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A priori estimates and a blow-up criterion for the incompressible ideal MHD equations with surface tension and a closed free surface

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Abstract

We establish the *a priori* estimates and prove a blow-up criterion for the three-dimensional free boundary incompressible ideal magnetohydrodynamics equations. The fluid occupies a bounded region with a free boundary that is a closed surface, without assumptions of simple connectedness or periodicity of the region (thus, Fourier transforms cannot be applied), nor the graph assumption for the free boundary. The fluid is under the influence of surface tension, and flattening the boundaries using local coordinates is insufficient to resolve this problem. This is because local coordinates fail to preserve curvature, as the mean curvature of a flat boundary degenerates to zero. To address these challenges and circumvent the intricate issue of spatial regularity in Lagrangian coordinates, we utilize reference surfaces to represent the free boundary and develop new energy functionals that both preserve the material derivative and incorporate spatial-temporal scaling $\partial_t \sim \nabla^{\frac{3}{2}}$ in Eulerian coordinates. This method enables us to establish both low-order and high-order regularity estimates without any loss of regularity. More importantly, we prove a blow-up criterion and provide a complete classification of blow-ups, including the self-intersection of the free boundary (which the graph assumption cannot handle), the breakdown of the mean curvature, and the blow-up of the normal velocity

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(which Lagrangian coordinates fail to capture). To the best of our knowledge, this is the first result addressing the *a priori* estimates and the blow-up criterion for free boundary problems with surface tension in general regions.

Keywords: free boundary problem, incompressible magnetohydrodynamics, blow-up, local regularity, surface tension

Mathematics Subject Classification numbers: 35Q35, 35R35, 35B44, 76B03, 76W05

1. Introduction

We consider the three-dimensional free boundary incompressible ideal magnetohydrodynamics (MHD) equations with surface tension in a bounded domain:

$$\begin{cases} \mathcal{D}_t v + \nabla p = H \cdot \nabla H, & \text{in } \Omega_t, \\ \mathcal{D}_t H = H \cdot \nabla v, & \text{in } \Omega_t, \\ \operatorname{div} v = 0, \quad \operatorname{div} H = 0, & \text{in } \Omega_t, \\ H \cdot \nu = 0, \quad p = \mathcal{A}_{\Gamma_t}, \quad v_n = V_{\Gamma_t}, & \text{on } \Gamma_t, \\ v(0,\cdot) = v_0, \quad H(0,\cdot) = H_0, & \text{in } \Omega_0, \end{cases} \tag{1.1}$$

where t represents the time, v the velocity, $\mathcal{D}_t := \partial_t + v \cdot \nabla$ the material derivative, H the magnetic field, and p the scalar total pressure. The moving domain $\Omega_t \subset \mathbb{R}^3$ is bounded with a closed surface $\Gamma_t := \partial\Omega_t$. ν denotes the unit outer normal, $\mathcal{A}_{\Gamma_t}$ the mean curvature, and V_{Γ_t} the normal velocity of Γ_t, which is equal to the normal component of the velocity $v_n := v \cdot \nu$. We specify the initial data v_0, H_0 and Ω_0, denoting $\Gamma_0 := \partial\Omega_0$. Additionally, the coefficient of surface tension is assumed to be 1 for simplicity.

In this paper, we establish *a priori* estimates and present a complete classification of the blow-up behaviour for system (1.1) in Sobolev spaces. To ensure the generality of our results, we impose no additional assumptions on the fluid region or the free boundary.

1.1. Energy functionals preserving the material derivative in Eulerian coordinates

Our analysis relies crucially on the new energy functionals constructed below in the Eulerian coordinates. For any integer $l \geqslant 1$, we define

$$\begin{aligned} e_l(t) :=& \frac{1}{2}\left(\|\mathcal{D}_t^{l+1} v\|_{L^2(\Omega_t)}^2 + \|\mathcal{D}_t^{l+1} H\|_{L^2(\Omega_t)}^2 + \|\bar{\nabla}\left(\mathcal{D}_t^l v \cdot \nu\right)\|_{L^2(\Gamma_t)}^2 \right) \\ &+ \frac{1}{2}\left(\|\nabla^{\lfloor \frac{3l+1}{2} \rfloor} \operatorname{curl} v\|_{L^2(\Omega_t)}^2 + \|\nabla^{\lfloor \frac{3l+1}{2} \rfloor} \operatorname{curl} H\|_{L^2(\Omega_t)}^2 \right), \end{aligned} \tag{1.2}$$

and we define the lower-order energy as $\bar{e}(t) = e_1(t) + e_2(t) + e_3(t)$, while the case $l \geqslant 4$ corresponds to the higher-order energy. In (1.2), $\lfloor \cdot \rfloor$ represents the integer part of a given number, $\bar{\nabla}$ denotes the tangential derivative, and $\operatorname{curl} F = \nabla F - (\nabla F)^\top$ applies to a vector field F. Additionally, we introduce the following energy functional:

$$\begin{aligned} E_l(t) :=& \sum_{k=0}^{l} \left(\|\mathcal{D}_t^{l+1-k} v\|_{H^{\frac{3}{2}k}(\Omega_t)}^2 + \|\mathcal{D}_t^{l+1-k} H\|_{H^{\frac{3}{2}k}(\Omega_t)}^2 \right) \\ &+ \|v\|_{H^{\lfloor \frac{3l+3}{2} \rfloor}(\Omega_t)}^2 + \|H\|_{H^{\lfloor \frac{3l+3}{2} \rfloor}(\Omega_t)}^2 + \|\bar{\nabla}\left(\mathcal{D}_t^l v \cdot \nu\right)\|_{L^2(\Gamma_t)}^2 + 1, \quad l \geqslant 1, \end{aligned} \tag{1.3}$$

where we take into account the spatial-temporal regularity. As before, the lower-order energy

$$\bar{E}(t) := \sum_{k=0}^{4} \left(\|\mathcal{D}_t^{4-k} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|\mathcal{D}_t^{4-k} H\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \right) + \sum_{k=1}^{3} \|\bar{\nabla}\left(\mathcal{D}_t^k v \cdot \nu\right)\|^2_{L^2(\Gamma_t)} + 1,$$

and we observe that $C_1(E_1+E_2+E_3) \leqslant \bar{E} \leqslant C_2(E_1+E_2+E_3)$ for some constants C_1, $C_2 > 0$.

The principle of reducing derivatives. The scaling $3/2$ in (1.3) is revealed in [37] that a second-order time derivative can be roughly equated to a third-order spatial differentiation, indicating the regularizing effect of the surface tension. From system (1.1), this scaling suggests that we can reduce '$1/2$-order' spatial regularity by substituting $\mathcal{D}_t v = -\nabla p + H \cdot \nabla H$ or $\mathcal{D}_t H = H \cdot \nabla v$. In this sense, we can also reduce '$1/2$-order' spatial regularity when the operators $\mathcal{D}_t$ and curl are combined (see lemma 2.5). These observations are crucial in deriving the optimal expressions for $\operatorname{div} \mathcal{D}_t^l v, \operatorname{curl} \mathcal{D}_t^l v$, the error terms, etc (see, e.g. lemmas 2.8 and 2.10) which allow us to control the higher-order energy (see lemma 6.3).

This principle will be consistently used throughout the paper.

1.2. Representation of the free boundary and the a priori assumptions

Let (v, H, p, Ω_t) be any solution to system (1.1) on $[0, T_0)$ for some $T_0 > 0$. We choose a smooth, compact reference surface Γ to represent the free boundary. Here, $\Gamma = \partial\Omega$, where Ω is a smooth, compact domain satisfying the uniform interior and exterior ball condition with radius $\mathcal{R} = \mathcal{R}(\Omega) > 0$.

The free boundary is represented as:

$$\Gamma_t = \{x + h(x,t)\nu_\Gamma(x) : x \in \Gamma\}, \quad t \in [0, T),$$

where the time $T \leqslant T_0$ and the height function $h : \Gamma \times [0, T) \to \mathbb{R}$ are characterized as follows:

$$\mathcal{M}_T := \mathcal{R} - \sup_{0 \leqslant t < T} \|h(\cdot, t)\|_{L^\infty(\Gamma)} > 0. \tag{1.4}$$

In other words, $h(\cdot, t)$ is well-defined in $[0, T)$ as long as $\mathcal{M}_T > 0$. The maximal representation interval $[0, T_r)$ for the reference surface Γ is defined as $T_r = \sup\{T \leqslant T_0 : (1.4) \text{ holds}\}$. It should be noted that one of the following three scenarios will occur as time approaches T_r.

(1) The free boundary Γ_t first self-intersects at time T_r ($T_r < T_0$ or $T_r = T_0$), resulting in a splash or splat singularity (see, e.g. [6]). That is, $\mathcal{R}(\Omega_t) > 0$ for $0 \leqslant t < T_r$ and $\mathcal{R}(\Omega_{T_r}) = 0$.
(2) $T_r = T_0$ and Γ_t does not self-intersect on $[0, T_0)$. In this scenario, we complete the representation of the free boundary throughout the existence of the solution.
(3) $T_r < T_0$ and Γ_t does not self-intersect on $[0, T_r)$. In this case, our reference surface is insufficient to represent the free boundary at time T_r, necessitating a switch to a new reference surface to continue the representation.

Having defined $\mathcal{M}_T$ to ensure the well-definedness of the height function, we introduce the following quantity to ensure the extension of the solution

$$\mathcal{N}_T := \sup_{0 \leqslant t < T} \left(\|h(\cdot, t)\|_{H^{3+\delta}(\Gamma)} + \|\nabla v\|_{H^3(\Omega_t)} + \|\nabla H\|_{H^3(\Omega_t)} + \|v_n\|_{H^4(\Gamma_t)} \right), \tag{1.5}$$

where $\delta > 0$ is a sufficiently small constant and $T \leqslant T_0$.

We mention that the requirements for the height function and the normal velocity are natural, as we do not fix the boundary using Lagrangian coordinates. These two parts precisely control the spatial and temporal regularity of the free boundary:

(1) $\|h\|_{H^{3+\delta}(\Gamma)}$ controls the tangential derivative of the height function. It also ensures that the second fundamental form B_{Γ_t} is uniformly bounded, i.e. $\|B_{\Gamma_t}\|_{L^\infty(\Gamma_t)} \leqslant C$.
(2) Note that $\partial_t h = v_n$, and therefore $\|v_n\|_{H^4(\Gamma_t)}$ controls the time derivative of the height function.

Moreover, $\|v\|_{L^2(\Omega_t)}$ and $\|H\|_{L^2(\Omega_t)}$ are not included, due to the energy conservation of system (1.1).

1.3. Main results

We make the following assumptions on the initial data throughout the paper. Let $v_0, H_0 \in H^6(\Omega_0;\mathbb{R}^3)$ be the initial divergence-free velocity and magnetic fields, satisfying $H_0 \cdot \nu_{\Gamma_0} = 0$ on Γ_0, where Ω_0 is the initial bounded domain, and the initial boundary $\Gamma_0 \in H^7$ is a non-self-intersecting closed surface. As discussed in section 1.2, we can choose a suitable reference surface $\Gamma = \partial\Omega$ with $\mathcal{R} = \mathcal{R}(\Omega) > 0$, and represent the free boundary. In particular, $\Gamma_0 = \{x + h_0(x)\nu_\Gamma(x) : x \in \Gamma\}$, where $\|h_0\|_{L^\infty(\Gamma)} < \mathcal{R}$.

Our main results are stated as follows.

Theorem 1.1. *Let (v, H, Ω_t) be any solution to system (1.1) on $[0,T)$ for some $T > 0$ with initial data (v_0, H_0, Ω_0), and satisfies the following the a priori assumptions:*

$$\mathcal{N}_T < \infty, \text{ and } \mathcal{M}_T > 0. \tag{1.6}$$

Then, we have the following results:

(1) *Lower-order quantitative regularity estimates:*

$$\sup_{0\leqslant t<T}\left(\bar{E}(t) + \sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|^2_{H^{\frac{3}{2}k+1}(\Omega_t)} + \|B_{\Gamma_t}\|^2_{H^5(\Gamma_t)}\right) \leqslant \bar{C}, \tag{1.7}$$

where $\bar{C}$ is a constant that depends only on $T, \mathcal{N}_T, \mathcal{M}_T, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$ and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$. Specifically, the following holds:

$$\sup_{0\leqslant t<T}\left[\sum_{k=0}^{4}\left(\|\partial_t^{4-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|\partial_t^{4-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_t)}\right) + \sum_{k=0}^{3}\|\partial_t^{3-k}p\|^2_{H^{\frac{3}{2}k+1}(\Omega_t)}\right] \leqslant \bar{C}, \tag{1.8}$$

where the constant $\bar{C}$ depends on the same quantities as in (1.7).

(2) *Higher-order regularity estimates for $l \geqslant 4$:*

$$\sup_{0\leqslant t<T} E_l(t) \leqslant C_l, \tag{1.9}$$

where C_l is a constant that depends on $l, T, \mathcal{N}_T, \mathcal{M}_T$, and $E_l(0)$. In particular, we have

$$\sup_{0\leqslant t<T}\left[\sum_{k=0}^{l}\left(\|\partial_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)}+\|\partial_t^{l+1-k}H\|^2_{H^{\frac{3}{2}k}(\Omega_t)}+\|\partial_t^{l-k}p\|^2_{H^{\frac{3}{2}k+1}(\Omega_t)}\right)\right.$$
$$\left.+\|v\|^2_{H^{\lfloor\frac{3(l+1)}{2}\rfloor}(\Omega_t)}+\|H\|^2_{H^{\lfloor\frac{3(l+1)}{2}\rfloor}(\Omega_t)}+\|B_{\Gamma_t}\|^2_{H^{\frac{3l+1}{2}}(\Gamma_t)}\right]\leqslant C_l, \tag{1.10}$$

where the constant C_1 depends on the same quantities as in (1.9).

(3) *There exists a time $T_0>0$ depending only on the initial quantities $\mathcal{M}_0, \|v_0\|_{H^6(\Omega_0)}$, $\|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$, such that the a priori assumptions (1.6) hold for $T=T_0$.*

Notably, if we consider a smooth solution on $[0,T)$, it will not develop singularities at time T and remains smooth with respect to both time and space, as long as the a priori assumptions (1.6) hold.

Next, we present the classification of blow-up for system (1.1), which fully captures the scenario of boundary self-intersection.

Theorem 1.2. *For any solution (v,H,Ω_t) to system (1.1) with initial data (v_0,H_0,Ω_0), define the maximal time interval of existence $[0,T_*)$, where T_* is the maximal time such that*

$$v,H\in C_t^0H^6(\Omega_t) \text{ and } \Gamma_t\in C_t^0H^7.$$

If the maximal time $T_*<\infty$, then one of the following scenarios must occur:

(1) *The free boundary Γ_t self-intersects for the first time at time T_*.*
(2) *Either the mean curvature does not belong to the $H^{1+\delta}$-class, or the free boundary Γ_t does not belong to the $H^{2+\varepsilon}$-class at time T_*, for some sufficiently small positive constants δ and ε.*
(3) *The normal velocity of the free boundary V_{Γ_t} does not belong to the H^4-class at time T_*.*
(4) *The breakdown of lower-order quantities on Ω_t, i.e.*

$$\sup_{0\leqslant t<T_*}\left(\|\nabla v\|_{H^3(\Omega_t)}+\|\nabla H\|_{H^3(\Omega_t)}\right)=\infty.$$

Remark 1.3. We assume that the initial data $v_0,H_0\in H^6$ is due to the consideration of a general bounded domain with a closed free surface. For a periodic flat initial region (e.g. $\mathbb{T}^2\times(0,1)$), we expect that the similar results of theorems 1.1 and 1.2 hold for initial data in $H^{\frac{9}{2}}$, as we can define the fractional derivative using the Fourier transform in this case.

1.4. History and background

In recent decades, there has been significant interest in studying the free boundary incompressible Euler equations, and substantial advancements have been made. Extensive research has been conducted for the irrotational case, especially the water wave equations. We refer readers to [12, 25, 28, 45] and the references therein. If the fluid flow exhibits vorticity, one may refer to [4, 5, 8, 9, 13, 30, 34, 36, 37, 43, 46] for results on the *a priori* estimates, the local well-posedness with or without surface tension, the zero surface tension limit, and more.

The investigation of free boundary problems for MHD equations has emerged relatively recently compared to the study of the Euler equations, mainly because of the strong interactions between the magnetic and velocity fields. We focus on the incompressible MHD equations. Hao and Luo [18] obtained *a priori* estimates for free boundary problems of the incompressible ideal MHD without surface tension under the Taylor-type sign condition. They considered the

case where the initial domain is homeomorphic to a ball. They also showed the ill-posedness of the problem if the Taylor-type sign condition is violated in the two-dimensional case [19]. Luo and Zhang [32] derived *a priori* estimates for the lower regular initial data in the initial domain of sufficiently small volume. In [15], a local existence result was provided, with a detailed proof in an initial flat domain $\mathbb{T}^2 \times (0,1)$. The local well-posedness for the incompressible ideal MHD equations with surface tension is established by Gu, Luo, and Zhang in [14], in the same initial domain setting, namely, $\mathbb{T}^2 \times (0,1)$. The nonlinear stability of the current-vortex sheet in the incompressible MHD equations was solved by Sun *et al* [39] under the Syrovatskij stability condition, assuming that the free boundaries are graphs in $\mathbb{T}^2 \times (-1,1)$. Wang and Xin [44] established the global well-posedness of a free interface problem for the incompressible inviscid resistive MHD under similar assumptions regarding the graph. We also refer to some related works [10, 17, 20, 29, 40–42] on the topics of the well-posedness, the current-vortex sheets problem, the breakdown criterion, the viscous splash singularity, and the compressible MHD.

It should be noted that the aforementioned well-posedness results for the incompressible MHD equations are primarily derived by applying the Lagrangian coordinates, which transform a moving domain into a fixed one. However, as indicated in [37, 38], the Lagrangian map lacks maximal regularity because all the variables are defined on an evolving domain. In fact, the moving surface can also be described using alternative methods, such as the study of the Euler equations with surface tension [36], the fluid interface problem [31, 38], the surface diffusion flow with elasticity [11], and the motion of charged liquid drop [26], among others.

Moreover, previous results on the incompressible MHD equations with surface tension predominantly apply to the flat periodic initial region $\mathbb{T}^2 \times (a,b)$ and rely on the graph assumption for the free boundary. However, the periodic assumptions and the graph assumptions have inherent limitations. In fact, it may be possible to reduce the problem of a general free boundary to the case of a graph by selecting local coordinates. However, this reduction is technically complicated and involves significant challenges. In the presence of surface tension, if we only select a portion of the free boundary and flatten it near a point, there is a risk of losing certain geometric characteristics of the free boundary, such as the evolution of its curvature. For the fluid in the flat domain $\mathbb{T}^2 \times (a,b)$, its initial mean curvature is evidently zero, as local coordinates fail to preserve the curvature. These facts highlight the necessity of making additional assumptions on the initial velocity on the boundary. For instance, in [33], the assumption $v_0 \in H^{3.5}(\mathbb{T}^2 \times (0,1)) \cap H^4(\mathbb{T}^2 \times \{1\})$ is made to obtain the *a priori* estimates; in [14], $v_0 \in H^{4.5}(\mathbb{T}^2 \times (0,1)) \cap H^5(\mathbb{T}^2 \times \{1\})$ is made to establish local existence. To the best of our knowledge, the local well-posedness for system (1.1) with surface tension remains open when Ω_t is a general bounded domain with a closed free surface.

In this paper, by constructing new energy functionals with spatial-temporal scaling $\partial_t \sim \nabla^{\frac{3}{2}}$ in Eulerian coordinates, we establish the *a priori* estimates on the general domain without any loss of regularity. We also eliminate the additional regularity requirement for the velocity on the initial boundary [14, 33] and our results highlight the effectiveness of employing the height function on the reference surface to analyse the evolution of curvature.

It is also natural and fundamentally important to consider the breakdown criterion of solutions to system (1.1), for which we are unaware of any relevant rigorous studies, although a few studies are available if we neglect the surface tension. Fu, along with both authors and Zhang, established a Beale-Kato-Majda continuation criterion for solutions to the free boundary incompressible ideal MHD equations without surface tension [10]. When the viscosity is taken into account, the authors proved the existence of finite-time splash singularities [20], while Hong *et al* also demonstrated the existence of such singularities [21]. Recently, Ifrim *et al*

established a low-regularity blow-up criterion for the incompressible ideal MHD equations without surface tension [23], inspired by the previous works [22, 24].

Based on the *a priori* estimates, we provide a complete classification of blow-up behaviour for solutions to system (1.1). In contrast to the graph assumption, which cannot capture non-graphical free boundaries, our method allows the analysis of free boundaries approaching self-intersection. Moreover, our energy functionals are defined in Eulerian coordinates, and the *a priori* assumptions—apart from the height function used to characterize the regularity of the boundary—are independent of the choice of coordinates. Therefore, our method remains unaffected by different coordinate choices as the free boundary approaches self-intersection.

1.5. Novelties and structure of the paper

The novelties of this study are as follows.

To the best of our knowledge, theorem 1.1 is the first result focusing on the regularity estimates of system (1.1) in a general bounded domain with a closed free surface, i.e. without imposing any periodicity or simple connectedness assumptions on the fluid region, or any graph assumptions on the free boundary.

a) Our *a priori* estimates are derived from an energy inequality of the following form, based on the *a priori* assumptions, without requiring smallness in time. That is,

$$\dot{\bar{E}} \lesssim_{\mathcal{N}_T,\mathcal{M}_T} C(\text{initial data})\,\bar{E}, \quad \dot{E}_l \lesssim_{\mathcal{N}_T,\mathcal{M}_T,\text{induction}} CE_l, \quad l \geqslant 4.$$

This is crucial for establishing a breakdown criterion [10, 22–24, 26, 34, 43]. If additional smallness in time were required, we could not establish a blow-up criterion, let alone a complete classification of blow-up behaviour. The common *a priori* estimates yield a polynomial of the energy, multiplied by time, such as $\sup_{[0,T]}\bar{E}(t) \leqslant C(\bar{E}(0)) + T^{\frac{1}{2}}\mathcal{P}(\sup_{[0,T]}\bar{E}(t))$. However, this inequality necessitates a sufficiently small time T to complete the energy estimates, making the breakdown criterion unattainable.

b) Our lower-order regularity results (1.7) extend the *a priori* estimates with an initial flat domain $\mathbb{T}^2\times(0,1)$ from [33] to a general domain without any loss of regularity. Moreover, we eliminate the additional regularity requirement for the velocity on the initial boundary (which was assumed in [33] as $v_0 \in H^{3.5}(\mathbb{T}^2\times(0,1))\cap H^4(\mathbb{T}^2\times\{1\})$) since our final estimate does not depend on this initial quantity. We also establish higher-order energy estimates without any loss of regularity.

c) We establish a distinct energy functional that preserves the material derivative $\mathcal{D}_t$ with a different spatial-temporal scaling ($\partial_t \sim \nabla^{\frac{3}{2}}$) in Eulerian coordinates, in contrast to the energy functional defined in the flat periodic domain using Lagrangian coordinates [14, 32, 33]. This strategy avoids destroying the structure of system (1.1) when separating ∂_t from $\mathcal{D}_t$, and the energy estimates are driven by the second fundamental form and pressure. We also eliminate the additional regularity requirement for the velocity on the initial boundary as in [14], i.e. the assumption $v_0 \in H^{4.5}(\mathbb{T}^2\times(0,1))\cap H^5(\mathbb{T}^2\times\{1\})$.

Theorem 1.2 provides the first comprehensive classification of blow-ups for solutions of (1.1).

a) In our classification, the first three types of singularities arise from the free boundary and are mutually distinct. These singularities can be effectively characterized using the height function: (1), (2), and (3) in theorem 1.2 correspond to the inability to choose a reference surface to define the height function, the blow-up of the tangential derivative of the

height function, and the blow-up of the time derivative of the height function, respectively. Therefore, each of these three types of singularities is indispensable.

b) The case where only the singularity in theorem 1.2 (1) arises, while the others in (2)–(4) do not occur, does exist. The singularity of boundary self-intersection, where the solution and free boundary remain smooth, exists in the presence of viscosity [20, 21]. For the free boundary incompressible ideal MHD equations with surface tension, it is conjectured in [6] that this singularity also exists.

c) If we consider the fixed boundary problem, our blow-up classification reduces to (4) in theorem 1.2, analogous to the remarkable Beale-Kato-Majda criterion for the Euler equations [1].

Our results hold without assuming that the free boundary is a graph. Analysing the evolution of a small region by selecting a portion of the closed surface and applying local coordinate flattening is insufficient to solve the problem. Moreover, the strategy for selecting the reference surface provides the following advantages compared to the graph assumption.

a) When the free boundary is represented by a graph function over the initial boundary $\mathbb{T}^2 \sim \mathbb{T}^2 \times \{1\}$, it corresponds to a specific height function. Choosing $\mathbb{T}^2 \times \{1\}$ as the reference surface with $(0,0,1)$ as the unit outer normal, the height function coincides with the graph function.

b) The height function enables direct computation of curvature evolution via tangential derivatives, whereas flattening the surface with local coordinates fails to preserve the intrinsic geometric properties of the moving surface.

c) We can continually select appropriate reference surfaces to represent the free boundary, particularly facilitating the characterization of the process by which the boundary develops self-intersection. However, the graph function fails when the moving surface boundary undergoes turning (see, e.g. the breakdown criterion for the free boundary Euler equations with surface tension in [34] and without surface tension in [43]).

We use reference surfaces to represent the free boundary, which offers advantages over fixing the boundary in Lagrangian coordinates for the following reasons.

a) It is more convenient to control the mean curvature and boundary regularity using the height function, as the regularity improvement of the free boundary is geometric [37], directly connected to the regularity of the mean curvature (see lemma A.2), and not entirely evident in the Lagrangian coordinates.

b) We avoid addressing the issue of spatial regularity of the flow map in Lagrangian coordinates.

c) A more precise estimation of the pressure can be obtained by analysing the normal velocity of the free boundary. In contrast, in Lagrangian coordinates, the normal velocity of the free boundary is implicit because the boundary is fixed.

The rest of this paper is organized as follows. In section 2, we calculate the commutators, the error terms, and additional terms to establish the energy estimates. In section 3, we compute the time derivative of the energy functional. In section 4, we will show that $\|p\|_{H^3(\Omega_t)}$ can be uniformly bounded within the time interval of existence. In section 5, we estimate the error terms that appeared in section 3. In section 6, we close the energy estimates and prove our main theorems. Finally, in section 7, we discuss the connection between the self-intersection and the curvature blow-up on the free boundary established in theorem 1.2.

2. Formulas for the energy estimates

Throughout the paper, we will use the Einstein summation convention and the notation $S\star T$ from [16] to denote a tensor formed by contracting certain indices of tensors S and T with constant coefficients. In particular, for $k,l\in\mathbb{N}=\{1,2,3,\cdots\}$ (we denote $\mathbb{N}_0=\{0,1,2,3,\cdots\}$), $\nabla^k f\star\nabla^l g$ represents a contraction of certain indices of tensors $\nabla^i f$ and $\nabla^j g$ for $0\leqslant i\leqslant k$ and $0\leqslant j\leqslant l$ with constant coefficients. Note that f and g can be vector fields, and we include the lower-order derivatives along with the function (or vector field) itself. However, we exclude the case of a single term $\nabla^i f$. Let $u:\Gamma\to\mathbb{R}$ and $F:\Gamma\to\mathbb{R}^3$ be a sufficiently regular function and vector field, respectively. Since the reference hypersurface Γ (embedded in $\mathbb{R}^3$) has a natural metric g induced by the Euclidean metric, (Γ,g) is a Riemannian manifold with connection $\hat{\nabla}$. For a function $u\in C^\infty(\Gamma)$ and a vector field F, $\hat{\nabla}_F u=Fu$.

We denote the normal part of F by $F_n:=F\cdot\nu_\Gamma$, and the tangential part by $F_\sigma:=F-F_n\nu_\Gamma$, where '$\cdot$' denotes the inner product. If Γ is smooth, we can extend both u and F to $\mathbb{R}^3$ and define the tangential differential by $\bar{\nabla}u:=(\nabla u)_\sigma$, the tangential gradient of F by $\bar{\nabla}F:=\nabla F-(\nabla F\nu)\otimes\nu$, i.e. $(\bar{\nabla}F)_{ij}=\partial_j F^i-\partial_l F^i\nu^l\nu_j$, and the tangential divergence by $\mathrm{div}_\sigma F:=\mathrm{Tr}(\bar{\nabla}F)$. The tangential gradient and covariant gradient are equivalent: for any vector field $\tilde{F}:\Gamma\to\mathbb{R}^3,\tilde{F}\cdot\nu=0$, we have $\hat{\nabla}_{\tilde{F}}u=\bar{\nabla}u\cdot\tilde{F}$. Additionally, the second fundamental form B and the mean curvature $\mathcal{A}$ can be written as $B=\bar{\nabla}\nu$ and $\mathcal{A}=\mathrm{div}_\sigma\nu$. The Beltrami-Laplacian is defined by $\Delta_B u:=\mathrm{div}_\sigma(\bar{\nabla}u)$, and it holds

$$\Delta_B u=\Delta u-\left(\nabla^2 u\nu\cdot\nu\right)-\mathcal{A}\partial_\nu u, \tag{2.1}$$

where ∂_ν denotes the outer normal derivative. We also recall the divergence theorem $\int_\Gamma \mathrm{div}_\sigma F\mathrm{d}S=\int_\Gamma\mathcal{A}_\Gamma(F\cdot\nu_\Gamma)\mathrm{d}S$, and the differentiation formula (see, e.g. [37])

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t} f\mathrm{d}S=\int_{\Gamma_t}\mathcal{D}_t f+f\mathrm{div}_\sigma v\mathrm{d}S. \tag{2.2}$$

We will fix our reference surface Γ, a boundary of a smooth, compact set Ω satisfying the uniform interior and exterior ball condition with radius $\mathcal{R}>0$. We denote its tubular neighbourhood $U(\mathcal{R},\Gamma)=\{x\in\mathbb{R}^3:\mathrm{dist}(x,\Gamma)<\mathcal{R}\}$. We say that $\Gamma_t=\partial\Omega_t$ (or Ω_t) is $H^s(\Gamma)$-regular, if $\Gamma_t=\{x+h(x,t)\nu_\Gamma(x):x\in\Gamma\}$, where $h(\cdot,t):\Gamma\to\mathbb{R}$ is $H^s(\Gamma)$-regular and $\|h(\cdot,t)\|_{L^\infty(\Gamma)}<\mathcal{R}$. Γ_t is called uniformly $H^s(\Gamma)$-regular if $\|h\|_{H^s(\Gamma)}\leqslant C$ and $\|h\|_{L^\infty(\Gamma)}\leqslant c\mathcal{R}$ for constants C and $c<1$ (see [26] for similar definitions). We can express the unit outer normal and the second fundamental form by the height function (see [35])

$$\nu_{\Gamma_t}=a_1\left(h(\cdot,t),\bar{\nabla}h(\cdot,t)\right),\quad B_{\Gamma_t}=a_2\left(h(\cdot,t),\bar{\nabla}h(\cdot,t)\right)\bar{\nabla}^2h(\cdot,t), \tag{2.3}$$

where $a_1,a_2\in C^\infty$. We extend ν to Ω via harmonic extension and denote it as $\tilde{\nu}$. We sometimes still denote the extended one by ν. From (1.6) and (2.3), $\|\tilde{\nu}\|_{H^{5/2+\delta}(\Omega_t)}\leqslant C$ for $\delta>0$ small.

From the definition $\mathrm{curl}\,F=\nabla F-(\nabla F)^\top$, a straightforward calculation yields:

Lemma 2.1. *Let $l,k\in\mathbb{N},F$, and G be smooth vector fields and f be a smooth function. Then, we have:*

(1) $\mathrm{curl}(F\cdot\nabla G)=\nabla G\nabla F-\nabla F^\top\nabla G^\top+(F\cdot\nabla)\,\mathrm{curl}\,G$ *and* $[\mathcal{D}_t,\mathrm{curl}]F=\nabla v^\top\nabla F^\top-\nabla F\nabla v$.
(2) $[\mathcal{D}_t^{l+1},\nabla^k]f=\mathcal{D}_t[\mathcal{D}_t^l,\nabla^k]f+[\mathcal{D}_t,\nabla^k]\mathcal{D}_t^l f$ *and* $[\mathcal{D}_t^l,\nabla^{k+1}]f=[\mathcal{D}_t^l,\nabla]\nabla^k f+\nabla[\mathcal{D}_t^l,\nabla^k]f$.

To derive a general formula for the commutators, we apply the following results. It is easy to verify that $\mathcal{D}_t a(\nu)=b(\nu)\bar{\nabla}v,\mathcal{D}_t\nabla\mathcal{D}_t^k v=\nabla\mathcal{D}_t^{k+1}v+\nabla v\star\nabla\mathcal{D}_t^k v,\mathcal{D}_t\bar{\nabla}\mathcal{D}_t^k v=\bar{\nabla}\mathcal{D}_t^{k+1}v+\bar{\nabla}v\star\bar{\nabla}\mathcal{D}_t^k v$ for $k\in\mathbb{N}$, where $a(\nu)$ and $b(\nu)$ denote the finite $\star$ product of ν.

Lemma 2.2. *Let $l,k\in\mathbb{N}, l\geqslant 2$ and $k\geqslant 3$. Then, we have:*

(1) $[\mathcal{D}_t,\nabla^2]f=\nabla v\star\nabla^2 f+\nabla^2 v\star\nabla f$.
(2) $[\mathcal{D}_t,\nabla^k]f=\sum_{|\alpha|\leqslant k-1}\nabla^{1+\alpha_1}v\star\nabla^{1+\alpha_2}f$.
(3) $[\mathcal{D}_t^l,\nabla]f=\sum_{2\leqslant m\leqslant l+1}\sum_{|\beta|\leqslant l+1-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}f$.
(4) $[\mathcal{D}_t^l,\nabla^2]f=\sum_{2\leqslant m\leqslant l+1}\sum_{|\alpha|\leqslant 1,|\beta|\leqslant l+1-m}\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_{m-1}}\mathcal{D}_t^{\beta_{m-1}}v\star\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}f$.

Roughly speaking, the leading term is $\nabla^k\mathcal{D}_t^{l-1}$ in the commutator $[\mathcal{D}_t^l,\nabla^k]$.

Proof. A direct calculation yields the first claim and the second claim can be found in [26, lemma 4.1]. We prove the third one by induction, and it is easy to verify the case of $l=2$. For the case of $l\geqslant 3$, from lemma 2.1 and the above formulas, it follows that

$$\begin{aligned}\left[\mathcal{D}_t^l,\nabla\right]f&=\mathcal{D}_t\left[\mathcal{D}_t^{l-1},\nabla\right]f+\nabla v\star\nabla\mathcal{D}_t^{l-1}f\\&=\mathcal{D}_t\left(\sum_{2\leqslant m\leqslant l}\sum_{|\beta|\leqslant l-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}f\right)+\nabla v\star\nabla\mathcal{D}_t^{l-1}f\\&=\sum_{2\leqslant m\leqslant l+1}\sum_{|\beta|\leqslant l+1-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}f.\end{aligned}$$

The last claim follows again by induction and we omit the proof. □

Let $a_\beta(\nu)$ and $a_{\alpha,\beta}(\nu,B)$ denote the finite $\star$ product of the tensors. We provide a more precise formulation of the quantities than those in [26, lemma 4.2].

Lemma 2.3. *Let $l\geqslant 1$ and we have the following results:*

(1) $[\mathcal{D}_t^l,\bar\nabla]f=\sum_{2\leqslant m\leqslant l+1}\sum_{|\beta|\leqslant l+1-m}\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla\mathcal{D}_t^{\beta_m}f$.
(2) $\mathcal{D}_t^l\nu=\sum_{1\leqslant m\leqslant l}\sum_{|\beta|\leqslant l-m}a_\beta(\nu)\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_m}v$.
(3) $\mathcal{D}_t^l B=\sum_{1\leqslant m\leqslant l}\sum_{|\beta|\leqslant l-m,|\alpha|\leqslant 1}a_{\alpha,\beta}(\nu,B)\bar\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v$.
(4) $[\mathcal{D}_t^l,\bar\nabla^2]f=\sum_{2\leqslant m\leqslant l+1}\sum_{|\beta|\leqslant l+1-m,|\alpha|\leqslant 1}a_{\alpha,\beta}(\nu,B)\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_{m-1}}\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}f$.

Proof. To prove the first claim, we recall $[\mathcal{D}_t,\bar\nabla]f=-(\bar\nabla v)^\top\bar\nabla f$ in lemma A.1. For the case of $l\geqslant 2$, we have by induction that

$$\begin{aligned}\left[\mathcal{D}_t^l,\bar\nabla\right]f&=\mathcal{D}_t\left[\mathcal{D}_t^{l-1},\bar\nabla\right]f+\left[\mathcal{D}_t,\bar\nabla\right]\mathcal{D}_t^{l-1}f\\&=\mathcal{D}_t\left(\sum_{2\leqslant m\leqslant l}\sum_{|\beta|\leqslant l-m}\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla\mathcal{D}_t^{\beta_m}f\right)+\bar\nabla v\star\bar\nabla\mathcal{D}_t^{l-1}f\\&=\sum_{2\leqslant m\leqslant l+1}\sum_{|\beta|\leqslant l+1-m}\bar\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\bar\nabla\mathcal{D}_t^{\beta_m}f.\end{aligned}$$

Similarly, we can obtain the last claim. For the second claim, we recall $\mathcal{D}_t\nu=\bar\nabla v\star\nu$, and for $l\geqslant 2$, it holds by induction. As for the third claim, we have for $l\geqslant 1$ that

$$\begin{aligned}\mathcal{D}_t^l B =& \left[\mathcal{D}_t^l, \bar{\nabla}\right]\nu + \bar{\nabla}\mathcal{D}_t^l\nu\\ =& \bar{\nabla}\left(\sum_{1\leqslant m\leqslant l}\sum_{|\beta|\leqslant l-m} a_\beta(\nu)\,\bar{\nabla}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar{\nabla}\mathcal{D}_t^{\beta_m}v\right)\\ &+\sum_{2\leqslant m\leqslant l+1}\sum_{|\beta|\leqslant l+1-m}\bar{\nabla}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar{\nabla}\mathcal{D}_t^{\beta_{m-1}}v\star\bar{\nabla}\mathcal{D}_t^{\beta_m}\nu =: I_1+I_2.\end{aligned}$$

It is clear that $I_1=\sum_{1\leqslant m\leqslant l}\sum_{|\beta|\leqslant l-m,|\alpha|\leqslant 1}a_{\alpha,\beta}(\nu,B)\bar{\nabla}^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar{\nabla}^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v$. For I_2, it follows that

$$\begin{aligned}I_2=&\sum_{2\leqslant m\leqslant l+1}\sum_{|\beta|\leqslant l+1-m}\bar{\nabla}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar{\nabla}\mathcal{D}_t^{\beta_{m-1}}v\\ &\star\left(\sum_{1\leqslant n\leqslant\beta_m}\sum_{|\lambda|\leqslant\beta_m-n,|\gamma|\leqslant 1}\bar{\nabla}^{1+\gamma_1}\mathcal{D}_t^{\lambda_1}v\star\cdots\star\bar{\nabla}^{1+\gamma_n}\mathcal{D}_t^{\lambda_n}v\right)\\ =&\sum_{2\leqslant m\leqslant l+1,|\beta|\leqslant l+1-m}\sum_{1\leqslant n\leqslant\beta_m}\sum_{|\lambda|\leqslant\beta_m-n,|\gamma|\leqslant 1}a_{\beta,\lambda,\gamma}(\nu,B)\,\bar{\nabla}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar{\nabla}\mathcal{D}_t^{\beta_{m-1}}v\\ &\star\bar{\nabla}^{1+\gamma_1}\mathcal{D}_t^{\lambda_1}v\star\cdots\star\bar{\nabla}^{1+\gamma_n}\mathcal{D}_t^{\lambda_n}v,\end{aligned}$$

which is also contained in $\sum_{1\leqslant m\leqslant l}\sum_{|\alpha|\leqslant 1,|\beta|\leqslant l-m}a_{\alpha,\beta}(\nu,B)\bar{\nabla}^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar{\nabla}^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v$. □

We denote the divergence of a matrix $A=(A_{ij})$ as $(\operatorname{div}A)_i:=\sum_j\partial_j A_{ij}$ and recall $\operatorname{curl}F=\nabla F-(\nabla F)^\top$. For later use, we recall [26, lemma 3.3]:

Lemma 2.4. *Let Ω be a bounded domain with $C^{1,\alpha}$ boundary. For any smooth vector field F, we have $\|F\|^2_{L^2(\Gamma)}\leqslant C(\|F_\tau\|^2_{L^2(\Gamma)}+\|F\|^2_{L^2(\Omega)}+\|\operatorname{div}F\|^2_{L^2(\Omega)}+\|\operatorname{curl}F\|^2_{L^2(\Omega)})$, where $\tau=n,\sigma$.*

To estimate energy, we begin with the following basic results. By the divergence-free condition, it is clear that $\operatorname{div}\mathcal{D}_t v=\partial_i v^j\partial_j v^i$ and we have

$$-\Delta p=\partial_i v^j\partial_j v^i-\partial_i H^j\partial_j H^i. \tag{2.4}$$

A direct calculation produces the following identities.

Lemma 2.5. *For the velocity and magnetic fields, we have*

(1) $\operatorname{curl}\mathcal{D}_t v=(\nabla H)^\top\operatorname{curl}H+\operatorname{curl}H\nabla H+(H\cdot\nabla)(\operatorname{curl}H),[\mathcal{D}_t,\operatorname{curl}]v=-(\nabla v)^\top\operatorname{curl}v-\operatorname{curl}v\nabla v.$
(2) $\operatorname{curl}\mathcal{D}_t H=\nabla v\nabla H-(\nabla H)^\top(\nabla v)^\top+(H\cdot\nabla)(\operatorname{curl}v),[\mathcal{D}_t,\operatorname{curl}]H=(\nabla v)^\top(\nabla H)^\top-\nabla H\nabla v.$

Next, we introduce some errors associated with the magnetic field. Denote $R^0_{\nabla H,H}:=0, R^0_{\nabla H,\nabla H}:=\nabla H\star\nabla H$, and for $k\geqslant 1$, we define

$$\begin{aligned}R^k_{\nabla H,H}:=&\sum_{3\leqslant m\leqslant k+2}\sum_{|\alpha|\leqslant 1,|\beta|\leqslant k+2-m}a_{\alpha,\beta}(\nabla v)\,\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_{m-2}}\mathcal{D}_t^{\beta_{m-2}}v\star\nabla^{\alpha_{m-1}}H\star H,\\ R^k_{\nabla H,\nabla H}:=&\sum_{3\leqslant m\leqslant k+2}\sum_{|\alpha|\leqslant 2,\alpha_i\leqslant 1,|\beta|\leqslant k+2-m}\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_{m-2}}\mathcal{D}_t^{\beta_{m-2}}v\star\nabla^{\alpha_{m-1}}H\star\nabla^{\alpha_m}H,\end{aligned}$$

where $a_{\alpha,\beta}(\nabla v)$ denotes the finite $\star$ product. In the case of $\beta_j = 0, \nabla\mathcal{D}_t^{\beta_j}$ can be absorbed into $a_{\alpha,\beta}(\nabla v)$. A direct calculation shows $\mathcal{D}_t(\nabla H\star\nabla H) = \nabla^2 v\star H\star\nabla H + \nabla v\star\nabla H\star\nabla H$ and $\mathcal{D}_t(\nabla H\star H) = \nabla^2 v\star H\star H + \nabla v\star\nabla H\star H$, and the following are the results for higher-order material derivatives.

Lemma 2.6. *Let $k\in\mathbb{N}$. We have $\mathcal{D}_t^k(\nabla H\star\nabla H) = R^k_{\nabla H,\nabla H}$ and $\mathcal{D}_t^k(\nabla H\star H) = R^k_{\nabla H,H}$.*

Proof. It is sufficient to consider the case of $k\geqslant 2$. We claim that given any $k\geqslant 2$, one has

$$\mathcal{D}_t^k(\nabla H\star\nabla H) = \sum_{2\leqslant m\leqslant k+2}\sum_{|\beta|\leqslant k+2-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-2}}v\star\nabla\mathcal{D}_t^{\beta_{m-1}}H\star\nabla\mathcal{D}_t^{\beta_m}H,$$
$$\mathcal{D}_t^k(\nabla H\star H) = \sum_{2\leqslant m\leqslant k+2}\sum_{|\beta|\leqslant k+2-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-2}}v\star\nabla\mathcal{D}_t^{\beta_{m-1}}H\star\mathcal{D}_t^{\beta_m}H.$$

In fact, from lemma 2.2, we see that

$$\begin{aligned}\mathcal{D}_t^k(\nabla H\star\nabla H) &= \nabla\mathcal{D}_t^k H\star\nabla H + \left[\mathcal{D}_t^k,\nabla\right]H\star\nabla H + \sum_{|\gamma|=k,\gamma_1,\gamma_2\geqslant 1}[\mathcal{D}_t^{\gamma_1},\nabla]H\star[\mathcal{D}_t^{\gamma_2},\nabla]H\\ &\quad + \nabla\mathcal{D}_t^{\gamma_1}H\star[\mathcal{D}_t^{\gamma_2},\nabla]H + \nabla\mathcal{D}_t^{\gamma_1}H\star\nabla\mathcal{D}_t^{\gamma_2}H\\ &= \sum_{2\leqslant m\leqslant k+2}\sum_{|\beta|\leqslant k+2-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-2}}v\star\nabla\mathcal{D}_t^{\beta_{m-1}}H\star\nabla\mathcal{D}_t^{\beta_m}H,\\ \mathcal{D}_t^k(\nabla H\star H) &= \nabla\mathcal{D}_t^k H\star H + \left[\mathcal{D}_t^k,\nabla\right]H\star H + \mathcal{D}_t^k H\star\nabla H\\ &\quad + \sum_{|\gamma|=k,\gamma_i\geqslant 1}[\mathcal{D}_t^{\gamma_1},\nabla]H\star\mathcal{D}_t^{\gamma_2}H + \nabla\mathcal{D}_t^{\gamma_1}H\star\mathcal{D}_t^{\gamma_2}H\\ &= \sum_{2\leqslant m\leqslant k+2}\sum_{|\beta|\leqslant k+2-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-2}}v\star\nabla\mathcal{D}_t^{\beta_{m-1}}H\star\mathcal{D}_t^{\beta_m}H.\end{aligned}$$

By substituting $\mathcal{D}_t H = H\cdot\nabla v$ and by induction, it is readily verified that

$$\mathcal{D}_t^j H = \sum_{1\leqslant m\leqslant j}\sum_{|\beta|\leqslant j-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_m}v\star H, \tag{2.5}$$
$$\nabla^i\mathcal{D}_t^j H = \sum_{1\leqslant m\leqslant j}\sum_{|\alpha|\leqslant i,|\beta|\leqslant j-m}\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v\star\nabla^{\alpha_{m+1}}H, \tag{2.6}$$

where $i,j\in\mathbb{N}$. These conclude the proof of the lemma. □

The above lemma shows that $\mathcal{D}_t^k(H\cdot\nabla H) = R^k_{\nabla H,H}$. Due to the divergence-free condition, it can be shown that taking the divergence does not increase the order of derivatives.

Lemma 2.7. *We have the following results:*

(1) $\operatorname{div}\mathcal{D}_t(H\cdot\nabla H) = \nabla^2 v\star\nabla H\star H + \nabla v\star\nabla H\star\nabla H + \nabla^2 H\star\nabla v\star H$.
(2) *For any integer $k\geqslant 2$, it holds* $\operatorname{div}\mathcal{D}_t^k(H\cdot\nabla H) = \partial_j\partial_l\mathcal{D}_t^{k-1}v^i H^l\partial_i H^j + \nabla^3\mathcal{D}_t^{k-2}v\star\nabla v\star H\star H + \mathrm{L.O.T.}$, *where* L.O.T. *stands for lower-order terms.*

Proof. By lemma A.1, a direct calculation gives the first result. For $k\geqslant 2$, the divergence-free condition implies that $\partial_j\mathcal{D}_t^\gamma\partial_i H^j = [\partial_j,\mathcal{D}_t^\gamma]\partial_i H^j$, and therefore

$$\begin{aligned}\operatorname{div}\mathcal{D}_t^k(H\cdot\nabla H)&=\partial_j\left(\mathcal{D}_t^k\partial_i H^j H^i\right)+\partial_j\left(\partial_i H^j\mathcal{D}_t^k H^i\right)+\partial_j\left(\sum_{|\gamma|=k,\gamma_i<k}\mathcal{D}_t^{\gamma_1}\partial_i H^j\mathcal{D}_t^{\gamma_2}H^i\right)\\&=\partial_j\mathcal{D}_t^k H^i\partial_i H^j+\left[\partial_j,\mathcal{D}_t^k\right]\partial_i H^j H^i+\left[\mathcal{D}_t^k,\nabla\right]H\star\nabla H+\nabla\mathcal{D}_t^{\gamma_1}H\star\nabla\mathcal{D}_t^{\gamma_2}H\\&\quad+[\mathcal{D}_t^{\gamma_1},\nabla]H\star\nabla\mathcal{D}_t^{\gamma_2}H+\sum_{|\gamma|=k,\gamma_i<k}[\partial_j,\mathcal{D}_t^{\gamma_1}]\partial_i H^j\mathcal{D}_t^{\gamma_2}H^i.\end{aligned}$$

In the above, it suffices to consider the most challenging term $\partial_j\mathcal{D}_t^k H^i\partial_i H^j$. Note that $\partial_j\mathcal{D}_t^k H^i=\partial_j\partial_l\mathcal{D}_t^{k-1}v^iH^l+\sum_{|\gamma|=k-1,\gamma_1<k-1}\partial_j\partial_l\mathcal{D}_t^{\gamma_1}v^i\mathcal{D}_t^{\gamma_2}H^l+\sum_{|\gamma|=k-1}\partial_j[\mathcal{D}_t^{\gamma_1},\partial_l]v^i\mathcal{D}_t^{\gamma_2}H^l$, and we find that

$$\begin{aligned}\operatorname{div}\mathcal{D}_t^k(H\cdot\nabla H)&=\partial_j\partial_l\mathcal{D}_t^{k-1}v^iH^l\partial_i H^j+\left[\nabla,\mathcal{D}_t^k\right]\nabla H\star H+\left[\mathcal{D}_t^k,\nabla\right]H\star\nabla H\\&\quad+\sum_{|\gamma|=k,\gamma_i<k}([\nabla,\mathcal{D}_t^{\gamma_1}]\nabla H\star\mathcal{D}_t^{\gamma_2}H+\nabla\mathcal{D}_t^{\gamma_1}H\star\nabla\mathcal{D}_t^{\gamma_2}H+[\mathcal{D}_t^{\gamma_1},\nabla]H\star\nabla\mathcal{D}_t^{\gamma_2}H)\\&\quad+\sum_{|\gamma|=k-1,\gamma_1<k-1}\nabla^2\mathcal{D}_t^{\gamma_1}v\star\mathcal{D}_t^{\gamma_2}H\star\nabla H+\sum_{|\gamma|=k-1}\nabla[\mathcal{D}_t^{\gamma_1},\nabla]v\star\mathcal{D}_t^{\gamma_2}H\star\nabla H\\&=:\partial_j\partial_l\mathcal{D}_t^{k-1}v^iH^l\partial_i H^j+R.\end{aligned}$$

Here, the highest-order term in R is $\nabla^2\mathcal{D}_t^{k-1}H\star\nabla v\star H$, resulting from $[\nabla,\mathcal{D}_t^k]\nabla H\star H$. To complete the proof, we replace the material derivative with the spatial derivative, resulting in $\nabla^3\mathcal{D}_t^{k-2}v\star\nabla v\star H\star H$, along with lower-order terms as shown in (2.6). □

To derive the energy estimates by applying the div-curl estimates, it is inevitable to compute $\operatorname{div}\mathcal{D}_t^l v,\operatorname{div}\mathcal{D}_t^l H,\operatorname{curl}\mathcal{D}_t^l v$, and $\operatorname{curl}\mathcal{D}_t^l H$. The following lemma is crucial for computing $\operatorname{curl}\mathcal{D}_t^l v$ (see lemma 2.10).

Lemma 2.8. *It holds* $\mathcal{D}_t((H\cdot\nabla)(\operatorname{curl}H))=\nabla^2\operatorname{curl}v\star H\star H+\nabla^2H\star\nabla v\star H+\nabla^2v\star\nabla H\star H$, *and*

$$\begin{aligned}\mathcal{D}_t^k((H\cdot\nabla)\operatorname{curl}H)&=\nabla^{k+1}\operatorname{curl}H\star\underbrace{H\star\cdots\star H}_{k\text{times}}+\sum_{|\alpha|,m\leqslant k+2,\alpha_i\leqslant k+1,F_j=v,H}\nabla^{\alpha_1}F_1\star\cdots\star\nabla^{\alpha_m}F_m\\&\quad+\sum_{\substack{|\alpha|+|\beta|\leqslant k+2,\alpha_i+\beta_i\leqslant k+1,\\ m\leqslant k+1,\beta_i\leqslant k-1,F_j=v,H}}\nabla^{\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{\alpha_{k-1}}\mathcal{D}_t^{\beta_{k-1}}v\star\nabla^{\alpha_k}F_k\star\cdots\star\nabla^{\alpha_m}F_m,\end{aligned}$$

if $k\geqslant 2$ *is even. For odd* $k\geqslant 3$, *we replace* $\nabla^{k+1}\operatorname{curl}H\star\underbrace{H\star\cdots\star H}_{k\text{times}}$ *by* $\nabla^{k+1}\operatorname{curl}v\star\underbrace{H\star\cdots\star H}_{k\text{times}}$.

Proof. First, we apply lemma 2.5 to obtain $\mathcal{D}_t[(H\cdot\nabla)(\operatorname{curl}H)]=\nabla^2\operatorname{curl}v\star H\star H+\nabla^2H\star\nabla v\star H+\nabla^2v\star\nabla H\star H$. In the case of $k=2$, one has

$$\begin{aligned}\mathcal{D}_t^2\left((H\cdot\nabla)(\operatorname{curl}H)\right)&=\partial_i\mathcal{D}_t^2\operatorname{curl}HH^i+\left[\mathcal{D}_t^2,\partial_i\right]\operatorname{curl}HH^i+\nabla^2H\star\mathcal{D}_t^2H\\&\quad+\mathcal{D}_t\nabla^2H\star\nabla v\star H=:I_1+I_2+I_3+I_4.\end{aligned}$$

We denote $I_1=(\nabla\operatorname{curl}\mathcal{D}_t(H\cdot\nabla v))\star H+\nabla([\mathcal{D}_t^2,\nabla]H)\star H=:I_{11}+I_{12}$. By lemma 2.5, it holds $\operatorname{curl}(H\cdot\nabla\mathcal{D}_t v)=\nabla\mathcal{D}_t v\star\nabla H+\nabla^2\operatorname{curl}H\star H\star H+\nabla^2H\star\nabla H\star H$, and using lemma A.1, it follows that

$$I_{11} = \nabla\left(\operatorname{curl}\left(\mathcal{D}_t H\cdot\nabla v\right)\right)\star H + \nabla\left(\operatorname{curl}\left(H\cdot\mathcal{D}_t\nabla v\right)\right)\star H$$
$$= \nabla^3\operatorname{curl} H\star H\star H + \nabla^2\mathcal{D}_t v\star\nabla H + \nabla\mathcal{D}_t v\star\nabla^2 H + \sum_{|\alpha|,m\leqslant 4,\alpha_i\leqslant 3,F_j=v,H}\nabla^{\alpha_1}F_1\star\cdots\star\nabla^{\alpha_m}F_m.$$

Applying lemma 2.2, we have $I_{12} = \nabla^2\mathcal{D}_t v\star\nabla H\star H + \nabla\mathcal{D}_t v\star\nabla^2 H\star H + \nabla^2 H\star\nabla v\star\nabla v\star H + \nabla^2 v\star\nabla H\star\nabla v\star H + \nabla^2 H\star\nabla v\star H + \nabla^2 v\star\nabla H\star H$, and $I_2 = \nabla\mathcal{D}_t v\star\nabla^2 H\star H + \nabla^3 v\star\nabla v\star H\star H + \nabla^2 H\star\nabla v\star\nabla v\star H + \nabla^2 v\star\nabla H\star\nabla v\star H + \nabla^2 H\star\nabla v\star H$.

To control the last two terms, (2.5) implies that $I_3 = \nabla\mathcal{D}_t v\star\nabla^2 H\star H + \nabla^2 H\star\nabla v\star\nabla v\star H + \nabla^2 H\star\nabla v\star H$, and lemma 2.2 together with (1.1) yields $I_4 = \nabla^3 v\star\nabla v\star H\star H + \nabla^2 H\star\nabla v\star\nabla v\star H + \nabla^2 v\star\nabla H\star\nabla v\star H$. We arrive at the following

$$\mathcal{D}_t^2\left((H\cdot\nabla)\operatorname{curl} H\right) = \nabla^3\operatorname{curl} H\star H\star H + \sum_{|\alpha|,m\leqslant 4,\alpha_i\leqslant 3,F_j=v,H}\nabla^{\alpha_1}F_1\star\cdots\star\nabla^{\alpha_m}F_m$$
$$+\sum_{\substack{|\alpha|+|\beta|\leqslant 4,\alpha_i+\beta_i\leqslant 3\\ \beta_i\leqslant 1,m\leqslant 3,F_j=v,H}}\nabla^{\alpha_1}\mathcal{D}_t^{\beta_1}v\star\nabla^{\alpha_2}F_2\star\cdots\star\nabla^{\alpha_m}F_m =: J_1 + J_2 + J_3.$$

As for $k=3$, to calculate $\mathcal{D}_t J_1$, we only focus on the most difficult term. Actually, it holds $\mathcal{D}_t\nabla^3\operatorname{curl} H = \nabla^4\operatorname{curl} v\star H + \sum_{|\alpha|\leqslant 5,\alpha_i\leqslant 4}\nabla^{\alpha_1}H\star\nabla^{\alpha_2}v$, from lemmas 2.2 and 2.5. With the help of lemma 2.2, $\mathcal{D}_t J_2$ and $\mathcal{D}_t J_3$ can be treated in the same fashion. Therefore, we obtain

$$\mathcal{D}_t^3\left((H\cdot\nabla)\operatorname{curl} H\right) = \nabla^4\operatorname{curl} v\star H\star H\star H + \sum_{|\alpha|,m\leqslant 5,\alpha_i\leqslant 4,F_j=v,H}\nabla^{\alpha_1}F_1\star\cdots\star\nabla^{\alpha_m}F_m$$
$$+\sum_{|\alpha|+|\beta|\leqslant 5,\beta_i\leqslant 2,\alpha_i+\beta_i\leqslant 4,m\leqslant 4,F_j=v,H}\nabla^{\alpha_1}\mathcal{D}_t^{\beta_1}v\star\nabla^{\alpha_2}\mathcal{D}_t^{\beta_2}v\star\nabla^{\alpha_3}F_3\star\cdots\star\nabla^{\alpha_m}F_m.$$

The other cases can be shown in the same way. □

From now on, we denote $R^0_{\nabla^2 H,H} := (H\cdot\nabla)\operatorname{curl} H$, and $R^k_{\nabla^2 H,H} := \mathcal{D}_t^k\left((H\cdot\nabla)\operatorname{curl} H\right)$ for $k\geqslant 1$. We proceed to introduce another two types of error terms. The first one is written in the form

$$R_I^0 = \nabla v\star\nabla v,\ R_I^l = \sum_{2\leqslant m\leqslant l+1}\ \sum_{|\beta|\leqslant l+2-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m}v, \tag{2.7}$$

for any $l\geqslant 1$. The second error term is denoted by

$$R_{II}^0 = \nabla v\star\mathcal{D}_t v + \nabla v\star\nabla v\star v,$$
$$R_{II}^l = \sum_{2\leqslant m\leqslant l+1,|\beta|\leqslant l,|\alpha|\leqslant 1} a_{\alpha,\beta}\left(\nabla v\right)\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+\beta_m}v, \tag{2.8}$$

where $l\geqslant 1$ and $a_{\alpha,\beta}(\nabla v)$ denotes the finite $\star$ product as before.

Lemma 2.9. *For $l\in\mathbb{N}_0$, we have $[\mathcal{D}_t^{l+1},\nabla]p = \sum_{i\leqslant l}\nabla\mathcal{D}_t^i v\star\nabla H\star H + R_{II}^l + R^l_{\nabla H,H}$.*

Proof. We prove this claim by induction. The case of $l=0$ follows directly. As for $l\geqslant 1$, by lemmas 2.1 and 2.2,

$$\left[\mathcal{D}_t^{l+1},\nabla\right]p = \mathcal{D}_t\left(\left[\mathcal{D}_t^l,\nabla\right]p\right) + \left[\mathcal{D}_t,\nabla\right]\mathcal{D}_t^l p = \mathcal{D}_t\left(\left[\mathcal{D}_t^l,\nabla\right]p\right) - \left(\nabla v\right)^{\top}\nabla\mathcal{D}_t^l p,$$

where $-\nabla\mathcal{D}_t^l p = [\mathcal{D}_t^l,\nabla]p + \mathcal{D}_t^l(\mathcal{D}_t v - H\cdot\nabla H) = [\mathcal{D}_t^l,\nabla]p + \mathcal{D}_t^{l+1}v - \mathcal{D}_t^l(H\cdot\nabla H)$. A direct computation shows that $\mathcal{D}_t R_{II}^{l-1} = R_{II}^l$ and $\mathcal{D}_t R_{\nabla H,H}^{l-1} = R_{\nabla H,H}^l$. These, combined with $[\mathcal{D}_t^l,\nabla]p = \sum_{i\leqslant l-1}\nabla\mathcal{D}_t^i v\star\nabla H\star H + R_{II}^{l-1} + R_{\nabla H,H}^{l-1}$ (also obtained by induction), yield that

$$\left[\mathcal{D}_t^{l+1},\nabla\right]p = \mathcal{D}_t\left(\sum_{i\leqslant l-1}\nabla\mathcal{D}_t^i v\star\nabla H\star H\right) + R_{II}^l + R_{\nabla H,H}^l = \sum_{i\leqslant l}\nabla\mathcal{D}_t^i v\star\nabla H\star H + R_{II}^l + R_{\nabla H,H}^l,$$

where in the last step, the lower-order terms have been absorbed into the terms R_{II}^l and $R_{\nabla H,H}^l$. □

Lemma 2.10. *Let $l\in\mathbb{N}$. We have*

$$\begin{aligned}
\mathcal{D}_t\nabla^l\operatorname{curl} v =\ & (H\cdot\nabla)\nabla^l\operatorname{curl} H + \nabla v\star\nabla^l\operatorname{curl} v + \nabla^{l+1}v\star\operatorname{curl} v\\
&+\sum_{|\beta|=l}\nabla^{1+\beta_1}H\star\nabla^{\beta_2}\operatorname{curl} H + \sum_{|\alpha|\leqslant l-1,\alpha_2\leqslant l-2}\nabla^{1+\alpha_1}v\star\nabla^{1+\alpha_2}\operatorname{curl} v,\\
\mathcal{D}_t\nabla^l\operatorname{curl} H =\ & (H\cdot\nabla)\nabla^l(\operatorname{curl} v) + \nabla v\star\nabla^l\operatorname{curl} H\\
&+\sum_{|\beta|=l}\nabla^{1+\beta_1}v\star\nabla^{1+\beta_2}H + \sum_{|\alpha|\leqslant l-1,\alpha_2\leqslant l-2}\nabla^{1+\alpha_1}v\star\nabla^{1+\alpha_2}\operatorname{curl} H.
\end{aligned}$$

Moreover, we can also write $\operatorname{div}\mathcal{D}_t^l v = R_I^{l-1}, \operatorname{curl}\mathcal{D}_t^l v = R_I^{l-1} + R_{\nabla H,\nabla H}^{l-1} + R_{\nabla^2 H,H}^{l-1}$, *and* $\operatorname{div}\mathcal{D}_t^{l+1}v = \operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^l v) + \operatorname{div} R_{II}^{l-1}$.

Proof. The first two claims are immediate consequences of lemmas 2.2 and 2.5. Regarding $\operatorname{curl}\mathcal{D}_t^l v$ and $\operatorname{div}\mathcal{D}_t^l v$ for $l\geqslant 2$. Noting that $(\mathcal{D}_t^l\nabla u)^\top = \mathcal{D}_t^l[(\nabla u)^\top]$ and applying lemmas 2.2 and 2.8, together with lemma 2.5, we obtain

$$\begin{aligned}
\operatorname{curl}\mathcal{D}_t^l v =\ & \left[\nabla,\mathcal{D}_t^{l-1}\right](\mathcal{D}_t v) - \left(\left[\nabla,\mathcal{D}_t^{l-1}\right](\mathcal{D}_t v)\right)^\top + \mathcal{D}_t^{l-1}\operatorname{curl}\mathcal{D}_t v\\
=\ & \sum_{2\leqslant m\leqslant l}\sum_{|\beta|\leqslant l-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\star\nabla\mathcal{D}_t^{\beta_m+1}v + \mathcal{D}_t^{l-1}(\nabla H\star\nabla H)\\
&+\mathcal{D}_t^{l-1}((H\cdot\nabla)(\operatorname{curl} H)) = R_I^{l-1} + R_{\nabla H,\nabla H}^{l-1} + R_{\nabla^2 H,H}^{l-1}.
\end{aligned}$$

Similarly, one has $\operatorname{div}\mathcal{D}_t^l v = R_I^{l-1}$ thanks to $\operatorname{div} v = 0$. For the last statement, we apply $[\mathcal{D}_t,\operatorname{div}]F = -\operatorname{div}(\nabla v F)$ and $\operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^l v) = \operatorname{div}(\nabla\mathcal{D}_t^l v v), l\geqslant 1$ (both can be easily computed). Then, we have

$$\operatorname{div}\mathcal{D}_t^2 v = \mathcal{D}_t\operatorname{div}(\nabla v v) + \operatorname{div}(\nabla v\mathcal{D}_t v) = \operatorname{div}\mathcal{D}_t(\nabla v v) - \operatorname{div}(\nabla v\nabla v v) + \operatorname{div} R_{II}^0,$$

and therefore,

$$\begin{aligned}
\operatorname{div}\mathcal{D}_t^2 v &= \operatorname{div}(\nabla\mathcal{D}_t v v) + \operatorname{div}([\mathcal{D}_t,\nabla]v v) - \operatorname{div}(\nabla v\nabla v v) + \operatorname{div} R_{II}^0\\
&= \operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t v) + \operatorname{div} R_{II}^0.
\end{aligned}$$

For $l\geqslant 2$, we argue by induction, i.e. $\operatorname{div}\mathcal{D}_t^{l+1}v = \mathcal{D}_t\operatorname{div}\mathcal{D}_t^l v - [\mathcal{D}_t,\operatorname{div}]\mathcal{D}_t^l v = \mathcal{D}_t\operatorname{div}(\nabla\mathcal{D}_t^{l-1}v v) + \mathcal{D}_t\operatorname{div} R_{II}^{l-2} + \operatorname{div}(\nabla v\mathcal{D}_t^l v)$. The proof is complete since $\mathcal{D}_t\operatorname{div} R_{II}^{l-2} = \operatorname{div} R_{II}^{l-1}, \operatorname{div}(\nabla v\mathcal{D}_t^l v) = \operatorname{div} R_{II}^{l-1}$ (direct calculations), and

$$\begin{aligned}\mathcal{D}_t \operatorname{div}\left(\nabla \mathcal{D}_t^{l-1} vv\right) &= \operatorname{div} \mathcal{D}_t\left(\nabla \mathcal{D}_t^{l-1} vv\right) + [\mathcal{D}_t, \operatorname{div}]\left(\nabla \mathcal{D}_t^{l-1} vv\right) \\ &= \operatorname{div}\left(\nabla v \star \nabla \mathcal{D}_t^{l-1} v \star v\right) + \operatorname{div}\left(\nabla \mathcal{D}_t^{l} vv + \mathcal{D}_t v \star \nabla \mathcal{D}_t^{l-1} v + v \star \nabla v \star \nabla \mathcal{D}_t^{l-1} v\right) \\ &= \operatorname{div}\operatorname{div}\left(v \otimes \mathcal{D}_t^{l} v\right) + \operatorname{div} R_{II}^{l-1}.\end{aligned}$$

□

Lemma 2.11. *Let $l \geqslant 1$. We have*

$$\begin{aligned}-\Delta \mathcal{D}_t p &= \operatorname{div}\operatorname{div}\left(v \otimes \mathcal{D}_t v\right) + \operatorname{div}\left(R_{II}^0 + \nabla v \star H \star \nabla H + H \cdot \nabla\left(H \cdot \nabla v\right)\right) \\ &= -\operatorname{div}\operatorname{div}\left(v \otimes \nabla p\right) + \operatorname{div} R_{II}^0 + \nabla^2 v \star \nabla H \star H + \nabla^2 H \star \nabla v \star H \\ &\quad + \nabla^2 H \star \nabla H \star v + \nabla v \star \nabla H \star \nabla H \\ -\Delta \mathcal{D}_t^{l+1} p &= \operatorname{div}\operatorname{div}\left(v \otimes \mathcal{D}_t^{l+1} v\right) - \operatorname{div} R_{\nabla^2 H, H}^{l+1} + \operatorname{div}\left(\sum_{i \leqslant l} \nabla \mathcal{D}_t^i v \star \nabla H \star H + R_{II}^l + R_{\nabla H, H}^l\right).\end{aligned}$$

Proof. From the divergence-free condition, lemmas 2.10 and 2.9, the first claim follows. The second claim follows by applying lemma 2.9 that

$$\begin{aligned}-\Delta \mathcal{D}_t^{l+1} p &= -\operatorname{div} \mathcal{D}_t^{l+1} \nabla p + \operatorname{div}\left[\mathcal{D}_t^{l+1}, \nabla\right] p \\ &= \operatorname{div} \mathcal{D}_t^{l+2} v - \operatorname{div} \mathcal{D}_t^{l+1}\left(H \cdot \nabla H\right) + \operatorname{div} R_{II}^l + \operatorname{div}\left(\sum_{i \leqslant l} \nabla \mathcal{D}_t^i v \star \nabla H \star H + R_{\nabla H, H}^l\right).\end{aligned}$$

□

From $p = \mathcal{A}$ and the identities (e.g. [37, section 3.1])

$$\mathcal{D}_t \mathcal{A} = -\Delta_B v_n - |B|^2 v_n + \bar{\nabla} \mathcal{A} \cdot v, \quad \Delta_B \nu = -|B|^2 \nu + \bar{\nabla} \mathcal{A}, \tag{2.9}$$

it holds on the free-boundary Γ_t that

$$\mathcal{D}_t p = -\Delta_B v \cdot \nu - 2B : \bar{\nabla} v = -\Delta_B v_n - |B|^2 v_n + \bar{\nabla} p \cdot v. \tag{2.10}$$

Finally, we introduce the error term R_p^l as described in [26]. We define

$$\begin{aligned}R_p^1 &= -|B|^2 \mathcal{D}_t v \cdot \nu + \bar{\nabla} p \cdot \mathcal{D}_t v + a_1(\nu, \nabla v) \star \nabla^2 v + a_2(\nu, \nabla v) \star B, \\ R_p^2 &= -|B|^2 \mathcal{D}_t^2 v \cdot \nu + \bar{\nabla} p \cdot \mathcal{D}_t^2 v + a_3(\nu, \nabla v) \star \nabla^2 \mathcal{D}_t v + a_4(\nu, \nabla v) \star \nabla \mathcal{D}_t v \star \nabla^2 v \\ &\quad + a_5(\nu, \nabla v) \star \nabla \mathcal{D}_t v \star B + a_6(\nu, \nabla v) \star \nabla^2 v + a_7(\nu, \nabla v) \star B, \\ R_p^3 &= -|B|^2 \mathcal{D}_t^3 v \cdot \nu + \bar{\nabla} p \cdot \mathcal{D}_t^3 v + a_8(\nu, \nabla v) \star \nabla^2 \mathcal{D}_t^2 v + a_9(\nu, \nabla v) \star \nabla \mathcal{D}_t^2 v \star \nabla^2 v \\ &\quad + a_{10}(\nu, \nabla v) \star \nabla \mathcal{D}_t^2 v \star B + a_{11}(\nu, \nabla v) \star \nabla^2 \mathcal{D}_t v \star \nabla \mathcal{D}_t v + a_{12}(\nu, \nabla v) \star \nabla^2 \mathcal{D}_t v \star B \\ &\quad + a_{13}(\nu, \nabla v) \star \nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t v \star \nabla^2 v + a_{14}(\nu, \nabla v) \star \nabla \mathcal{D}_t v \star \nabla \mathcal{D}_t v \star B + \text{L.O.T.}, \\ R_p^l &= -|B|^2 \mathcal{D}_t^l v \cdot \nu + \bar{\nabla} p \cdot \mathcal{D}_t^l v + \sum_{|\alpha| \leqslant 1, |\beta| \leqslant l-1} a_{\alpha,\beta}(\nu, B) \nabla^{1+\alpha_1} \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_{l+1}} \mathcal{D}_t^{\beta_{l+1}} v,\end{aligned}$$

where $l \geqslant 4$, $a_i(\nu, \nabla v)$ and $a_{\alpha,\beta}(\nu, B)$ denote the finite $\star$ product.

Lemma 2.12. *On the free-boundary Γ_t, we have $\mathcal{D}_t^{l+1} p = -\Delta_B(\mathcal{D}_t^l v \cdot \nu) + R_p^l$ for $l \in \mathbb{N}$.*

Proof. For $l=1$, we differentiate (2.10) to obtain $\mathcal{D}_t^2 p = -\mathcal{D}_t\Delta_B v\cdot\nu - \Delta_B v\cdot\mathcal{D}_t\nu - 2\mathcal{D}_t B : \bar{\nabla} v - 2B : \mathcal{D}_t\bar{\nabla} v$. Recalling the formulas for $[\mathcal{D}_t,\Delta_B],\mathcal{D}_t\nu$ and $\mathcal{D}_t B$ in lemma A.1, it holds $\mathcal{D}_t^2 p = -\Delta_B\mathcal{D}_t v\cdot\nu - 2B:\bar{\nabla}\mathcal{D}_t v + a_1(\nu,\nabla v)\star\nabla^2 v + a_2(\nu,\nabla v)\star B$.

For $l=2$, we differentiate $\mathcal{D}_t^2 p$ and calculate $[\mathcal{D}_t,\Delta_B]\mathcal{D}_t v = \bar{\nabla}^2\mathcal{D}_t v\star\nabla v - \bar{\nabla}\mathcal{D}_t v\cdot\Delta_B v + B_\Gamma\star\nabla v\star\bar{\nabla}\mathcal{D}_t v, \mathcal{D}_t B = a_1(\nu,\nabla v)\star B + a_2(\nu,\nabla v)\star\nabla^2 v, \mathcal{D}_t\bar{\nabla}\mathcal{D}_t v = \bar{\nabla}\mathcal{D}_t^2 v + \bar{\nabla} v\star\bar{\nabla}\mathcal{D}_t v, \mathcal{D}_t a(\nu,\nabla v) = b(\nu,\nabla v)\star\nabla\mathcal{D}_t v, \mathcal{D}_t\nabla^2 v = \nabla^2 v\star\nabla v + \nabla^2\mathcal{D}_t v$ to obtain $\mathcal{D}_t^3 p = -\Delta_B\mathcal{D}_t^2 v\cdot\nu - 2B:\bar{\nabla}\mathcal{D}_t^2 v + a_3(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v + a_4(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star\nabla^2 v + a_5(\nu,\nabla v)\star\nabla\mathcal{D}_t v\star B + a_6(\nu,\nabla v)\star\nabla^2 v + a_7(\nu,\nabla v)\star B$. We can obtain the case of $l=3$ in the same way and the remaining proof is similar to [26, lemma 4.7]. □

3. Time derivatives of the energy functionals

In this section, we compute the time derivative of the energy functional $e_l(t)$ by applying Reynolds transport theorem and (2.2). The main result in this section is the following proposition.

Proposition 3.1. *Assume that the a priori assumptions (1.6) hold for some $T>0$. Then, we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bar{e}(t) \leqslant C\sum_{l=1}^{3}\left(\|R_I^l\|^2_{H^{1/2}(\Omega_t)} + \|R_{II}^l\|^2_{L^2(\Omega_t)} + \|R^l_{\nabla H,H}\|^2_{L^2(\Omega_t)} + \|R_p^l\|^2_{H^{1/2}(\Gamma_t)}\right)\\ + C\left(1+\|\nabla^2 p\|^2_{L^2(\Omega_t)}\right)\bar{E}(t),\end{aligned}$$

where the constant C depends on $T,\mathcal{N}_T$, and $\mathcal{M}_T$.

Moreover, we further assume that $\sup_{0\leqslant t<T}E_{l-1}(t)\leqslant C$ for $l\geqslant 4$. Then, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}e_l(t) \leqslant C\left(E_l(t) + \|R_I^l\|^2_{H^{1/2}(\Omega_t)} + \|R_{II}^l\|^2_{L^2(\Omega_t)} + \|R^l_{\nabla H,H}\|^2_{L^2(\Omega_t)} + \|R_p^l\|^2_{H^{1/2}(\Gamma_t)}\right),$$

for $l\geqslant 4$, where the constant C depends on $T,\mathcal{N}_T,\mathcal{M}_T$, and $\sup_{0\leqslant t<T}E_{l-1}(t)$.

Denote $I_1^l(t) = \frac{1}{2}\|\mathcal{D}_t^{l+1}v\|^2_{L^2(\Omega_t)}, I_2^l(t) = \frac{1}{2}\|\mathcal{D}_t^{l+1}H\|^2_{L^2(\Omega_t)}, I_3^l(t) = \frac{1}{2}\|\bar{\nabla}(\mathcal{D}_t^l v\cdot\nu)\|^2_{L^2(\Gamma_t)}, I_4^l(t) = \frac{1}{2}\|\nabla^{\lfloor\frac{3l+1}{2}\rfloor}\operatorname{curl} v\|^2_{L^2(\Omega_t)}$ and $I_5^l(t) = \frac{1}{2}\|\nabla^{\lfloor\frac{3l+1}{2}\rfloor}\operatorname{curl} H\|^2_{L^2(\Omega_t)}$. We will apply Reynolds transport theorem and (2.2) several times and we start with $I_1^l(t)$. From (1.1) and the divergence theorem,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_1^l(t) &= -\int_{\Omega_t}\mathcal{D}_t^{l+1}\nabla p\cdot\mathcal{D}_t^{l+1}v\,\mathrm{d}x + \int_{\Omega_t}\mathcal{D}_t^{l+1}(H\cdot\nabla H)\cdot\mathcal{D}_t^{l+1}v\,\mathrm{d}x\\ &= -\int_{\Omega_t}\nabla\mathcal{D}_t^{l+1}p\cdot\mathcal{D}_t^{l+1}v\,\mathrm{d}x - \int_{\Omega_t}\left[\mathcal{D}_t^{l+1},\nabla\right]p\cdot\mathcal{D}_t^{l+1}v\,\mathrm{d}x + \int_{\Omega_t}\mathcal{D}_t^{l+1}\left(H^j\partial_j H_i\right)\mathcal{D}_t^{l+1}v^i\,\mathrm{d}x\\ &= -\int_{\Omega_t}\operatorname{div}\left(\mathcal{D}_t^{l+1}p\,\mathcal{D}_t^{l+1}v\right)\mathrm{d}x + \int_{\Omega_t}\mathcal{D}_t^{l+1}p\operatorname{div}\mathcal{D}_t^{l+1}v\,\mathrm{d}x\\ &\quad -\int_{\Omega_t}\left[\mathcal{D}_t^{l+1},\nabla\right]p\cdot\mathcal{D}_t^{l+1}v\,\mathrm{d}x + \int_{\Omega_t}\mathcal{D}_t^{l+1}\left(H^j\partial_j H_i\right)\mathcal{D}_t^{l+1}v^i\,\mathrm{d}x\end{aligned}$$

$$\leqslant \underbrace{\int_{\Omega_t} H^j\partial_j\left(\mathcal{D}_t^{l+1}H_i\right)\mathcal{D}_t^{l+1}v^i\mathrm{d}x}_{=:I_1^l(t)} \underbrace{-\int_{\Gamma_t}\mathcal{D}_t^{l+1}p\left(\mathcal{D}_t^{l+1}v\cdot\nu\right)\mathrm{d}S}_{=:K_1^l(t)} + \|\mathcal{D}_t^{l+1}v\|^2_{L^2(\Omega_t)}$$
$$+ \underbrace{\int_{\Omega_t}\mathcal{D}_t^{l+1}p\,\mathrm{div}\,\mathcal{D}_t^{l+1}v\mathrm{d}x}_{=:I_{11}^l(t)} + \underbrace{\|\left[\mathcal{D}_t^{l+1},\nabla\right]p\|^2_{L^2(\Omega_t)}}_{=:I_{12}^l(t)}$$
$$+ \underbrace{\sum_{k=0}^{l}\int_{\Omega_t}\mathcal{D}_t^kH^j\left[\mathcal{D}_t^{l+1-k},\partial_j\right]H_i\mathcal{D}_t^{l+1}v^i\mathrm{d}x}_{=:I_{13}^l(t)} + \underbrace{\sum_{k=1}^{l+1}\int_{\Omega_t}\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{l+1-k}H_i\mathcal{D}_t^{l+1}v^i\mathrm{d}x}_{=:I_{14}^l(t)},$$

where we have used the fact that

$$\mathcal{D}_t^{l+1}\left(H^j\partial_jH_i\right)\mathcal{D}_t^{l+1}v^i = H^j\partial_j\left(\mathcal{D}_t^{l+1}H_i\right)\mathcal{D}_t^{l+1}v^i + \sum_{k=0}^{l}\mathcal{D}_t^kH^j\left[\mathcal{D}_t^{l+1-k},\partial_j\right]H_i\mathcal{D}_t^{l+1}v^i + \sum_{k=1}^{l+1}\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{l+1-k}H_i\mathcal{D}_t^{l+1}v^i.$$

Similarly, for the magnetic field, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}I_2^l(t) = \underbrace{\int_{\Omega_t}H^j\partial_j\left(\mathcal{D}_t^{l+1}v^i\right)\mathcal{D}_t^{l+1}H_i\mathrm{d}x}_{=:J_2^l(t)} + \underbrace{\sum_{k=0}^{l}\int_{\Omega_t}\mathcal{D}_t^kH^j\left[\mathcal{D}_t^{l+1-k},\partial_j\right]v^i\mathcal{D}_t^{l+1}H_i\mathrm{d}x}_{=:I_{21}^l(t)}$$
$$+ \underbrace{\sum_{k=1}^{l+1}\int_{\Omega_t}\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{l+1-k}v^i\mathcal{D}_t^{l+1}H_i\mathrm{d}x}_{=:I_{22}^l(t)}.$$

Recalling the divergence-free condition and $H\cdot\nu=0$ on Γ_t, it is clear that $J_1^l(t)+J_2^l(t)=0$, and we obtain $\frac{\mathrm{d}}{\mathrm{d}t}(I_1^l(t)+I_2^l(t)) \leqslant K_1^l(t)+\sum_{i=1}^4 I_{1i}^l(t)+I_{21}^l(t)+I_{22}^l(t)+\|\mathcal{D}_t^{l+1}v\|^2_{L^2(\Omega_t)}$.

To control the third term, we apply lemma A.1 to deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}I_3^l(t) = \int_{\Gamma_t} -\left(\bar{\nabla}v\right)^\top\bar{\nabla}\left(\mathcal{D}_t^lv\cdot\nu\right)\cdot\bar{\nabla}\left(\mathcal{D}_t^lv\cdot\nu\right)\mathrm{d}S + \frac{1}{2}\int_{\Gamma_t}|\bar{\nabla}\left(\mathcal{D}_t^lv\cdot\nu\right)|^2\,\mathrm{div}_\sigma\, v\mathrm{d}S$$
$$+ \int_{\Gamma_t}\bar{\nabla}\left(\mathcal{D}_t^{l+1}v\cdot\nu\right)\cdot\bar{\nabla}\left(\mathcal{D}_t^lv\cdot\nu\right)\mathrm{d}S + \int_{\Gamma_t}\bar{\nabla}\left(\mathcal{D}_t^lv\cdot\mathcal{D}_t\nu\right)\cdot\bar{\nabla}\left(\mathcal{D}_t^lv\cdot\nu\right)\mathrm{d}S$$
$$\leqslant \underbrace{-\int_{\Gamma_t}\left(\mathcal{D}_t^{l+1}v\cdot\nu\right)\cdot\Delta_B\left(\mathcal{D}_t^lv\cdot\nu\right)\mathrm{d}S}_{=:K_3^l(t)} + \underbrace{\|\bar{\nabla}\left(\mathcal{D}_t^lv\cdot\mathcal{D}_t\nu\right)\|^2_{L^2(\Gamma_t)}}_{=:I_{31}^l(t)}$$
$$+ C\left(\|\bar{\nabla}v\|_{L^\infty(\Gamma_t)}+1\right)\|\bar{\nabla}\left(\mathcal{D}_t^lv\cdot\nu\right)\|^2_{L^2(\Gamma_t)}.$$

Finally, to compute the last two terms involving the curl, we denote $\mu_l := \lfloor(3l+1)/2\rfloor$. We then utilize the divergence-free condition and the fact that $H\cdot\nu=0$ on Γ_t

to obtain $\int_{\Omega_t}\sum_{|\alpha|=l}(H\cdot\nabla)(\nabla^\alpha \operatorname{curl} H:\nabla^\alpha \operatorname{curl} v+\nabla^\alpha \operatorname{curl} v:\nabla^\alpha \operatorname{curl} H)\mathrm{d}x=0$. Therefore, from lemma 2.10, it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}I_4^l(t)-\int_{\Omega_t}\sum_{|\alpha|=l}(H\cdot\nabla)\nabla^\alpha \operatorname{curl} H:\nabla^\alpha \operatorname{curl} v\mathrm{d}x\\
&\leqslant C\left(\|\nabla v\|_{L^\infty(\Omega_t)}+1\right)\|\nabla^{\mu_l+1}v\|^2_{L^2(\Omega_t)}+\|\nabla H\|^2_{L^\infty(\Omega_t)}\|\operatorname{curl} H\|^2_{H^{\mu_l}(\Omega_t)}\\
&\quad+\|\operatorname{curl} H\|^2_{L^\infty(\Omega_t)}\|\nabla H\|^2_{H^{\mu_l}(\Omega_t)}+\|\nabla v\|^2_{L^\infty(\Omega_t)}\|\nabla v\|^2_{H^{\mu_l}(\Omega_t)},\\
&\frac{\mathrm{d}}{\mathrm{d}t}I_5^l(t)-\int_{\Omega_t}\sum_{|\alpha|=l}(H\cdot\nabla)\nabla^\alpha \operatorname{curl} v:\nabla^\alpha \operatorname{curl} H\mathrm{d}x\\
&\leqslant C\left(\|\nabla v\|_{L^\infty(\Omega_t)}+1\right)\|\nabla^{\mu_l}\operatorname{curl} H\|^2_{L^2(\Omega_t)}+\|\nabla v\|^2_{H^{\mu_l}(\Omega_t)}\|\nabla H\|^2_{L^\infty(\Omega_t)}\\
&\quad+\|\nabla H\|^2_{H^{\mu_l}(\Omega_t)}\|\nabla v\|^2_{L^\infty(\Omega_t)}.
\end{aligned}$$

Proof of proposition 3.1. By (1.6), one has $\|\bar{\nabla}v\|_{L^\infty(\Gamma_t)}\leqslant C\|\nabla v\|_{L^\infty(\Omega_t)}\leqslant C$. This, combined with the above calculations and applying lemma 2.12, $\|\nabla H\|_{L^\infty(\Omega_t)}\leqslant C$ by (1.6), together with the definition of $\bar{E}(t)$, we obtain $K_1^l(t)+K_3^l(t)=-\int_{\Gamma_t}R_p^l(\mathcal{D}_t^{l+1}v\cdot\nu)\mathrm{d}S$, and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bar{e}(t)\leqslant C\bar{E}(t)+C\sum_{l=1}^{3}\left(-\int_{\Gamma_t}R_p^l\left(\mathcal{D}_t^{l+1}v\cdot\nu\right)\mathrm{d}S+\sum_{i=1}^{4}I_{1i}^l(t)+I_{31}^l(t)+I_{21}^l(t)+I_{22}^l(t)\right),\\
&\frac{\mathrm{d}}{\mathrm{d}t}e_l(t)\leqslant CE_l(t)+C\left(-\int_{\Gamma_t}R_p^l\left(\mathcal{D}_t^{l+1}v\cdot\nu\right)\mathrm{d}S+\sum_{i=1}^{4}I_{1i}^l(t)+I_{31}^l(t)+I_{21}^l(t)+I_{22}^l(t)\right),\quad l\geqslant 4.
\end{aligned}$$

We divide the remaining proof into six steps.

Step 1. We control $I_{14}^l(t)$ and $I_{22}^l(t)$. We omit the case of $l=1$, and assume $F=v,G=H$ or $F=H,G=v$ respectively. In the case of $l=2$, from the fact that

$$\|\nabla\mathcal{D}_tH\|^2_{L^2(\Omega_t)}\leqslant\|\nabla(H\cdot\nabla v)\|^2_{L^2(\Omega_t)}\leqslant C, \tag{3.1}$$

$$\|\nabla\mathcal{D}_tv\|^2_{L^2(\Omega_t)}\leqslant\|\nabla(H\cdot\nabla H)\|^2_{L^2(\Omega_t)}+\|\nabla^2p\|^2_{L^2(\Omega_t)}\leqslant C\left(1+\|\nabla^2p\|^2_{L^2(\Omega_t)}\right), \tag{3.2}$$

it follows that

$$\begin{aligned}
&\sum_{k=1}^{3}\int_{\Omega_t}\mathcal{D}_t^kH^j\partial_j\mathcal{D}_t^{3-k}F_i\mathcal{D}_t^3G^i\mathrm{d}x\\
&\leqslant C\Big(E_2(t)+\|H\cdot\nabla v\|^2_{L^2(\Omega_t)}\|\mathcal{D}_t^2F\|^2_{H^3(\Omega_t)}+\|\mathcal{D}_t^2H\|^2_{H^2(\Omega_t)}\|\nabla\mathcal{D}_tF\|^2_{L^2(\Omega_t)}\\
&\quad+\|\mathcal{D}_t^3H\|^2_{L^2(\Omega_t)}\|\nabla F\|^2_{L^\infty(\Omega_t)}\Big)\leqslant C\left(1+\|\nabla^2p\|^2_{L^2(\Omega_t)}\right)\bar{E}(t).
\end{aligned}$$

As for $l=3$, again by (3.1) and (3.2), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{4}\int_{\Omega_t}\mathcal{D}_t^k H^j\partial_j\mathcal{D}_t^{4-k}F_i\,\mathcal{D}_t^4 G^i\mathrm{d}x\\
&\quad\leqslant C\Big(E_3(t)+\|H\cdot\nabla v\|_{L^6(\Omega_t)}^2\|\mathcal{D}_t^3F\|_{H^{3/2}(\Omega_t)}^2+\|\mathcal{D}_t^2H\|_{L^2(\Omega_t)}^2\|\mathcal{D}_t^2F\|_{H^3(\Omega_t)}^2\\
&\qquad+\|\mathcal{D}_t^2(H\cdot\nabla v)\|_{L^\infty(\Omega_t)}^2\|\nabla\mathcal{D}_tF\|_{L^2(\Omega_t)}^2+\|\mathcal{D}_t^4H\|_{L^2(\Omega_t)}^2\|\nabla F\|_{L^\infty(\Omega_t)}^2\Big)\\
&\quad\leqslant C\Big(1+\|\nabla^2p\|_{L^2(\Omega_t)}^2\Big)\bar{E}(t),
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\|\mathcal{D}_t^2H\|_{L^2(\Omega_t)}^2&\leqslant C\Big(\|\mathcal{D}_tH\star\nabla v\|_{L^2(\Omega_t)}^2+\|H\star\mathcal{D}_t\nabla v\|_{L^2(\Omega_t)}^2\Big)\leqslant C\Big(1+\|\nabla^2p\|_{L^2(\Omega_t)}^2\Big),\\
\|\mathcal{D}_t^3H\|_{L^\infty(\Omega_t)}^2&\leqslant\|\mathcal{D}_t^2H\star\nabla v\|_{L^\infty(\Omega_t)}^2+\|\mathcal{D}_tH\star\mathcal{D}_t\nabla v\|_{L^\infty(\Omega_t)}^2+\|H\star\mathcal{D}_t^2\nabla v\|_{L^\infty(\Omega_t)}^2\\
&\leqslant C\Big(\|\mathcal{D}_t^2H\|_{L^\infty(\Omega_t)}^2+\|[\mathcal{D}_t,\nabla]v\|_{L^\infty(\Omega_t)}^2+\|\nabla\mathcal{D}_tv\|_{L^\infty(\Omega_t)}^2\\
&\qquad+\|[\mathcal{D}_t^2,\nabla]v\|_{L^\infty(\Omega_t)}^2+\|\nabla\mathcal{D}_t^2v\|_{L^\infty(\Omega_t)}^2\Big)\leqslant C\bar{E}(t),
\end{aligned}
\tag{3.3}
$$

by utilizing (1.6), lemmas A.1 and 2.2. Additionally, one order material derivative has been substituted with the spatial derivative of the velocity field. As $l\geqslant 4$, we use the hypotheses $E_{l-1}(t)\leqslant C$ to obtain

$$
\begin{aligned}
&\sum_{k=1}^{l+1}\int_{\Omega_t}\mathcal{D}_t^k H^j\partial_j\mathcal{D}_t^{l+1-k}F_i\,\mathcal{D}_t^{l+1}G^i\mathrm{d}x\\
&\quad\leqslant C\left(\sum_{k=2}^{l}\|\mathcal{D}_t^kH\|_{H^1(\Omega_t)}^2\|\mathcal{D}_t^{l+1-k}F\|_{H^{3/2}(\Omega_t)}^2+\|\mathcal{D}_tH^j\partial_j\mathcal{D}_t^lF\|_{L^2(\Omega_t)}^2+E_l(t)\right)\\
&\quad\leqslant CE_l(t)E_{l-1}(t)+CE_l(t)+C\|\mathcal{D}_tH\|_{L^6(\Omega_t)}^2\|\nabla\mathcal{D}_t^lF\|_{L^3(\Omega_t)}^2\leqslant CE_l(t).
\end{aligned}
$$

Step 2. We control $I_{13}^l(t)$ and $I_{21}^l(t)$. As before, we assume $F=v,G=H$ or $F=H,G=v$. We only consider the case of $l\geqslant 3$. In fact, from $[\mathcal{D}_t^j,\nabla]$ in lemma 2.2, (3.1)–(3.3), it holds

$$
\begin{aligned}
&\sum_{k=0}^{3}\int_{\Omega_t}\mathcal{D}_t^kH^j\left[\mathcal{D}_t^{4-k},\partial_j\right]F_i\,\mathcal{D}_t^4G^i\mathrm{d}x\\
&\quad\leqslant C\Big(E_3(t)+\|\mathcal{D}_t^3H^j\partial_jv^k\partial_kF\|_{L^2(\Omega_t)}^2\\
&\qquad+\|\mathcal{D}_t^2H\star(\nabla v\star\nabla F+\nabla\mathcal{D}_tv\star\nabla F+\nabla v\star\nabla\mathcal{D}_tF+\nabla v\star\nabla v\star\nabla F)\|_{L^2(\Omega_t)}^2\\
&\qquad+\|\mathcal{D}_tH\star\big(\nabla\mathcal{D}_t^2v\star\nabla F+\nabla\mathcal{D}_tv\star\nabla\mathcal{D}_tF+\nabla v\star\nabla\mathcal{D}_t^2F\\
&\qquad+\nabla\mathcal{D}_tv\star\nabla v\star\nabla F+\nabla v\star\nabla v\star\nabla\mathcal{D}_tF+\mathrm{L.O.T.}\big)\|_{L^2(\Omega_t)}^2\\
&\qquad+\|H\star\big(\nabla\mathcal{D}_t^3v\star\nabla F+\nabla\mathcal{D}_t^2v\star\nabla\mathcal{D}_tF+\nabla\mathcal{D}_tv\star\nabla\mathcal{D}_t^2F\\
&\qquad+\nabla v\star\nabla\mathcal{D}_t^3F+\mathrm{L.O.T.}\big)\|_{L^2(\Omega_t)}^2\Big)\leqslant C\Big(1+\|\nabla^2p\|_{L^2(\Omega_t)}^2\Big)\bar{E}(t).
\end{aligned}
$$

For $l\geqslant 4$, from lemma 2.2 and the assumption $E_{l-1}(t)\leqslant C$, we deduce

$$
\begin{aligned}
&\sum_{k=0}^{l}\int_{\Omega_t}\mathcal{D}_t^k H^j\left[\mathcal{D}_t^{l+1-k},\partial_j\right]F_i\,\mathcal{D}_t^{l+1}G^i\mathrm{d}x\\
&\quad\leqslant C\Big(E_l(t)+\|\mathcal{D}_t^l H^j\partial_j v^k\partial_k F\|_{L^2(\Omega_t)}^2\\
&\qquad+\sum_{k=0}^{l-1}\|\mathcal{D}_t^k H\star\sum_{2\leqslant m\leqslant l+2-k}\ \sum_{|\beta|\leqslant l+2-k-m}\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{m-1}}v\\
&\qquad\star\nabla\mathcal{D}_t^{\beta_m}F\|_{L^2(\Omega_t)}^2\Big)\\
&\quad\leqslant CE_{l-1}(t)E_l(t)+CE_l(t)\leqslant CE_l(t).
\end{aligned}
$$

Step 3. To estimate $\int_{\Gamma_t}R_p^l(\mathcal{D}_t^{l+1}v\cdot\nu)\mathrm{d}S$, we apply lemma 2.10 and the normal trace theorem (e.g. [2, theorem 3.1]) to obtain $\|\mathcal{D}_t^{l+1}v\cdot\nu\|_{H^{-1/2}(\Gamma_t)}\leqslant C(\|\mathcal{D}_t^{l+1}v\|_{L^2(\Omega_t)}+\|\operatorname{div}\mathcal{D}_t^{l+1}v\|_{H^{-1}(\Omega_t)})$. Therefore, it follows that

$$
\begin{aligned}
&\left|\int_{\Gamma_t}R_p^l\left(\mathcal{D}_t^{l+1}v\cdot\nu\right)\mathrm{d}S\right|\leqslant C\left(\bar{E}(t)+\|R_I^l\|_{L^2(\Omega_t)}^2+\|R_p^l\|_{H^{1/2}(\Gamma_t)}^2\right),\ l\leqslant 3,\\
&\left|\int_{\Gamma_t}R_p^l\left(\mathcal{D}_t^{l+1}v\cdot\nu\right)\mathrm{d}S\right|\leqslant C\left(E_l(t)+\|R_I^l\|_{L^2(\Omega_t)}^2+\|R_p^l\|_{H^{1/2}(\Gamma_t)}^2\right),\ l\geqslant 4.
\end{aligned}
$$

Step 4. We estimate $I_{31}^l(t)$. We only present estimates for $l\geqslant 3$, and the cases of $l\leqslant 2$ are easier. Actually, by the *a priori* assumptions (1.6) and the trace theorem, one has

$$
\begin{aligned}
&\|\bar{\nabla}\left(\mathcal{D}_t^3 v\cdot\mathcal{D}_t\nu\right)\|_{L^2(\Gamma_t)}^2\leqslant\|\bar{\nabla}\mathcal{D}_t^3 v\star\mathcal{D}_t\nu\|_{L^2(\Gamma_t)}^2+\|\mathcal{D}_t^3 v\star\bar{\nabla}\mathcal{D}_t\nu\|_{L^2(\Gamma_t)}^2\\
&\leqslant C\left(\|\mathcal{D}_t\nu\|_{L^\infty(\Gamma_t)}^2\|\mathcal{D}_t^3 v\|_{H^{3/2}(\Omega_t)}^2+\underbrace{\|\mathcal{D}_t^3 v\star\bar{\nabla}^2 v\star\nu\|_{L^2(\Gamma_t)}^2}_{=:L_{31}^3(t)}+\|\mathcal{D}_t^3 v\star\bar{\nabla}v\star\bar{\nabla}\nu\|_{L^2(\Gamma_t)}^2\right)\leqslant C\bar{E}(t).
\end{aligned}
$$

Above, we have applied the Sobolev embedding, i.e. for $p^{-1}+q^{-1}=2^{-1}, p=2\delta^{-1}$ with $\delta>0$ small enough, it holds $L_{31}^3(t)\leqslant C\|\mathcal{D}_t^3 v\|_{H^{1-\delta}(\Gamma_t)}^2\|\bar{\nabla}^2 v\|_{H^\delta(\Gamma_t)}^2$, and $\|\mathcal{D}_t^3 v\|_{H^{1-\delta}(\Gamma_t)}^2\|\bar{\nabla}^2 v\|_{H^\delta(\Gamma_t)}^2\leqslant\|\mathcal{D}_t^3 v\|_{H^{3/2-\delta}(\Omega_t)}^2\|v\|_{H^{5/2+\delta}(\Omega_t)}^2\leqslant C\bar{E}(t)$, by using the trace theorem. As for $l\geqslant 4$, it follows that

$$
\begin{aligned}
\|\bar{\nabla}\left(\mathcal{D}_t^l v\cdot\mathcal{D}_t\nu\right)\|_{L^2(\Gamma_t)}^2&\leqslant C\left(\|\mathcal{D}_t\nu\|_{L^\infty(\Gamma_t)}^2\|\mathcal{D}_t^l v\|_{H^1(\Gamma_t)}^2+\|\mathcal{D}_t\nu\|_{W^{1,4}(\Gamma_t)}^2\|\mathcal{D}_t^l v\|_{L^4(\Gamma_t)}^2\right)\\
&\leqslant C\left(\|\mathcal{D}_t^l v\|_{H^{3/2}(\Omega_t)}^2+E_{l-1}(t)\|\mathcal{D}_t^l v\|_{H^1(\Omega_t)}^2\right)\leqslant CE_l(t),
\end{aligned}
$$

where we have used $\mathcal{D}_t\nu=\bar{\nabla}v\star\nu$ from lemma A.1 and $\|\nu\|_{H^{2+\delta}(\Gamma_t)}\leqslant C$ by (1.6) together with (2.3).

Step 5. For $I_{12}^l(t)$, we recall that it holds $[\mathcal{D}_t^{l+1},\nabla]p=\sum_{\beta_1\leqslant l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H+R_{II}^l+R_{\nabla H,H}^l$ by lemma 2.9. Clearly, we have $\|\sum_{\beta_1\leqslant l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H\|_{L^2(\Omega_t)}^2\leqslant C\bar{E}(t)$ for $l\leqslant 3$, and $\|\sum_{\beta_1\leqslant l}\nabla\mathcal{D}_t^{\beta_1}v\star\nabla H\star H\|_{L^2(\Omega_t)}^2\leqslant CE_l(t)$ as $l\geqslant 4$.

Step 6. Finally, controlling $I_{11}^l(t)$ is trickier. Let u be a solution to

$$
\begin{cases}
-\Delta u=\operatorname{div}\mathcal{D}_t^{l+1}v, & \text{in }\Omega_t,\\
u=0, & \text{on }\Gamma_t,
\end{cases}
$$

where $l \geqslant 1$. We first recall the elliptic estimates (see, e.g. [26, proposition 3.8])

$$\|\partial_\nu u\|_{H^1(\Gamma_t)} + \|\nabla u\|_{H^{3/2}(\Omega_t)} \leqslant C\|\operatorname{div} \mathcal{D}_t^{l+1} v\|_{H^{1/2}(\Omega_t)}. \tag{3.4}$$

Then, we integrate by parts to obtain $I_{11}^l(t) = -\int_{\Omega_t} \Delta \mathcal{D}_t^{l+1} p u \mathrm{d}x - \int_{\Gamma_t} \mathcal{D}_t^{l+1} p \partial_\nu u \mathrm{d}S =: I_{111}^l(t) + I_{112}^l(t)$. Again by integration by parts, lemma 2.11 and the divergence theorem, it follows that

$$\begin{aligned} I_{111}^l(t) =& \int_{\Omega_t} \left(v \otimes \mathcal{D}_t^{l+1} v\right) : \nabla^2 u \mathrm{d}x - \int_{\Omega_t} \left(R_{II}^l + R_{\nabla H, H}^l + \sum_{\beta_1 \leqslant l} \nabla \mathcal{D}_t^{\beta_1} v \star \nabla H \star H \right) \cdot \nabla u \mathrm{d}x \\ & - \int_{\Omega_t} \operatorname{div} \mathcal{D}_t^{l+1} (H \cdot \nabla H) u \mathrm{d}x - \int_{\Gamma_t} v^i \mathcal{D}_t^{l+1} v^j \partial_i u \nu_j \mathrm{d}S \\ \leqslant & C\left(\|u\|_{H^2(\Omega_t)}^2 + E_l(t) + \|R_{II}^l\|_{L^2(\Omega_t)}^2 + \|R_{\nabla H, H}^l\|_{L^2(\Omega_t)}^2 \right) \\ & + \underbrace{\int_{\Omega_t} \operatorname{div}\left(v^i \mathcal{D}_t^{l+1} v \partial_i u\right) \mathrm{d}x}_{=: L_{1111}^l(t)} \underbrace{- \int_{\Omega_t} \operatorname{div} \mathcal{D}_t^{l+1} (H \cdot \nabla H) u \mathrm{d}x}_{=: L_{1112}^l(t)}. \end{aligned}$$

We estimate the first term by using lemma 2.10. Indeed, it holds

$$\begin{aligned} |L_{1111}^l(t)| &= \Big| \int_{\Omega_t} \nabla v \star \mathcal{D}_t^{l+1} v \star \nabla u + v \star \operatorname{div} \mathcal{D}_t^{l+1} v \star \nabla u + v \star \mathcal{D}_t^{l+1} v \star \nabla^2 u \mathrm{d}x \Big| \\ &\leqslant C\left(\|u\|_{H^2(\Omega_t)}^2 + E_l(t) + \|R_I^l\|_{L^2(\Omega_t)}^2 \right). \end{aligned}$$

To control $L_{1112}^l(t)$, it is important to note that the integration by parts method used previously is not applicable. However, as indicated in lemmas 2.6 and 2.7, a one-order material derivative can be substituted for a one-order spatial derivative due to the divergence-free condition. In fact, we have from lemma 2.7 that $\operatorname{div} \mathcal{D}_t^{l+1}(H \cdot \nabla H) = \partial_i \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i + \nabla^3 \mathcal{D}_t^{l-1} v \star H \star H + \text{L.O.T.}$, and

$$\begin{aligned} |L_{1112}^l(t)| &\leqslant \left| \int_{\Omega_t} \partial_i \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i u \mathrm{d}x \right| + C\|u\|_{L^2(\Omega_t)}^2 + C\|\nabla^3 \mathcal{D}_t^{l-1} v \star H \star H\|_{L^2(\Omega_t)}^2 + M_{1112}^l(t) \\ &\leqslant C\|u\|_{H^1(\Omega_t)}^2 + CE_l(t) + M_{1112}^l(t), \end{aligned}$$

where we have used $H \cdot \nu = 0$, and

$$\begin{aligned} \left| \int_{\Omega_t} \partial_i \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i u \mathrm{d}x \right| &= \left| \int_{\Omega_t} \partial_m \mathcal{D}_t^l v^j \partial_i \partial_j H^m H^i u + \partial_m \mathcal{D}_t^l v^j \partial_j H^m H^i \partial_i u \mathrm{d}x \right| \\ &\leqslant CE_l(t) + C\|u\|_{H^1(\Omega_t)}^2, \end{aligned}$$

by integration by parts. Also, $M_{1112}^l(t)$ contains lower-order terms (at most $\nabla^2 \mathcal{D}_t^{l-1}$) which can be controlled in the same fashion as before. These, together with the fact $\|u\|_{H^2(\Omega_t)}^2 \leqslant \|\operatorname{div} \mathcal{D}_t^{l+1} v\|_{L^2(\Omega_t)}^2 \leqslant C\|R_I^l\|_{L^2(\Omega_t)}^2$, it holds $|I_{111}^l(t)| \leqslant C(\bar{E}(t) + \|R_I^l\|_{L^2(\Omega_t)}^2 + \|R_{II}^l\|_{L^2(\Omega_t)}^2 + \|R_{\nabla H, H}^l\|_{L^2(\Omega_t)}^2)$ for $l \leqslant 3$, and for $l \geqslant 4, |I_{111}^l(t)| \leqslant C(E_l(t) + \|R_I^l\|_{L^2(\Omega_t)}^2 + \|R_{II}^l\|_{L^2(\Omega_t)}^2 + \|R_{\nabla H, H}^l\|_{L^2(\Omega_t)}^2)$.

We are left with $I_{112}^l(t)$. Applying lemma 2.12 and integration by parts, one has $\int_{\Gamma_t} \mathcal{D}_t^{l+1} p \partial_\nu \mathrm{d}S u = \int_{\Gamma_t} \bar{\nabla}(\mathcal{D}_t^l v \cdot \nu) \cdot \bar{\nabla} \partial_\nu u \mathrm{d}S + \int_{\Gamma_t} R_p^l \partial_\nu u \mathrm{d}S$. Then, we use (3.4) to deduce

$$|I_{112}^l(t)| \leqslant C\left(\|\bar{\nabla}\left(\mathcal{D}_t^l v\cdot\nu\right)\|_{L^2(\Gamma_t)}^2+\|\partial_\nu u\|_{H^1(\Gamma_t)}^2+\|R_p^l\|_{L^2(\Gamma_t)}^2\right)$$
$$\leqslant C\left(\bar{E}(t)+\|R_I^l\|_{H^{1/2}(\Omega_t)}^2+\|R_p^l\|_{L^2(\Gamma_t)}^2\right),\quad l\leqslant 3.$$

Similarly, $|I_{112}^l(t)|\leqslant C(E_l(t)+\|R_I^l\|_{H^{1/2}(\Omega_t)}^2+\|R_p^l\|_{L^2(\Gamma_t)}^2)$ for $l\geqslant 4$. This completes the proof. □

4. Estimates for the pressure

In this section, we treat the pressure and will show that

$$\sup_{t\in[0,T]}\|p\|_{H^3(\Omega_t)}\leqslant C, \tag{4.1}$$

where the constant C depends on the time $T>0$, the *a priori* assumptions $\mathcal{N}_T,\mathcal{M}_T$, and the initial data $\|v_0\|_{H^6(\Omega_0)},\|H_0\|_{H^6(\Omega_0)}$ and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$. For this purpose, we assume the *a priori* assumptions (1.6) for some $T>0$. As a result, it follows that $\sup_{0\leqslant t<T}\|h\|_{H^{3+\delta}(\Gamma)}\leqslant C$ and $\sup_{0\leqslant t<T}\|B\|_{H^{1+\delta}(\Gamma_t)}\leqslant C$. In particular, we have $\|p\|_{H^{1+\delta}(\Gamma_t)}\leqslant C$ and

$$\int_0^T\|p\|_{H^1(\Gamma_t)}^2\mathrm{d}t\leqslant C(\mathcal{N}_T,\mathcal{M}_T)\,T. \tag{4.2}$$

Recalling we define $H^{1/2}(\Gamma_t)$ via the harmonic extension. From lemma 2.4 and (A1), we obtain

$$\begin{aligned}\|\partial_\nu p\|_{L^2(\Gamma_t)}^2&\leqslant C\left(\|\bar{\nabla}p\|_{L^2(\Gamma_t)}^2+\|\nabla p\|_{L^2(\Omega_t)}^2+\|\Delta p\|_{L^2(\Omega_t)}^2\right)\\&\leqslant C\left(\|\bar{\nabla}p\|_{L^2(\Gamma_t)}^2+\|p\|_{H^{1/2}(\Gamma_t)}^2+\|\Delta p\|_{L^2(\Omega_t)}^2\right)\\&\leqslant C\left(\|p\|_{H^1(\Gamma_t)}^2+\|\Delta p\|_{L^2(\Omega_t)}^2\right)\leqslant C(\mathcal{N}_T,\mathcal{M}_T)(1+T).\end{aligned} \tag{4.3}$$

For higher-order derivatives, we have the following results.

Proposition 4.1. *Assume that Γ_t is uniformly $H^{3+\delta}(\Gamma)$-regular for $\delta>0$ sufficiently small. For smooth function f, it holds*

$$\|\nabla^2 f\|_{L^2(\Gamma_t)}^2\leqslant C\left(\|\Delta f\|_{H^1(\Omega_t)}^2+\|f\|_{H^2(\Gamma_t)}^2\right), \tag{4.4}$$
$$\|\nabla^3 f\|_{L^2(\Gamma_t)}^2\leqslant C\left(\|\Delta f\|_{H^2(\Omega_t)}^2+\|f\|_{H^3(\Gamma_t)}^2\right). \tag{4.5}$$

Proof. For any $k\in\{1,2,3\}$, it holds $\|\nabla\partial_k f\|_{L^2(\Gamma_t)}^2\leqslant C(\|\bar{\nabla}\partial_k f\|_{L^2(\Gamma_t)}^2+\|\nabla^2 f\|_{L^2(\Omega_t)}^2+\|\nabla\Delta f\|_{L^2(\Omega_t)}^2)$ by applying lemma 2.4. Recall that we extend the unit outer normal ν to Ω_t by the harmonic extension and $\|\tilde{\nu}\|_{H^{5/2+\delta}(\Omega_t)}\leqslant C$. This, combined with lemmas A.1 and 2.4 implies that

$$\begin{aligned}\|\bar{\nabla}\partial_k f\|^2_{L^2(\Gamma_t)} &\leqslant C\left(\|\nabla\bar{\nabla}f\|^2_{L^2(\Gamma_t)} + \|\nabla f\star\nabla\tilde{\nu}\star\tilde{\nu}\|^2_{L^2(\Gamma_t)}\right)\\ &\leqslant C\Big(\|\bar{\nabla}^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla\Delta f\|^2_{L^2(\Omega_t)} + \|\nabla f\|^2_{H^1(\Omega_t)}\\ &\quad + \|\nabla f\star\nabla\tilde{\nu}\star\nabla\tilde{\nu}\|^2_{L^2(\Omega_t)} + \|\nabla^2 f\star\nabla\tilde{\nu}\|^2_{L^2(\Omega_t)}\Big)\\ &\leqslant C\left(\|\bar{\nabla}^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla\Delta f\|^2_{L^2(\Omega_t)} + \|\nabla f\|^2_{H^1(\Omega_t)}\right),\end{aligned}$$

and $\|\nabla\partial_k f\|^2_{L^2(\Gamma_t)} \leqslant C(\|\bar{\nabla}^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla\Delta f\|^2_{L^2(\Omega_t)} + \|\nabla f\|^2_{H^1(\Omega_t)})$ as a consequence. Next, we apply (A1) and lemma A.7 to find that

$$\begin{aligned}\|\nabla f\|^2_{H^1(\Omega_t)} &\leqslant C\left(\|\partial_\nu f\|^2_{H^{1/2}(\Gamma_t)} + \|\nabla f\|^2_{L^2(\Omega_t)} + \|\Delta f\|^2_{L^2(\Omega_t)}\right)\\ &\leqslant C\left(\|\partial_\nu f\|^2_{H^{1/2}(\Gamma_t)} + \|f\|^2_{H^{1/2}(\Gamma_t)} + \|\Delta f\|^2_{L^2(\Omega_t)}\right).\end{aligned}$$

To control $\|\partial_\nu f\|^2_{H^{1/2}(\Gamma_t)}$, using lemma 2.4 and by interpolation, one has

$$\|\partial_\nu f\|^2_{H^{1/2}(\Gamma_t)} \leqslant \varepsilon\left(\|\nabla^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla f\|^2_{H^1(\Omega_t)}\right) + C_\varepsilon\left(\|\bar{\nabla} f\|^2_{L^2(\Gamma_t)} + \|f\|^2_{H^{1/2}(\Gamma_t)} + \|\Delta f\|^2_{L^2(\Omega_t)}\right),$$

where $\varepsilon > 0$ is sufficiently small. We conclude that

$$\|\nabla f\|^2_{H^1(\Omega_t)} \leqslant \varepsilon\|\nabla^2 f\|^2_{L^2(\Gamma_t)} + C\left(\|f\|^2_{H^1(\Gamma_t)} + \|\Delta f\|^2_{L^2(\Omega_t)}\right), \tag{4.6}$$

and then (4.4) follows.

To prove the second claim, by lemma 2.4, it follows that $\|\nabla\partial_k\partial_l f\|^2_{L^2(\Gamma_t)} \leqslant C(\|\bar{\nabla}\partial_k\partial_l f\|^2_{L^2(\Gamma_t)} + \|\nabla^3 f\|^2_{L^2(\Omega_t)} + \|\nabla^2\Delta f\|^2_{L^2(\Omega_t)}), k\in\{1,2,3\}$. To estimate $\|\nabla^3 f\|^2_{L^2(\Omega_t)}$, from lemma A.7, we obtain $\|\partial_i f\|^2_{H^2(\Omega_t)} \leqslant C(\|\partial_\nu\partial_i f\|^2_{H^{1/2}(\Gamma_t)} + \|\nabla f\|^2_{L^2(\Omega_t)} + \|\nabla\Delta f\|^2_{L^2(\Omega_t)})$ for $i\in\{1,2,3\}$. Then, we obtain $\|\partial_\nu\partial_i f\|^2_{H^{1/2}(\Gamma_t)} \leqslant \varepsilon\|\bar{\nabla}\partial_\nu\partial_i f\|^2_{L^2(\Gamma_t)} + C_\varepsilon\|\partial_\nu\partial_i f\|^2_{L^2(\Gamma_t)}$ by interpolation, where $\varepsilon > 0$ is small enough. These, combined with (4.4), (A1) and the fact that $\|\tilde{\nu}\|_{H^{5/2+\delta}(\Omega_t)} \leqslant C$, yield

$$\begin{aligned}\|\nabla f\|^2_{H^2(\Omega_t)} &\leqslant \varepsilon\left(\|\nabla^3 f\|^2_{L^2(\Gamma_t)} + \|\nabla^2 f\star\nabla\tilde{\nu}\|^2_{L^2(\Gamma_t)}\right) + \|f\|^2_{H^2(\Gamma_t)} + \|\Delta f\|^2_{H^1(\Omega_t)}\\ &\leqslant \varepsilon\|\nabla^3 f\|^2_{L^2(\Gamma_t)} + \|f\|^2_{H^2(\Gamma_t)} + \|\Delta f\|^2_{H^1(\Omega_t)}.\end{aligned} \tag{4.7}$$

Then, we control $\|\bar{\nabla}\partial_k\partial_l f\|^2_{L^2(\Gamma_t)}$ by lemma 2.4 and the fact that $\Delta\tilde{\nu} = 0$

$$\begin{aligned}\|\bar{\nabla}\partial_k\partial_l f\|^2_{L^2(\Gamma_t)} &\leqslant C\left(\|\bar{\nabla}^2\partial_l f\|^2_{L^2(\Gamma_t)} + \|\nabla\bar{\nabla}\nabla f\|^2_{L^2(\Omega_t)} + \|\Delta\bar{\nabla}\nabla f\|^2_{L^2(\Omega_t)} + \|\nabla^2 f\|^2_{L^2(\Gamma_t)}\right)\\ &\leqslant C\left(\|\bar{\nabla}^2\partial_l f\|^2_{L^2(\Gamma_t)} + \|\nabla^2 f\|^2_{H^1(\Omega_t)} + \|\Delta f\|^2_{H^2(\Omega_t)} + \|f\|^2_{H^2(\Gamma_t)}\right).\end{aligned}$$

Again by (4.4) and lemma 2.4, we obtain

$$\begin{aligned}&\|\bar{\nabla}^2\partial_l f\|^2_{L^2(\Gamma_t)}\\ &\quad\leqslant \|\bar{\nabla}^3 f\|^2_{L^2(\Gamma_t)} + \|\nabla^3 f\|^2_{L^2(\Omega_t)} + \|\nabla^3 f\star\nabla\tilde{\nu}\|^2_{L^2(\Omega_t)} + \|\nabla^2 f\star\nabla\tilde{\nu}\|^2_{L^2(\Omega_t)} + \|\nabla^2\Delta f\|^2_{L^2(\Omega_t)}\\ &\qquad + \|\nabla f\star\nabla^2\tilde{\nu}\|^2_{L^2(\Omega_t)} + \|\nabla^2 f\|^2_{L^2(\Gamma_t)} + \|\nabla^2 f\star\nabla^2\tilde{\nu}\|^2_{L^2(\Omega_t)} + \|\nabla f\|^2_{H^{3/2+\delta}(\Omega_t)}\\ &\quad\leqslant C\left(\|f\|^2_{H^3(\Gamma_t)} + \|\nabla f\|^2_{H^2(\Omega_t)} + \|\Delta f\|^2_{H^2(\Omega_t)}\right).\end{aligned}$$

Recalling (4.7), we conclude that $\|\nabla^3 f\|^2_{L^2(\Gamma_t)} \leqslant \varepsilon\|\nabla^3 f\|^2_{L^2(\Gamma_t)} + C(\|f\|^2_{H^3(\Gamma_t)} + \|\Delta f\|^2_{H^2(\Omega_t)})$, and this completes the proof. □

We will proceed with the estimates for the pressure.

Lemma 4.2. *Assume that (1.6) holds for some $T>0$. Then, we have*

$$\sup_{t\in[0,T]} \|\nabla p\|^2_{L^2(\Omega_t)} \leqslant e^{C(\mathcal{N}_T,\mathcal{M}_T)(1+T)}\left(1+\|\nabla p\|^2_{L^2(\Omega_0)}\right).$$

Proof. From lemma A.1, Reynolds transport theorem, and the divergence-free condition, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega_t}|\nabla p|^2\mathrm{d}x = \int_{\Omega_t}\nabla\mathcal{D}_t p\cdot\nabla p\mathrm{d}x + \int_{\Omega_t}\nabla v\star\nabla p\star\nabla p\mathrm{d}x =: I_1(t)+I_2(t).$$

Clearly, (1.6) implies $|I_2(t)|\leqslant C\|\nabla p\|^2_{L^2(\Omega_t)}$. For $|I_1(t)|$, by (2.10), (4.3), and the divergence theorem, we have $|I_1(t)|\leqslant\int_{\Gamma_t}\mathcal{D}_t p\partial_\nu p\mathrm{d}S - \int_{\Omega_t}\mathcal{D}_t p\Delta p\mathrm{d}x \leqslant C(1+\|p\|^2_{H^1(\Gamma_t)}) - \int_{\Omega_t}\mathcal{D}_t p\Delta p\mathrm{d}x$. To control $\int_{\Omega_t}\mathcal{D}_t p\Delta p\mathrm{d}x$, we consider the following elliptic equation

$$\begin{cases} -\Delta u = \Delta p, & \text{in } \Omega_t,\\ u = 0, & \text{on } \Gamma_t. \end{cases}$$

Then, we see that $-\int_{\Omega_t}\mathcal{D}_t p\Delta p\mathrm{d}x = \int_{\Omega_t}\Delta\mathcal{D}_t pu\mathrm{d}x + \int_{\Gamma_t}\mathcal{D}_t p\partial_\nu u\mathrm{d}S =: I_{11}(t)+I_{12}(t)$. Note that (2.4) implies $|\Delta p|\leqslant C$, and we have $\|u\|_{H^1(\Omega_t)}\leqslant C$. Also, we get $\|\nabla u\|^2_{L^2(\Gamma_t)}\leqslant C$ and $|I_{12}(t)|\leqslant\|\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + \|\partial_\nu u\|^2_{L^2(\Gamma_t)} \leqslant C(1+\|p\|^2_{H^1(\Gamma_t)})$ from lemma 2.4. We are left with $I_{11}(t)$, for which one can repeat the argument in [26, Propsition 6.3] to deduce $\|u\|^2_{H^2(\Omega_t)}\leqslant C(1+\|p\|^2_{H^1(\Gamma_t)})$. Then, by (1.1), (1.6), lemma 2.11, (2.8) and (4.3), we integrate by parts to obtain $I_{11}(t)\leqslant C(1+\|p\|^2_{H^1(\Gamma_t)}+\|\nabla p\|^2_{L^2(\Omega_t)})$. Combining the above calculations, it follows that $I_1(t)+I_2(t)\leqslant C(1+\|p\|^2_{H^1(\Gamma_t)}+\|\nabla p\|^2_{L^2(\Omega_t)})$. With the help of estimate (4.2), the proof is complete. □

Lemma 4.3. *Assume that (1.6) holds for some $T>0$. Then, we have*

$$\int_0^T\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)}\mathrm{d}t \leqslant C(\mathcal{N}_T,\mathcal{M}_T)(1+T).$$

Proof. We define $I(t):=\int_{\Gamma_t}\bar\nabla p\cdot\bar\nabla(\nabla v\nu\cdot\nu)\mathrm{d}S$, and from the hypothesis (1.6) and (4.2), we see that $|I(t)|\leqslant C\|\bar\nabla p\|^2_{L^2(\Gamma_t)} + C\|\nabla^2 v\|^2_{L^2(\Gamma_t)} + C\|\nabla v\star B\|^2_{L^2(\Gamma_t)}\leqslant C$. Again by (1.6), the divergence theorem, lemma A.1 and (2.2), we deduce for sufficiently small $\varepsilon>0$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(t) &\leqslant C|I(t)| + \int_{\Gamma_t}\mathcal{D}_t\bar\nabla p\cdot\bar\nabla(\nabla v\nu\cdot\nu) + \bar\nabla p\cdot\mathcal{D}_t\bar\nabla(\nabla v\nu\cdot\nu)\mathrm{d}S\\ &\leqslant C_\varepsilon + \varepsilon\|\bar\nabla\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + \int_{\Gamma_t}\bar\nabla p\cdot\bar\nabla\mathcal{D}_t(\nabla v\nu\cdot\nu)\mathrm{d}S\\ &\leqslant C_\varepsilon + \varepsilon\|\bar\nabla\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} - \int_{\Gamma_t}\Delta_B p\mathcal{D}_t(\nabla v\nu\cdot\nu)\mathrm{d}S =: C_\varepsilon + \varepsilon I_1(t) + I_2(t).\end{aligned}$$

By (1.6), (2.10) and (4.2), it holds $|I_1(t)|\leqslant C(1+\|v_n\|^2_{H^3(\Gamma_t)} + \|\bar\nabla B\star B\star v_n\|^2_{L^2(\Gamma_t)} + \|\bar\nabla p\|^2_{H^1(\Gamma_t)})\leqslant C(1+\|\bar\nabla^2 p\|^2_{L^2(\Gamma_t)})$. For $|I_2(t)|$, from (1.6), lemma (A.1) and the divergence

theorem, we have

$$\begin{aligned}
|I_2(t)| &\leqslant -\int_{\Gamma_t} \Delta_B p\left(\nabla \mathcal{D}_t v \nu\cdot\nu\right) dS + C\|\bar{\nabla} p\|_{L^1(\Gamma_t)} \\
&= -\int_{\Gamma_t} \Delta_B p\left(\nabla\left(-\nabla p + H\cdot\nabla H\right)\nu\cdot\nu\right) dS + \varepsilon\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon \\
&\leqslant \int_{\Gamma_t} \Delta_B p\left(\nabla^2 p\nu\cdot\nu\right) \mathrm{d}S - \int_{\Gamma_t} \Delta_B p \star \nabla^2 H \star \nabla H \star \nu \star \nu \mathrm{d}S + \varepsilon\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon \\
&\leqslant \int_{\Gamma_t} \Delta_B p\left(\nabla^2 p\nu\cdot\nu\right) \mathrm{d}S + \varepsilon\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon .
\end{aligned}$$

Recalling $|\Delta p| \leqslant C$ and by (2.1), (4.3), the divergence theorem, for $\varepsilon > 0$ small enough, we deduce

$$\begin{aligned}
\int_{\Gamma_t} \Delta_B p\left(\nabla^2 p\nu\cdot\nu\right) \mathrm{d}S &= \int_{\Gamma_t} \Delta_B p\Delta p - \Delta_B p\Delta_B p - \Delta_B p\mathcal{A}\partial_\nu p \mathrm{d}S \\
&\leqslant C + \frac{\varepsilon}{2}\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon\|p\|^2_{H^1(\Gamma_t)} - \int_{\Gamma_t} |\bar{\nabla}^2 p|^2 \mathrm{d}S \\
&\quad + \|\Delta_B p\|_{L^2(\Gamma_t)}\|\partial_\nu p\|_{L^2(\Gamma_t)}\|p\|_{L^\infty(\Gamma_t)} \leqslant -\frac{3}{4}\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon .
\end{aligned}$$

Above, we have applied [11, remark 2.4] that $\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} \leqslant \|\Delta_B p\|^2_{L^2(\Gamma_t)} + C\int_{\Gamma_t} |B|^2|\bar{\nabla} p|^2 \mathrm{d}S$. Combining the above calculations, the proof is complete since $\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leqslant -\frac{1}{2}\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C$. □

Lemma 4.4. *Assume that (1.6) holds for some $T > 0$. Then, we have*

$$\sup_{t\in[0,T]} \|\nabla^2 p\|^2_{L^2(\Omega_t)} \leqslant e^{C(\mathcal{N}_T, \mathcal{M}_T)(1+T)}\left(1 + \|\nabla^2 p\|^2_{L^2(\Omega_0)}\right).$$

Proof. We differentiate and apply lemma 2.2 to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega_t} |\nabla^2 p|^2 \mathrm{d}x &= \int_{\Omega_t} \nabla^2 \mathcal{D}_t p : \nabla^2 p \mathrm{d}x + \int_{\Omega_t} \nabla^2 v \star \nabla p \star \nabla^2 p + \nabla v \star \nabla^2 p \star \nabla^2 p \mathrm{d}x \\
&=: I_1(t) + I_2(t) .
\end{aligned}$$

From (1.6), (2.4) and using lemma 4.2, we have

$$\begin{aligned}
I_1(t) &\leqslant \int_{\Omega_t} \sum_{i,j} \partial_i\left(\partial_j \mathcal{D}_t p \partial_i \partial_j p\right) \mathrm{d}x - \int_{\Omega_t} \nabla \mathcal{D}_t p \cdot \nabla \Delta p \mathrm{d}x \\
&\leqslant \int_{\Gamma_t} \sum_j \partial_j \mathcal{D}_t p \partial_\nu \partial_j p \mathrm{d}S + \int_{\Omega_t} \Delta \mathcal{D}_t p \Delta p \mathrm{d}x - \int_{\Gamma_t} \partial_\nu \mathcal{D}_t p \Delta p \mathrm{d}S \\
&\leqslant C\sum_j \|\partial_\nu \partial_j p\|^2_{L^2(\Gamma_t)} + C\|\partial_\nu \mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + C\|\Delta \mathcal{D}_t p\|^2_{L^2(\Omega_t)} =: I_{11}(t) + I_{12}(t) + I_{13}(t) , \\
I_2(t) &\leqslant C\left(\|v\|^2_{H^{7/2}(\Omega_t)}\|\nabla p\|^2_{L^6(\Omega_t)} + \|\nabla^2 p\|^2_{L^2(\Omega_t)}\right) \leqslant C\left(1 + \|\nabla^2 p\|^2_{L^2(\Omega_t)}\right).
\end{aligned}$$

We apply lemmas 2.11 and 4.2, and (4.4) to obtain $|I_{13}(t)| \leqslant C(1 + \|\nabla^2 p\|^2_{L^2(\Omega_t)})$ and $|I_{11}(t)| \leqslant C(1 + \|p\|^2_{H^2(\Gamma_t)})$. Finally, (1.6), lemmas 2.11 and 2.4, and (A1) imply that

$$
\begin{aligned}
|I_{12}(t)| &\leqslant C\left(\|\bar{\nabla}\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + \|\nabla\mathcal{D}_t p\|^2_{L^2(\Omega_t)} + \|\Delta\mathcal{D}_t p\|^2_{L^2(\Omega_t)}\right)\\
&\leqslant C\left(\|\bar{\nabla}\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + \|\mathcal{D}_t p\|^2_{H^{1/2}(\Gamma_t)} + \|\Delta\mathcal{D}_t p\|^2_{L^2(\Omega_t)}\right)\\
&\leqslant C\left(1 + \|p\|^2_{H^2(\Gamma_t)} + \|\nabla^2 p\|^2_{L^2(\Omega_t)}\right).
\end{aligned}
$$

Combined with (4.2) and lemma 4.3, the proof is complete. □

Lemma 4.5. *Assume that (1.6) holds for some $T>0$. Then, we have*

$$
\sup_{t\in[0,T]}\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + \int_0^T \|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)}\mathrm{d}t \leqslant C\left(T,\mathcal{N}_T,\mathcal{M}_T,\|\bar{\nabla}^2 p\|_{L^2(\Gamma_0)},\|\nabla p\|_{H^1(\Omega_0)}\right).
$$

Proof. We define

$$
I(t) := \int_{\Gamma_t}\bar{\nabla}^2 p : \bar{\nabla}^2(\nabla v\nu\cdot\nu)\,\mathrm{d}S + \varepsilon\int_{\Gamma_t}|\bar{\nabla}^2 p|^2\mathrm{d}S =: I_1(t) + \varepsilon I_2(t),
$$

where $\varepsilon>0$ will be chosen later. From (1.6), (4.2), lemmas 4.3 and A.6, we have

$$
\begin{aligned}
|I_1(t)| &\leqslant C_\varepsilon\left(\|\nabla^3 v\|^2_{L^2(\Gamma_t)} + \|\nabla^2 v\star B\|^2_{L^2(\Gamma_t)} + \|\nabla v\star\bar{\nabla}B\|^2_{L^2(\Gamma_t)}\right)\\
&\quad + \frac{\varepsilon}{2}\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} \leqslant \frac{\varepsilon}{2}\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon,
\end{aligned}
$$

and therefore, $I(t)\geqslant -C_\varepsilon + \frac{\varepsilon}{2}\|\bar{\nabla}^2 p(\cdot,t)\|^2_{L^2(\Gamma_t)}$. We differentiate and use (1.6), (4.2), the divergence theorem, lemmas A.1 and A.6 to obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I_1(t) &\leqslant C|I_1(t)| + \int_{\Gamma_t}\mathcal{D}_t\bar{\nabla}^2 p : \bar{\nabla}^2(\nabla v\nu\cdot\nu) + \bar{\nabla}^2 p : \mathcal{D}_t\bar{\nabla}^2(\nabla v\nu\cdot\nu)\,\mathrm{d}S\\
&\leqslant C_\varepsilon + \varepsilon\left(\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + \|\bar{\nabla}^2\mathcal{D}_t p\|^2_{L^2(\Gamma_t)}\right) + \int_{\Gamma_t}\bar{\nabla}^2 p : \bar{\nabla}^2\mathcal{D}_t(\nabla v\nu\cdot\nu)\,\mathrm{d}S\\
&\leqslant \varepsilon\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + C_\varepsilon + \varepsilon\underbrace{\|\bar{\nabla}^2\mathcal{D}_t p\|^2_{L^2(\Gamma_t)}}_{=:I_{11}(t)} \underbrace{-\int_{\Gamma_t}\bar{\nabla}\Delta_B p\cdot\bar{\nabla}\mathcal{D}_t(\nabla v\nu\cdot\nu)\,\mathrm{d}S}_{=:I_{12}(t)}.
\end{aligned}
$$

The first term can be controlled by (1.6), (2.10), (4.2) and lemma A.6, i.e. $|I_{11}(t)|\leqslant C(1+\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + \|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)})$. As for $I_{12}(t)$, applying (1.6), lemma (A.1) and the divergence theorem, it follows that

$$
\begin{aligned}
|I_{12}(t)| &\leqslant -\int_{\Gamma_t}\bar{\nabla}\Delta_B p\cdot\bar{\nabla}(\nabla\mathcal{D}_t v\nu\cdot\nu)\,\mathrm{d}S + C\left(\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + 1\right)\\
&= -\int_{\Gamma_t}\bar{\nabla}\Delta_B p\cdot\bar{\nabla}\left(\nabla\left(-\nabla p + H\cdot\nabla H\right)\nu\cdot\nu\right)\mathrm{d}S + C\left(\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + 1\right)\\
&\leqslant \int_{\Gamma_t}\bar{\nabla}\Delta_B p\cdot\bar{\nabla}\left(\nabla^2 p\nu\cdot\nu\right)\mathrm{d}S - \int_{\Gamma_t}\bar{\nabla}\Delta_B p\cdot\bar{\nabla}\left(\nabla^2 H\star H\star\nu\star\nu\right)\mathrm{d}S + C\left(\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + 1\right)\\
&\leqslant \int_{\Gamma_t}\bar{\nabla}\Delta_B p\cdot\bar{\nabla}\left(\nabla^2 p\nu\cdot\nu\right)\mathrm{d}S + \varepsilon\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)} + C\left(\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} + 1\right).
\end{aligned}
$$

To estimate $\int_{\Gamma_t}\bar{\nabla}\Delta_B p\cdot\bar{\nabla}(\nabla^2 p\nu\cdot\nu)dS$, by (1.6), (2.1), (2.4), (4.3), lemma A.7 and the divergence theorem, it holds

$$\begin{aligned}\int_{\Gamma_t}\bar{\nabla}\Delta_{\mathcal{B}}p\cdot\bar{\nabla}\left(\nabla^2 p\nu\cdot\nu\right)\mathrm{d}S &= \int_{\Gamma_t}\bar{\nabla}\Delta_{\mathcal{B}}p\cdot\bar{\nabla}\Delta p-\bar{\nabla}\Delta_{\mathcal{B}}p\cdot\bar{\nabla}\Delta_{\mathcal{B}}p-\bar{\nabla}\Delta_{\mathcal{B}}p\cdot\bar{\nabla}\left(\mathcal{A}\partial_\nu p\right)\mathrm{d}S\\ &\leqslant C_\varepsilon\|\Delta p\|^2_{H^{3/2}(\Omega_t)}+\varepsilon\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)}+C_\varepsilon\|p\|^2_{H^2(\Gamma_t)}-\frac{7}{8}\int_{\Gamma_t}|\bar{\nabla}^3 p|^2\mathrm{d}S\\ &\quad+\|\bar{\nabla}^3 p\|_{L^2(\Gamma_t)}\left(\|\bar{\nabla}\partial_\nu p\|_{L^2(\Gamma_t)}\|p\|_{L^\infty(\Gamma_t)}+\|\partial_\nu p\|_{L^4(\Gamma_t)}\|\bar{\nabla}p\|_{L^4(\Gamma_t)}\right)\\ &\leqslant C_\varepsilon-\frac{3}{4}\int_{\Gamma_t}|\bar{\nabla}^3 p|^2\mathrm{d}S+C_\varepsilon\|\nabla\partial_\nu p\|^2_{L^2(\Gamma_t)}+\|\bar{\nabla}^3 p\|_{L^2(\Gamma_t)}\|\nabla p\|^2_{H^1(\Omega_t)}\\ &\leqslant C_\varepsilon-\frac{1}{2}\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)}+C\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)}.\end{aligned}$$

Above, we have used lemma A.1 and (4.4) to deduce $\|\nabla\partial_\nu p\|^2_{L^2(\Gamma_t)}\leqslant C(1+\|p\|^2_{H^2(\Gamma_t)}+\|\Delta p\|^2_{H^1(\Omega_t)})$, and the result [11, lemma 2.3], i.e. $\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)}\leqslant\|\bar{\nabla}\Delta_{\mathcal{B}}p\|^2_{L^2(\Gamma_t)}+C\|p\|^2_{H^2(\Gamma_t)}$. Similarly, we can obtain $\frac{\mathrm{d}}{\mathrm{d}t}I_2(t)\leqslant C(1+\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)}+\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)})$.

Combined the above calculations and by choosing suitable $\varepsilon>0$, one has $\frac{\mathrm{d}}{\mathrm{d}t}I(t)\leqslant-\frac{1}{4}\|\bar{\nabla}^3 p\|^2_{L^2(\Gamma_t)}+C\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)}+C$. Integrating the above over $[0,t]$ with $0<t\leqslant T$ and recalling (4.2) together with $I(t)\geqslant-C_\varepsilon+\frac{\varepsilon}{2}\|\bar{\nabla}^2 p(\cdot,t)\|^2_{L^2(\Gamma_t)}$, the lemma follows. □

Lemma 4.6. *Assume that (1.6) holds for some $T>0$. Then, we have*

$$\sup_{t\in[0,T]}\|p\|^2_{H^3(\Omega_t)}\leqslant C\left(\mathcal{N}_T,\mathcal{M}_T,\|\bar{\nabla}^2 p\|_{L^2(\Gamma_0)},\|\nabla p\|_{H^2(\Omega_0)},T\right).$$

Proof. We differentiate and apply lemma 2.2 to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega_t}|\nabla^3 p|^2\mathrm{d}x\\ &\quad=\int_{\Omega_t}\sum_{ijk}\partial_{ijk}\mathcal{D}_t p\partial_{ijk}p\mathrm{d}x+\int_{\Omega_t}\nabla^3 v\star\nabla p\star\nabla^3 p+\nabla^2 v\star\nabla^2 p\star\nabla^3 p+\nabla v\star\nabla^3 p\star\nabla^3 p\mathrm{d}x\\ &\quad=:I_1(t)+I_2(t).\end{aligned}$$

From (1.6), (2.4) and lemma 2.11, we have

$$\begin{aligned}|I_1(t)|&\leqslant\int_{\Omega_t}\sum_{i,j,k}\partial_i\left(\partial_{jk}\mathcal{D}_t p\partial_{ijk}p\right)\mathrm{d}x-\int_{\Omega_t}\sum_{j,k}\partial_{jk}\mathcal{D}_t p\partial_{jk}\Delta p\mathrm{d}x\\ &\leqslant\int_{\Gamma_t}\sum_{j,k}\partial_{jk}\mathcal{D}_t p\partial_\nu\partial_{jk}p\mathrm{d}S+\int_{\Omega_t}\sum_k\partial_k\Delta\mathcal{D}_t p\partial_k\Delta p\mathrm{d}x-\int_{\Gamma_t}\sum_k\partial_\nu\partial_k\mathcal{D}_t p\partial_k\Delta p\mathrm{d}S\\ &\leqslant C\sum_{j,k}\|\partial_\nu\partial_{jk}p\|^2_{L^2(\Gamma_t)}+C\sum_{j,k}\|\partial_{jk}\mathcal{D}_t p\|^2_{L^2(\Gamma_t)}+C\left(1+\|\nabla p\|^2_{H^2(\Omega_t)}\right),\end{aligned}$$

and $|I_2(t)|\leqslant C(1+\|\nabla p\|^2_{H^2(\Omega_t)})$. Applying (4.4) and (4.5), we obtain

$$\begin{aligned}\|\partial_\nu\partial_{jk}p\|^2_{L^2(\Gamma_t)}+\|\partial_{jk}\mathcal{D}_t p\|^2_{L^2(\Gamma_t)}&\leqslant C\left(\|\Delta p\|^2_{H^2(\Omega_t)}+\|\nabla p\|^2_{H^2(\Omega_t)}+\|\mathcal{D}_t p\|^2_{H^2(\Gamma_t)}+\|p\|^2_{H^3(\Gamma_t)}\right)\\ &\leqslant C\left(1+\|\nabla p\|^2_{H^2(\Omega_t)}+\|p\|^2_{H^3(\Gamma_t)}\right),\end{aligned}$$

for any indices j,k. The claim follows from lemma A.7 and the previous pressure estimates ((4.2), lemmas 4.2, 4.3, 4.4 and 4.5), since

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega_t}|\nabla^3 p|^2 \leqslant C\left(1+\|\nabla p\|^2_{H^2(\Omega_t)}+\|p\|^2_{H^3(\Gamma_t)}\right).$$

□

We conclude this section by controlling the initial quantities $\bar{E}(0)$ and $\sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|^2_{H^{3k/2+1}(\Omega_0)}$.

Proposition 4.7. *Assume that Ω_0 is a smooth and $\mathcal{M}_0 := \mathcal{R} - \|h_0\|_{L^\infty(\Gamma)} > 0$. Then, we have*

$$\bar{E}(0)+\sum_{k=0}^{3}\|\mathcal{D}_t^{3-k}p\|^2_{H^{3k/2+1}(\Omega_0)} \leqslant C\left(\mathcal{M}_0, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}, \|\mathcal{A}\|_{H^5(\Gamma_0)}\right).$$

The result remains valid when the initial time is replaced with any $t\in(0,T)$, provided $\|h(\cdot,t)\|_{L^\infty(\Gamma)} < \mathcal{R}$.

Proof. We only need to consider the case when $t=0$ and we divide the proof into three steps.

Step 1. We control $\|\mathcal{D}_t^{4-k}H\|^2_{H^{3k/2}(\Omega_0)}$ by the lower-order velocity terms using (2.5) and (2.6). For $k=0$, from $\|H\|_{L^\infty(\Omega_0)} \leqslant C\|H\|_{H^6(\Omega_0)}$, we apply (2.5) to obtain

$$\begin{aligned}\|\mathcal{D}_t^4 H\|^2_{L^2(\Omega_0)} &\leqslant C\left(\sum_{|\beta|\leqslant 3}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{L^2(\Omega_0)}+\sum_{|\beta|\leqslant 2}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{L^3(\Omega_0)}\|\nabla\mathcal{D}_t^{\beta_2}v\|^2_{L^6(\Omega_0)}\right.\\ &\qquad\left.+\sum_{|\beta|\leqslant 1}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{L^6(\Omega_0)}\|\nabla\mathcal{D}_t^{\beta_2}v\|^2_{L^6(\Omega_0)}\|\nabla\mathcal{D}_t^{\beta_3}v\|^2_{L^6(\Omega_0)}+\|v\|^8_{H^3(\Omega_0)}\right)\\ &\leqslant C\|\mathcal{D}_t^3 v\|^2_{H^1(\Omega_0)}+C\left(1+\|\mathcal{D}_t^2 v\|^2_{H^2(\Omega_0)}\right)\left(1+\|\mathcal{D}_t v\|^2_{H^2(\Omega_0)}\right).\end{aligned}$$

We claim that

$$\begin{aligned}\sum_{k=1}^{3}\|\mathcal{D}_t^{4-k}H\|^2_{H^{3k/2}(\Omega_0)} &\leqslant C\left(\|v\|_{H^4(\Omega_0)}, \|H\|_{H^4(\Omega_0)}\right)\\ &\quad\times\left(1+\sum_{k=1}^{3}\|\mathcal{D}_t^{4-k}v\|^2_{H^{(3k-1)/2}(\Omega_0)}+\|v\|^2_{H^{11/2}(\Omega_0)}+\|H\|^2_{H^{9/2}(\Omega_0)}\right).\end{aligned} \tag{4.8}$$

Indeed, by (2.5), it follows that

$$\begin{aligned}\|\mathcal{D}_t^3 H\|^2_{H^{3/2}(\Omega_0)} &\leqslant C\sum_{|\beta|\leqslant 2}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{H^{3/2}(\Omega_0)}\|H\|^2_{H^2(\Omega_0)}+C\sum_{|\beta|\leqslant 1}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{H^2(\Omega_0)}\\ &\qquad\cdot\|\nabla\mathcal{D}_t^{\beta_m}v\|^2_{H^2(\Omega_0)}\|H\|^2_{H^2(\Omega_0)}+C\|v\|^6_{H^3(\Omega_0)}\|H\|^2_{H^2(\Omega_0)}\\ &\leqslant C\left(\|v\|_{H^4(\Omega_0)}, \|H\|_{H^4(\Omega_0)}\right)\left(1+\|\mathcal{D}_t^2 v\|^2_{H^{5/2}(\Omega_0)}+\|\mathcal{D}_t v\|^2_{H^{5/2}(\Omega_0)}\right).\end{aligned}$$

Again from (2.5) and lemma A.5, we see that

$$\begin{aligned}\|\mathcal{D}_t^2 H\|_{H^3(\Omega_0)}^2 &\leqslant C\|\nabla\mathcal{D}_t v\|_{H^3(\Omega_0)}^2\|H\|_{H^3(\Omega_0)}^2 + C\|\nabla v\|_{H^3(\Omega_0)}^4\|H\|_{H^3(\Omega_0)}^2\\ &\leqslant C\left(\|v\|_{H^4(\Omega_0)}, \|H\|_{H^4(\Omega_0)}\right)\left(1+\|\mathcal{D}_t v\|_{H^4(\Omega_0)}^2\right),\\ \|\mathcal{D}_t H\|_{H^{9/2}(\Omega_0)}^2 &\leqslant C\left(\|H\|_{L^\infty(\Omega_0)}^2\|v\|_{H^{11/2}(\Omega_0)}^2 + \|H\|_{H^{9/2}(\Omega_0)}^2\|v\|_{L^\infty(\Omega_0)}^2\right)\\ &\leqslant C\left(\|v\|_{H^4(\Omega_0)}, \|H\|_{H^4(\Omega_0)}\right)\left(\|v\|_{H^{11/2}(\Omega_0)}^2 + \|H\|_{H^{9/2}(\Omega_0)}^2\right).\end{aligned}$$

Step 2. We control $\|\mathcal{D}_t^{4-k}v\|_{H^{3k/2}(\Omega_0)}^2$ by the pressure terms. Note that

$$\|\mathcal{D}_t v\|_{H^{9/2}(\Omega_0)}^2 \leqslant C\left(\|p\|_{H^{11/2}(\Omega_0)}^2 + \|H\|_{H^{11/2}(\Omega_0)}^2\|H\|_{H^{9/2}(\Omega_0)}^2\right) \leqslant C\left(\|p\|_{H^{11/2}(\Omega_0)}^2 + 1\right),$$

and by lemma 2.6, we have

$$\|\mathcal{D}_t^2 v\|_{H^3(\Omega_0)}^2 \leqslant \|\nabla\mathcal{D}_t p\|_{H^3(\Omega_0)}^2 + \|[\mathcal{D}_t,\nabla]p\|_{H^3(\Omega_0)}^2 + C \leqslant C\left(\|\nabla\mathcal{D}_t p\|_{H^3(\Omega_0)}^2 + \|p\|_{H^4(\Omega_0)}^2 + 1\right).$$

Similarly, applying lemmas 2.8 and 2.9, we obtain

$$\begin{aligned}\|\mathcal{D}_t^3 v\|_{H^{3/2}(\Omega_0)}^2 &\leqslant C\left(\|\nabla\mathcal{D}_t^2 p\|_{H^{3/2}(\Omega_0)}^2 + \|\nabla\mathcal{D}_t p\|_{H^{3/2}(\Omega_0)}^2 + \|p\|_{H^{9/2}(\Omega_0)}^2 + 1\right),\\ \|\mathcal{D}_t^4 v\|_{L^2(\Omega_0)}^2 &\leqslant C\left(\|\nabla\mathcal{D}_t^3 p\|_{L^2(\Omega_0)}^2 + \|\nabla\mathcal{D}_t^2 p\|_{L^2(\Omega_0)}^2 + \|\nabla\mathcal{D}_t p\|_{H^2(\Omega_0)}^2 + \|p\|_{H^{9/2}(\Omega_0)}^2\right).\end{aligned}$$

Step 3. We show that $\sum_{k=0}^3 \|\mathcal{D}_t^{3-k}p\|_{H^{3k/2+1}(\Omega_0)}^2 \leqslant C$. Consider the following elliptic equation

$$\begin{cases} -\Delta p = \partial_i v^j \partial_j v^i - \partial_i H^j \partial_j H^i, & \text{in } \Omega_0,\\ p = \mathcal{A}_{\Gamma_0}, & \text{on } \Gamma_0. \end{cases}$$

We find that $\|p\|_{H^{11/2}(\Omega_0)} \leqslant C(\|\partial_i v^j\partial_j v^i - \partial_i H^j\partial_j H^i\|_{H^{7/2}(\Omega_0)} + \|\mathcal{A}\|_{H^5(\Gamma_0)}) \leqslant C$ from the standard elliptic estimates. Again by the elliptic estimates, it holds $\|\mathcal{D}_t p\|_{H^4(\Omega_0)} \leqslant C(\|\Delta\mathcal{D}_t p\|_{H^2(\Omega_0)} + \|\mathcal{D}_t p\|_{H^{7/2}(\Gamma_0)})$, and $\|\mathcal{D}_t^2 p\|_{H^{5/2}(\Omega_0)} \leqslant C(\|\Delta\mathcal{D}_t^2 p\|_{H^{1/2}(\Omega_0)} + \|\mathcal{D}_t^2 p\|_{H^2(\Gamma_0)})$. Also, by (A2), $\|\mathcal{D}_t^3 p\|_{H^1(\Omega_0)} \leqslant C(\|\Delta\mathcal{D}_t^3 p\|_{L^2(\Omega_0)} + \|\mathcal{D}_t^3 p\|_{H^{1/2}(\Gamma_0)})$. The calculations of the remaining terms on the right-hand side are direct applications of lemmas 2.11 and 2.12, and (2.10), since we have $\|p\|_{H^{11/2}(\Omega_0)} \leqslant C$.

Finally, for $1 \leqslant j \leqslant 3$, $\|\bar{\nabla}(\mathcal{D}_t^j v\cdot\nu)\|_{L^2(\Gamma_0)}^2$ can be estimated by the trace theorem due to the regularity of the boundary. Using the mean curvature bound, we apply lemma A.6 to obtain $\|B\|_{H^2(\Gamma_0)} \leqslant C$ and therefore $\|\bar{\nabla}(\mathcal{D}_t^j v\cdot\nu)\|_{L^2(\Gamma_0)}^2 \leqslant C(\|\bar{\nabla}\mathcal{D}_t^j v \star \nu\|_{L^2(\Gamma_0)}^2 + \|\mathcal{D}_t^j v\star B\|_{L^2(\Gamma_0)}^2) \leqslant C$. This concludes the proof of the proposition. □

5. Estimates for the error terms

In this section, we estimate the error terms by the energy functional and the pressure. We start with the following results.

Lemma 5.1. *Assume that (1.6) holds for $T>0$. Then, we have $\|B\|_{H^{5/2}(\Gamma_t)} \leqslant C$, and $\|B\|_{H^k(\Gamma_t)} \leqslant C(1+\|p\|_{H^k(\Gamma_t)})$ for $k\in\mathbb{N}/2, k\leqslant 9/2$. Assume further that $\sup_{0\leqslant t<T} E_{l-1}(t) \leqslant C$ for $l\geqslant 4$. Then, it holds $\|B\|_{H^{3l/2-1}(\Gamma_t)} \leqslant C$, and $\|B\|_{H^k(\Gamma_t)} \leqslant C(1+\|p\|_{H^k(\Gamma_t)})$ for $k\in\mathbb{N}/2, k\leqslant 3l/2+1$.*

Proof. We recall (4.1) that $\|p\|_{H^3(\Omega_t)} \leqslant C$ by the results in section 4. Since Γ_t is uniformly $H^{3+\delta}(\Gamma)$-regular, it holds $\|B\|_{L^\infty(\Gamma_t)} + \|B\|_{H^1(\Gamma_t)} \leqslant C$. Applying lemma A.6, for $k \in \mathbb{N}/2, k \leqslant 3$, we see that $\|B\|_{H^k(\Gamma_t)} \leqslant C(1 + \|\mathcal{A}\|_{H^k(\Gamma_t)}) \leqslant C(1 + \|p\|_{H^k(\Gamma_t)})$, and $\|B\|_{H^{5/2}(\Gamma_t)} \leqslant C$. Again by lemma A.6, the first claim follows. As for $l \geqslant 4$, the assumption implies that

$$\begin{aligned}\|p\|^2_{H^{3l/2-1}(\Gamma_t)} &\leqslant C\left(1 + \|\nabla p\|^2_{H^{3(l-1)/2}(\Omega_t)}\right)\\ &\leqslant C\left(1 + \|\mathcal{D}_t v\|^2_{H^{3(l-1)/2}(\Omega_t)} + \|H\cdot\nabla H\|^2_{H^{\lfloor 3l/2-1\rfloor}(\Omega_t)}\right) \leqslant C.\end{aligned}$$

For $l=4$, we have $\|p\|_{H^5(\Gamma_t)} \leqslant C$ and $\|B\|_{H^{9/2}(\Gamma_t)} \leqslant C$ by the first claim. Moreover, by lemma A.6, it implies $\|B\|_{H^5(\Gamma_t)} \leqslant C(1 + \|p\|_{H^5(\Gamma_t)}) \leqslant C$, i.e. $\|B\|_{H^{3l/2-1}(\Gamma_t)} \leqslant C$ in this case. Therefore, it holds $\|B\|_{H^k(\Gamma_t)} \leqslant C(1 + \|\mathcal{A}\|_{H^k(\Gamma_t)}) \leqslant (1 + \|p\|_{H^k(\Gamma_t)}), k \in \mathbb{N}/2, k \leqslant 3l/2 + 1$. Using a similar argument, the second claim follows for $l \geqslant 5$. □

Lemma 5.2. *Assume that (1.6) holds for $T>0$. We have $\|R^l_I\|^2_{H^{1/2}(\Omega_t)} \leqslant C(1 + \|\nabla^2 p\|^2_{H^{1/2}(\Omega_t)})\bar{E}(t)$ for $l \leqslant 3$. Assume further that $\sup_{0\leqslant t<T} E_{l-1}(t) \leqslant C$ for $l \geqslant 4$. Then, we have $\|R^l_I\|^2_{H^{1/2}(\Omega_t)} \leqslant CE_l(t)$, and there exists a constant $\varepsilon > 0$ small enough such that $\|R^{l-k}_I\|^2_{H^{3k/2-1}(\Omega_t)} \leqslant \varepsilon E_l(t) + C_\varepsilon$ for $k \in \mathbb{N}, 1 \leqslant k \leqslant l$.*

Proof. Thanks to the regularity of the free boundary in lemma 5.1, it is feasible to extend functions in $H^2(\Omega_t)$ to the entire space $\mathbb{R}^3$ (e.g. [26, proposition 2.1]) and then apply lemma A.4. To simplify the notation, we will not distinguish between the original function and its extension.

It suffices to estimate $R^3_I = \sum_{2\leqslant m\leqslant 4}\sum_{|\beta|\leqslant 5-m} \nabla\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla\mathcal{D}_t^{\beta_{m-1}} v \star \nabla\mathcal{D}_t^{\beta_m} v$ defined in (2.7) since R^1_I and R^2_I are easier to handle. We deal with the case of $m=2$, i.e. $\sum_{|\beta|\leqslant 3} \nabla\mathcal{D}_t^{\beta_1} v \star \nabla\mathcal{D}_t^{\beta_2} v$ and we only show the estimates when $|\beta| = 3$. From (1.6) and lemma A.4, we see that

$$\begin{aligned}\|\nabla v \star \nabla\mathcal{D}_t^3 v\|_{H^{1/2}(\Omega_t)} &\leqslant C\left(\|\nabla v\|_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^3 v\|_{H^{1/2}(\Omega_t)} + \|\nabla v\|_{W^{1/2,6}(\Omega_t)}\|\nabla\mathcal{D}_t^3 v\|_{L^3(\Omega_t)}\right)\\ &\leqslant C\left(\|\nabla v\|_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^3 v\|_{H^{1/2}(\Omega_t)} + \|v\|_{H^{5/2}(\Omega_t)}\|\mathcal{D}_t^3 v\|_{H^{3/2}(\Omega_t)}\right) \leqslant C\bar{E}(t)^{1/2},\\ \|\nabla\mathcal{D}_t^2 v \star \nabla\mathcal{D}_t v\|_{H^{1/2}(\Omega_t)} &\leqslant C\left(\|\nabla\mathcal{D}_t v\|_{H^{1/2}(\Omega_t)}\|\nabla\mathcal{D}_t^2 v\|_{L^\infty(\Omega_t)} + \|\nabla\mathcal{D}_t v\|_{L^3(\Omega_t)}\|\nabla\mathcal{D}_t^2 v\|_{W^{1/2,6}(\Omega_t)}\right)\\ &\leqslant C\left(1 + \|\nabla^2 p\|_{H^{1/2}(\Omega_t)}\right)\bar{E}(t)^{1/2}.\end{aligned}$$

If $l \geqslant 4$, the assumption $E_{l-1}(t) \leqslant C$ also ensures that the functions in $H^{3l/2+1}(\Omega_t)$ can be extended by lemma 5.1 and the extension theorem (e.g. [26, proposition 2.1]). Then, it follows that $\|\nabla v \star \nabla\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)} \leqslant C(\|\nabla v\|_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)} + \|v\|_{H^{5/2}(\Omega_t)}\|\mathcal{D}_t^l v\|_{H^{3/2}(\Omega_t)}) \leqslant CE_l(t)^{1/2}$. For $1 \leqslant j \leqslant l-j \leqslant l-1$, we have $j \leqslant \lfloor l/2\rfloor \leqslant l-2$ due to $l \geqslant 4$, and obtain

$$\begin{aligned}&\|\nabla\mathcal{D}_t^j v \star \nabla\mathcal{D}_t^{l-j} v\|_{H^{1/2}(\Omega_t)}\\ &\quad\leqslant C\left(\|\nabla\mathcal{D}_t^j v\|_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^{l-j} v\|_{H^{1/2}(\Omega_t)} + \|\mathcal{D}_t^j v\|_{H^{5/2}(\Omega_t)}\|\nabla\mathcal{D}_t^{l-j} v\|_{H^{3/2}(\Omega_t)}\right) \leqslant CE_l(t)^{1/2},\end{aligned}$$

where we have used the fact that $\|\mathcal{D}_t^j v\|_{H^{5/2+\varepsilon}(\Omega_t)} \leqslant E_{l-1}(t) \leqslant C$. Again from the hypothesis $E_{l-1}(t) \leqslant C$, the terms involving the product of more than three items can be controlled since we will have fewer material derivatives in this case.

To prove the last claim, we first estimate that $\|R^0_I\|^2_{H^{3l/2-1}(\Omega_t)} \leqslant C\|\nabla v\|^2_{L^\infty(\Omega_t)}\|\nabla v\|^2_{H^{3l/2-1}(\Omega_t)} \leqslant C\|\nabla v\|^2_{H^{3l/2-1}(\Omega_t)}$. By interpolation, we have $\|R^0_I\|^2_{H^{3l/2-1}(\Omega_t)}$

$\leqslant \varepsilon E_l(t) + C_\varepsilon$ for $l = 5,7,\cdots$, and $\|R_I^0\|^2_{H^{3l/2-1}(\Omega_t)} \leqslant CE_{l-1}(t) \leqslant C$ for $l = 4,6,\cdots$ Then we control the case of $k=1$. When $l \geqslant 5$, applying the previous estimates, it holds $\|R_I^{l-1}\|^2_{H^{1/2}(\Omega_t)} \leqslant CE_{l-1}(t) \leqslant C$. If $l=4$, one has $\|R_I^{l-1}\|^2_{H^{1/2}(\Omega_t)} \leqslant C(1+\|\nabla^2 p\|^2_{H^{1/2}(\Omega_t)})E_{l-1}(t) \leqslant C$, since $\|\nabla^2 p\|^2_{H^{1/2}(\Omega_t)} \leqslant \|\nabla(H\cdot\nabla H - \mathcal{D}_t v)\|^2_{H^{1/2}(\Omega_t)} \leqslant C$.

We are left with the case of $2 \leqslant k \leqslant l-1$. Note that $R_I^{l-k} = \sum_{2\leqslant m\leqslant l-k+1} \sum_2 |\beta| \leqslant l-k+2-m \nabla\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla\mathcal{D}_t^{\beta_{m-1}} v \star \nabla\mathcal{D}_t^{\beta_m} v$. We only estimate the case of $k=m=2$, i.e. $\nabla\mathcal{D}_t^{l-2-j} v \star \nabla\mathcal{D}_t^j v$. As before, we assume that $0 \leqslant j \leqslant l-2-j \leqslant l-2$ and it holds $j \leqslant \lfloor (l-2)/2 \rfloor \leqslant l-2, l=4$, and $j \leqslant \lfloor (l-2)/2 \rfloor \leqslant l-3, l \geqslant 5$. We deal with the first case, i.e. $\|\nabla\mathcal{D}_t v \star \nabla\mathcal{D}_t v\|^2_{H^2(\Omega_t)} + \|\nabla\mathcal{D}_t^2 v \star \nabla v\|^2_{H^2(\Omega_t)}$, since the same arguments work for $l \geqslant 5$ ($j \leqslant l-3$ in this case). We deduce that

$$\begin{aligned}
\|\nabla v \star \nabla\mathcal{D}_t^2 v\|^2_{H^2(\Omega_t)} &\leqslant C\left(\|\nabla v\|^2_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^2 v\|^2_{H^2(\Omega_t)} + \|\nabla v\|^2_{H^3(\Omega_t)}\|\nabla\mathcal{D}_t^2 v\|^2_{H^{5/2}(\Omega_t)}\right)\\
&\leqslant \varepsilon\|\nabla\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)} + C\|\nabla\mathcal{D}_t^2 v\|^2_{H^2(\Omega_t)} \leqslant \varepsilon E_l(t) + C_\varepsilon,\\
\|\nabla\mathcal{D}_t v \star \nabla\mathcal{D}_t v\|^2_{H^2(\Omega_t)} &\leqslant C\|\nabla\mathcal{D}_t v\|^2_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t v\|^2_{H^2(\Omega_t)} \leqslant CE_{l-1}(t) \leqslant C.
\end{aligned}$$

The proof is complete. □

Lemma 5.3. *Assume that (1.6) holds for $T>0$. For $l \leqslant 3$, we have*

$$\|R^l_{\nabla H,H}\|^2_{H^{1/2}(\Omega_t)} + \|R^l_{\nabla H,\nabla H}\|^2_{H^{1/2}(\Omega_t)} + \|R^l_{\nabla^2 H,H}\|^2_{H^{1/2}(\Omega_t)} \leqslant C\left(1+\|\nabla^2 p\|^2_{H^{1/2}(\Omega_t)}\right)\bar{E}(t).$$

Assume further that $\sup_{0\leqslant t<T} E_{l-1}(t) \leqslant C$ *for $l \geqslant 4$, then we have*

$$\begin{aligned}
&\|R^l_{\nabla H,H}\|^2_{H^{1/2}(\Omega_t)} + \|R^l_{\nabla H,\nabla H}\|^2_{H^{1/2}(\Omega_t)} + \|R^l_{\nabla^2 H,H}\|^2_{H^{1/2}(\Omega_t)} \leqslant CE_l(t),\\
&\|R^0_{\nabla H,H}\|^2_{H^{3l/2-1}(\Omega_t)} + \|R^0_{\nabla H,\nabla H}\|^2_{H^{3l/2-1}(\Omega_t)} \leqslant \varepsilon E_l(t) + C_\varepsilon,\\
&\|R^0_{\nabla^2 H,H}\|^2_{H^{3k/2-1}(\Omega_t)} \leqslant C\|\operatorname{curl} H\|^2_{H^{\lfloor 3l/2+1/2\rfloor}(\Omega_t)},\\
&\|R^{l-k}_{\nabla H,H}\|^2_{H^{3k/2-1}(\Omega_t)} + \|R^{l-k}_{\nabla H,\nabla H}\|^2_{H^{3k/2-1}(\Omega_t)} + \|R^{l-k}_{\nabla^2 H,H}\|^2_{H^{3k/2-1}(\Omega_t)} \leqslant \varepsilon E_l(t) + C_\varepsilon, \qquad (5.1)
\end{aligned}$$

for $k \in \mathbb{N}, 1 \leqslant k < l$. In the above, $\varepsilon > 0$ is a constant small enough.

Proof. We note that $R^l_{\nabla^2 H,H}$ contains all the highest-order terms in $R^l_{\nabla H,H}$ and $R^l_{\nabla H,\nabla H}$ and we focus on the estimate for $R^l_{\nabla^2 H,H}$. To control $R^3_{\nabla^2 H,H}$ in the case of $l \leqslant 3$, we recall $R^3_{\nabla^2 H,H}$ in lemma 2.8. From (1.6), we have $\|\nabla^4 \operatorname{curl} H \star H \star \cdots \star H\|^2_{H^{1/2}(\Omega_t)} \leqslant C\|H\|_{H^6(\Omega_t)} \leqslant C\bar{E}(t)$, and $\|\nabla^2\mathcal{D}_t^2 v \star \nabla F_2 \star F_3\|^2_{H^{1/2}(\Omega_t)} \leqslant C\bar{E}(t)$, as in lemma 5.2. The leading terms in $R^3_{\nabla^2 H,H}$ have been controlled, and the estimates of the lower-order terms follow from the same arguments as in lemma 5.2.

As for $l \geqslant 4$, to prove the first result, it is sufficient to bound $\nabla^{l+1}\operatorname{curl} v \star H \star \cdots \star H$ and $\nabla^{l+1}\operatorname{curl} H \star H \star \cdots \star H$ since the other terms are either simpler or have already been estimated in lemma 5.2. From the assumption, $\|v\|_{H^{\lfloor 3l/2\rfloor}(\Omega_t)} + \|H\|_{H^{\lfloor 3l/2\rfloor}(\Omega_t)} \leqslant C$. As before, we extend the functions and estimate as in lemma 5.2 to obtain

$$\begin{aligned}
&\|\nabla^{l+1}\operatorname{curl} v \star H \star \cdots \star H\|_{H^{1/2}(\Omega_t)}\\
&\quad\leqslant C\left(\|H\star\cdots\star H\|_{L^\infty(\Omega_t)}\|\nabla^{l+1}\operatorname{curl} v\|_{H^{1/2}(\Omega_t)} + \|H\star\cdots\star H\|_{W^{1/2,6}(\Omega_t)}\|\nabla^{l+1}\operatorname{curl} v\|_{L^3(\Omega_t)}\right)\\
&\quad\leqslant C\|v\|_{H^{l+5/2}(\Omega_t)} \leqslant CE_l(t)^{1/2}.
\end{aligned}$$

In the last step, the condition $l \geqslant 4$ implies $5/2+l \leqslant \lfloor 3(l+1)/2 \rfloor$ and therefore, it holds $\|v\|_{H^{l+5/2}(\Omega_t)} \leqslant \|v\|_{H^{\lfloor 3(l+1)/2 \rfloor}(\Omega_t)}$.

Next, to verify (5.1), we shall control $\|R^{l-k}_{\nabla^2 H,H}\|^2_{H^{3k/2-1}(\Omega_t)}$ for $1 \leqslant k \leqslant l-1$. We concentrate on the estimate for $\|\nabla^{l-k+1} \operatorname{curl} v \star H \star \cdots \star H\|^2_{H^{3k/2-1}(\Omega_t)}$. This time we obtain for $1 \leqslant k < l$ that

$$\|\nabla^{l-k+1} \operatorname{curl} v \star H \star \cdots \star H\|^2_{H^{3k/2-1}(\Omega_t)} \leqslant C\|v\|_{H^{3l/2+1/2}(\Omega_t)}.$$

By interpolation, it holds $\|v\|^2_{H^{3l/2+1/2}(\Omega_t)} \leqslant \varepsilon \|v\|^2_{H^{\lfloor 3(l+1)/2 \rfloor}(\Omega_t)} + C_\varepsilon \|v\|^2_{H^{\lfloor 3l/2 \rfloor}(\Omega_t)} \leqslant \varepsilon E_l(t) + C_\varepsilon$. Finally, to obtain the last two estimates, we only need to bound the most difficult term $R^0_{\nabla^2 H,H} = (H \cdot \nabla) \operatorname{curl} H$. Since $l \geqslant 4$, we have $\|R^0_{\nabla^2 H,H}\|^2_{H^{3l/2-1}(\Omega_t)} \leqslant C\|H\|^2_{H^{\lfloor 3l/2 \rfloor}(\Omega_t)} \|\operatorname{curl} H\|^2_{H^{\lfloor 3l/2+1/2 \rfloor}(\Omega_t)} \leqslant C\|\operatorname{curl} H\|^2_{H^{\lfloor 3l/2+1/2 \rfloor}(\Omega_t)}$, and the proof is complete. □

Lemma 5.4. *Assume that (1.6) holds for $T>0$. We have $\|R^l_{II}\|^2_{L^2(\Omega_t)} \leqslant C(1+\|\nabla p\|^2_{H^{3/2}(\Omega_t)})\bar{E}(t)$ for $l \leqslant 3$. Assume further that $\sup_{0\leqslant t<T} E_{l-1}(t) \leqslant C$ for $l \geqslant 4$. Then, it follows that $\|R^l_{II}\|^2_{L^2(\Omega_t)} \leqslant CE_l(t)$, and $\|R^{l-k}_{II}\|^2_{H^{3(k-1)/2}(\Omega_t)} \leqslant C$ for $k \in \mathbb{N}, 1 \leqslant k \leqslant l-1$.*

Proof. To prove the first claim, we estimate

$$R^3_{II} = \sum_{1\leqslant m\leqslant 4} \sum_{\substack{|\beta|\leqslant 3, |\alpha|\leqslant 1 \\ \beta_1,\cdots,\beta_{m-1}\geqslant 1}} a_{\alpha,\beta}(\nabla v) \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{m-1}} v \star \nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2+\beta_m} v.$$

If $m=1$, we consider the case of $|\beta|=\beta_1=3$ and $|\alpha|=1$. We should control $a(\nabla v)\mathcal{D}_t^4 v + b(\nabla v)\nabla \mathcal{D}_t^3 v$. From the hypothesis (1.6), it is clear that $\|a(\nabla v)\mathcal{D}_t^4 v\|^2_{L^2(\Omega_t)} + \|b(\nabla v)\nabla \mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} \leqslant C\bar{E}(t)$. For $m=2, |\beta|=3$ and $|\alpha|=1$, we show the estimates of $a(\nabla v)\nabla \mathcal{D}_t v \star \mathcal{D}_t^3 v$ and $b(\nabla v)\nabla \mathcal{D}_t^2 v \star \mathcal{D}_t^2 v$. Choosing $1/p+1/q=1/2, p=3/\delta$ with $\delta>0$ small enough, we see that $\|\nabla^2 H\|^2_{L^q(\Omega_t)} \leqslant C\|H\|^2_{H^{5/2+\delta}(\Omega_t)}$,

$$\begin{aligned}\|a(\nabla v)\nabla \mathcal{D}_t v \star \mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} &\leqslant C\|\nabla^2 p + \nabla H \star \nabla H + H \star \nabla^2 H\|^2_{L^q(\Omega_t)} \|\mathcal{D}_t^3 v\|^2_{H^{3/2}(\Omega_t)} \\ &\leqslant C\left(1+\|\nabla^2 p\|^2_{H^{1/2}(\Omega_t)}\right)\bar{E}(t),\end{aligned}$$

and $\|a(\nabla v)\nabla \mathcal{D}_t^2 v \star \mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)} \leqslant C\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)}\bar{E}(t)$. To control $\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)}$, from the boundedness $\|\Delta p\|_{H^1(\Omega_t)} \leqslant C$ and using (2.4), (2.11), (2.8), together with (A1), we obtain

$$\begin{aligned}\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)} &\leqslant \|\nabla \mathcal{D}_t p\|^2_{L^2(\Omega_t)} + \|[\mathcal{D}_t, \nabla] p\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t H \star \nabla H + H \star \mathcal{D}_t \nabla H\|^2_{L^2(\Omega_t)} \\ &\leqslant \|\Delta \mathcal{D}_t p\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t p\|^2_{H^{1/2}(\Gamma_t)} + \|\nabla v \star \nabla p\|^2_{L^2(\Omega_t)} \\ &\quad + \|H \star \nabla v \star \nabla H + H \star \nabla v \star \nabla H + H \star \nabla^2 v \star H\|^2_{L^2(\Omega_t)} \\ &\leqslant \|\operatorname{div}\operatorname{div}(v \otimes \nabla p)\|^2_{L^2(\Omega_t)} + \|\nabla p\|^2_{L^2(\Omega_t)} + C \\ &\quad + \|\operatorname{div} R^0_{II} + \nabla^2 v \star \nabla H \star H + \nabla v \star \nabla H \star \nabla H \\ &\quad + \nabla^2 H \star \nabla v \star H + v \star \nabla^2 H \star \nabla H\|^2_{L^2(\Omega_t)} \\ &\leqslant \|\partial_j \partial_i \left(v^i \partial_j p\right)\|^2_{L^2(\Omega_t)} + \|\nabla p\|^2_{H^1(\Omega_t)} + C \leqslant C\left(1+\|\nabla p\|^2_{H^1(\Omega_t)}\right).\end{aligned}$$

In the case of $m=3$ and $m=4$, we estimate in the same fashion, and obtain $\|R_{II}^{l}\|^2_{L^2(\Omega_t)} \leqslant C(1+\|\nabla p\|^2_{H^{3/2}(\Omega_t)})\bar{E}(t)$, as desired.

To control R_{II}^{l} for $l \geqslant 4$, we focus on the case of $|\beta|=l$ and $|\alpha|=1$. If $m=1$, it holds $\|a(\nabla v)\mathcal{D}_t^{l+1}v + b(\nabla v)\nabla\mathcal{D}_t^{l}v\|^2_{L^2(\Omega_t)} \leqslant CE_{l-1}(t) \leqslant C$. Next, we handle the product of functions as follows. We simply assume $\alpha_2=1$ since the material derivative $\mathcal{D}_t$ is $1/2$-higher than the spatial derivative. If $1 \leqslant j \leqslant l+1-j \leqslant l$, it follows that $1 \leqslant j \leqslant \lfloor (l+1)/2 \rfloor \leqslant l-2$, and we have $\|a(\nabla v)\nabla\mathcal{D}_t^{j}v \star \mathcal{D}_t^{l+1-j}v\|^2_{L^2(\Omega_t)} \leqslant C\|\nabla\mathcal{D}_t^{j}v\|^2_{L^\infty(\Omega_t)}\|\mathcal{D}_t^{l+1-j}v\|^2_{L^2(\Omega_t)} \leqslant CE_l(t)$. If $1 \leqslant l+1-j < j \leqslant l$, we find that $\lfloor (l+1)/2 \rfloor + 1 \leqslant j$ and $1 \leqslant l+1-j \leqslant l-2$. Then, we obtain $\|a(\nabla v)\nabla\mathcal{D}_t^{j}v \star \mathcal{D}_t^{l+1-j}v\|^2_{L^2(\Omega_t)} \leqslant C\|\mathcal{D}_t^{j}v\|^2_{H^1(\Omega_t)}\|\mathcal{D}_t^{l+1-j}v\|^2_{H^{3/2+\varepsilon}(\Omega_t)} \leqslant CE_l(t)$. The others can be estimated in the same way.

We are left with the last claim. For $k=1$, it follows by applying the above estimates with $l-1$ if $l \geqslant 5$. As $k=1$ and $l=4$, it follows from the hypothesis that $E_3(t) \leqslant C$. Therefore, $\|\nabla p\|^2_{H^1(\Omega_t)} \leqslant C\|H\cdot\nabla H - \mathcal{D}_t v\|^2_{H^1(\Omega_t)} \leqslant C$. This concludes the proof for $k=1$. Assume that $2 \leqslant k \leqslant l-1$ and we shall control $\|R_{II}^{l-k}\|^2_{H^{3(k-1)/2}(\Omega_t)}$ defined in (2.8):

$$R_{II}^{l-k} = \sum_{1\leqslant m\leqslant l-k+1} \sum_{|\beta|\leqslant l-k, |\alpha|\leqslant 1, \beta_1,\cdots,\beta_{m-1}\geqslant 1} a_{\alpha,\beta}(\nabla v)\,\nabla\mathcal{D}_t^{\beta_1}v \star \cdots \star \nabla\mathcal{D}_t^{\beta_{m-1}}v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+\beta_m}v.$$

If $m=1, |\beta|=l-k$ and $|\alpha|=1$, it is clear that $\|a(\nabla v)\mathcal{D}_t^{l+1-k}v + b(\nabla v)\nabla\mathcal{D}_t^{l-k}v\|^2_{H^{3(k-1)/2}(\Omega_t)} \leqslant CE_{l-1}(t) \leqslant C$. To bound the product of functions, e.g. $m=2, |\beta|=l-k, |\alpha|=1$ and $1 \leqslant j \leqslant l-k-j \leqslant l-k-1$, we note that $1 \leqslant j \leqslant \lfloor (l-k)/2 \rfloor$ and

$$\|a(\nabla v)\|^2_{H^{3k/2-1/2}(\Omega_t)} \leqslant C\|\nabla v\|^2_{L^\infty(\Omega_t)}\cdots\|\nabla v\|^2_{L^\infty(\Omega_t)}\|v\|^2_{\lfloor 3l/2\rfloor(\Omega_t)} \leqslant C.$$

This, combined with the Sobolev embedding and lemma A.4, we deduce that

$$\begin{aligned}
&\|a(\nabla v)\nabla\mathcal{D}_t^{j}v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1}v\|^2_{H^{3(k-1)/2}(\Omega_t)}\\
&\quad\leqslant C\|a(\nabla v)\|^2_{W^{3(k-1)/2,6}(\Omega_t)}\|\nabla\mathcal{D}_t^{j}v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1}v\|^2_{L^3(\Omega_t)}\\
&\qquad + C\|a(\nabla v)\|^2_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t^{j}v \star \nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1}v\|^2_{H^{3(k-1)/2}(\Omega_t)}\\
&\quad\leqslant C\|\nabla\mathcal{D}_t^{j}v\|^2_{H^{3k/2-1/2}(\Omega_t)}\|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1}v\|^2_{L^3(\Omega_t)}\\
&\qquad + C\|\nabla\mathcal{D}_t^{j}v\|^2_{L^\infty(\Omega_t)}\|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-k-1}v\|^2_{H^{3(k-1)/2}(\Omega_t)} \leqslant C,
\end{aligned}$$

where we have used the fact that $\|\nabla\mathcal{D}_t^{j}v\|^2_{H^{3k/2-1/2}(\Omega_t)} + \|\nabla\mathcal{D}_t^{j}v\|^2_{L^\infty(\Omega_t)} \leqslant C(\|\mathcal{D}_t^{j}v\|^2_{H^{3k/2+1/2}(\Omega_t)} + \|\mathcal{D}_t^{j}v\|^2_{H^{5/2+\varepsilon}(\Omega_t)})$ for $\varepsilon>0$ small enough. Thus, the proof is complete since the other terms can be estimated by using similar arguments. □

Lemma 5.5. *Assume that (1.6) holds for $T>0$. We have $\|R_p^{l}\|^2_{H^{1/2}(\Gamma_t)} \leqslant C\left(1+\|\nabla p\|^2_{H^2(\Omega_t)}\right)\bar{E}(t)$ for $l \leqslant 3$. Assume further that $\sup_{0\leqslant t<T}E_{l-1}(t) \leqslant C$ for $l \geqslant 4$. Then it follows that $\|R_p^{l}\|^2_{H^{1/2}(\Gamma_t)} \leqslant CE_l(t)$, and $\|R_p^{l-k}\|^2_{H^{3k/2-1}(\Gamma_t)} \leqslant \varepsilon E_l(t) + C_\varepsilon$ for $k\in\mathbb{N}, 1\leqslant k\leqslant l-1$ with $\varepsilon>0$ small enough.*

Proof. It is sufficient to show the estimate for $l=3$ since the other cases are easier. Recall the definition of R_p^3, we have

$$\begin{aligned}
\|\bar{\nabla}p\cdot\mathcal{D}_t^3 v\|^2_{H^{1/2}(\Gamma_t)} &\leqslant C\left(\|\bar{\nabla}p\|^2_{W^{1/2,4}(\Gamma_t)}\|\mathcal{D}_t^3 v\|^2_{L^4(\Gamma_t)} + \|\bar{\nabla}p\|^2_{L^4(\Gamma_t)}\|\mathcal{D}_t^3 v\|^2_{W^{1/2,4}(\Gamma_t)}\right)\\
&\leqslant C\|\nabla p\|^2_{H^{3/2}(\Omega_t)}\bar{E}(t),
\end{aligned}$$

we have used the fact that $\|\bar{\nabla}^2 p\|^2_{L^2(\Gamma_t)} \leqslant C(\|\nabla p\|^2_{H^1(\Gamma_t)} + \|\nabla p \star B\|^2_{L^2(\Gamma_t)}) \leqslant C\|\nabla p\|^2_{H^1(\Gamma_t)}$, and the trace theorem. Similarly, to deal with the term $-|B|^2\mathcal{D}_t^3 v\cdot\nu$, we have $\|\mathcal{D}_t^3 v\cdot\nu\|^2_{L^2(\Gamma_t)} \leqslant C\|\mathcal{D}_t^3 v\|^2_{H^1(\Omega_t)} \leqslant C\bar{E}(t)$, and $\|-|B|^2\mathcal{D}_t^3 v\cdot\nu\|^2_{H^{1/2}(\Gamma_t)} \leqslant C\||B|^2\|^2_{H^1(\Gamma_t)}\|\mathcal{D}_t^3 v\cdot\nu\|^2_{H^1(\Gamma_t)} \leqslant C(1+\|\nabla p\|^2_{H^1(\Omega_t)})\bar{E}(t)$ by (1.6). Again from (1.6), it follows that

$$
\begin{aligned}
&\|a_8(\nu,\nabla v)\star\nabla^2\mathcal{D}_t^2 v\|^2_{H^{1/2}(\Gamma_t)} \leqslant C\left(\|\nabla^2\mathcal{D}_t^2 v\|^2_{H^{1/2}(\Gamma_t)} + \|a_8(\nu,\nabla v)\|^2_{W^{1/2,4}(\Gamma_t)}\|\nabla^2\mathcal{D}_t^2 v\|^2_{H^{1/2}(\Gamma_t)}\right),\\
&\|a_9(\nu,\nabla v)\star\nabla\mathcal{D}_t^2 v\star B\|^2_{H^{1/2}(\Gamma_t)} \leqslant C\left(\|\nabla\mathcal{D}_t^2 v\|^2_{W^{1/2,4}(\Gamma_t)}\|B\|^2_{L^4(\Gamma_t)} + \|B\|^2_{H^{1/2}(\Gamma_t)}\|\nabla\mathcal{D}_t^2 v\|^2_{L^\infty(\Gamma_t)}\right),\\
&\|a_{10}(\nu,\nabla v)\star\nabla\mathcal{D}_t^2 v\star\nabla^2 v\|^2_{H^{1/2}(\Gamma_t)} \leqslant C\|\nabla^2 v\|^2_{H^{1/2}(\Gamma_t)}\left(\|\nabla\mathcal{D}_t^2 v\|^2_{W^{1/2,4}(\Gamma_t)} + \|\nabla\mathcal{D}_t^2 v\|^2_{L^\infty(\Gamma_t)}\right),
\end{aligned}
$$

and they can be controlled by $C\bar{E}(t)$. Moreover,

$$
\begin{aligned}
&\|a_{11}(\nu,\nabla v)\star\nabla^2\mathcal{D}_t v\star\nabla\mathcal{D}_t v\|^2_{H^{1/2}(\Gamma_t)}\\
&\quad\leqslant C\left(\|\nabla^2\mathcal{D}_t v\|^2_{L^\infty(\Omega_t)}\|\nabla\mathcal{D}_t v\|^2_{H^1(\Omega_t)} + \|\nabla^2\mathcal{D}_t v\|^2_{W^{1,6}(\Omega_t)}\|\nabla\mathcal{D}_t v\|^2_{L^3(\Omega_t)}\right)\\
&\quad\leqslant C\|\nabla(-\nabla p + H\cdot\nabla H)\|^2_{H^1(\Omega_t)}\|\mathcal{D}_t v\|^2_{H^4(\Omega_t)} \leqslant C\left(1+\|\nabla^2 p\|^2_{H^1(\Omega_t)}\right)\bar{E}(t),
\end{aligned}
$$

and the other terms can be estimated in the same way. For $l\geqslant 4$, the proof is similar to [26, lemma 5.8], and we omit the details. □

Applying the above error estimates and recalling proposition 3.1 as well as (4.1), we conclude this section by presenting the following improved version of proposition 3.1.

Proposition 5.6. *Assume that (1.6) holds for $T>0$. Then, we have $\frac{\mathrm{d}}{\mathrm{d}t}\bar{e}(t) \leqslant C\bar{E}(t)$, where C depends on $T,\mathcal{N}_T,\mathcal{M}_T,\|v_0\|_{H^6(\Omega_0)},\|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$. For $l\geqslant 4$, assume further that $\sup_{0\leqslant t<T}E_{l-1}(t)\leqslant C$, then we have $\frac{\mathrm{d}}{\mathrm{d}t}e_l(t)\leqslant CE_l(t)$, where the constant C depends on $T,\mathcal{N}_T,\mathcal{M}_T$, and $\sup_{0\leqslant t<T}E_{l-1}(t)$.*

6. Closing the energy estimates and proving the main theorems

In this section, we close the energy estimates and prove theorem 1.1. We introduce the energy functional

$$
\begin{aligned}
\tilde{e}(t) :=& \frac{1}{2}\sum_{k=1}^{3}\left(\|\mathcal{D}_t^{k+1}v\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t^{k+1}H\|^2_{L^2(\Omega_t)} + \|\bar{\nabla}\left(\mathcal{D}_t^k v\cdot\nu\right)\|^2_{L^2(\Gamma_t)}\right)\\
&+\frac{1}{2}\left(\|\operatorname{curl} v\|^2_{H^5(\Omega_t)} + \|\operatorname{curl} H\|^2_{H^5(\Omega_t)}\right)+1,\\
\tilde{e}_l(t) :=& \frac{1}{2}\left(\|\mathcal{D}_t^{l+1}v\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t^{l+1}H\|^2_{L^2(\Omega_t)} + \|\bar{\nabla}\left(\mathcal{D}_t^l v\cdot\nu\right)\|^2_{L^2(\Gamma_t)}\right)\\
&+\frac{1}{2}\left(\|\operatorname{curl} v\|^2_{H^{\lfloor(3l+1)/2\rfloor}(\Omega_t)} + \|\operatorname{curl} H\|^2_{H^{\lfloor(3l+1)/2\rfloor}(\Omega_t)}\right)+1,\quad l\geqslant 4.
\end{aligned}
$$

Note that from the *a priori* assumptions (1.6), it holds $\|\operatorname{curl} v\|^2_{L^2(\Omega_t)} + \|\operatorname{curl} H\|^2_{L^2(\Omega_t)} \leqslant C$. By interpolation, we have $\tilde{e}(t)\leqslant C(\bar{e}(t)+1)$ and $\tilde{e}_l(t)\leqslant C(e_l(t)+1)$ for $l\geqslant 4$.

We will apply the following div-curl estimates in [26, section 3.1].

Lemma 6.1. *Let the integer $l \geqslant 2$ and assume that $\|B_\Gamma\|_{H^{3l/2-1}(\Gamma)} \leqslant C$. Let $j \in \{5/2, 3, 7/2, 4, \cdots, 3l/2\}$ and $k \in \{3/2, 5/2, 3, 7/2, 4, \cdots, 3l/2\}$. Then, for all smooth vector fields F, it holds*

$$\|F\|_{H^k(\Omega)} \leqslant C\left(\|F_n\|_{H^{k-1/2}(\Gamma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}(\Omega)}\right), \tag{6.1}$$

$$\|F\|_{H^j(\Omega)} \leqslant C\left(\|\Delta_\Gamma F_n\|_{H^{j-5/2}(\Gamma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{H^{j-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{j-1}(\Omega)}\right), \tag{6.2}$$

$$\begin{aligned}\|F\|_{H^{\lfloor (3(l+1))/2 \rfloor}(\Omega)} \leqslant C\Big(&\|\Delta_B F_n\|_{H^{\lfloor (3l-2)/2 \rfloor}(\Gamma)} + \left(1 + \|B\|_{H^{3l/2}(\Gamma)}\right)\|F\|_{L^\infty(\Omega)} \\ &+ \|\operatorname{div} F\|_{H^{\lfloor (3l+1)/2 \rfloor - 1}(\Omega)} + \|\operatorname{curl} F\|_{H^{\lfloor (3l+1)/2 \rfloor - 1}(\Omega)}\Big).\end{aligned} \tag{6.3}$$

Proposition 6.2. *Assume that $\Gamma_t \in H^{3+\delta}(\Gamma)$ with $\delta > 0$ small enough. Assume that $\|p\|_{H^3(\Omega_t)} + \|v\|_{H^4(\Omega_t)} + \|H\|_{H^4(\Omega_t)} \leqslant C_0$. Then we have $\bar{E}(t) + \|B\|^2_{H^{9/2}(\Gamma_t)} \leqslant C(1 + \bar{e}(t))$, where the constant C depends on $\mathcal{M}_t, \|h(\cdot, t)\|_{H^{3+\delta}(\Gamma)}, \|p\|_{H^3(\Omega_t)}, \|v\|_{H^4(\Omega_t)}$, and $\|H\|_{H^4(\Omega_t)}$.*

Proof. We shall show that $\bar{E}(t) \leqslant C\tilde{e}(t)$. We need to control $\|\mathcal{D}_t^{4-k} v\|^2_{H^{3k/2}(\Omega_t)}, \|\mathcal{D}_t^{4-k} H\|^2_{H^{3k/2}(\Omega_t)}$, $k \leqslant 3, \|v\|^2_{H^6(\Omega_t)}$, and $\|H\|^2_{H^6(\Omega_t)}$. Recalling (4.8), it is sufficient to control $\|\mathcal{D}_t^3 v\|^2_{H^{3/2}(\Omega_t)}$, $\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)}, \|\mathcal{D}_t v\|^2_{H^{9/2}(\Omega_t)}, \|v\|^2_{H^6(\Omega_t)}$, and $\|H\|^2_{H^6(\Omega_t)}$. We divide the proof into three steps.

Step 1. We control $\|\mathcal{D}_t^3 v\|^2_{H^{3/2}(\Omega_t)}$. Recalling that $\|\tilde{\nu}\|_{H^{5/2+\delta}(\Omega_t)} \leqslant C$, we have

$$\begin{aligned}&\|\mathcal{D}_t^3 v \cdot \nu\|^2_{L^2(\Gamma_t)} \\ &\quad\leqslant \left|\int_{\Omega_t} \left(\mathcal{D}_t^3 v \cdot \nu\right) \operatorname{div} \mathcal{D}_t^3 v \mathrm{d}x\right| + \left|\int_{\Omega_t} \nabla \mathcal{D}_t^3 v \star \mathcal{D}_t^3 v \mathrm{d}x\right| + \left|\int_{\Omega_t} \mathcal{D}_t^3 v \star \nabla \nu \star \mathcal{D}_t^3 v \mathrm{d}x\right| \\ &\quad\leqslant C\left(\|\mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} + \|\operatorname{div} \mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} + \|\nabla \mathcal{D}_t^3 v\|_{L^2(\Omega_t)} \|\mathcal{D}_t^3 v\|_{L^2(\Omega_t)}\right) \\ &\quad\leqslant \varepsilon \|\nabla \mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} + C_\varepsilon \tilde{e}(t) + C\|\operatorname{div} \mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)}.\end{aligned}$$

This, combined with lemmas 2.10 and 5.2, and (6.1), it follows that $\|\mathcal{D}_t^3 v\|^2_{H^{3/2}(\Omega_t)} \leqslant C(\|\mathcal{D}_t^3 v \cdot \nu\|^2_{H^1(\Gamma_t)} + \|\mathcal{D}_t^3 v\|^2_{L^2(\Omega_t)} + \|\operatorname{div} \mathcal{D}_t^3 v\|^2_{H^{1/2}(\Omega_t)} + \|\operatorname{curl} \mathcal{D}_t^3 v\|^2_{H^{1/2}(\Omega_t)})$, and therefore,

$$\|\mathcal{D}_t^3 v\|^2_{H^{3/2}(\Omega_t)} \leqslant C\left(\tilde{e}(t) + \|R_I^2\|^2_{H^{1/2}(\Omega_t)} + \|R^2_{\nabla H, \nabla H}\|^2_{H^{1/2}(\Omega_t)} + \|R^2_{\nabla^2 H, H}\|^2_{H^{1/2}(\Omega_t)}\right).$$

To control $\|R^2_{\nabla H, \nabla H}\|^2_{H^{1/2}(\Omega_t)}$, we estimate as follows. Indeed, by the assumption, applying Young's inequality and lemma A.4, we obtain $\|\nabla^2 \mathcal{D}_t v \star \nabla H \star H\|^2_{H^{1/2}(\Omega_t)} + \|\nabla \mathcal{D}_t v \star \nabla^2 H \star H\|^2_{H^{1/2}(\Omega_t)} \leqslant C\|\mathcal{D}_t v\|^2_{H^3(\Omega_t)} \|H\|^4_{H^3(\Omega_t)}$ and

$$\|\mathcal{D}_t v\|^2_{H^3(\Omega_t)} \|H\|^4_{H^3(\Omega_t)} \leqslant \varepsilon \bar{E}(t) + C_\varepsilon \|p\|^2_{H^1(\Omega_t)} + C_\varepsilon \|H \cdot \nabla H\|^2_{L^2(\Omega_t)} \leqslant \varepsilon \bar{E}(t) + C_\varepsilon.$$

As for $\|R^2_{\nabla^2 H, H}\|^2_{H^{1/2}(\Omega_t)}$, we recall lemma 2.8, and we handle the most difficult term, i.e. $\|\nabla^3 \operatorname{curl} H \star H \star H \star H\|^2_{H^{1/2}(\Omega_t)} \leqslant C\|\operatorname{curl} H\|^2_{H^4(\Omega_t)} \leqslant C\tilde{e}(t)$. Again by the Young's inequality and lemma A.4, we can control $\|R_I^2\|^2_{H^{1/2}(\Omega_t)}$. In fact, we have

$$\begin{aligned}
&\|\nabla\mathcal{D}_t^2 v\star\nabla v\|_{H^{1/2}(\Omega_t)}^2+\|\nabla\mathcal{D}_t v\star\nabla\mathcal{D}_t v\|_{H^{1/2}(\Omega_t)}^2\\
&\quad\leqslant C\|v\|_{H^2(\Omega_t)}^2\|\mathcal{D}_t^2 v\|_{H^2(\Omega_t)}^2+C\|\nabla\mathcal{D}_t v\|_{L^3(\Omega_t)}^2\|\nabla\mathcal{D}_t v\|_{H^{3/2}(\Omega_t)}^2\\
&\quad\leqslant\left(\varepsilon\|\mathcal{D}_t v\|_{H^{9/2}(\Omega_t)}^2+C_\varepsilon\|\mathcal{D}_t v\|_{L^2(\Omega_t)}^2\right)\left(\|\nabla^2 p\|_{L^3(\Omega_t)}^2+\|\nabla\left(H\cdot\nabla H\right)\|_{L^3(\Omega_t)}^2\right)\\
&\qquad+\varepsilon\|\mathcal{D}_t^2 v\|_{H^3(\Omega_t)}^2+C_\varepsilon\|\mathcal{D}_t^2 v\|_{L^2(\Omega_t)}^2\leqslant C_\varepsilon\tilde{e}\left(t\right)+\varepsilon\bar{E}\left(t\right).
\end{aligned}$$

Combining the above estimates, we obtain $\|\mathcal{D}_t^3 v\|_{H^{3/2}(\Omega_t)}^2\leqslant\varepsilon\bar{E}(t)+C_\varepsilon\tilde{e}(t)$.

Step 2. We estimate $\|\mathcal{D}_t^2 v\|_{H^3(\Omega_t)}^2$ and $\|\mathcal{D}_t v\|_{H^{9/2}(\Omega_t)}^2$. Applying lemma 2.10 and (6.2), it holds

$$\begin{aligned}
\|\mathcal{D}_t v\|_{H^{9/2}(\Omega_t)}^2&\leqslant C\tilde{e}\left(t\right)+C\Big(\|\Delta_B\left(\mathcal{D}_t v\cdot\nu\right)\|_{H^2(\Gamma_t)}^2+\|\nabla v\star\nabla v\|_{H^{7/2}(\Omega_t)}^2\\
&\qquad+\|\nabla H\star\nabla H\|_{H^{7/2}(\Omega_t)}^2+\|H\star\nabla\operatorname{curl}H\|_{H^{7/2}(\Omega_t)}^2\Big)\\
&\leqslant C\tilde{e}\left(t\right)+C\|\Delta_B\left(\mathcal{D}_t v\cdot\nu\right)\|_{H^2(\Gamma_t)}^2,\\
\|\mathcal{D}_t^2 v\|_{H^3(\Omega_t)}^2&\leqslant C\Big(\|\Delta_B\left(\mathcal{D}_t^2 v\cdot\nu\right)\|_{H^{1/2}(\Gamma_t)}^2+\|R_I^1\|_{H^2(\Omega_t)}^2+\|R_{\nabla H,\nabla H}^1\|_{H^2(\Omega_t)}^2\\
&\qquad+\|R_{\nabla^2 H,H}^1\|_{H^2(\Omega_t)}^2\Big)+C\tilde{e}\left(t\right).
\end{aligned}$$

We control $\|R_I^1\|_{H^2(\Omega_t)}^2$ by the bilinear inequality, $\|\nabla\mathcal{D}_t v\star\nabla v\|_{H^2(\Omega_t)}^2\leqslant C\|\mathcal{D}_t v\|_{H^3(\Omega_t)}^2\|v\|_{H^3(\Omega_t)}^2$ $\leqslant\varepsilon\bar{E}(t)+C_\varepsilon\tilde{e}(t)$. For $\|R_{\nabla H,\nabla H}^1\|_{H^2(\Omega_t)}^2$, it holds that $\|\nabla^2 v\star\nabla H\star H\|_{H^2(\Omega_t)}^2+\|\nabla v\star$ $\nabla H\star\nabla H\|_{H^2(\Omega_t)}^2\leqslant C\|v\|_{H^4(\Omega_t)}^2\|H\|_{H^3(\Omega_t)}^4$ from the assumption. Then, the estimate for $\|R_{\nabla^2 H,H}^1\|_{H^2(\Omega_t)}^2$ follows since $\|\nabla^2\operatorname{curl}v\star H\star H\|_{H^2(\Omega_t)}^2\leqslant C\tilde{e}(t)$.

We are left with $\|\Delta_B(\mathcal{D}_t^2 v\cdot\nu)\|_{H^{1/2}(\Gamma_t)}^2$ and $\|\Delta_B(\mathcal{D}_t v\cdot\nu)\|_{H^2(\Gamma_t)}^2$. We focus on the estimate of $\|\Delta_B(\mathcal{D}_t^2 v\cdot\nu)\|_{H^{1/2}(\Gamma_t)}^2$. Recalling that from lemma 2.12, we have $\mathcal{D}_t^3 p=-\Delta_B(\mathcal{D}_t^2 v\cdot\nu)+R_p^2$. Since $\|R_p^2\|_{H^{1/2}(\Gamma_t)}^2$ is easier to control than $\|\mathcal{D}_t^3 p\|_{H^{1/2}(\Gamma_t)}^2$, we only bound $\|\mathcal{D}_t^3 p\|_{H^{1/2}(\Gamma_t)}^2$. By the definition of $H^{1/2}(\Gamma)$, it holds $\|\mathcal{D}_t^3 p\|_{H^{1/2}(\Gamma_t)}^2\leqslant C\|\mathcal{D}_t^3 p\|_{L^2(\Gamma_t)}^2+C\|\nabla\mathcal{D}_t^3 p\|_{L^2(\Omega_t)}^2$. Applying (3) in lemma 2.3, for the first term, we have

$$\|\mathcal{D}_t^3 p\|_{L^2(\Gamma_t)}^2\leqslant C\|\sum_{1\leqslant m\leqslant 3}\sum_{|\beta|\leqslant 3-m,|\alpha|\leqslant 1}a_{\alpha,\beta}\left(\nu,B\right)\bar{\nabla}^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\bar{\nabla}^{1+\alpha_m}\mathcal{D}_t^{\beta_m}v\|_{L^2(\Gamma_t)}^2.$$

For $m=1$, from $\|B\|_{L^\infty(\Gamma_t)}\leqslant C$, we control $a(\nu,B)\bar{\nabla}^2\mathcal{D}_t^2 v$ by the trace theorem and interpolation: $\|a(\nu,B)\bar{\nabla}^2\mathcal{D}_t^2 v\|_{L^2(\Gamma_t)}^2\leqslant C\|\mathcal{D}_t^2 v\|_{H^{5/2}(\Omega_t)}^2\leqslant\varepsilon\bar{E}(t)+C_\varepsilon\tilde{e}(t)$. The other cases are either simpler or similar. As for $\|\nabla\mathcal{D}_t^3 p\|_{L^2(\Omega_t)}^2$, it follows that $\|\nabla\mathcal{D}_t^3 p\|_{L^2(\Omega_t)}^2\leqslant C\tilde{e}(t)+C\|\mathcal{D}_t^3(H\cdot$ $\nabla H)\|_{L^2(\Omega_t)}^2+C\|[\nabla,\mathcal{D}_t^3]p\|_{L^2(\Omega_t)}^2$. To control $\|\mathcal{D}_t^3(H\cdot\nabla H)\|_{L^2(\Omega_t)}^2$, again by interpolation, we see that $\|\nabla^2\mathcal{D}_t^2 v\star H\star H\|_{L^2(\Omega_t)}^2+\|\nabla^2\mathcal{D}_t v\star H\star H\|_{L^2(\Omega_t)}^2\leqslant\varepsilon\bar{E}(t)+C_\varepsilon\tilde{e}(t)$, and we estimate $\|[\nabla,\mathcal{D}_t^3]p\|_{L^2(\Omega_t)}^2$ as follows

$$\begin{aligned}
&\|\nabla\mathcal{D}_t^2 v\star\nabla p\|_{L^2(\Omega_t)}^2+\|\nabla\mathcal{D}_t v\star\nabla\mathcal{D}_t p\|_{L^2(\Omega_t)}^2+\|\nabla v\star\nabla\mathcal{D}_t^2 p\|_{L^2(\Omega_t)}^2\\
&\quad\leqslant C\Big(\|\mathcal{D}_t^2 v\|_{H^2(\Omega_t)}^2\|p\|_{H^{3/2}(\Omega_t)}^2+\|\nabla\left(H\cdot\nabla H\right)\star\nabla\mathcal{D}_t p\|_{L^2(\Omega_t)}^2\\
&\qquad+\|\nabla^2 p\star\nabla\mathcal{D}_t p\|_{L^2(\Omega_t)}^2+\|\nabla\mathcal{D}_t^2 p\|_{L^2(\Omega_t)}^2\Big)\\
&\quad\leqslant C\|\mathcal{D}_t^2 v\|_{H^2(\Omega_t)}^2+C\|\nabla\mathcal{D}_t p\|_{L^3(\Omega_t)}^2+C\|\nabla\mathcal{D}_t^2 p\|_{L^2(\Omega_t)}^2.
\end{aligned}$$

Note that $\|\nabla\mathcal{D}_t^2 p\|^2_{L^2(\Omega_t)}$ and $\|\nabla\mathcal{D}_t p\|^2_{L^3(\Omega_t)}$ have fewer material derivatives than $\|\nabla\mathcal{D}_t^3 p\|^2_{L^2(\Omega_t)}$. Therefore, it can be estimated as $I_8(t)$ in the same fashion, and we can obtain $\|\mathcal{D}_t^3 p\|^2_{H^{1/2}(\Gamma_t)} \leqslant C\tilde{e}(t)+\varepsilon\bar{E}(t)$. Similarly, it holds $\|\mathcal{D}_t^2 p\|^2_{H^2(\Gamma_t)} \leqslant C\tilde{e}(t)+\varepsilon\bar{E}(t)$. Combining the above estimates, we conclude that $\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)} + \|\mathcal{D}_t v\|^2_{H^{9/2}(\Omega_t)} \leqslant C\tilde{e}(t)+\varepsilon\bar{E}(t)$.

Step 3. Finally, we bound $\|v\|^2_{H^6(\Omega_t)}$ and $\|H\|^2_{H^6(\Omega_t)}$. From (6.3), we see that $\|v\|^2_{H^6(\Omega_t)} \leqslant C(\tilde{e}(t)+\|\Delta_B v_n\|^2_{H^{7/2}(\Gamma_t)}+\|B\|^2_{H^{9/2}(\Gamma_t)})$ and $\|H\|^2_{H^6(\Omega_t)} \leqslant C(\tilde{e}(t)+\|B\|^2_{H^{9/2}(\Gamma_t)})$. Recalling lemma 5.1 and by the trace theorem, it follows that $\|B\|^2_{H^{9/2}(\Gamma_t)} \leqslant C(1+\|p\|^2_{H^{9/2}(\Gamma_t)}) \leqslant \varepsilon\bar{E}(t)+\|H\|^2_{H^5(\Omega_t)}+C_\varepsilon$.

Again by (6.3), we can estimate in $H^5(\Omega_t)$ and deduce $\|H\|^2_{H^5(\Omega_t)} \leqslant C\tilde{e}(t)+\|B\|^2_{H^{7/2}(\Gamma_t)}$. Similarly, it holds $\|B\|^2_{H^{7/2}(\Gamma_t)} \leqslant \varepsilon\bar{E}(t)+\|H\|^2_{H^4(\Omega_t)}+C_\varepsilon \leqslant \varepsilon\bar{E}(t)+C_\varepsilon$. Thus, $\|B\|^2_{H^{9/2}(\Gamma_t)} \leqslant \varepsilon\bar{E}(t)+C_\varepsilon$, $\|p\|^2_{H^{9/2}(\Gamma_t)} \leqslant \varepsilon\bar{E}(t)+C_\varepsilon$, and $\|H\|^2_{H^6(\Omega_t)} \leqslant \varepsilon\bar{E}(t)+C_\varepsilon$.

We are left with the term $\|\Delta_B v_n\|^2_{H^{7/2}(\Gamma_t)}$. From (2.10) and by the above calculations, it follows that

$$\begin{aligned}\|\Delta_B v_n\|^2_{H^{7/2}(\Gamma_t)} &\leqslant C\|\mathcal{D}_t p\|^2_{H^{7/2}(\Gamma_t)} + C\||B|^2 v_n\|^2_{H^{7/2}(\Gamma_t)} + C\|\bar{\nabla}p\cdot v\|^2_{H^{7/2}(\Gamma_t)}\\ &\leqslant C\|v\|^2_{H^4(\Omega_t)}\|B\|^2_{L^\infty(\Gamma_t)}\|B\|^2_{H^{7/2}(\Gamma_t)} + \frac{\varepsilon}{2}\bar{E}(t) + C_\varepsilon\tilde{e}(t) \leqslant C_\varepsilon\tilde{e}(t)+\varepsilon\bar{E}(t),\end{aligned}$$

where we have used the fact that

$$\begin{aligned}\|\mathcal{D}_t p\|^2_{H^{7/2}(\Gamma_t)} &\leqslant C\left(\|\mathcal{D}_t p\|^2_{L^2(\Gamma_t)} + \|\nabla\mathcal{D}_t p\|^2_{H^3(\Omega_t)}\right)\\ &\leqslant C\left(1+\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)} + \|\mathcal{D}_t(H\cdot\nabla H)\|^2_{H^3(\Omega_t)} + \|\nabla v\star(H\cdot\nabla H-\mathcal{D}_t v)\|^2_{H^3(\Omega_t)}\right)\\ &\leqslant C_\varepsilon\tilde{e}(t)+\frac{\varepsilon}{2}\bar{E}(t),\end{aligned}$$

and interpolation arguments since $\|\mathcal{D}_t H\|^2_{H^4(\Omega_t)}$ and $\|\mathcal{D}_t^2 v\|^2_{H^3(\Omega_t)}$ have already been controlled. This completes the proof. □

Proposition 6.3. *Let $l\geqslant 4$. Assume that (1.6) holds for some $T>0$ and $\sup_{0\leqslant t<T}E_{l-1}(t)\leqslant C$. Then, we have $E_l(t)\leqslant C(1+e_l(t))$, where the constant C depends on $l,T,\mathcal{N}_T,\mathcal{M}_T$ and $\sup_{0\leqslant t<T}E_{l-1}(t)$.*

Proof. We will show that $E_l(t)\leqslant C\tilde{e}_l(t)$ and we divide the proof into three steps.

Step 1. We claim that it is sufficient to bound $\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{3k/2}(\Omega_t)}, k\in\{1,2,\cdots,l\}$, $\|v\|^2_{H^{\lfloor 3(l+1)/2\rfloor}(\Omega_t)}$ and $\|H\|^2_{H^{\lfloor 3(l+1)/2\rfloor}(\Omega_t)}$. Indeed, $\|\mathcal{D}_t^{l+1-k}H\|^2_{H^{3k/2}(\Omega_t)}$ can be controlled by these quantities. Starting with the case of $2\leqslant k\leqslant l-1$, from the hypothesis, (2.5) and (2.6), we have

$$\|\mathcal{D}_t^{l+1-k}H\|^2_{H^{3k/2}(\Omega_t)} \leqslant C\sum_{\substack{1\leqslant m\leqslant l+1-k\\ |\beta|\leqslant l+1-k-m}}\|\nabla\mathcal{D}_t^{\beta_1}v\|^2_{H^{3k/2}(\Omega_t)}\cdots\|\nabla\mathcal{D}_t^{\beta_m}v\|^2_{H^{3k/2}(\Omega_t)}\|H\|^2_{H^{3k/2}(\Omega_t)}.$$

If $m=1$, we see that $\|\mathcal{D}_t^{l+1-k}H\|^2_{H^{3k/2}(\Omega_t)} \leqslant C(\|\mathcal{D}_t^{l+1-(k+1)}v\|^2_{H^{3(k+1)/2}(\Omega_t)}+1)$, since $\|H\|^2_{H^{3k/2}(\Omega_t)} \leqslant CE_{l-1}(t)\leqslant C$. For $m\geqslant 2$, it holds $\|\mathcal{D}_t^{l+1-k}H\|^2_{H^{3k/2}(\Omega_t)} \leqslant CE_{l-1}(t)\cdots E_{l-1}(t) \leqslant C$.

Next, we deal with the case of $k=1$, and it follows that

$$\begin{aligned}\|\mathcal{D}_t^l H\|_{H^{3/2}(\Omega_t)}^2 &\leqslant C\sum_{\beta_1\leqslant l-1}\|\nabla\mathcal{D}_t^{\beta_1}v\|_{H^{3/2}(\Omega_t)}^2\|H\|_{H^2(\Omega_t)}^2\\ &\quad + C\sum_{2\leqslant m\leqslant l,|\beta|\leqslant l-m}\|\nabla\mathcal{D}_t^{\beta_1}v\|_{H^2(\Omega_t)}^2\cdots\|\nabla\mathcal{D}_t^{\beta_m}v\|_{H^2(\Omega_t)}^2\|H\|_{H^2(\Omega_t)}^2\\ &\leqslant C\left(\|\nabla\mathcal{D}_t^{l-1}v\|_{H^{3/2}(\Omega_t)}^2+1\right)\leqslant C\left(\|\mathcal{D}_t^{l+1-2}v\|_{H^3(\Omega_t)}^2+1\right).\end{aligned}$$

Finally, for an even integer $k=l$, one has $\|\mathcal{D}_t H\|_{H^{3l/2}(\Omega_t)}^2\leqslant C\|H\|_{H^{\lfloor 3l/2\rfloor}(\Omega_t)}^2\|v\|_{H^{\lfloor 3l/2+1\rfloor}(\Omega_t)}^2\leqslant C\|v\|_{H^{\lfloor 3l/2+1\rfloor}(\Omega_t)}^2$ from $E_{l-1}(t)\leqslant C$, and if $k=l$ is odd, we have by lemma A.5 that

$$\begin{aligned}\|\mathcal{D}_t H\|_{H^{3l/2}(\Omega_t)}^2 &\leqslant C\left(\|H\|_{L^\infty(\Omega_t)}^2\|v\|_{H^{3l/2+1}(\Omega_t)}^2+\|H\|_{H^{3l/2}(\Omega_t)}^2\|v\|_{L^\infty(\Omega_t)}^2\right)\\ &\leqslant C\|v\|_{H^{\lfloor 3(l+1)/2\rfloor}(\Omega_t)}^2+C\|H\|_{H^{\lfloor 3(l+1)/2\rfloor}(\Omega_t)}^2.\end{aligned}$$

Step 2. We claim that $\|\mathcal{D}_t^l v\|_{H^{3/2}(\Omega_t)}^2\leqslant\varepsilon E_l(t)+C\tilde{e}_l(t)$. Due to the fact that $\|\nu\|_{H^{5/2+\delta}(\Omega_t)}\leqslant C$ and the assumption $E_{l-1}(t)\leqslant C$, we have

$$\begin{aligned}\|\mathcal{D}_t^l v\cdot\nu\|_{L^2(\Gamma_t)}^2 &\leqslant\left|\int_{\Omega_t}\left(\mathcal{D}_t^l v\cdot\nu\right)\operatorname{div}\mathcal{D}_t^l v\mathrm{d}x\right|+\left|\int_{\Omega_t}\nabla\mathcal{D}_t^l v\star\mathcal{D}_t^l v\mathrm{d}x\right|+\left|\int_{\Omega_t}\mathcal{D}_t^l v\star\nabla\nu\star\mathcal{D}_t^l v\mathrm{d}x\right|\\ &\leqslant C\left(\left\|\mathcal{D}_t^l v\right\|_{L^2(\Omega_t)}^2+\left\|\operatorname{div}\mathcal{D}_t^l v\right\|_{L^2(\Omega_t)}^2+\|\nabla\mathcal{D}_t^l v\|_{L^2(\Omega_t)}\|\mathcal{D}_t^l v\|_{L^2(\Omega_t)}\right)\\ &\leqslant\varepsilon\|\nabla\mathcal{D}_t^l v\|_{L^2(\Omega_t)}^2+C\left(1+\|\operatorname{div}\mathcal{D}_t^l v\|_{L^2(\Omega_t)}^2\right).\end{aligned}$$

This, combined with (6.1), we see that

$$\begin{aligned}\|\mathcal{D}_t^l v\|_{H^{3/2}(\Omega_t)}^2 &\leqslant C\left(\|\mathcal{D}_t^l v\cdot\nu\|_{H^1(\Gamma_t)}^2+\|\mathcal{D}_t^l v\|_{L^2(\Omega_t)}^2+\|\operatorname{div}\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)}^2+\|\operatorname{curl}\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)}^2\right)\\ &\leqslant C\left(\varepsilon\|\mathcal{D}_t^l v\|_{H^1(\Omega_t)}^2+1+E_{l-1}(t)+\|\bar{\nabla}\left(\mathcal{D}_t^l v\cdot\nu\right)\|_{L^2(\Gamma_t)}^2\right.\\ &\qquad\left.+\|\operatorname{div}\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)}^2+\|\operatorname{curl}\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)}^2\right).\end{aligned}$$

Then, it follows that $\|\mathcal{D}_t^l v\|_{H^{3/2}(\Omega_t)}^2\leqslant C(\tilde{e}_l(t)+\|\operatorname{div}\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)}^2+\|\operatorname{curl}\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)}^2)$. Applying lemmas 2.10, 5.2 and 5.3, we arrive at

$$\begin{aligned}&\|\operatorname{div}\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)}^2+\|\operatorname{curl}\mathcal{D}_t^l v\|_{H^{1/2}(\Omega_t)}^2\\ &\quad\leqslant C\left(\|R_l^{l-1}\|_{H^{1/2}(\Omega_t)}^2+\|R_{\nabla H,\nabla H}^{l-1}\|_{H^{1/2}(\Omega_t)}^2+\|R_{\nabla^2 H,H}^{l-1}\|_{H^{1/2}(\Omega_t)}^2\right)\leqslant\varepsilon E_l(t)+C_\varepsilon,\end{aligned}$$

where $\varepsilon>0$ is sufficiently small. This concludes the claim.

Step 3. We claim that for $2\leqslant k\leqslant l$, it holds

$$\|\mathcal{D}_t^{l+1-k}v\|_{H^{3k/2}(\Omega_t)}^2\leqslant C\|\mathcal{D}_t^{l+3-k}v\|_{H^{3k/2-3}(\Omega_t)}^2+\varepsilon E_l(t)+C_\varepsilon\tilde{e}_l(t).\tag{6.4}$$

Once we have these estimates, it follows that $\|\mathcal{D}_t^{l-1}v\|_{H^3(\Omega_t)}^2\leqslant\varepsilon E_l(t)+C_\varepsilon\tilde{e}_l$. This, combined with Step 2, will control $\|\mathcal{D}_t^{l+1-k}v\|_{H^{3k/2}(\Omega_t)}^2$ for any $3\leqslant k\leqslant l$. To prove (6.4), from lemmas 2.10, 5.2 and 5.3, and (6.2), it holds

$$\begin{aligned}\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{3k/2}(\Omega_t)} &\leqslant C\Big(\|\Delta_B\left(\mathcal{D}_t^{l+1-k}v\cdot\nu\right)\|^2_{H^{(3k-5)/2}(\Gamma_t)}+\|R_I^{l-k}\|^2_{H^{(3k-2)/2}(\Omega_t)}\\ &\quad+\|R^{l-k}_{\nabla H,\nabla H}\|^2_{H^{(3k-2)/2}(\Omega_t)}+\|R^{l-k}_{\nabla^2 H,H}\|^2_{H^{(3k-2)/2}(\Omega_t)}+E_{l-1}(t)\Big)\\ &\leqslant C\|\Delta_B\left(\mathcal{D}_t^{l+1-k}v\cdot\nu\right)\|^2_{H^{(3k-2)/2}(\Gamma_t)}+\varepsilon E_l(t)+C_\varepsilon.\end{aligned}$$

Lemmas 2.12 and 5.5 give $\mathcal{D}_t^{l+2-k}p=-\Delta_B(\mathcal{D}_t^{l+1-k}v\cdot\nu)+R_p^{l+1-k}$, and $\|R_p^{l+1-k}\|^2_{H^{(3k-5)/2}(\Gamma_t)}\leqslant\varepsilon E_l(t)+C_\varepsilon$. Then, we obtain $\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{3k/2}(\Omega_t)}\leqslant C\|\mathcal{D}_t^{l+2-k}p\|^2_{H^{(3k-5)/2}(\Gamma_t)}+\varepsilon E_l(t)+C_\varepsilon$. By (A1), we see that $\|\mathcal{D}_t^{l+2-k}p\|^2_{H^{(3k-5)/2}(\Gamma_t)}\leqslant\|\mathcal{D}_t^{l+2-k}p\|^2_{H^{(3k-6)/2}(\Gamma_t)}+\|\nabla\mathcal{D}_t^{l+2-k}p\|^2_{H^{(3k-6)/2}(\Omega_t)}$.

The first term can be controlled by lemma 2.3 as in proposition 6.2, i.e. $\|\mathcal{D}_t^{l+2-k}p\|^2_{H^{(3k-6)/2}(\Gamma_t)}\leqslant\varepsilon E_l(t)+C_\varepsilon$. For the second term, by (1.1), lemmas 5.3 and 5.4, it holds

$$\begin{aligned}\|\nabla\mathcal{D}_t^{l+2-k}p\|^2_{H^{(3k-6)/2}(\Omega_t)} &\leqslant \|\mathcal{D}_t^{l+3-k}v\|^2_{H^{(3k-6)/2}(\Omega_t)}+\|\sum_{\beta\leqslant l+1-k}\nabla\mathcal{D}_t^{\beta}v\star\nabla H\star H\|^2_{H^{(3k-6)/2}(\Omega_t)}\\ &\quad+\|R_{II}^{l+1-k}\|^2_{H^{(3k-6)/2}(\Omega_t)}+\|R^{l+2-k}_{\nabla H,H}\|^2_{H^{(3k-6)/2}(\Omega_t)}\\ &\leqslant \|\mathcal{D}_t^{l+3-k}v\|^2_{H^{(3k-6)/2}(\Omega_t)}+\varepsilon E_l(t)+C_\varepsilon.\end{aligned}$$

Combining the above estimates, (6.4) follows.

It remains to verify that $\|v\|^2_{H^{\lfloor 3(l+1)/2\rfloor}(\Omega_t)}+\|H\|^2_{H^{\lfloor 3(l+1)/2\rfloor}(\Omega_t)}\leqslant\varepsilon E_l(t)+C_\varepsilon\tilde{e}_l(t)$. Note that from lemma 5.1 with $l\geqslant 4$, one has $\|B\|_{H^{3l/2-1}(\Gamma_t)}\leqslant C$ and $\|B\|_{H^k(\Gamma_t)}\leqslant C(1+\|p\|_{H^k(\Gamma_t)})$ for $k\in\mathbb{N}/2, k\leqslant 3l/2$. Then, we can apply the argument as in proposition 6.2. This completes the proof. □

We are ready to prove the main results.

Proof of theorem 1.1. We divide the proof into three parts.

Step 1. We prove the first two statements in theorem 1.1. Assume that the *a priori* assumptions (1.6) hold for some $T>0$.

Recalling the estimates in section 4 that $\bar{E}(0)+\sup_{0\leqslant t<T}\|p\|^2_{H^3(\Omega_t)}\leqslant C$, where C depends on $T,\mathcal{N}_T,\mathcal{M}_T,\|v_0\|_{H^6(\Omega_0)},\|H_0\|_{H^6(\Omega_0)}$, and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$. Then, the assumptions of proposition 6.2 hold for any $0\leqslant t<T$, and propositions 5.6 and 6.2 allow us to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{e}(t)\leqslant C\bar{E}(t)\leqslant C(1+\bar{e}(t)),\quad 0\leqslant t<T. \tag{6.5}$$

Integrating over $(0,t)$, we have $\sup_{0\leqslant t<T}\bar{e}(t)\leqslant C(1+\bar{e}(0))e^{CT}$. Again by proposition 6.2, we see that

$$\sup_{0\leqslant t<T}\bar{E}(t)\leqslant C+C(1+\bar{e}(0))e^{CT}\leqslant C+C\bar{E}(0)e^{CT}\leqslant\bar{C}_0, \tag{6.6}$$

where $\bar{C}_0=\bar{C}_0\left(T,\mathcal{N}_T,\mathcal{M}_T,\|v_0\|_{H^6(\Omega_0)},\|H_0\|_{H^6(\Omega_0)},\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}\right)$.

With $\sup_{0\leqslant t<T}(\bar{E}(t)+\|p\|^2_{H^3(\Omega_t)})\leqslant\bar{C}_0$, applying lemma 5.1 and the trace theorem, it follows that $\|B\|^2_{H^{9/2}(\Gamma_t)}\leqslant C(1+\|H\cdot\nabla H-\mathcal{D}_t v\|^2_{H^4(\Omega_t)})\leqslant C(\bar{C}_0)$, giving $\|B\|^2_{H^{9/2}(\Gamma_t)}+\|p\|^2_{H^5(\Omega_t)}\leqslant C(\bar{C}_0)$. We proceed to find that

$$\|p\|^2_{H^{11/2}(\Omega_t)}\leqslant C\left(1+\|\nabla p\|^2_{H^{9/2}(\Omega_t)}\right)\leqslant C\left(1+\|H\cdot\nabla H-\mathcal{D}_t v\|^2_{H^{9/2}(\Omega_t)}\right)\leqslant C(\bar{C}_0),$$

and we utilize lemma A.6 to obtain $\|B\|^2_{H^5(\Gamma_t)} \leqslant C(1+\|p\|^2_{H^5(\Gamma_t)}) \leqslant C(\bar{C}_0)$. In particular, it follows that $\|\mathcal{A}\|^2_{H^5(\Gamma_t)} \leqslant C(\bar{C}_0)$, and proposition 4.7 yields $\sum_{k=0}^{3} \|\mathcal{D}_t^{3-k} p\|^2_{H^{3k/2+1}(\Omega_t)} \leqslant C$, where $C = C\left(\mathcal{R} - \|h(\cdot,t)\|_{L^\infty(\Gamma)}, \|v\|_{H^6(\Omega_t)}, \|H\|_{H^6(\Omega_t)}, \|\mathcal{A}\|_{H^5(\Gamma_t)}\right)$. Combining the above estimates, (1.7) follows. Then, from the definitions of the material derivative and $\bar{E}(t)$, we can also verify (1.8).

To prove the second result, for $l \geqslant 4$, we apply propositions 5.6 and 6.3 by induction to find that: if $\sup_{0\leqslant t<T} E_{l-1}(t) \leqslant C$, then it follows that $\frac{\mathrm{d}}{\mathrm{d}t} e_l(t) \leqslant C E_l(t) \leqslant C(1+e_l(t))$. Similarly, we integrate over $(0,t)$ and use proposition 6.3 again to obtain $\sup_{0\leqslant t<T} e_l(t) \leqslant C(1+e_l(0))e^{CT}$, and

$$\sup_{0\leqslant t<T} E_l(t) \leqslant C + C(1+e_l(0))\, e^{CT} \leqslant C_l\left(l, T, \mathcal{N}_T, \mathcal{M}_T, \sup_{0\leqslant t<T} E_{l-1}(t), e_l(0)\right). \tag{6.7}$$

However, the induction argument implies that (6.7) holds for all l and the constant C_l which depends on $l, T, \mathcal{N}_T, \mathcal{M}_T, e_l(0)$ and $\bar{e}(0)$ from (6.6). Note that $\bar{e}(0) + e_l(0) \leqslant C E_l(0)$, and the constant C_l in fact depends on $l, T, \mathcal{N}_T, \mathcal{M}_T$, and $E_l(0)$. This completes the proof of our claim. Again by the definition of the material derivative, (1.10) follows.

Step 2. We prove the last statement in theorem 1.1, i.e. the *a priori* assumptions (1.6) hold for some time $T_0 \geqslant c_0 > 0$, where c_0 depends on $\mathcal{M}_0, \|v_0\|_{H^6(\Omega_0)}, \|H_0\|_{H^6(\Omega_0)}$ and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$. To this aim, we define

$$I(t) := \|B\|^2_{H^3(\Gamma_t)} + \|p\|^2_{H^3(\Omega_t)} + \|v\|^2_{H^4(\Omega_t)} + \|H\|^2_{H^4(\Omega_t)} + 1, \quad t \geqslant 0.$$

Suppose that it holds $I(t) \leqslant 2I(0)$ and $\mathcal{M}_t \geqslant \mathcal{M}_0/2$ for some $t > 0$, where $\mathcal{M}_0 = \mathcal{R} - \|h_0\|_{L^\infty(\Gamma)}$. Then we have $\|\mathcal{A}_{\Gamma_t}\|^2_{H^3(\Gamma_t)} \leqslant C(I(0))$. Thus, applying lemma A.2, one has $\|h(\cdot,t)\|_{H^{3+\delta}(\Gamma)} \leqslant C$, for $\delta > 0$ small enough, where the constant C depends on $\|\mathcal{A}_{\Gamma_t}\|_{H^{1+\delta}(\Gamma_t)}$, and hence on $I(0)$. An application of proposition 6.2 allows us to obtain that there exists a constant C, depending on $I(0)$ and $\mathcal{M}_0$ such that

$$\bar{E}(t) \leqslant C(1+\bar{e}(t)). \tag{6.8}$$

From the above argument, we define $T_0 \in (0,1]$ to be the largest number such that

$$[0,T_0] \subset \{t \in [0,1] : I(t) \geqslant I(0)/2, \mathcal{M}_t \geqslant \mathcal{M}_0/2, \text{and } \bar{e}(t) \leqslant 1 + \bar{e}(0)\}. \tag{6.9}$$

Here, we assume that $T_0 < 1$, since the claim would be trivial otherwise. We note that the last condition together with (6.8) implies that

$$\sup_{0\leqslant t\leqslant T_0} \bar{E}(t) \leqslant C(1+\bar{e}(t)) \leqslant C(2+\bar{e}(0)) \leqslant C\bar{E}(0). \tag{6.10}$$

Also, we observe that satisfies $\mathcal{N}^2_{T_0} \leqslant C \sup_{0\leqslant t<T_0} \bar{E}(t)$, thanks to the curvature bound $\|B\|_{H^3(\Gamma_t)} \leqslant 2I(0)$. Indeed, from $\bar{\nabla} v_n = \bar{\nabla} v \cdot \nu - v \star B$, we can bound $\|v_n\|_{H^4(\Gamma_t)}$ by using $\|v\|_{H^4(\Gamma_t)}$ and $\|B\|_{H^3(\Gamma_t)}$.

The estimate (6.10) ensures that the *a priori* assumptions (1.6) hold for time $T = T_0$, and the claim follows once we show that T_0 specified in (6.9) has a lower bound $c_0 > 0$. From the definition of T_0, at least one of the three conditions has equality. Assume that $I(T_0) = 2I(0)$. Then, it holds $\bar{E}(t) \leqslant C\bar{E}(0)$, for all $t \leqslant T_0$ by (6.10). We will show that

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leqslant C\bar{E}(t) I(t) \leqslant C\bar{E}(0) I(t). \tag{6.11}$$

We focus on the computation of the highest-order terms. In fact, lemma 2.2 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla^4 v\|^2_{L^2(\Omega_t)}+\|\nabla^4 H\|^2_{L^2(\Omega_t)}\right)\leqslant \int_{\Omega_t}\nabla^4\mathcal{D}_t v\star\nabla^4 v+\sum_{|\alpha|\leqslant 3}\nabla^{1+\alpha_1}v\star\nabla^{1+\alpha_2}v\star\nabla^4 v\mathrm{d}x$$
$$+\int_{\Omega_t}\nabla^4\mathcal{D}_t H\star\nabla^4 H+\sum_{|\alpha|\leqslant 3}\nabla^{1+\alpha_1}v\star\nabla^{1+\alpha_2}H\star\nabla^4 H\mathrm{d}x$$
$$\leqslant C\bar{E}(t)I(t).$$

Applying lemmas 2.2 and 2.6, it is easy to deduce $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^3 p\|^2_{L^2(\Omega_t)}\leqslant C\bar{E}(t)I(t)$. Similarly, we can obtain by lemma A.1 that $\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\nabla}^3 B\|^2_{L^2(\Gamma_t)}\leqslant C\bar{E}(t)I(t)$. By integrating (6.11) over $(0,T_0)$ and using $I(T_0)=2I(0)$, we obtain $\ln 2=\ln I(T_0)-\ln I(0)\leqslant CT_0\bar{E}(0)$. Then we have $T_0\geqslant C/\bar{E}(0)=c_0$, where the constant c_0 depends on $I(0),\mathcal{M}_0$, and $\bar{E}(0)$. Moreover, by lemma 5.1 and proposition 4.7, the constant c_0 depends only on $\mathcal{M}_0,\|v_0\|_{H^6(\Omega_0)},\|H_0\|_{H^6(\Omega_0)}$ and $\|\mathcal{A}_{\Gamma_0}\|_{H^5(\Gamma_0)}$.

A similar argument applies if we have an equality in the third condition, i.e. $\bar{e}(T_0)=1+\bar{e}(0)$. In fact, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\bar{e}(t)\leqslant C\bar{E}(t)\leqslant C\bar{E}(0)$ by (6.5) and (6.10), and we integrate over $(0,T_0)$ to obtain $1=\bar{e}(T_0)-\bar{e}(0)\leqslant C\bar{E}(0)T_0$. This results in $T_0\geqslant c_0>0$ again.

Finally, we assume that $\mathcal{M}_{T_0}=\mathcal{M}_0/2$. Recalling that $\mathcal{M}_T=\mathcal{R}-\sup_{0\leqslant t<T}\|h(\cdot,t)\|_{L^\infty(\Gamma)}$, and $\mathcal{M}_0>0$, we define $0<T_1\leqslant T_0$ by $\mathcal{M}_{T_0}=\mathcal{R}-\|h(\cdot,T_1)\|_{L^\infty(\Gamma)}$. It is clear that $\|v_n\|^2_{L^\infty(\Omega_t)}\leqslant C\bar{E}(t)\leqslant C\bar{E}(0)$ by using (6.10). Recalling the fact that $\partial_t h=v_n$, we have by the fundamental theorem of calculus that

$$\mathcal{M}_{T_0}=\mathcal{R}-\|h(\cdot,T_1)\|_{L^\infty(\Gamma)}\geqslant\mathcal{R}-\|h_0\|_{L^\infty(\Gamma)}-\int_0^{T_1}\|v_n\|_{L^\infty(\Omega_t)}\mathrm{d}t\geqslant\mathcal{M}_0-C\bar{E}(0)^{\frac{1}{2}}T_1,$$

which means $T_0\geqslant T_1\geqslant C\mathcal{M}_0/\bar{E}(0)^{1/2}>0$. This concludes the claim.

Step 3. Finally, we prove that the smooth solution does not develop singularities at time T. According to the *a priori* assumptions, the estimates (1.7) and (1.9) hold. In particular, we conclude by lemmas 5.1 and A.2 that the regularity of the curvature implies the regularity of the free boundary, i.e. $\Gamma_T\in C^\infty$. Additionally, the quantitative regularity estimates show that the time derivatives of arbitrary order of the velocity and magnetic field are smooth, i.e. belong to $\in C^\infty(\Omega_T)$. This completes the proof of the theorem. □

Finally, we prove the blow-up classification in theorem 1.2.

Proof of theorem 1.2. We prove this by contradiction. Assume that $T_*<\infty$, i.e. $v(\cdot,T_*),H(\cdot,T_*)\notin H^6(\Omega_{T_*})$ or $\Gamma_{T_*}\notin H^7$. Assume further that none of (1)–(4) hold. That is, $\inf_{0\leqslant t<T_*}\mathcal{R}(\Omega_t)>0,\Gamma_t\in H^{3+\delta},0\leqslant t\leqslant T_*$, and $\sup_{0\leqslant t<T_*}(\|\nabla v\|_{H^3(\Omega_t)}+\|\nabla H\|_{H^3(\Omega_t)}+\|v_n\|_{H^4(\Gamma_t)})<\infty$, where we have applied lemma A.2 and the fact that $v_n=V_{\Gamma_t}$. In particular, $\mathcal{R}(\Omega_{T_*})>0$ and we choose $\Gamma_{T_*}=\partial\Omega_{T_*}$ as the reference surface to represent the free boundary over a short time interval before T_*. More precisely, the height function $h(\cdot,t)$ is well-defined on $[T_*-\varepsilon,T_*)$ for sufficiently small $\varepsilon>0$ and one has $\sup_{[T_*-\varepsilon,T_*)}\|h\|_{H^{3+\delta}(\Gamma_{T_*})}<\infty$. Therefore, it holds that

$$\sup_{T_*-\varepsilon\leqslant t<T_*}\left(\|h\|_{H^{3+\delta}(\Gamma_{T_*})}+\|\nabla v\|_{H^3(\Omega_t)}+\|\nabla H\|_{H^3(\Omega_t)}+\|v_n\|_{H^4(\Gamma_t)}\right)<\infty.$$

Applying the low-order estimates in theorem 1.1, it follows that $v(\cdot,T_*),H(\cdot,T_*)\in H^6(\Omega_{T_*})$ and $\Gamma_{T_*}\in H^7$ and the solution can be extended for some time. This leads to a contradiction and the proof is complete. □

7. Further discussions of theorem 1.2

In the blow-up classification given in theorem 1.2, the first two scenarios concern the geometric behaviour of the free boundary. In the final section of this manuscript, we explore the connection between the self-intersection singularity in case (1) and the curvature blow-up in case (2).

To quantitatively characterize how close the free boundary is to self-intersection, we adopt the concept of the injectivity radius ι_0 of the normal exponential map, as introduced in [4]. Specifically, $\iota_0(t)$ is defined as the largest positive number such that the map

$$\Gamma_t \times (-\iota_0(t), \iota_0(t)) \to \left\{ y \in \mathbb{R}^3 : \operatorname{dist}(y, \Gamma_t) < \iota_0(t) \right\} \text{ given by } (x, \iota) \mapsto x + \iota \nu(x),$$

is an injection.

By combining a lower bound on the injectivity radius $\iota_0(t)$ with an upper bound on the second fundamental form B_{Γ_t}, which measures the curvature, one can derive a positive lower bound for the uniform interior and exterior ball radius via [13, lemma 1]. Specifically, if there exists a constant $K > 0$ such that

$$\frac{1}{\iota_0(t)} + \|B_{\Gamma_t}\|_{L^\infty(\Gamma_t)} \leqslant K, \tag{7.1}$$

then there exists $r = r(K) > 0$ such that $\mathcal{R}(\Omega_t) \geqslant r$. Consequently, if condition (7.1) holds uniformly for all $t \in [0, T_*)$, i.e.

$$\sup_{t \in [0, T_*)} \left(\frac{1}{\iota_0(t)} + \|B_{\Gamma_t}\|_{L^\infty(\Gamma_t)} \right) \leqslant K,$$

then the self-intersection singularity will be excluded.

However, a uniform upper bound on the second fundamental form alone does not, in general, guarantee a uniform positive lower bound for the injectivity radius $\inf_{t \in [0, T_*)} \iota_0(t)$ or for the uniform interior and exterior ball radius $\inf_{t \in [0, T_*)} \mathcal{R}(\Omega_t)$.

In fact, there exist surfaces whose curvature remains uniformly bounded while their injectivity radius tends to zero. Such configurations were employed by Coutand and Shkoller [7] to construct initial domains for the viscous water wave equations that lie sufficiently close to self-intersection (see figures 2 and 3 in [7]), together with divergence-free initial velocity fields that drive the boundary toward self-intersection. Notably, the curvature either remains unchanged or undergoes only minimal variation during the deformation that leads to the self-intersection in finite time. Similar constructions were later developed by Hong *et al* [21] in the context of the viscous and non-resistive incompressible MHD equations.

Moreover, there exist surfaces for which the curvature becomes unbounded while the injectivity radius simultaneously tends to zero. To illustrate this, consider a dumbbell-shaped surface whose connecting neck is gradually squeezed and thinned. As this constriction intensifies, the curvature tends to infinity, and the interior ball radius approaches zero. A natural and interesting question is whether one can construct a class of regular solutions to system (1.1) based on such special geometric configurations, where the curvature of the free boundary blows up and the boundary simultaneously approaches self-intersection within a finite time.

Data availability statement

No new data were created or analysed in this study.

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Conflict of interest

The authors declare that there is no conflict of interest.

Appendix. Some estimates and formulas

Lemma A.1. *For a smooth function f, it holds*

(1) $[\mathcal{D}_t,\partial_i]f=-\partial_i v^k\partial_k f,[\mathcal{D}_t,\bar\nabla]f=-(\bar\nabla v)^\top\bar\nabla f,[\mathcal{D}_t,\bar\nabla^2]f=\bar\nabla^2 v\star\bar\nabla f+\bar\nabla v\star\bar\nabla^2 f,[\bar\nabla,\nabla]f=\nabla f\star\nabla\nu\star\nu,[\mathcal{D}_t,\Delta_B]f=\bar\nabla^2 f\star\nabla v-\bar\nabla f\cdot\Delta_B v+B\star\nabla v\star\bar\nabla f,[\partial_\nu,\partial_k]u=-\nabla u\cdot\partial_k\nu.$
(2) $\mathcal{D}_t\nu=-(\bar\nabla v)^\top\nu=-\bar\nabla v_n+Bv_\sigma,\bar\nabla v_n=\bar\nabla v^\top\nu+B_\Gamma v_\sigma,\mathcal{D}_t B=-\bar\nabla^2 v\star\nu-\bar\nabla v\star B.$

Proof. Most formulas can be found in [37, section 3.1] and the others follow from direct calculations. □

Lemma A.2 ([37, proposition A.2]). *Let $\Omega\subset\mathbb{R}^3$ be a domain such that $\partial\Omega\in H^{s_0},s_0>2$. Suppose $\|\mathcal{A}\|_{H^{s-2}(\Gamma_t)}\leqslant C$ with $s\geqslant s_0$, then $\partial\Omega\in H^s$.*

Let $u\in L^2(\Gamma)$. We define the space $H^{1/2}(\Gamma)$ via the harmonic extension:

$$\|u\|_{H^{1/2}(\Gamma)}:=\|u\|_{L^2(\Gamma)}+\inf\left\{\|\nabla w\|_{L^2(\Omega)}:w\in H^1(\Omega)\text{ and }w=u\text{on }\Gamma\right\}=\|u\|_{L^2(\Gamma)}+\|\nabla v\|_{L^2(\Omega)},$$

where $v\in H^1(\Omega)$ such that $v|_\Gamma=u$ in the trace sense and $\Delta v=0$ in the weak sense. We note that for $u\in H^1(\Omega)$, it holds $\|u\|_{H^{1/2}(\Gamma)}\leqslant\|u\|_{L^2(\Gamma)}+\|\nabla u\|_{L^2(\Omega)}$. Moreover, for $u\in H^2(\Omega)$ and $v\in H^1(\Omega)$ such that $u|_\Gamma$ is the trace of v on Γ, we have $\|\nabla u\|^2_{L^2(\Omega)}\leqslant\|\nabla(u-v)\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\leqslant\|(u-v)\Delta u\|_{L^1(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\leqslant\varepsilon\|u-v\|^2_{L^2(\Omega)}+C_\varepsilon\|\Delta u\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}$, and therefore

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega)}&\leqslant\varepsilon\|\nabla(u-v)\|^2_{L^2(\Omega)}+C_\varepsilon\|\Delta u\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\\&\leqslant\varepsilon\|\nabla u\|^2_{L^2(\Omega)}+C\left(\|\Delta u\|^2_{L^2(\Omega)}+\|u\|^2_{H^{1/2}(\Gamma)}\right),\end{aligned}$$

where we have used the fact that $v-u\in H^1_0(\Omega)$ and Poincaré's inequality. Therefore, we obtain

$$\|\nabla u\|_{L^2(\Omega)}\leqslant C\left(\|\Delta u\|_{L^2(\Omega)}+\|u\|_{H^{1/2}(\Gamma)}\right).\tag{A1}$$

Moreover, if v is the harmonic extension of $u|_\Gamma$, it holds $\|v\|_{H^1(\Omega)}\leqslant C\|u\|_{H^{1/2}(\Gamma)}$, and we have

$$\|u\|_{H^1(\Omega)}\leqslant C\left(\|\Delta u\|_{L^2(\Omega)}+\|u\|_{H^{1/2}(\Gamma)}\right).\tag{A2}$$

Lemma A.3 ([26, corollary 2.9]). *Let $m \in \mathbb{N}_0$ and $\Gamma \subset \mathbb{R}^3$ be a compact 2-dimensional hypersurface which is $C^{1,\alpha}$-regular such that $\Gamma = \partial\Omega$ and satisfies the condition (H_m), i.e.*

$$\|B\|_{L^4(\Gamma)} \leqslant C, \text{ if } m = 2, \quad \|B\|_{L^\infty(\Gamma)} + \|B\|_{H^{m-2}(\Gamma)} \leqslant C, \text{ if } m > 2. \tag{A3}$$

Then for all $k,l \in \mathbb{N}/2$ with $k < l \leqslant m$ and for $q \in [1,\infty]$, it holds $\|u\|_{H^k(\Gamma)} \leqslant C\|u\|^{\theta}_{H^l(\Gamma)}\|u\|^{1-\theta}_{L^q(\Gamma)}$, where $\theta \in [0,1]$ is given by $1 = k - \theta(l-1) + (2-2\theta)/q$, and $\|u\|_{H^k(\Omega)} \leqslant C\|u\|^{\theta}_{H^l(\Omega)}\|u\|^{1-\theta}_{L^q(\Omega)}$, where $\theta \in [0,1]$ is given by $1/2 = k/3 + \theta(1/2 - l/3) + (1-\theta)/q$. Moreover, for $k,l \in \mathbb{N}_0$ with $k < l \leqslant m, p \in [1,\infty), q \in [1,\infty]$, it holds $\|\nabla^k u\|_{L^p(\Omega)} \leqslant C\|u\|^{\theta}_{H^l(\Omega)}\|u\|^{1-\theta}_{L^q(\Omega)}$, where $\theta \in [0,1]$ is given by $1/p = k/3 + \theta(1/2 - l/3) + (1-\theta)/q$.

Lemma A.4 ([3, 27]). *For $f,g \in C_0^\infty(\mathbb{R}^n)$ and $2 \leqslant p_1, q_2 < \infty, 2 \leqslant p_2, q_1 \leqslant \infty$ with $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1/2$, we have for all $k \in \mathbb{N}/2$, $\|fg\|_{H^k} \leqslant C(\|f\|_{W^{k,p_1}}\|g\|_{L^{q_1}} + \|g\|_{W^{k,q_2}}\|f\|_{L^{p_2}})$.*

Lemma A.5 ([26, proposition 2.10]). *Assume $\partial\Omega$ is $C^{1,\alpha}$-regular and satisfies (H_m) defined in (A3). Then for all $k \in \mathbb{N}/2, k \leqslant m$, it holds $\|fg\|_{H^k(\partial\Omega)} \leqslant C(\|f\|_{H^k(\partial\Omega)}\|g\|_{L^\infty(\partial\Omega)} + \|f\|_{L^\infty(\partial\Omega)}\|g\|_{H^k(\partial\Omega)})$, and $\|fg\|_{H^k(\Omega)} \leqslant C(\|f\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{H^k(\Omega)})$. Moreover, assume that $\|B\|_{L^4} \leqslant C$ and $k \in \mathbb{N}_0$. Then for $p_1,p_2,q_1,q_2 \in [2,\infty]$ with $p_1,q_2 < \infty, 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1/2$, we have $\|fg\|_{H^k(\Gamma)} \leqslant C(\|f\|_{W^{k,p_1}(\Gamma)}\|g\|_{L^{q_1}(\Gamma)} + \|f\|_{L^{p_2}(\Gamma)}\|g\|_{W^{k,q_2}(\Gamma)})$.*

Lemma A.6 ([26, proposition 2.12]). *Assume that Γ is $C^{1,\alpha}$-regular. For every $p \in (1,\infty)$, it holds $\|B_\Gamma\|_{L^p(\Gamma)} \leqslant C(1 + \|\mathcal{A}_\Gamma\|_{L^p(\Gamma)})$. If in addition $\|B_\Gamma\|_{L^4(\Gamma)} \leqslant C$, then for $k = 1/2, 1, 2$, it holds $\|B_\Gamma\|_{H^k(\Gamma)} \leqslant C(1 + \|\mathcal{A}_\Gamma\|_{H^k(\Gamma)})$. Finally, let $m \in \mathbb{N}/2, m \geqslant 3$, and assume additionally $\|B\|_{L^\infty(\Gamma)} + \|B\|_{H^{m-2}(\Gamma)} \leqslant C$. Then the above estimate holds for all $k \in \mathbb{N}/2$ with $k \leqslant m$.*

Lemma A.7 ([26, lemma 3.5]). *Let Ω be a bounded domain with $\partial\Omega \in C^1$ and $\|B\|_{L^4} \leqslant C$. Then $\|u\|_{H^2(\Omega)} \leqslant C(\|\partial_\nu u\|_{H^{1/2}(\partial\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)})$. Moreover, $\|\nabla u\|_{L^2(\Omega)}$ can be replaced by $\|u\|_{L^2(\Omega)}$.*

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