 + \log^+ E_4)(E_4 + 1)\left(1 + \sum_{s=0}^{3} E_s\right).\end{aligned}$$

Proof. Since $v = v_1 + v_2$, $v_1|_{\partial\Omega} = 0$ and $\beta\cdot N|_{\partial\Omega} = 0$, by Lemma 4.1, we get

$$\|\nabla v_1\|_{L^\infty(\Omega)} + \|\nabla\beta\|_{L^\infty(\Omega)} \le C\big(\big(1 + \log^+ E_4\big)\mathscr{A} + 1\big).$$

Since v_2 is harmonic, by the maximum principle, we have $\|\nabla v_2\|_{L^\infty(\Omega)} \le \mathscr{A}(t)$. Therefore, by Proposition 3.1 and the Hölder inequality, $E_4(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}E_4 \le C\big(K,\varepsilon^{-1}\big)\Big(1 + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q\|^2_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big)\\ \cdot\big(1 + \log^+ E_4\big)(E_4 + 1)\left(1 + \sum_{s=0}^{3} E_s\right),\end{aligned}$$

and similarly for $r = 1, 2, 3$

$$\frac{d}{dt}E_r \le C\big(K, \delta^{-1}\big)\Big(1 + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q\|^2_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big) \cdot (1 + \log^+ E_3)\sum_{s=0}^{r} E_s.$$

Finally, we can prove the main result. □

Proof of Theorem 1.1. By Corollary 4.2, we have that

$$\frac{d}{dt}E_4 \le C\big(K, \varepsilon^{-1}\big)\Big(1 + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \|\nabla p\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q\|^2_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big) \cdot (1 + \log^+ E_4)(E_4 + 1)\left(1 + \sum_{s=0}^{3} E_s\right),$$

where E_r is defined by (3.2). Since $\frac{d}{dt}(E_4 + 1) = \frac{d}{dt}E_4$, we set that $y_s = E_s + 1$ for $s = 0, \ldots, 4$, we have

$$\frac{d}{dt}y_4 \le C\big(K, \varepsilon^{-1}\big)\Big(1 + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q\|^2_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big) \cdot (1 + \log^+ y_4)y_4\sum_{s=0}^{3} y_s,$$

and by Corollary 4.2

$$\frac{d}{dt}\sum_{s=0}^{3} y_s \le C\big(K, \varepsilon^{-1}\big)\Big(1 + \mathscr{A} + \|\nabla_N D_t q\|_{L^\infty(\partial\Omega)} + \|\nabla q\|_{L^\infty(\partial\Omega)} + \|\nabla_N D_t q\|^2_{L^\infty(\partial\Omega)} + \mathscr{A}^2\Big) \cdot \left(1 + \log^+ \sum_{s=0}^{3} y_s\right)\sum_{s=0}^{3} y_s.$$

On the other hand, since T^* is the largest time for which (1.8) has a solution in the space (1.9), by Proposition 3.2, we have $\limsup_{t \nearrow T^*} y_4(t) = \infty$ which implies the desired results by Gronwall's inequality, Corollary 4.2 and the induction argument. □

ACKNOWLEDGMENTS

C. Hao, S. Yang, and W. Zhang were partially supported by NSF of China under Grant No. 12171460. C. Hao was also partially supported by CAS Project for Young Scientists in Basic Research under Grant No. YSBR-031 and National Key R & D Program of China under Grant No. 2021YFA1000800.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Jie Fu: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal). **Chengchun Hao**: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal). **Siqi Yang**: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal). **Wei Zhang**: Data curation (equal); Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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