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ABSTRACT. We establish the first complete classification of finite-time blow-up scenarios for strong solutions to the three-dimensional incompressible Euler equations with surface tension in a bounded domain possessing a closed, moving free boundary. Uniquely, we make *no* assumptions on symmetry, periodicity, graph representation, or domain topology (simple connectivity). At the maximal existence time  $T < \infty$ , up to which the velocity field and the free boundary can be continued in  $H^3 \times H^4$ , blow-up must occur in at least one of five mutually exclusive ways: (i) self-intersection of the free boundary for the first time; (ii) loss of mean curvature regularity in  $H^{\frac{3}{2}}$ , or the free boundary regularity in  $H^{2+\varepsilon}$  (for any sufficiently small constant  $\varepsilon > 0$ ); (iii) loss of  $H^{\frac{5}{2}}$  regularity for the normal boundary velocity; (iv) the  $L_t^1 L^{\infty}$ -blow-up of the tangential velocity gradient on the boundary; or (v) the  $L_t^1 L^{\infty}$ -blow-up of the full velocity gradient in the interior. Furthermore, for simply connected domains, blow-up scenario (v) simplifies to a vorticity-based Beale-Kato-Majda criterion, and in particular, irrotational flows admit blow-up only at the free boundary.

### 1. INTRODUCTION

We consider the three-dimensional incompressible Euler equations with surface tension in a bounded domain  $\Omega_t \subset \mathbb{R}^3$  evolving under a closed free surface  $\partial \Omega_t$  (see Fig. 1):

$$\int \mathfrak{D}_t v + \nabla p = 0, \qquad \text{in } \Omega_t, \qquad (1.1a)$$

$$\nabla \cdot v = 0, \qquad \text{in } \Omega_t, \qquad (1.1b)$$

$$p = \mathscr{H}_{\partial\Omega_t}, \quad v_n = \mathscr{V}_{\partial\Omega_t}, \quad \text{on } \partial\Omega_t,$$
 (1.1c)

$$\mathbf{l} \ v(\cdot, 0) = v_0, \qquad \qquad \text{in } \Omega_0, \qquad (1.1d)$$

where t > 0 denotes time,  $\mathfrak{D}_t \coloneqq \partial_t + v \cdot \nabla$  the material derivative, v = v(x,t) the velocity field, p = p(x,t) the scalar pressure,  $v \cdot \nabla$  the directional derivative. On  $\partial \Omega_t$ ,  $n = (n^1, n^2, n^3)^{\top}$  represents the unit outer normal,  $\mathscr{H}_{\partial\Omega_t}$  the mean curvature, and  $\mathscr{V}_{\partial\Omega_t}$  the normal velocity of  $\partial\Omega_t$  which is equal to the normal component of the velocity field  $v_n = v \cdot n$ . The initial domain  $\Omega_0$  has  $H^4$ -regular boundary, and the initial velocity field  $v_0$  lies in  $H^3(\Omega_0)$ , and the surface tension coefficient is normalized to unity for simplicity.



FIG. 1. Bounded fluid domain with a closed free surface.

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The free boundary problem for the incompressible Euler equations has witnessed substantial theoretical advances across several decades, with progress characterized by increasingly refined analyses of rotational effects, interfacial conditions, and singularity formation. Foundational results established well-posedness for irrotational flows without surface tension [2, 3, 28, 34, 36, 40, 52-54], while subsequent studies incorporated surface tension as a regularizing mechanism, demonstrating wellposedness [1, 23, 34, 35] and rigorous convergence to zero surface tension limits [4, 5]. For rotational flows, Christodoulou and Lindblad [15] pioneered a priori estimates, with local well-posedness proved later in [20, 41, 55] without surface tension. When surface tension interacts with vorticity, we refer to [20, 25, 45-47, 49]. For more recent references, see [6, 32, 33, 44].

Beyond well-posedness theory, the free boundary of regular solutions can develop finite-time selfintersecting singularities under suitable initial conditions, causing loss of injectivity in the flow map. These singularities manifest as either pointwise *splash singularities* or higher-dimensional *splat singularities* — arcwise in 2D or surfacial in 3D — as illustrated in Fig. 2. Crucially, when represented parametrically, the surface maintains regularity in parameter space throughout self-intersection formation. First observed in 2D water waves without surface tension [11, 16], such singularities persist even with surface tension [10]. Subsequent studies extended these findings to rotational Euler equations [21] and Navier-Stokes systems [12, 22], with further advances in this type of singularity documented in [17, 18, 24, 37]. The relationship between self-intersection mechanisms and curvature dynamics is analyzed in Section 1.2.



FIG. 2. Some cases of self-intersection of the free boundary.

The formation of these self-intersecting singularities typically preserves solution regularity, with blow-up criteria for the incompressible rotational Euler equations systematically established when surface tension is neglected: the graph-based framework for bounded domains [51], the result for initial domains diffeomorphic to a ball [29], and particularly the sharp blow-up characterization for general bounded domains without simple-connectedness assumptions [33]. Conversely, when surface tension is incorporated in (1.1c), related blow-up results have been obtained in [38] (at vanishing electric fields), [42] and [31] (under zero magnetic fields).

Since the present work focuses on surface tension effects, we briefly summarize key blow-up criteria from [31, 38, 42]. For general bounded domains, [38] represents the free boundary through a height function h defined on a fixed smooth reference surface  $\Gamma$ , establishing that finite-time singularity formation requires either:

- (i) topological breakdown of the height function representation, or
- (ii) divergence of the composite norm:

$$\sup_{0 \le t < T^{\dagger}} \left( \|h(\cdot, t)\|_{C^{1,\alpha}(\Gamma)} + \|\nabla v(\cdot, t)\|_{L^{\infty}(\Omega_t)} + \|v_n(\cdot, t)\|_{H^2(\partial\Omega_t)} \right) = \infty.$$

Critically, the  $L^{\infty}$ -type blow-up condition

$$\sup_{0 \leqslant t < T^{\dagger}} \left\| \nabla v(\cdot, t) \right\|_{L^{\infty}(\Omega_t)} = \infty$$

imposes a stronger regularity requirement in time than its integral counterpart

$$\int_0^{T^{\dagger}} \|\nabla v(\cdot, t)\|_{L^{\infty}(\Omega_t)} dt = \infty,$$

as the boundedness of the sup-norm implies that of the  $L^1$ -norm due to  $T^{\dagger} \in (0, \infty)$ .

In [42], a blow-up criterion for solutions  $(v, \Omega_t)$  to a free-boundary Euler equations was established in the periodic simply-connected graph domain

$$\Omega_t = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \mathbb{T}^2, -b < x_3 \leqslant \psi(t, x_1, x_2) \right\},\$$

with regularity  $(v, \Omega_t) \in H^s \times H^{s+1}$  for  $s > \frac{9}{2}$ . If the maximal existence time  $T^{\dagger} < \infty$ , then at least one of the following scenarios must occur:

$$\int_0^{T^{\dagger}} \|(\partial^{\varphi} \times v)(t)\|_{W^{1,\infty}} dt = \infty, \qquad (1.2)$$

$$\limsup_{t \to T^{\dagger}} \|\psi(t)\|_{C^{3}} + \|\partial_{t}\psi(t)\|_{C^{3}} + \|\partial_{tt}^{2}\psi(t)\|_{H^{3/2}} = \infty,$$
(1.3)

$$\limsup_{t \to T^{\dagger}} \int_{0}^{t} \| (v_{1}(s), v_{2}(s)) \|_{\dot{W}^{1,\infty}} \, ds + \| (v_{1}(t), v_{2}(t)) \|_{L^{\infty}} = \infty, \tag{1.4}$$

$$\limsup_{t \to T^{\dagger}} \left( \frac{1}{\partial_3 \varphi(t)} + \frac{1}{b - \|\psi(t)\|_{L^{\infty}}} \right) = \infty, \text{ or turning occurs on } \partial\Omega_{T^{\dagger}}, \qquad (1.5)$$

where  $\varphi(t, x_1, x_2, x_3) = x_3 + \chi(x_3)\psi(t, x_1, x_2)$  with some cut-off function  $\chi \in C_0^{\infty}(-b, 0]$  (cf. [42, (1.5)]),  $\partial_a^{\varphi} = \partial_a - \frac{\partial_a \varphi}{\partial_3 \varphi} \partial_3$  for a = 1, 2, and  $\partial_3^{\varphi} = \frac{1}{\partial_3 \varphi} \partial_3$ . On the free boundary, (1.3) and (1.5) involve somewhat complicated quantities and require the second-order time derivative of the graph function.

Moreover, the graph-based blow-up criteria in [42] and [51] (without surface tension) are fundamentally limited by their reliance on a global single-valued height function  $\psi$ , which becomes geometrically inapplicable when the free boundary undergoes *turning* or *self-intersection*, as these scenarios violate the monotonic  $x_3$ -dependence assumption; consequently, such formulations can only describe *topbottom contact singularities* ( $\psi \rightarrow -b$ ) while failing to capture generic boundary evolution like folding or pinch-off, which require non-graph representations, as illustrated in Fig. 3.



FIG. 3. Two scenarios under the graph assumption on the torus  $\mathbb{T}^2$ : on the left, boundary turning occurs, and on the right, the upper free boundary contacts the bottom.

To characterize self-intersection of the free boundary, the first and third authors introduced a dynamically updated reference surface methodology [31], which preserves non-degenerate coordinate mapping even as the interface approaches self-contact. This framework establishes a blow-up criterion for high-regularity solutions ( $H^6$ -class), capturing both boundary and interior singularities. However, the interior singularity condition (cf. (4) in [31, Theorem 1.2]) still fundamentally relies on

$$\sup_{0 \leqslant t < T^{\dagger}} \left\| \nabla v(\cdot, t) \right\|_{H^{3}(\Omega_{t})},$$

indicating that supremum-based blow-up detection remains intrinsic to the formulation.

In this paper, we establish a complete classification of finite-time blow-up for  $H^3$  solutions in general bounded domains with closed free boundaries, eliminating conventional assumptions of periodicity, symmetry, simply connectivity, or graph-based boundary representations. We develop a minimalquantity characterization framework requiring the weakest possible regularity conditions — notably avoiding time derivatives — while fully capturing self-intersection singularities. Furthermore, subject to simple connectivity, we demonstrate that interior blow-up remains governed by vorticity dynamics.

### 1.1. Main results. The following presents the main results of this paper.

**Theorem 1.1.** Let  $v_0 \in H^3(\Omega_0; \mathbb{R}^3)$  be the initial divergence-free velocity field, where  $\Omega_0 \subset \mathbb{R}^3$  is the initial bounded domain (need not be simply connected) with non-self-intersecting  $H^4$ -class closed boundary. Let  $(v, \Omega_t)$  be the solution to free boundary problem (1.1) with initial data  $(v_0, \Omega_0)$  with maximal existence interval  $[0, T^{\dagger})$  satisfying

$$v \in C([0, T^{\dagger}); H^3(\Omega_t))$$
 and  $\partial \Omega_t \in C([0, T^{\dagger}); H^4)$ 

If  $T^{\dagger} < \infty$ , then at least one of the following scenarios must necessarily occur:

- (1) Geometric singularity: First self-intersection of  $\partial \Omega_t$  at  $t = T^{\dagger}$ .
- (2) Boundary regularity loss: For any sufficiently small constant  $\varepsilon > 0$  independent of  $T^{\dagger}$ ,

$$\limsup_{t \nearrow T^{\dagger}} \left\{ \|\mathscr{H}_{\partial\Omega_{t}}(t)\|_{H^{3/2}(\partial\Omega_{t})} + \|\partial\Omega_{t}\|_{H^{2+\varepsilon}} \right\} = \infty.$$

(3) Kinematic breakdown:

$$\limsup_{t \nearrow T^{\dagger}} \|v_n(t)\|_{H^{5/2}(\partial\Omega_t)} = \infty.$$

(4) Tangential gradient blow-up:

$$\int_0^{T^{\dagger}} \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial \Omega_t)} dt = \infty.$$

(5) Interior accumulation of velocity gradients:

$$\int_0^{T^\dagger} \|\nabla v\|_{L^\infty(\Omega_t)} dt = \infty.$$
(1.6)

For problem (1.1), the a priori estimates were derived in [47, 48], and the existence of solutions  $(v \in H^3 \text{ and } \partial \Omega \in H^4)$  was established in [49, Theorem B]. To the best of our knowledge, Theorem 1.1 establishes the first systematic classification of finite-time blow-up scenarios for strong solutions to system (1.1) in general bounded domains with a closed free surface, eliminating periodicity, symmetry, simple connectivity, and graph assumptions. Aiming for a theoretically optimal blow-up classification, we avoid involving any time derivatives.

**Remark 1.2.** Distinct self-intersection scenarios for Case (1) are visualized in Fig. 2, while Cases (2)-(3) exhibit possible boundary regularity loss (Fig. 4). The boundary regularity requirements in Case (2) stem from two fundamental constraints:

- (i) The  $H^2$ -regularity of  $\partial \Omega_t$  is minimal for mean curvature to be classically defined in (1.1c);
- (ii) Boundary regularity recovery from the mean curvature via Lemma 2.7 requires the marginally stronger  $H^{2+\varepsilon}$ -regularity ( $\varepsilon > 0$ ).

This  $\varepsilon$ -gap reflects the intrinsic difference between elliptic regularity (Lemma 2.7) and geometric measure requirements.

**Remark 1.3.** The distinction between Cases (4) and (5) in Theorem 1.1 requires careful attention. While  $H^3(\Omega_t) \hookrightarrow C^{1,\frac{1}{2}-\delta}(\overline{\Omega_t})$  in 3D (Sobolev embedding with  $0 < \delta \ll 1$ ), this does not guarantee the continuity of  $\nabla v$  up to the boundary, nor the pointwise boundedness of its second derivatives. Consequently:

- (i) The tangential gradient  $\overline{\nabla}v$  may not admit a continuous extension to  $\overline{\Omega_t}$ ;
- (ii) The inequality  $\|\overline{\nabla}v\|_{L^{\infty}(\partial\Omega_t)} \leq \|\nabla v\|_{L^{\infty}(\Omega_t)}$  fails in general.

Case (4) thus remains essential to characterize boundary-driven singularities independent of the interior regularity, particularly when  $\nabla v$  develops singularities localized near  $\partial \Omega_t$  (e.g., boundary layer separation).

**Remark 1.4.** For fixed boundary problems where  $\Omega_t \equiv \Omega_0$  and  $v_n \equiv 0$  on  $\partial \Omega_0$  for all t, the following simplifications occur in Theorem 1.1:



FIG. 4. The regularity loss of the free boundary and its normal velocity.

- (i) Cases (1)–(3) become trivial: Geometric evolution quantities (self-intersection, mean curvature, normal velocity  $v_n$ ) are time-independent by assumption.
- (ii) Lemma 2.5 no longer requires tracking mean curvature evolution, and the boundary energy term

$$\int_{\partial\Omega_t} \left| \overline{\nabla} \left( \mathfrak{D}_t v \cdot n \right) \right|^2 dS$$

vanishes from the proof of (3.11).

 (iii) Case (4) (tangential gradient blow-up) does not manifest, as boundary-tangential dynamics decouple from interior evolution.

The blow-up criterion thus reduces exclusively to the accumulation of interior velocity gradients:

$$\int_0^{T^{\dagger}} \|\nabla v\|_{L^{\infty}(\Omega_0)} dt = \infty.$$
(1.7)

If we impose the assumption of simple connectivity, scenario (5) can be refined.

**Theorem 1.5.** Let  $T^{\dagger} < \infty$  denote the maximal existence time defined in Theorem 1.1, and assume  $\Omega_t$  is simply connected. Then the interior blow-up criterion (1.6) in Theorem 1.1 admits a vorticity-based refinement:

$$\limsup_{t \to T^{\dagger}} \|\nabla \times v\|_{L^{2}(\Omega_{t})} + \int_{0}^{T^{\dagger}} \|\nabla \times v\|_{L^{\infty}(\Omega_{t})} dt = \infty.$$
(1.8)

Furthermore, in the irrotational case ( $\nabla \times v \equiv 0$ ), singularities must manifest exclusively as boundary phenomena — corresponding to Cases (1)–(4) of Theorem 1.1 — since (1.8) vanishes.

In the fixed-boundary case revisited through the lens of (1.8), the sole blow-up scenario (1.7) refines to the vorticity-dominated condition:

$$\limsup_{t \to T^{\dagger}} \|\nabla \times v\|_{L^{2}(\Omega_{0})} + \int_{0}^{T^{\dagger}} \|\nabla \times v\|_{L^{\infty}(\Omega_{0})} dt = \infty,$$
(1.9)

which differs from the classical Beale-Kato-Majda (BKM) criterion [8, 26]

$$\int_0^{T^{\dagger}} \|\nabla \times v\|_{L^{\infty}(\Omega_0)} dt = \infty$$
(1.10)

by incorporating enstrophy dynamics. Significantly, no existing blow-up theory for (1.1) in general simply connected domains achieves the reduction to (1.10). This arises from our deliberate trade-off: to secure sharp characterization of free-boundary singularities (developed in Section 1.2), we relax full interior control — consequently, (1.8) lacks the precision needed to collapse completely to (1.10) in the fixed-boundary setting.

Identifying finite-time blow-up in free-boundary flows necessitates balancing boundary and interior singularity detection, where for instance in simply connected graph domains, the criterion (1.2)-(1.5) in [42] reduces to the classical BKM condition (1.10) for fixed boundaries, achieved at the expense of a significantly more involved blow-up criterion on the free boundary. In contrast, in our Theorem 1.1,

the regularity requirements in Cases (2) and (3) are strictly lower than  $\|\psi(t)\|_{C^3}$  and  $\|\partial_t \psi(t)\|_{C^3}$  in (1.3), respectively. Moreover, our analysis does not rely on quantities such as

$$\|\partial_{tt}^2 \psi(t)\|_{H^{1.5}}, \quad \int_0^t \|(v_1(s), v_2(s))\|_{L^{\infty}} ds, \quad \|(v_1(t), v_2(t))\|_{L^{\infty}},$$

nor on the condition (1.5) associated with the graph-specific constraints.

1.2. Sharp blow-up classification on the free boundary. We assert that Cases (1)-(4) in Theorem 1.1 provide a sharp and complete classification of free-boundary singularities, respectively capturing: topological self-intersection, geometric regularity loss (curvature blow-up), normal velocity breakdown, and tangential velocity gradient cascade. Crucially, these represent orthogonal singularity mechanisms — geometric regularity (encoded in curvature) depends solely on tangential boundary smoothness, while normal velocity governs boundary evolution kinematics, with no intrinsic coupling between normal and tangential dynamics.

Regarding Case (1), self-intersection singularities may develop independently of geometric regularity loss: even when mean curvature remains bounded (Case (2) excluded), boundary self-contact can occur through geometric measure concentration without curvature blow-up, as demonstrated by cusp formation in collapsing cavities where  $\|\mathscr{H}_{\partial\Omega_t}\|_{H^{3/2}}$  remains bounded while injectivity fails.

Self-intersection singularities exhibit distinct curvature behaviors depending on their formation mechanism. In gravity-driven collisions (e.g., ocean wave impact), the upper fluid layer descends onto the lower domain without significant curvature amplification prior to contact. This produces *splash/splat singularities* where self-intersection occurs with bounded curvature variation, as constructed for 3D Euler equations under Rayleigh-Taylor stability in [21] (cf. Fig. 5). Crucially, curvature norms may grow subcritically (without blow-up) during intersection—verified for 2D water waves with/without surface tension [10, 11].



FIG. 5. Self-intersection with bounded curvature variation.

Conversely, in *squeeze singularities* (Fig. 6, 7), boundary self-intersection may coincide with curvature blow-up when fluid is extruded through narrowing channels (Fig. 6). Although no results are currently available for the construction of solutions in fluid domains of the type shown in Fig. 6, we expect that a construction may be carried out using domain-perturbation methods developed in [21,22]. Notably, squeeze-induced intersection can also preserve curvature regularity when separation occurs along flat interfaces (Fig. 7), demonstrating decoupled curvature evolution.

Therefore, Cases (1)–(4) capture mutually independent phenomena, and each singularity type must be treated as an essential and distinct possibility in the classification, and they are also independent from the interior blow-up (Case (5)).

1.3. Advantages relative to graph-based, boundary-flattening, and fixed-boundary approaches. Our analytical framework overcomes three fundamental limitations inherent in conventional approaches to free-boundary problems:

(1) Elimination of geometric distortion: Graph-based assumptions and local boundary flattening (via coordinate mappings F) introduce artificial geometric constraints. Flattening  $\partial \Omega_t$  destroys intrinsic curvature properties — a critical limitation since the mean curvature explicitly







FIG. 7. The free boundary approaches self-intersection without curvature blow-up.

governs the condition in (1.1c). Consequently, such approaches either yield degenerate physical models (applying (1.1c) to curvature-erased surfaces), or generate analytically intractable systems (transforming full equations through F introduces uncontrolled nonlinearities). Our methodology preserves geometric integrity by directly analyzing curved boundaries.

- (2) Dynamic boundary compatibility: Fixed-boundary formulations fundamentally misrepresent free-surface kinematics. While Lagrangian coordinates freeze domain evolution, our Eulerian framework naturally couples curvature evolution to fluid dynamics (Lemma 2.7), embeds 3/2scaling laws in energy functionals via material derivatives (Remark 3.6), and explicitly resolves boundary-driven singularities (Cases (1)-(4)) unobtainable in domain-constrained analyses.
- (3) Comprehensive singularity detection: Boundary-flattening obscures self-intersection mechanisms (Case (1)), while fixed-boundary methods eliminate normal/tangential velocity blow-up (Cases (3)-(4)). Our Eulerian approach uniquely captures the full singularity spectrum.

1.4. **Paper structure.** This paper develops as follows: Section 2 establishes specialized analytical foundations central to our framework; Section 3 proves Theorem 1.1 by introducing uniform exterior and interior ball radii as geometric criteria for free-boundary self-intersection; finally, Section 4 extends these results to simply connected domains, demonstrating Theorem 1.5's vorticity-driven blow-up characterization under topological constraints.

# 2. TAILORED AUXILIARY RESULTS

In this section, we present some fundamental results. We will adopt the Einstein summation convention and utilize the notation \* to denote the contraction of certain indices of tensors with constant coefficients (see, e.g., [30, 43]).

Using the unit outer normal vector n, we define the tangential derivative of a scalar function f by

$$\overline{\nabla}f \coloneqq \nabla f - (\nabla f \cdot n) \, n. \tag{2.1}$$

Similarly, for a vector field  $F = (F^1, F^2, F^3)$ , the tangential gradient and the tangential divergence are defined by

$$\overline{\nabla}F = \nabla F - \nabla F(n \otimes n), \quad \overline{\nabla} \cdot F \coloneqq \operatorname{Tr}\left(\overline{\nabla}F\right).$$

The (i, j)-th component of the tangential gradient (with  $i, j \in \{1, 2, 3\}$ ) is given by

$$\overline{\nabla}_j F^i = \left(\overline{\nabla}F\right)_{ij} = \partial_j F^i - \partial_l F^i n^l n_j.$$

With these notations, the mean curvature of  $\partial \Omega_t$  is expressed as

$$\mathscr{H}_{\partial\Omega_t} = \overline{\nabla} \cdot n$$

Furthermore, fix a point  $x \in \partial \Omega_t$ , and let  $X, Y \in T_x \partial \Omega_t$ . The second fundamental form is defined by<sup>1</sup>

$$\mathbf{II}(X,Y) = \nabla_X n \cdot Y = X^j \partial_j n \cdot Y$$

Since  $X \cdot n = 0$ , it follows that

$$\mathbf{II}(X,Y) = \left(\partial_j n^i - \partial_l n^i n^l n_j\right) X^j Y_i = \mathbf{II} X \cdot Y = X^\top \mathbf{II} Y,$$

where  $i, j, l \in \{1, 2, 3\}$  and II is defined by the tangential gradient of n:

$$\mathrm{II} = \overline{\nabla}n, \quad \mathrm{II}_{ij} = \partial_j n^i - \partial_l n^i n^l n_j.$$

Following the convention in [27], we refer to the matrix II as the second fundamental form.

**Lemma 2.1.** Let f be a smooth function, and  $i, j \in \{1, 2, 3\}$ . Then, the following holds

$$\begin{split} & [\mathfrak{D}_t, \nabla] f = -(\nabla v)^\top \nabla f, \quad [\mathfrak{D}_t, \partial_i] f = -\partial_i v^k \partial_k f, \\ & [\mathfrak{D}_t, \nabla^2] f = \nabla v * \nabla^2 f + \nabla^2 v * \nabla f, \\ & [\mathfrak{D}_t, \overline{\nabla}] f = -(\overline{\nabla} v)^\top \overline{\nabla} f, \quad [\mathfrak{D}_t, \overline{\nabla}_i] f = -\overline{\nabla}_i v^k \overline{\nabla}_k f, \\ & [\mathfrak{D}_t, \overline{\nabla}^2] f = \overline{\nabla} v * \overline{\nabla}^2 f + \overline{\nabla}^2 v * \overline{\nabla} f, \\ & \mathfrak{D}_t n = -(\overline{\nabla} v)^\top n, \quad \mathfrak{D}_t n_i = -\overline{\nabla}_i v^j n_j, \\ & \Delta_{\mathrm{II}} n = -|\mathrm{II}|^2 n + \overline{\nabla} \mathscr{H}, \\ & \mathfrak{D}_t \mathrm{II}_{ij} = -\overline{\nabla}_j \overline{\nabla}_i v^k n_k - \overline{\nabla}_i v^k \mathrm{II}_{kj} - \overline{\nabla}_j v^k \mathrm{II}_{ik}, \\ & \mathfrak{D}_t \mathscr{H} = -\Delta_{\mathrm{II}} v_n - |\mathrm{II}|^2 v_n + \overline{\nabla} \mathscr{H} \cdot v, \end{split}$$

where  $[\cdot, \cdot]$  denotes the Lie bracket,  $\Delta_{II} \coloneqq \overline{\nabla} \cdot \overline{\nabla}$  represents the Beltrami-Laplacian operator, and we abbreviate  $\mathscr{H} = \mathscr{H}_{\partial\Omega_t}$ .

*Proof.* Most of the above formulas can be found in [47, Section 3.1]. The remaining ones follow from direct calculations. For example, since  $II_{ij} = \overline{\nabla}_j n_i$ , we have

$$\begin{aligned} \mathfrak{D}_{t}\Pi_{ij} &= \overline{\nabla}_{j}\mathfrak{D}_{t}n_{i} + [\mathfrak{D}_{t}, \overline{\nabla}_{j}]n_{i} \\ &= \overline{\nabla}_{j}\left(-\overline{\nabla}_{i}v^{k}n_{k}\right) - \overline{\nabla}_{j}v^{k}\overline{\nabla}_{k}n_{i} \\ &= -\overline{\nabla}_{j}\overline{\nabla}_{i}v^{k}n_{k} - \overline{\nabla}_{i}v^{k}\overline{\nabla}_{j}n_{k} - \overline{\nabla}_{j}v^{k}\overline{\nabla}_{k}n_{i} \\ &= -\overline{\nabla}_{j}\overline{\nabla}_{i}v^{k}n_{k} - \overline{\nabla}_{i}v^{k}\Pi_{kj} - \overline{\nabla}_{j}v^{k}\Pi_{ik}. \end{aligned}$$

By the divergence-free condition (1.1b), it follows that

$$\nabla \cdot \mathfrak{D}_t v = \partial_i v^j \partial_j v^i, \tag{2.2}$$

and therefore, from (1.1a), we obtain

$$-\Delta p = \partial_i v^j \partial_j v^i. \tag{2.3}$$

Applying Lemma 2.1 to the commutator for the pressure yields the following result.

<sup>&</sup>lt;sup>1</sup>The second fundamental form, when defined via the shape operator, typically depends on the choice of orientation [50, Section 5.2], and a minus sign may be included accordingly. We have chosen not to include the minus sign.

**Lemma 2.2.** For pressure p and velocity v, the commutators with spatial derivatives satisfy:

$$[\mathfrak{D}_t,\partial_j]p = \partial_j v_i \mathfrak{D}_t v^i, \quad [\mathfrak{D}_t^2,\partial_j]p = 2\partial_j v_i \mathfrak{D}_t^2 v^i + \partial_j \mathfrak{D}_t v_i \mathfrak{D}_t v^i, \tag{2.4}$$

for j = 1, 2, 3.

*Proof.* The first identity follows directly from the commutator  $[\mathfrak{D}_t, \partial_j]$  in Lemma 2.1 and the Euler equation (1.1a). For the second identity, applying the first result and again utilizing the commutator  $[\mathfrak{D}_t, \partial_j]$  from Lemma 2.1, we obtain

$$\begin{split} [\mathfrak{D}_{t}^{2},\partial_{j}]p &= \mathfrak{D}_{t}\left([\mathfrak{D}_{t},\partial_{j}]p\right) + [\mathfrak{D}_{t},\partial_{j}]\mathfrak{D}_{t}p \\ &= \mathfrak{D}_{t}\left(\partial_{j}v_{i}\mathfrak{D}_{t}v^{i}\right) - \partial_{j}v^{k}\partial_{k}\mathfrak{D}_{t}p \\ &= \partial_{j}v_{i}\mathfrak{D}_{t}^{2}v^{i} + \mathfrak{D}_{t}\partial_{j}v_{i}\mathfrak{D}_{t}v^{i} - \partial_{j}v^{k}[\partial_{k},\mathfrak{D}_{t}]p - \partial_{j}v^{k}\mathfrak{D}_{t}\partial_{k}p \\ &= \partial_{j}v_{i}\mathfrak{D}_{t}^{2}v^{i} + [\mathfrak{D}_{t},\partial_{j}]v_{i}\mathfrak{D}_{t}v^{i} + \partial_{j}\mathfrak{D}_{t}v_{i}\mathfrak{D}_{t}v^{i} + \partial_{j}v^{k}\partial_{k}v^{i}\mathfrak{D}_{t}v_{i} + \partial_{j}v^{k}\mathfrak{D}_{t}^{2}v_{k} \\ &= 2\partial_{j}v_{i}\mathfrak{D}_{t}^{2}v^{i} - \partial_{j}v^{k}\partial_{k}v_{i}\mathfrak{D}_{t}v^{i} + \partial_{j}\mathfrak{D}_{t}v_{i}\mathfrak{D}_{t}v^{i} + \partial_{j}v^{k}\partial_{k}v^{i}\mathfrak{D}_{t}v_{i} \\ &= 2\partial_{j}v_{i}\mathfrak{D}_{t}^{2}v^{i} + \partial_{j}\mathfrak{D}_{t}v_{i}\mathfrak{D}_{t}v^{i}. \end{split}$$

This completes the proof.

For a vector field F, we define

$$\nabla \times F \coloneqq \nabla F - (\nabla F)^\top.$$

A straightforward calculation also yields the following identity:

$$[\mathfrak{D}_t, \nabla \times]F = (\nabla v)^\top (\nabla F)^\top - \nabla F \nabla v.$$
(2.5)

To establish the energy estimate in Section 3, it is necessary to compute the exact expressions for the following quantities.

**Lemma 2.3.** For the vorticity  $\nabla \times v$ , we have

$$\mathfrak{D}_t \nabla^2 \left( \nabla \times v \right) = \nabla v * \nabla^2 \left( \nabla \times v \right) + \left( \nabla \times v \right) * \nabla^3 v + \nabla^2 v * \nabla \left( \nabla \times v \right).$$

where the symbol \* denotes the contraction of specific tensor indices as previously stated. Moreover, the following identities are valid:

$$\nabla \cdot \mathfrak{D}_t^2 v = 3\partial_i v^j \partial_j \mathfrak{D}_t v^i - 2\partial_i v^j \partial_j v^k \partial_k v^i,$$
  
$$\nabla \cdot \mathfrak{D}_t^3 v = 4\partial_i v^j \partial_j \mathfrak{D}_t^2 v^i + 3\partial_i \mathfrak{D}_t v^j \partial_j \mathfrak{D}_t v^i - 12\partial_i v^l \partial_l v^j \partial_j \mathfrak{D}_t v^i + 6\partial_i v^l \partial_l v^j \partial_j v^k \partial_k v^i.$$

*Proof.* Since by (1.1a)

$$\nabla \times \mathfrak{D}_t v = 0,$$

it follows that

$$\mathfrak{D}_t \left( \nabla \times v \right) = \mathfrak{D}_t \nabla \times v - \nabla \times \mathfrak{D}_t v = [\mathfrak{D}_t, \nabla \times] v$$

and by applying the identity (2.5), we obtain

$$\mathfrak{D}_t (\nabla \times v) = (\nabla v)^\top (\nabla v)^\top - \nabla v \nabla v$$
$$= -(\nabla v)^\top (\nabla \times v) - (\nabla \times v) \nabla v.$$
(2.6)

Then, the first claim follows immediately from the commutator  $[\mathfrak{D}_t, \nabla^2]$  in Lemma 2.1. Indeed, applying (2.6), we obtain

$$\begin{split} \mathfrak{D}_t \nabla^2 \left( \nabla \times v \right) &= \nabla^2 \mathfrak{D}_t \left( \nabla \times v \right) + \left[ \mathfrak{D}_t, \nabla^2 \right] \left( \nabla \times v \right) \\ &= \nabla^2 \left[ - (\nabla v)^\top \left( \nabla \times v \right) - (\nabla \times v) \nabla v \right] + \nabla v * \nabla^2 \left( \nabla \times v \right) + \nabla^2 v * \nabla \left( \nabla \times v \right) \\ &= \nabla v * \nabla^2 \left( \nabla \times v \right) + \left( \nabla \times v \right) * \nabla^3 v + \nabla^2 v * \nabla \left( \nabla \times v \right). \end{split}$$

Next, using the divergence-free condition (1.1b), the commutator  $[\mathfrak{D}_t, \partial_j]$  from Lemma 2.1, and the definition of the material derivative  $\mathfrak{D}_t = \partial_t + v^k \partial_k$ , we obtain

$$\nabla \cdot \mathfrak{D}_t^2 v = \partial_i \left( \partial_t \mathfrak{D}_t v^i + v^j \partial_j \mathfrak{D}_t v^i \right)$$
$$= \partial_i \left[ \partial_t \left( \partial_t v^i + v^k \partial_k v^i \right) + v^j \partial_j \left( \partial_t v^i + v^k \partial_k v^i \right) \right]$$

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$$\begin{split} &=\partial_{i}\left[\partial_{t}^{2}v^{i}+\partial_{t}v^{k}\partial_{k}v^{i}+v^{k}\partial_{t}\partial_{k}v^{i}+v^{j}\left(\partial_{t}\partial_{j}v^{i}+\partial_{j}v^{k}\partial_{k}v^{i}+v^{k}\partial_{jk}^{2}v^{i}\right)\right]\\ &=\partial_{t}\partial_{i}v^{k}\partial_{k}v^{i}+\partial_{i}v^{k}\partial_{t}\partial_{k}v^{i}+\partial_{i}v^{j}\left(\partial_{t}\partial_{j}v^{i}+\partial_{j}v^{k}\partial_{k}v^{i}+v^{k}\partial_{jk}^{2}v^{i}\right)\\ &+v^{j}\left(\partial_{ij}^{2}v^{k}\partial_{k}v^{i}+\partial_{i}v^{k}\partial_{jk}^{2}v^{i}\right)\\ &=3\partial_{t}\partial_{j}v^{i}\partial_{i}v^{j}+3\partial_{i}v^{j}v^{k}\partial_{jk}v^{i}+\partial_{i}v^{j}\partial_{j}v^{k}\partial_{k}v^{i}\\ &=3\partial_{i}v^{j}\mathfrak{D}_{t}\partial_{j}v^{i}+\partial_{i}v^{j}\partial_{j}v^{k}\partial_{k}v^{i}\\ &=3\partial_{i}v^{j}\partial_{j}\mathfrak{D}_{t}v^{i}+3\partial_{i}v^{j}[\mathfrak{D}_{t},\partial_{j}]v^{i}+\partial_{i}v^{j}\partial_{j}v^{k}\partial_{k}v^{i}\\ &=3\partial_{i}v^{j}\partial_{j}\mathfrak{D}_{t}v^{i}-2\partial_{i}v^{j}\partial_{j}v^{k}\partial_{k}v^{i}.\end{split}$$

Finally, from the above formula and again using the commutator  $[\mathfrak{D}_t, \partial_j]$  in Lemma 2.1, it follows that

$$\begin{split} \nabla \cdot \mathfrak{D}_{t}^{3} v &= \mathfrak{D}_{t} \nabla \cdot \mathfrak{D}_{t}^{2} v + [\nabla \cdot, \mathfrak{D}_{t}] \mathfrak{D}_{t}^{2} v \\ &= \mathfrak{D}_{t} \left[ 3 \partial_{i} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} - 2 \partial_{i} v^{j} \partial_{j} v^{k} \partial_{k} v^{i} \right] + [\partial_{j}, \mathfrak{D}_{t}] \mathfrak{D}_{t}^{2} v^{j} \\ &= 3 \mathfrak{D}_{t} \left( \partial_{i} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} \right) - 6 \left( \mathfrak{D}_{t} \partial_{i} v^{j} \right) \partial_{j} v^{k} \partial_{k} v^{i} + \partial_{j} v^{i} \partial_{i} \mathfrak{D}_{t}^{2} v^{j} \\ &= 3 \mathfrak{D}_{t} \partial_{i} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} + 3 \partial_{i} v^{j} \mathfrak{D}_{t} \partial_{j} \mathfrak{D}_{t} v^{i} - 6 \partial_{i} \mathfrak{D}_{t} v^{j} \partial_{j} v^{k} \partial_{k} v^{i} + 6 \partial_{i} v^{l} \partial_{l} v^{j} \partial_{j} v^{k} \partial_{k} v^{i} \\ &+ \partial_{j} v^{i} \partial_{i} \mathfrak{D}_{t}^{2} v^{j} \\ &= 3 \left( \partial_{i} \mathfrak{D}_{t} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} - \partial_{i} v^{l} \partial_{l} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} \right) + 3 \left( \partial_{i} v^{j} \partial_{j} \mathfrak{D}_{t}^{2} v^{i} - \partial_{i} v^{j} \partial_{j} v^{l} \partial_{l} \mathfrak{D}_{t} v^{i} \right) \\ &- 6 \partial_{i} \mathfrak{D}_{t} v^{j} \partial_{j} v^{k} \partial_{k} v^{i} + 6 \partial_{i} v^{l} \partial_{l} v^{j} \partial_{j} v^{k} \partial_{k} v^{i} + \partial_{j} v^{i} \partial_{i} \mathfrak{D}_{t}^{2} v^{j} \\ &= 4 \partial_{i} v^{j} \partial_{j} \mathfrak{D}_{t}^{2} v^{i} + 3 \partial_{i} \mathfrak{D}_{t} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} - 12 \partial_{i} v^{l} \partial_{l} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} + 6 \partial_{i} v^{l} \partial_{l} v^{j} \partial_{j} v^{k} \partial_{k} v^{i}, \end{split}$$

where we have utilized the following identities:

$$\begin{aligned} \mathfrak{D}_t \partial_i v^j \partial_j \mathfrak{D}_t v^i &= \partial_i \mathfrak{D}_t v^j \partial_j \mathfrak{D}_t v^i - \partial_i v^l \partial_l v^j \partial_j \mathfrak{D}_t v^i, \\ \partial_i v^j \mathfrak{D}_t \partial_j \mathfrak{D}_t v^i &= \partial_i v^j \partial_j \mathfrak{D}_t^2 v^i - \partial_i v^j \partial_j v^l \partial_l \mathfrak{D}_t v^i. \end{aligned}$$

This concludes the proof.

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To control  $I_1(t)$  in Proposition 3.11, we extract the divergence part of  $-\Delta \mathfrak{D}_t^2 p$ .

Lemma 2.4. The following identity holds:

$$-\Delta \mathfrak{D}_t^2 p = 4\partial_i v^j \partial_j \mathfrak{D}_t^2 v^i + 3\partial_i \mathfrak{D}_t v^j \partial_j \mathfrak{D}_t v^i - 12\partial_i v^l \partial_l v^j \partial_j \mathfrak{D}_t v^i + 6\partial_i v^l \partial_l v^j \partial_j v^k \partial_k v^i + \sum_j \partial_j \left( 2\partial_j v_i \mathfrak{D}_t^2 v^i + \partial_j \mathfrak{D}_t v_i \mathfrak{D}_t v^i \right).$$

*Proof.* This result is obtained by applying (1.1a), (2.4), and Lemma 2.3:

$$\begin{split} -\Delta \mathfrak{D}_{t}^{2}p &= -\nabla \cdot \nabla \mathfrak{D}_{t}^{2}p \\ &= -\nabla \cdot \mathfrak{D}_{t}^{2} \nabla p - \nabla \cdot [\nabla, \mathfrak{D}_{t}^{2}]p \\ &= \nabla \cdot \mathfrak{D}_{t}^{3}v + \sum_{j} \partial_{j} [\mathfrak{D}_{t}^{2}, \partial_{j}]p \\ &= 4\partial_{i}v^{j}\partial_{j}\mathfrak{D}_{t}^{2}v^{i} + 3\partial_{i}\mathfrak{D}_{t}v^{j}\partial_{j}\mathfrak{D}_{t}v^{i} - 12\partial_{i}v^{l}\partial_{l}v^{j}\partial_{j}\mathfrak{D}_{t}v^{i} + 6\partial_{i}v^{l}\partial_{l}v^{j}\partial_{j}v^{k}\partial_{k}v^{i} \\ &+ \sum_{j} \partial_{j} \left( 2\partial_{j}v_{i}\mathfrak{D}_{t}^{2}v^{i} + \partial_{j}\mathfrak{D}_{t}v_{i}\mathfrak{D}_{t}v^{i} \right). \end{split}$$

To handle the boundary energy, it is necessary to track the evolution of the pressure on the free boundary and isolate the error terms.

**Lemma 2.5.** On the free boundary  $\partial \Omega_t$ , we have

$$\mathfrak{D}_t^2 p = -\Delta_{\mathrm{II}}(\mathfrak{D}_t v \cdot n) + \mathfrak{R},$$

with the error terms

$$\mathfrak{R} = -\left|\mathrm{II}\right|^2 \mathfrak{D}_t v \cdot n + \overline{\nabla} p \cdot \mathfrak{D}_t v + \overline{\nabla}^2 v * \overline{\nabla} v * n + \overline{\nabla} v * \overline{\nabla} v * \mathrm{II}.$$

*Proof.* From the boundary condition (1.1c),  $II_{ij} = \overline{\nabla}_j n_i$  and the identities

$$\mathfrak{D}_t \mathscr{H} = -\Delta_{\mathrm{II}} v_n - |\mathrm{II}|^2 v_n + \overline{\nabla} \mathscr{H} \cdot v, \quad \Delta_{\mathrm{II}} n = -|\mathrm{II}|^2 n + \overline{\nabla} \mathscr{H}, \tag{2.7}$$

in Lemma 2.1, it follows that

$$\begin{aligned} \mathfrak{D}_t p &= -\Delta_{\mathrm{II}} v_n - |\mathrm{II}|^2 v_n + \overline{\nabla} \mathscr{H} \cdot v \\ &= -\Delta_{\mathrm{II}} v_n + \Delta_{\mathrm{II}} n \cdot v \\ &= -\Delta_{\mathrm{II}} v \cdot n - 2\mathrm{II} : \overline{\nabla} v, \end{aligned}$$

where ":" denotes the Frobenius inner product for tensors, e.g.,  $A : B = A_j^i B_i^j = tr(A^{\top}B)$  for the matrices A and B. We then differentiate to obtain

$$\mathfrak{D}_t^2 p = -\mathfrak{D}_t \Delta_{\mathrm{II}} v \cdot n - \Delta_{\mathrm{II}} v \cdot \mathfrak{D}_t n - 2\mathfrak{D}_t \mathrm{II} : \overline{\nabla} v - 2\mathrm{II} : \mathfrak{D}_t \overline{\nabla} v.$$

From the formulas for  $\mathfrak{D}_t n$ ,  $\mathfrak{D}_t \Pi$ ,  $[\mathfrak{D}_t, \overline{\nabla}]$ , and  $[\mathfrak{D}_t, \overline{\nabla}^2]$  in Lemma 2.1, along with  $\Pi_{ij} = \overline{\nabla}_j n_i$ , it follows that

$$\begin{split} \mathfrak{D}_{t}^{2}p &= -\Delta_{\mathrm{II}}\mathfrak{D}_{t}v^{i}n_{i} - [\mathfrak{D}_{t},\Delta_{\mathrm{II}}]v^{i}n_{i} - \Delta_{\mathrm{II}}v^{i}\mathfrak{D}_{t}n_{i} - 2\mathfrak{D}_{t}\mathrm{II}_{ij}\overline{\nabla}_{j}v^{i} \\ &- 2\sum_{j}\mathrm{II}_{ij}\overline{\nabla}_{j}\mathfrak{D}_{t}v^{i} - 2\sum_{j}\mathrm{II}_{ij}:[\mathfrak{D}_{t},\overline{\nabla}_{j}]v^{i} \\ &= -\sum_{j}\overline{\nabla}_{j}\overline{\nabla}_{j}\mathfrak{D}_{t}v^{i}n_{i} - 2\sum_{j}\overline{\nabla}_{j}\mathfrak{D}_{t}v^{i}\overline{\nabla}_{j}n_{i} - [\mathfrak{D}_{t},\Delta_{\mathrm{II}}]v^{i}n_{i} + \Delta_{\mathrm{II}}v^{i}\overline{\nabla}_{i}v^{j}n_{j} \\ &+ 2\sum_{j}\left(\overline{\nabla}_{j}\overline{\nabla}_{i}v^{k}n_{k} + \overline{\nabla}_{i}v^{k}\mathrm{II}_{kj} + \overline{\nabla}_{j}v^{k}\mathrm{II}_{kk}\right)\overline{\nabla}_{j}v^{i} + 2\sum_{j}\mathrm{II}_{ij}\overline{\nabla}_{j}v^{k}\overline{\nabla}_{k}v^{i} \\ &= -\Delta_{\mathrm{II}}\left(\mathfrak{D}_{t}v^{i}n_{i}\right) + \mathfrak{D}_{t}v^{i}\Delta_{\mathrm{II}}n_{i} + \overline{\nabla}^{2}v * \overline{\nabla}v * n + \Delta_{\mathrm{II}}v^{i}\overline{\nabla}_{i}v^{j}n_{j} \\ &+ 2\sum_{j}\overline{\nabla}_{j}\overline{\nabla}_{i}v^{k}\overline{\nabla}_{j}v^{i}n_{k} + 2\sum_{j}\overline{\nabla}_{j}v^{k}\overline{\nabla}_{j}v^{i}\mathrm{II}_{ik} + 4\sum_{j}\overline{\nabla}_{j}v^{k}\overline{\nabla}_{k}v^{i}\mathrm{II}_{ij} \\ &= -\Delta_{\mathrm{II}}\left(\mathfrak{D}_{t}v \cdot n\right) + \mathfrak{D}_{t}v^{i}\left(-|\mathrm{II}|^{2}n_{i} + \overline{\nabla}_{i}\mathscr{H}\right) + \overline{\nabla}^{2}v * \overline{\nabla}v * n + \overline{\nabla}v * \overline{\nabla}v * \mathrm{II}, \end{split}$$

where (2.7) has been used in the last step, and the proof is complete.

$$\square$$

For 
$$u \in L^2(\partial\Omega)$$
, we define  $u \in H^{\frac{1}{2}}(\partial\Omega)$  if  
 $\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \coloneqq \|u\|_{L^2(\partial\Omega)} + \inf \left\{ \|\nabla w\|_{L^2(\Omega)} : w \in H^1(\Omega) \text{ and } w|_{\partial\Omega} \right\}$ 

Observe that for any  $u \in H^1(\Omega)$ , the trace inequality holds:

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq \|u\|_{L^{2}(\partial\Omega)} + \|\nabla u\|_{L^{2}(\Omega)}.$$
(2.8)

 $=u\Big\}<\infty.$ 

The relationship between the regularity of the mean curvature and that of the second fundamental form is expressed as follows:

**Lemma 2.6.** Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with  $C^{1,\alpha}$  boundary  $(\alpha \in (0,1))$ , the second fundamental form II and mean curvature  $\mathscr{H}$  satisfy: For every  $p \in (1,\infty)$ ,

$$\|\mathrm{II}\|_{L^{p}(\partial\Omega)} \leq C\left(1 + \|\mathscr{H}\|_{L^{p}(\partial\Omega)}\right)$$

If, in addition,  $\|II\|_{L^4(\partial\Omega)} \leq C$  for some positive constant C, then for  $k \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ , we have

$$\|\mathrm{II}\|_{H^{k}(\partial\Omega)} \leq C\left(1 + \|\mathscr{H}\|_{H^{k}(\partial\Omega)}\right).$$

*Proof.* The  $L^p$  estimate and the  $H^k$  estimate for  $k = \frac{1}{2}, 1, 2$  can be found in [38, Proposition 2.12], while the case for  $k = \frac{3}{2}$  can be obtained via Sobolev interpolation  $H^{\frac{3}{2}} = [H^1, H^2]_{\frac{1}{2}}$ .

The regularity of the free boundary is fully controlled by its mean curvature through elliptic regularity:

**Lemma 2.7.** Let  $\Omega \subset \mathbb{R}^3$  be a domain with  $\partial \Omega \in H^{s_0}$  for some  $s_0 > 2$ . If the mean curvature satisfies  $\|\mathscr{H}\|_{H^{s-2}(\partial \Omega)} < \infty$  for  $s > s_0$ ,

then the boundary regularity lifts to  $\partial \Omega \in H^s$ .

*Proof.* See [47, Proposition A.2].

This implies that for a time-dependent free boundary  $\partial \Omega_t$ , the uniform bound

 $\sup_{t\in I} \|\mathscr{H}\|_{H^{s-2}(\partial\Omega_t)} < \infty, \text{ on the time interval } I,$ 

guarantees uniform control of the boundary regularity:  $\partial \Omega_t \in H^s, \forall t \in I$ , provided  $\partial \Omega_t \in H^{s_0}$ .

The following div-curl estimates play a pivotal role in the subsequent section.

**Lemma 2.8.** Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with  $\partial \Omega \in H^{2+\varepsilon}$  for some  $\varepsilon > 0$  and  $\|\mathrm{II}\|_{H^{\frac{3}{2}}(\partial \Omega)} \leq C$  for some positive constant C. Then, for any smooth vector field F and for  $k \in \{1, \frac{3}{2}, 3\}$ , the following holds:

$$\|F\|_{H^{k}(\Omega)} \leq C\left(\|F_{n}\|_{H^{k-\frac{1}{2}}(\partial\Omega)} + \|F\|_{L^{2}(\Omega)} + \|\nabla \cdot F\|_{H^{k-1}(\Omega)} + \|\nabla \times F\|_{H^{k-1}(\Omega)}\right),$$
(2.9)

where  $F_n = F \cdot n$ .

*Proof.* The  $H^{2+\varepsilon}$ -regularity of the free boundary implies  $C^{1,\alpha}$ -regularity for some sufficiently small  $\alpha = \alpha(\varepsilon) > 0$ .

Case k = 1: Follows from [38, Theorem 3.6] via standard div-curl theory.

Case k = 3: By Lemma 2.7,  $\|II\|_{H^{\frac{3}{2}}(\partial\Omega)} \leq C$  implies  $\partial\Omega \in H^{7/2}$ . Adapting [38, Theorem 3.1] (based on [14, Theorem 1.3]):

$$\begin{aligned} \|F\|_{H^{3}(\Omega)} &\leq C\Big(\left\|\overline{\nabla}F_{n}\right\|_{H^{\frac{3}{2}}(\partial\Omega)} + (1 + \|\Pi\|_{H^{\frac{3}{2}}(\partial\Omega)}) \|F\|_{L^{\infty}(\Omega)} \\ &+ \|\nabla \cdot F\|_{H^{2}(\Omega)} + \|\nabla \times F\|_{H^{2}(\Omega)}\Big) \\ &\leq C\left(\|F_{n}\|_{H^{\frac{5}{2}}(\partial\Omega)} + \|F\|_{L^{\infty}(\Omega)} + \|\nabla \cdot F\|_{H^{2}(\Omega)} + \|\nabla \times F\|_{H^{2}(\Omega)}\Big). \end{aligned}$$

By interpolation, we obtain the following estimate:

$$\left\|F\right\|_{L^{\infty}(\Omega)} \leqslant \varepsilon \left\|F\right\|_{H^{3}(\Omega)} + C_{\varepsilon} \left\|F\right\|_{L^{2}(\Omega)},$$

for sufficiently small  $\varepsilon > 0$ . Therefore, we have

$$\|F\|_{H^{3}(\Omega)} \leq C\left(\|F_{n}\|_{H^{\frac{5}{2}}(\partial\Omega)} + \|F\|_{L^{2}(\Omega)} + \|\nabla \cdot F\|_{H^{2}(\Omega)} + \|\nabla \times F\|_{H^{2}(\Omega)}\right)$$

Case  $k = \frac{3}{2}$ : Follows by interpolation between  $H^1$  and  $H^3$  cases.

Finally, we list the specific elliptic estimates that will be utilized in subsequent sections.

**Lemma 2.9.** Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with  $\partial \Omega \in C^{1,\alpha}$  for some  $\alpha \in (0,1)$ , and  $\|\mathrm{II}\|_{H^{\frac{1}{2}}(\partial \Omega)} \leq C$  for some positive constant C. Let u be the solution to the following Dirichlet problem:

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial \Omega \end{cases}$$

Then, it holds

$$\left\|\partial_{n}u\right\|_{H^{1}(\partial\Omega)}+\left\|\nabla u\right\|_{H^{\frac{3}{2}}(\Omega)}\leqslant C\left\|f\right\|_{H^{\frac{1}{2}}(\Omega)}$$

where  $\partial_n$  denotes the outer normal derivative.

Moreover, if the function f can be expressed as  $f = \nabla \cdot F$  for some vector field F, then we have

$$||u||_{H^1(\Omega)} \leq C ||F||_{L^2(\Omega)}.$$

*Proof.* See [38, Proposition 3.8] for the first elliptic estimate. We now demonstrate the second one. Using integration by parts, we obtain

$$\int_{\Omega} |\nabla u|^2 \, dx = -\int_{\Omega} \Delta u u dx = -\int_{\Omega} \nabla \cdot F u dx = \int_{\Omega} F^i \partial_i u dx.$$

It follows that

$$\|\nabla u\|_{L^{2}(\Omega)}^{2} \leq \|\nabla u\|_{L^{2}(\Omega)} \|F\|_{L^{2}(\Omega)}$$

and hence

$$\left\|\nabla u\right\|_{L^{2}(\Omega)} \leqslant \left\|F\right\|_{L^{2}(\Omega)}.$$

Therefore, the  $H^1$  estimate follows by Poincaré's inequality.

3. Proof of Theorem 1.1

We establish Theorem 1.1 by contradiction.

Suppose that the maximal time  $T^{\dagger} < \infty$ . Then, either the velocity field  $v(\cdot, T^{\dagger}) \notin H^3(\Omega_{T^{\dagger}})$ , or the free boundary  $\partial \Omega_{T^{\dagger}} \notin H^4$ .

Assume that none of the five scenarios in Theorem 1.1 holds. Then, there exists a positive constant  $\mathscr{C}_{\dagger}$  such that

$$\inf_{0 \leqslant t < T^{\dagger}} \mathscr{R}(\Omega_t) > \mathscr{C}_{\dagger}^{-1}, \quad \partial \Omega_t \in H^{2+\varepsilon}, \quad 0 \leqslant t < T^{\dagger},$$
(3.1)

$$\sup_{0 \leq t < T^{\dagger}} \left( \left\| \mathscr{H}_{\partial\Omega_{t}} \right\|_{H^{\frac{3}{2}}(\partial\Omega_{t})} + \left\| v_{n} \right\|_{H^{\frac{5}{2}}(\partial\Omega_{t})} \right) + \int_{0}^{T^{\dagger}} \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial\Omega_{t})} dt \leq \mathscr{C}_{\dagger}, \tag{3.2}$$

$$\int_{0}^{T^{\dagger}} \left\|\nabla v\right\|_{L^{\infty}(\Omega_{t})} dt \leqslant \mathscr{C}_{\dagger},\tag{3.3}$$

$$\|\nabla \times v_0\|_{L^2(\Omega_0)}^2 + \frac{1}{2} \|v_0\|_{L^2(\Omega_0)}^2 + \mathcal{H}^2(\partial\Omega_0) \leqslant \mathscr{C}_{\dagger}.$$
(3.4)

Here,  $\mathcal{H}^2(\partial\Omega_0) := \int_{\partial\Omega_0} dS$  is the surface area — the 2D Hausdorff measure — of  $\partial\Omega_0$ ,  $\mathscr{R}(\Omega_t)$  denotes the uniform exterior and interior ball radius of  $\Omega_t$ , defined as:

 $\mathscr{R}(\Omega_t) := \sup \left\{ r > 0 : \forall x \in \partial \Omega_t, \ \exists B_r(y) \subset \Omega_t, \ B_r(z) \subset \Omega_t^c \text{ with } x \in \partial B_r(y) \cap \partial B_r(z) \right\}.$ 

Note that  $\mathscr{R}(\Omega_t) > 0$  excludes singularities like cusps, corners, or boundary self-intersection (Case (1)), see Fig. 2 for an illustration of the case where the radius  $\mathscr{R} = 0$ , in which the boundary self-intersects in at least one point.

We recall the following Reynolds transport theorem (see, e.g., [47]).

**Lemma 3.1.** For a smooth function f defined on the moving domain  $\Omega_t$ , the following holds:

$$\frac{d}{dt} \int_{\Omega_t} f dx = \int_{\Omega_t} \mathfrak{D}_t f dx,$$
$$\frac{d}{dt} \int_{\partial\Omega_t} f dS = \int_{\partial\Omega_t} (\mathfrak{D}_t f + f \overline{\nabla} \cdot v) dS$$

Applying this to the energy components of (1.1) yields

$$\frac{d}{dt} \left( \frac{1}{2} \| v \|_{L^2(\Omega_t)}^2 \right) = \int_{\Omega_t} v \cdot (-\nabla p) dx = -\int_{\partial \Omega_t} p v_n dS,$$
$$\frac{d}{dt} \mathcal{H}^2(\partial \Omega_t) = \frac{d}{dt} \int_{\partial \Omega_t} dS = \int_{\partial \Omega_t} \overline{\nabla} \cdot v dS = \int_{\partial \Omega_t} \mathscr{H}_{\partial \Omega_t} v_n dS,$$

where the last equality arises from differential geometry for a closed surface, i.e.,  $\int_{\partial \Omega_t} \overline{\nabla} \cdot v_{tan} dS = 0$  for the tangential velocity  $v_{tan} = v - v_n n$ . The pressure-curvature coupling  $p = \mathscr{H}_{\partial \Omega_t}$  then yields exact energy conservation:

$$\frac{1}{2} \|v\|_{L^2(\Omega_t)}^2 + \mathcal{H}^2(\partial\Omega_t) \equiv \frac{1}{2} \|v\|_{L^2(\Omega_0)}^2 + \mathcal{H}^2(\partial\Omega_0), \quad 0 \leqslant t < T^{\dagger}.$$

Thus from (3.4), we obtain the uniform bound:

$$\sup_{0 \leq t < T^{\dagger}} \left( \frac{1}{2} \| v \|_{L^{2}(\Omega_{t})}^{2} + \mathcal{H}^{2}(\partial \Omega_{t}) \right) \leq \mathscr{C}_{\dagger}.$$
(3.5)

**Remark 3.2.** The constant in the following will primarily depend on  $C_{\dagger}$  and  $C_{\dagger}^{-1}$  (the lower bound of the uniform ball radius), and we will denote it simply as  $C(C_{\dagger})$ .

Next, we establish uniform vorticity control:

$$\sup_{0 \leqslant t < T^{\dagger}} \|\nabla \times v\|_{L^{2}(\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger})$$

In fact, by applying Lemma 3.1 to the vorticity energy and using (2.6), we obtain

$$\begin{split} \frac{d}{dt} \left( \frac{1}{2} \left\| \nabla \times v \right\|_{L^{2}(\Omega_{t})}^{2} \right) &= \int_{\Omega_{t}} \mathfrak{D}_{t} \left( \nabla \times v \right) : \left( \nabla \times v \right) dx \\ &= -\int_{\Omega_{t}} \left[ (\nabla v)^{\top} \left( \nabla \times v \right) + \left( \nabla \times v \right) \nabla v \right] : \left( \nabla \times v \right) dx \\ &\leqslant 2 \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} \left\| \nabla \times v \right\|_{L^{2}(\Omega_{t})}^{2}, \quad 0 < t < T^{\dagger}. \end{split}$$

By (3.3) and (3.4), for any  $0 \leq t < T^{\dagger}$ , Grönwall's inequality then yields:

$$\begin{aligned} \|\nabla \times v\|_{L^{2}(\Omega_{t})}^{2} &\leqslant \|\nabla \times v_{0}\|_{L^{2}(\Omega_{0})}^{2} \exp\left(4\int_{0}^{t} \|\nabla v(s)\|_{L^{\infty}(\Omega_{s})} \, ds\right) \\ &\leqslant \|\nabla \times v_{0}\|_{L^{2}(\Omega_{0})}^{2} \exp\left(4\int_{0}^{T^{\dagger}} \|\nabla v(t)\|_{L^{\infty}(\Omega_{t})} \, dt\right) \\ &\leqslant C(\mathscr{C}_{\dagger}). \end{aligned}$$

$$(3.6)$$

We note that, by (3.1), the free boundary  $\partial \Omega_t$  belongs to the  $C^{1,\alpha}$ -class for some  $\alpha \in (0, 1)$  throughout the time interval  $[0, T^{\dagger})$ . Combining with the curvature bound in (3.2) and Sobolev's embedding, we apply Lemma 2.6 to conclude that the second fundamental form II satisfies

$$\|\mathrm{II}\|_{L^4(\partial\Omega_t)} \leqslant C\left(1 + \|\mathscr{H}\|_{L^4(\partial\Omega_t)}\right) \leqslant C(\mathscr{C}_{\dagger}), \quad 0 \leqslant t < T^{\dagger}.$$

Then, from Lemma 2.6 again, it follows that

$$\|\mathrm{II}\|_{H^{\frac{3}{2}}(\partial\Omega_{t})} \leq C(\mathscr{C}_{\dagger}) \left(1 + \|\mathscr{H}\|_{H^{\frac{3}{2}}(\partial\Omega_{t})}\right) \leq C(\mathscr{C}_{\dagger}), \quad 0 \leq t < T^{\dagger}.$$
(3.7)

Since, by (3.1), the free boundary  $\partial \Omega_t$  belongs to the  $H^{2+\varepsilon}$ -class throughout the time interval  $[0, T^{\dagger})$ , and by applying Lemma 2.7, we deduce that the free boundary  $\partial \Omega_t \in H^{\frac{7}{2}}$  uniformly on  $[0, T^{\dagger})$ , as the constant  $C(\mathscr{C}_{\dagger})$  in (3.7) is independent of time. Additionally, the unit outer normal vector  $n \in H^{\frac{5}{2}}$ uniformly on  $[0, T^{\dagger})$ , since  $\mathbf{II} = \overline{\nabla} n$ . Using the boundary condition (1.1c), we conclude that

$$\inf_{\substack{0 \leq t < T^{\dagger}}} \mathscr{R}(\Omega_{t}) > \mathscr{C}_{\dagger}^{-1}, \quad \partial \Omega_{t} \in H^{\frac{1}{2}}, \quad 0 \leq t < T^{\dagger}, \\
\sup_{\substack{0 \leq t < T^{\dagger}}} \left( \left\| n \right\|_{H^{\frac{5}{2}}(\partial \Omega_{t})} + \left\| \mathrm{II} \right\|_{H^{\frac{3}{2}}(\partial \Omega_{t})} + \left\| v_{n} \right\|_{H^{\frac{5}{2}}(\partial \Omega_{t})} \right) + \int_{0}^{T^{\dagger}} \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial \Omega_{t})} dt \leq C(\mathscr{C}_{\dagger}), \quad (3.8)$$

$$\sup_{\substack{0 \leq t < T^{\dagger}}} \left( \left\| \nabla \times v \right\|_{L^{2}(\Omega_{t})} + \left\| v \right\|_{L^{2}(\Omega_{t})} \right) \leq C(\mathscr{C}_{\dagger}).$$

**Remark 3.3.** The (3.1)–(3.4) are equivalent to (3.3) and (3.8). However, in the absence of (3.3), we can derive all the other assumptions in (3.8), except the  $L^2$ -bound on the vorticity given by (3.6), i.e.,

$$\sup_{0 \leqslant t < T^{\dagger}} \|\nabla \times v\|_{L^{2}(\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger})$$

may not be valid since the exponential growth factor in Grönwall's estimate depends critically on  $\int_0^{T^{\dagger}} \|\nabla v(t)\|_{L^{\infty}(\Omega_t)} dt.$ 

We extend the unit outer normal n to  $\Omega_t$  using harmonic extension. By elliptic regularity theory for Dirichlet problems and the uniform domain characteristics in (3.8), the extended n (retaining the same notation) satisfies:

$$\sup_{0 \leqslant t < T^{\dagger}} \|n\|_{H^{3}(\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger}) \sup_{0 \leqslant t < T^{\dagger}} \|n\|_{H^{5/2}(\partial\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger}).$$
(3.9)

Under (3.8), we will derive the energy estimates (3.25) and (3.30) on the time interval  $(0, T^{\dagger})$ .

We now list some inequalities that will be frequently used. The following Gagliardo-Nirenberg inequality can be found in [9].

**Lemma 3.4.** Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain. For any  $s_1, s_2 \ge 0$ ,  $1 \le p_1, p_2 \le \infty$ ,  $\theta \in (0, 1)$  with

$$s = \theta s_1 + (1 - \theta) s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}$$

the Gagliardo-Nirenberg inequality holds:

$$\|f\|_{W^{s,p}(\Omega)} \leq C \|f\|_{W^{s_1,p_1}(\Omega)}^{\theta} \|f\|_{W^{s_2,p_2}(\Omega)}^{1-\theta}$$

provided either

- (1)  $s_1 \neq s_2$  and  $\nexists$  integer k such that  $s_2 = k$  and  $p_2 = 1$  with  $s_1 1/p_1 \ge s_2 1$ , or
- (2)  $s_1 = s_2$  (reducing to Hölder's inequality).

The following Kato-Ponce type inequality can be found in [14], [39], and [38, Proposition 2.10].

**Lemma 3.5.** Let  $\Omega \subset \mathbb{R}^3$  be bounded with  $C^{1,\alpha}$  boundary  $(\alpha > 0)$  and  $\|\mathrm{II}\|_{H^{\frac{3}{2}}(\partial\Omega)} \leq C$ . Let  $k \in \frac{1}{2}\mathbb{N}, l \in \mathbb{N}$ , with  $k, l \leq 3$  and let  $p_1, p_2, q_1, q_2 \in [2, \infty]$  with  $p_1, q_2 < \infty$  satisfying

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{2}.$$

Then, the following product estimates hold:

$$\begin{split} \|fg\|_{H^{k}(\partial\Omega)} &\leqslant C\left(\|f\|_{H^{k}(\partial\Omega)} \|g\|_{L^{\infty}(\partial\Omega)} + \|f\|_{L^{\infty}(\partial\Omega)} \|g\|_{H^{k}(\partial\Omega)}\right), \\ \|fg\|_{H^{l}(\partial\Omega)} &\leqslant C\left(\|f\|_{W^{l,p_{1}}(\partial\Omega)} \|g\|_{L^{q_{1}}(\partial\Omega)} + \|f\|_{L^{p_{2}}(\partial\Omega)} \|g\|_{W^{l,q_{2}}(\partial\Omega)}\right), \\ \|fg\|_{H^{k}(\Omega)} &\leqslant C\left(\|f\|_{W^{k,p_{1}}(\Omega)} \|g\|_{L^{q_{1}}(\Omega)} + \|f\|_{L^{p_{2}}(\Omega)} \|g\|_{W^{k,q_{2}}(\Omega)}\right). \end{split}$$

We will also apply the following bilinear inequality, which can be found in [7, Lemma 2.5]:

$$\|fg\|_{H^{s}(\Omega)} \leq C \,\|f\|_{H^{r}(\Omega)} \,\|g\|_{H^{s}(\Omega)} \,, \quad r > \frac{3}{2}, \ r \geqslant s \geqslant 0.$$
(3.10)

The primary energy functional encodes critical dynamics:

$$\mathscr{E}(t) \coloneqq \frac{1}{2} \left( \int_{\Omega_t} \left| \mathfrak{D}_t^2 v \right|^2 dx + \int_{\partial \Omega_t} \left| \overline{\nabla} \left( \mathfrak{D}_t v \cdot n \right) \right|^2 dS + \int_{\Omega_t} \left| \nabla^2 \left( \nabla \times v \right) \right|^2 dx \right).$$
(3.11)

The composite energy functional integrates spatial regularity:

$$E(t) \coloneqq \left\|\mathfrak{D}_{t}^{2}v\right\|_{L^{2}(\Omega_{t})}^{2} + \left\|\mathfrak{D}_{t}v\right\|_{H^{\frac{3}{2}}(\Omega_{t})}^{2} + \left\|v\right\|_{H^{3}(\Omega_{t})}^{2} + \left\|\mathfrak{D}_{t}v\cdot n\right\|_{H^{1}(\partial\Omega_{t})}^{2} + 1.$$
(3.12)

**Remark 3.6.** The 3/2-growth in the Sobolev regularity in (3.12), as identified in [31, 38, 47], reflects that a material derivative is effectively comparable to a 3/2-order spatial derivative, capturing the intrinsic smoothing effect induced by surface tension. The energy functional is constructed using the material derivative without separating the time derivative, a formulation particularly suited for energy estimates in the Eulerian framework: the local-in-time a priori estimate for system (1.1) is obtained by applying energy functional (3.12) in [38] by neglecting the electric field therein. This type of functional leverages the structure of system (1.1), allowing for a gain of 1/2-order spatial derivative through direct substitution from (1.1a); see [31] for detailed discussions.

We compute the time derivative of the energy functional (3.11).

**Lemma 3.7.** Under the consolidated estimates (3.8), for any  $t \in (0, T^{\dagger})$ , it holds

$$\frac{d\mathscr{E}}{dt} \leq C(\mathscr{C}_{\dagger}) \left( \|\nabla v\|_{L^{\infty}(\Omega_{t})} + \|\overline{\nabla}v\|_{L^{\infty}(\partial\Omega_{t})} + \|\nabla p\|_{H^{1}(\Omega_{t})} \right) E(t) 
- \int_{\Omega_{t}} [\mathfrak{D}_{t}^{2}, \nabla] p \cdot \mathfrak{D}_{t}^{2} v dx + \int_{\Omega_{t}} \mathfrak{D}_{t}^{2} p \nabla \cdot \mathfrak{D}_{t}^{2} v dx - \int_{\partial\Omega_{t}} \mathfrak{R}(\mathfrak{D}_{t}^{2} v \cdot n) dS,$$
(3.13)

where the error  $\Re$  is defined in Lemma 2.5.

Proof. We define

$$\mathscr{E}(t) = \frac{1}{2} \int_{\Omega_t} \left| \mathfrak{D}_t^2 v \right|^2 dx + \frac{1}{2} \int_{\partial \Omega_t} \left| \overline{\nabla} (\mathfrak{D}_t v \cdot n) \right|^2 dS + \frac{1}{2} \int_{\Omega_t} \left| \nabla^2 \left( \nabla \times v \right) \right|^2 dx$$
$$=: \Lambda_1(t) + \Lambda_2(t) + \Lambda_3(t),$$

and we will apply the Reynolds transport theorem (Lemma 3.1) several times without further mention. From (1.1a) and the divergence theorem, we obtain

$$\begin{split} \frac{d\Lambda_1}{dt} &= \int_{\Omega_t} \mathfrak{D}_t^3 v \cdot \mathfrak{D}_t^2 v dx \\ &= -\int_{\Omega_t} \mathfrak{D}_t^2 \nabla p \cdot \mathfrak{D}_t^2 v dx \\ &= -\int_{\Omega_t} \nabla \mathfrak{D}_t^2 p \cdot \mathfrak{D}_t^2 v dx - \int_{\Omega_t} [\mathfrak{D}_t^2, \nabla] p \cdot \mathfrak{D}_t^2 v dx \\ &= \underbrace{-\int_{\partial\Omega_t} \mathfrak{D}_t^2 p(\mathfrak{D}_t^2 v \cdot n) dS}_{=:J_1(t)} + \int_{\Omega_t} \mathfrak{D}_t^2 p \nabla \cdot \mathfrak{D}_t^2 v dx - \int_{\Omega_t} [\mathfrak{D}_t^2, \nabla] p \cdot \mathfrak{D}_t^2 v dx \end{split}$$

To control the boundary energy  $\Lambda_2$ , we apply the commutator  $[\mathfrak{D}_t, \overline{\nabla}]$  from Lemma 2.1 along with the following divergence theorem, namely,

$$\int_{\partial\Omega_t} \overline{\nabla} f \cdot \overline{\nabla} g dS = -\int_{\partial\Omega_t} \Delta_{\mathrm{II}} f g dS,$$

to deduce

$$\begin{split} \frac{d\Lambda_2}{dt} &= \int_{\partial\Omega_t} \mathfrak{D}_t \overline{\nabla}(\mathfrak{D}_t v \cdot n) \cdot \overline{\nabla}(\mathfrak{D}_t v \cdot n) dS + \frac{1}{2} \int_{\partial\Omega_t} \left| \overline{\nabla}(\mathfrak{D}_t v \cdot n) \right|^2 \overline{\nabla} \cdot v dS \\ &= \int_{\partial\Omega_t} [\mathfrak{D}_t, \overline{\nabla}](\mathfrak{D}_t v \cdot n) \cdot \overline{\nabla}(\mathfrak{D}_t v \cdot n) dS + \int_{\partial\Omega_t} \overline{\nabla}\mathfrak{D}_t(\mathfrak{D}_t v \cdot n) \cdot \overline{\nabla}(\mathfrak{D}_t v \cdot n) dS \\ &+ \frac{1}{2} \int_{\partial\Omega_t} \left| \overline{\nabla}(\mathfrak{D}_t v \cdot n) \right|^2 \overline{\nabla} \cdot v dS \\ &= - \int_{\partial\Omega_t} \left[ (\overline{\nabla} v)^\top \overline{\nabla}(\mathfrak{D}_t v \cdot n) \right] \cdot \overline{\nabla}(\mathfrak{D}_t v \cdot n) dS + \frac{1}{2} \int_{\partial\Omega_t} \left| \overline{\nabla}(\mathfrak{D}_t v \cdot n) \right|^2 \overline{\nabla} \cdot v dS \\ &+ \int_{\partial\Omega_t} \overline{\nabla}(\mathfrak{D}_t^2 v \cdot n) \cdot \overline{\nabla}(\mathfrak{D}_t v \cdot n) dS + \int_{\partial\Omega_t} \overline{\nabla}(\mathfrak{D}_t v \cdot \mathfrak{D}_t n) \cdot \overline{\nabla}(\mathfrak{D}_t v \cdot n) dS \\ &\leq - \int_{\partial\Omega_t} (\mathfrak{D}_t^2 v \cdot n) \Delta_{\mathrm{II}}(\mathfrak{D}_t v \cdot n) dS + C \left\| \overline{\nabla} v \right\|_{L^\infty(\partial\Omega_t)} \left\| \overline{\nabla}(\mathfrak{D}_t v \cdot n) \right\|_{L^2(\partial\Omega_t)}^2 \\ &+ \underbrace{\int_{\partial\Omega_t} \overline{\nabla}(\mathfrak{D}_t v \cdot \mathfrak{D}_t n) \cdot \overline{\nabla}(\mathfrak{D}_t v \cdot n) dS }_{=:W(t)}. \end{split}$$

Again, using the formula for  $\mathfrak{D}_t n$  in Lemma 2.1, the trace theorem, and the regularity of the normal vector in (3.8), we have

$$\begin{split} |W(t)| &\leq \int_{\partial\Omega_{t}} \left| \overline{\nabla}\mathfrak{D}_{t}v * \mathfrak{D}_{t}n * \overline{\nabla}(\mathfrak{D}_{t}v \cdot n) \right| dS + \int_{\partial\Omega_{t}} \left| \mathfrak{D}_{t}v * \overline{\nabla}\mathfrak{D}_{t}n * \overline{\nabla}(\mathfrak{D}_{t}v \cdot n) \right| dS \\ &\leq C \left\| \overline{\nabla}\mathfrak{D}_{t}v \right\|_{L^{2}(\partial\Omega_{t})} \left\| \mathfrak{D}_{t}n \right\|_{L^{\infty}(\partial\Omega_{t})} \left\| \overline{\nabla}(\mathfrak{D}_{t}v \cdot n) \right\|_{L^{2}(\partial\Omega_{t})} \\ &+ C \left( \left\| \mathfrak{D}_{t}v * \overline{\nabla}^{2}v * n \right\|_{L^{2}(\partial\Omega_{t})} + \left\| \mathfrak{D}_{t}v * \overline{\nabla}v * \overline{\nabla}n \right\|_{L^{2}(\partial\Omega_{t})} \right) \left\| \overline{\nabla}(\mathfrak{D}_{t}v \cdot n) \right\|_{L^{2}(\partial\Omega_{t})} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \overline{\nabla}v \right\|_{L^{\infty}(\partial\Omega_{t})} \left( \left\| \overline{\nabla}\mathfrak{D}_{t}v \right\|_{L^{2}(\partial\Omega_{t})}^{2} + \left\| \overline{\nabla}(\mathfrak{D}_{t}v \cdot n) \right\|_{L^{2}(\partial\Omega_{t})}^{2} \right) \\ &+ C(\mathscr{C}_{\dagger}) \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} \left\| v \right\|_{H^{3}(\Omega_{t})} \left\| \overline{\nabla}(\mathfrak{D}_{t}v \cdot n) \right\|_{L^{2}(\partial\Omega_{t})} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \overline{\nabla}v \right\|_{L^{\infty}(\partial\Omega_{t})} \left( \left\| v \right\|_{H^{3}(\Omega_{t})}^{2} + \left\| \overline{\nabla}(\mathfrak{D}_{t}v \cdot n) \right\|_{L^{2}(\partial\Omega_{t})}^{2} \right) \\ &+ C(\mathscr{C}_{\dagger}) \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} \left( \left\| v \right\|_{H^{3}(\Omega_{t})}^{2} + \left\| \overline{\nabla}(\mathfrak{D}_{t}v \cdot n) \right\|_{L^{2}(\partial\Omega_{t})}^{2} \right) \\ &\leq C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla}v \right\|_{L^{\infty}(\partial\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} \right) E(t), \end{split}$$

where we have used the following results in the third inequality:

$$\begin{split} \left\| \mathfrak{D}_{t} v * \overline{\nabla}^{2} v * n \right\|_{L^{2}(\partial \Omega_{t})} &\leqslant \left\| \mathfrak{D}_{t} v \right\|_{L^{4}(\partial \Omega_{t})} \left\| \overline{\nabla}^{2} v \right\|_{L^{4}(\partial \Omega_{t})} \left\| n \right\|_{L^{\infty}(\partial \Omega_{t})} \\ &\leqslant C \left\| \mathfrak{D}_{t} v \right\|_{H^{\frac{1}{2}}(\partial \Omega_{t})} \left\| v \right\|_{H^{\frac{5}{2}}(\partial \Omega_{t})} \\ &\leqslant C \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} \left\| v \right\|_{H^{3}(\Omega_{t})}, \\ \\ \left\| \mathfrak{D}_{t} v * \overline{\nabla} v * \overline{\nabla} n \right\|_{L^{2}(\partial \Omega_{t})} &\leqslant \left\| \mathfrak{D}_{t} v \right\|_{L^{4}(\partial \Omega_{t})} \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial \Omega_{t})} \left\| \overline{\nabla} n \right\|_{L^{4}(\partial \Omega_{t})} \\ &\leqslant C \left\| \mathfrak{D}_{t} v \right\|_{H^{\frac{1}{2}}(\partial \Omega_{t})} \left\| v \right\|_{H^{\frac{5}{2}}(\partial \Omega_{t})} \left\| n \right\|_{H^{\frac{3}{2}}(\partial \Omega_{t})} \\ &\leqslant C(\mathscr{C}_{\dagger}) \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} \left\| v \right\|_{H^{3}(\Omega_{t})}, \end{split}$$

which can be proved using the Sobolev embedding theorem, the trace theorem, (1.1a), and the regularity of the normal vector in (3.8).

Therefore, we obtain

$$\frac{d\Lambda_2}{dt} \leqslant \underbrace{-\int_{\partial\Omega_t} (\mathfrak{D}_t^2 v \cdot n) \Delta_{\mathrm{II}}(\mathfrak{D}_t v \cdot n) dS}_{=:J_2(t)} + C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial\Omega_t)} + \left\| \nabla p \right\|_{H^1(\Omega_t)} \right) E(t).$$

To compute the last term involving the curl,  $\Lambda_3$ , we utilize Lemma 2.3 to obtain

$$\begin{split} \frac{d\Lambda_3}{dt} &= \int_{\Omega} \mathfrak{D}_t \nabla^2 \left( \nabla \times v \right) * \nabla^2 \left( \nabla \times v \right) dx \\ &= \int_{\Omega_t} \nabla v * \nabla^2 \left( \nabla \times v \right) * \nabla^2 \left( \nabla \times v \right) + \nabla^3 v * \left( \nabla \times v \right) * \nabla^2 \left( \nabla \times v \right) \\ &+ \nabla^2 v * \nabla \left( \nabla \times v \right) * \nabla^2 \left( \nabla \times v \right) dx \\ &\leqslant C \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| \nabla^2 v \right\|_{H^1(\Omega_t)}^2 + \left\| \nabla^2 v * \nabla \left( \nabla \times v \right) * \nabla^2 \left( \nabla \times v \right) \right\|_{L^1(\Omega_t)} \right) \\ &\leqslant C \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \| v \|_{H^3(\Omega_t)}^2 + \left\| \left| \nabla^2 v \right|_{L^2(\Omega_t)}^2 \right\|_{L^2(\Omega_t)} \right). \end{split}$$

Note that we have

$$\left\|\nabla^{2} v\right\|^{2} = \left\|\nabla^{2} v\right\|^{2}_{L^{4}(\Omega_{t})} \leq C \left\|\nabla v\right\|^{2}_{W^{1,4}(\Omega_{t})}$$

and by Gagliardo-Nirenberg inequality in Lemma 3.4

$$\|\nabla v\|_{W^{1,4}(\Omega_t)} \leq C \|\nabla v\|_{H^2(\Omega_t)}^{\frac{1}{2}} \|\nabla v\|_{L^{\infty}(\Omega_t)}^{\frac{1}{2}}.$$
(3.14)

Thus, it holds

$$\left\| \left| \nabla^2 v \right|^2 \right\|_{L^2(\Omega_t)} \left\| \nabla^3 v \right\|_{L^2(\Omega_t)} \leqslant C \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^3(\Omega_t)}^2$$

and

$$\frac{d\Lambda_3}{dt} \leqslant C \left\|\nabla v\right\|_{L^{\infty}(\Omega_t)} E(t).$$

Combining with the above calculations and applying Lemma 2.5, we obtain

$$J_{1}(t) + J_{2}(t) = -\int_{\partial\Omega_{t}} \mathfrak{D}_{t}^{2} p(\mathfrak{D}_{t}^{2} v \cdot n) dS - \int_{\partial\Omega_{t}} (\mathfrak{D}_{t}^{2} v \cdot n) \Delta_{\mathrm{II}}(\mathfrak{D}_{t} v \cdot n) dS$$
$$= -\int_{\partial\Omega_{t}} \mathfrak{R}(\mathfrak{D}_{t}^{2} v \cdot n) dS.$$

Therefore, the claim (3.13) follows.

The error term  $\mathfrak{R}$  can be estimated in the following lemma.

Lemma 3.8. Under (3.8), we have

$$\|\mathfrak{R}\|_{H^{\frac{1}{2}}(\partial\Omega_{t})} \leq C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla}v \right\|_{L^{\infty}(\partial\Omega_{t})} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} + 1 \right) \sqrt{E(t)}, \quad t \in (0, T^{\dagger}).$$
(3.15)

*Proof.* Form Lemma 2.5, the error  $\Re$  can be expressed as

$$\mathfrak{R} = -|\mathrm{II}|^2 \,\mathfrak{D}_t v \cdot n + \overline{\nabla} p \cdot \mathfrak{D}_t v + \overline{\nabla}^2 v * \overline{\nabla} v * n + \overline{\nabla} v * \overline{\nabla} v * \mathrm{II}.$$

For the second term, recalling the curvature bound provided in (3.8), we proceed by applying Kato-Ponce estimates from Lemma 3.5. It follows from (1.1a) and the definition of the tangential derivative in (2.1) that  $\overline{\Sigma} = \overline{\Sigma} = \overline{$ 

$$\overline{\nabla}p\cdot\mathfrak{D}_t v = -\overline{\nabla}p\cdot\nabla p = -\overline{\nabla}p\cdot\left(\overline{\nabla}p + (\nabla p\cdot n)n\right) = -\overline{\nabla}p\cdot\overline{\nabla}p,$$

since  $\overline{\nabla}p \cdot n$  vanishes. Next, by the extension of the normal vector satisfying  $||n||_{H^3(\Omega_t)} \leq C(\mathscr{C}_{\dagger})$  as given in (3.9), and by the trace theorem, we obtain

$$\left\|\overline{\nabla}p\cdot\mathfrak{D}_{t}v\right\|_{H^{\frac{1}{2}}(\partial\Omega_{t})} = \left\|\overline{\nabla}p\cdot\overline{\nabla}p\right\|_{H^{\frac{1}{2}}(\partial\Omega_{t})} \leqslant C\left\|\overline{\nabla}p\cdot\overline{\nabla}p\right\|_{H^{1}(\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger})\left\|\left|\nabla p\right|^{2}\right\|_{H^{1}(\Omega_{t})}.$$

We then apply (1.1a), Lemma 3.5, and the Sobolev embedding theorem to obtain

$$\begin{split} \left\| |\nabla p|^2 \right\|_{H^1(\Omega_t)} &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla p \right\|_{L^6(\Omega_t)} \left\| \nabla p \right\|_{W^{1,3}(\Omega_t)} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla p \right\|_{H^1(\Omega_t)} \left\| \nabla p \right\|_{H^{\frac{3}{2}}(\Omega_t)} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla p \right\|_{H^1(\Omega_t)} \sqrt{E(t)}. \end{split}$$

For the first term, again utilizing the curvature bound in (3.8), the bilinear inequality (3.10), and the trace theorem, we have

$$\left\| |\mathrm{II}|^2 \,\mathfrak{D}_t v \cdot n \right\|_{H^{\frac{1}{2}}(\partial\Omega_t)} \leqslant C(\mathscr{C}_{\dagger}) \, \|\mathrm{II}\|_{H^{\frac{3}{2}}(\partial\Omega_t)}^2 \, \|\mathfrak{D}_t v \cdot n\|_{H^{\frac{1}{2}}(\partial\Omega_t)} \leqslant C(\mathscr{C}_{\dagger}) \sqrt{E(t)}.$$

For the remaining terms, owing to the regularity of the normal vector and the second fundamental form as stated in (3.8), and by the bilinear inequality (3.10), it suffices to show that

$$\left\|\overline{\nabla}^{2}v \ast \overline{\nabla}v\right\|_{H^{\frac{1}{2}}(\partial\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger})\left(\left\|\overline{\nabla}v\right\|_{L^{\infty}(\partial\Omega_{t})} + \left\|\nabla v\right\|_{L^{\infty}(\Omega_{t})}\right)\sqrt{E(t)}$$

Indeed, by applying (2.8), one obtains

$$\left\|\overline{\nabla}^{2}v \ast \overline{\nabla}v\right\|_{H^{\frac{1}{2}}(\partial\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger})\left(\left\|\overline{\nabla}^{2}v \ast \overline{\nabla}v\right\|_{L^{2}(\partial\Omega_{t})} + \left\|\nabla\left(\overline{\nabla}^{2}v \ast \overline{\nabla}v\right)\right\|_{L^{2}(\Omega_{t})}\right).$$
(3.16)

To control the second term, a straightforward calculation reveals that the following tensors can be expressed in terms of the \*-products:

$$\begin{split} \overline{\nabla}v &= \nabla v + \nabla v * n * n, \\ \nabla \overline{\nabla}v &= \nabla^2 v + \nabla^2 v * n * n + \nabla v * \nabla n * n, \\ \overline{\nabla}^2 v &= \nabla^2 v + \nabla^2 v * n * n + \nabla^2 v * n * n * n * n \\ &+ \nabla v * \nabla n * n + \nabla v * \nabla n * n * n * n * n, \\ \nabla \overline{\nabla}^2 v &= \nabla^3 v + \nabla^3 v * n * n + \nabla^3 v * n * n * n * n \\ &+ \nabla^2 v * \nabla n * n * n * n + \nabla^2 v * \nabla n * n \\ &+ \nabla v * \nabla^2 n * n * n * n + \nabla v * \nabla^2 n * n * n * n \\ &+ \nabla v * \nabla^2 n * n + \nabla v * \nabla^2 n * n * n * n. \end{split}$$

Then, we proceed to estimate

$$\left\|\nabla\left(\overline{\nabla}^{2}v\ast\overline{\nabla}v\right)\right\|_{L^{2}(\Omega_{t})} \leqslant C\left(\underbrace{\left\|\nabla\overline{\nabla}^{2}v\ast\overline{\nabla}v\right\|_{L^{2}(\Omega_{t})}}_{=:\Pi_{1}} + \underbrace{\left\|\overline{\nabla}^{2}v\ast\nabla\overline{\nabla}v\right\|_{L^{2}(\Omega_{t})}}_{=:\Pi_{2}}\right).$$

To estimate  $\Pi_1$ , for sufficiently small  $\varepsilon > 0$ , by the bilinear inequality (3.10) and (3.9), we obtain  $\left\| (\nabla v * n * n) * (\nabla^3 v * n * n * n * n) \right\|_{L^2(\Omega_t)} \leq C(\mathscr{C}_{\dagger}) \|\nabla v\|_{L^{\infty}(\Omega_t)} \|v\|_{H^3(\Omega_t)}.$  Similarly, for sufficiently small  $\varepsilon > 0$ , we obtain

$$\begin{split} \left\| \left( \nabla v * n * n \right) * \left( \nabla^2 v * \nabla n * n * n * n \right) \right\|_{L^2(\Omega_t)} &\leq C \left\| n \right\|_{H^{\frac{5}{2} + \varepsilon}(\Omega_t)}^5 \left\| n \right\|_{H^{\frac{5}{2} + \varepsilon}(\Omega_t)} \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^{3}(\Omega_t)} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^{3}(\Omega_t)} , \\ \left\| \left( \nabla v * n * n \right) * \left( \nabla v * \nabla^2 n * n * n * n \right) \right\|_{L^2(\Omega_t)} &\leq C \left\| n \right\|_{H^{\frac{5}{2} + \varepsilon}(\Omega_t)}^5 \left\| \nabla^2 n \right\|_{L^{3}(\Omega_t)} \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^{3}(\Omega_t)} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^{3}(\Omega_t)} , \\ \left\| \left( \nabla v * n * n \right) * \left( \nabla v * \nabla n * \nabla n * n * n \right) \right\|_{L^2(\Omega_t)} &\leq C \left\| n \right\|_{H^{\frac{3}{2} + \varepsilon}(\Omega_t)}^4 \left\| \nabla n \right\|_{H^{\frac{3}{2} + \varepsilon}(\Omega_t)} \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^{3}(\Omega_t)} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^{3}(\Omega_t)} , \end{split}$$

and the remaining estimates in  $\Pi_1$  are straightforward.

To handle  $\Pi_2$ , it suffices to control the product term

$$\left(\nabla^2 v \ast n \ast n\right) \ast \left(\nabla^2 v \ast n \ast n \ast n \ast n \ast n\right),$$

since the remaining terms are either straightforward or have already been computed above. By Lemma 3.4 and (3.9), it follows that

$$\left\| \left( \nabla^2 v * n * n \right) * \left( \nabla^2 v * n * n * n * n \right) \right\|_{L^2(\Omega_t)} \leqslant C(\mathscr{C}_{\dagger}) \left\| \nabla^2 v \right\|_{L^4(\Omega_t)}^2$$
  
 
$$\leqslant C \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| \nabla v \right\|_{H^2(\Omega_t)}.$$

We conclude that

$$\left\| \nabla \left( \overline{\nabla}^2 v * \overline{\nabla} v \right) \right\|_{L^2(\Omega_t)} \leqslant C(\mathscr{C}_{\dagger}) \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^3(\Omega_t)},$$

and thus, from (3.16) and the trace theorem, it follows that

$$\begin{split} \left\| \overline{\nabla}^2 v * \overline{\nabla} v \right\|_{H^{\frac{1}{2}}(\partial \Omega_t)} &\leqslant C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial \Omega_t)} \left\| \overline{\nabla}^2 v \right\|_{L^{2}(\partial \Omega_t)} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| v \right\|_{H^{3}(\Omega_t)} \right) \\ &\leqslant C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial \Omega_t)} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \right) \sqrt{E(t)}. \end{split}$$

The estimate for  $\|\overline{\nabla}v * \overline{\nabla}v\|_{H^{\frac{1}{2}}(\partial\Omega_t)}$  can be derived similarly, and we omit the details. Therefore, (3.15) follows.

Then, the terms  $\int_{\partial\Omega_t} \Re(\mathfrak{D}_t^2 v \cdot n) dS$  and  $\int_{\Omega_t} [\mathfrak{D}_t^2, \nabla] p \cdot \mathfrak{D}_t^2 v dx$  in (3.13) can be controlled as follows. Lemma 3.9. Under (3.8), it holds

$$\begin{aligned} \left| \int_{\partial\Omega_{t}} \Re(\mathfrak{D}_{t}^{2}v \cdot n) dS \right| + \left| \int_{\Omega_{t}} [\mathfrak{D}_{t}^{2}, \nabla] p \cdot \mathfrak{D}_{t}^{2} v dx \right| \\ &\leq C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla}v \right\|_{L^{\infty}(\partial\Omega_{t})} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} + 1 \right) E(t), \quad t \in (0, T^{\dagger}). \end{aligned}$$
(3.17)

*Proof.* For the first term, it holds

$$\left|\int_{\partial\Omega_t} \Re(\mathfrak{D}_t^2 v \cdot n) dS\right| \leqslant C \left\|\mathfrak{D}_t^2 v \cdot n\right\|_{H^{-\frac{1}{2}}(\partial\Omega_t)} \left\|\mathfrak{R}\right\|_{H^{\frac{1}{2}}(\partial\Omega_t)}$$

Note that, from (3.8), the free boundary  $\partial \Omega_t \in H^{\frac{7}{2}}$ , and thus, we can apply the normal trace theorem (see, e.g., [13, Theorem 3.1] and [19, Lemma 5.1]).

$$\left\|\mathfrak{D}_{t}^{2}v\cdot n\right\|_{H^{-\frac{1}{2}}(\partial\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger})\left(\left\|\mathfrak{D}_{t}^{2}v\right\|_{L^{2}(\Omega_{t})}+\left\|\nabla\cdot\mathfrak{D}_{t}^{2}v\right\|_{H^{-1}(\Omega_{t})}\right),$$

and we have

$$\begin{aligned} \left\| \nabla \cdot \mathfrak{D}_{t}^{2} v \right\|_{H^{-1}(\Omega_{t})} &\leqslant \sup \left\{ \left| \int_{\Omega_{t}} \nabla \cdot \mathfrak{D}_{t}^{2} v F dx \right| : F \in H_{0}^{1}(\Omega_{t}), \|F\|_{H_{0}^{1}(\Omega_{t})} \leqslant 1 \right\} \\ &\leqslant \sup \left\{ \left| \int_{\Omega_{t}} \mathfrak{D}_{t}^{2} v \cdot \nabla F dx \right| : F \in H_{0}^{1}(\Omega_{t}), \|F\|_{H_{0}^{1}(\Omega_{t})} \leqslant 1 \right\} \\ &\leqslant \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})}. \end{aligned}$$
(3.18)

We then use (3.15) to obtain

$$\left| \int_{\partial\Omega_t} \Re(\mathfrak{D}_t^2 v \cdot n) dS \right| \leqslant C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial\Omega_t)} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} + \left\| \nabla p \right\|_{H^1(\Omega_t)} + 1 \right) E(t).$$

For the second term, by applying (1.1a), (2.4), and the Sobolev embedding theorem, we obtain

$$\begin{split} \left| \int_{\Omega_{t}} [\mathfrak{D}_{t}^{2}, \nabla] p \cdot \mathfrak{D}_{t}^{2} v dx \right| &\leq C \left\| [\mathfrak{D}_{t}^{2}, \nabla] p \right\|_{L^{2}(\Omega_{t})} \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})} \\ &\leq C \left( \left\| \nabla v \ast \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})} + \left\| \nabla \mathfrak{D}_{t} v \ast \mathfrak{D}_{t} v \right\|_{L^{2}(\Omega_{t})} \right) \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})} \\ &\leq C \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})} + \left\| \nabla \mathfrak{D}_{t} v \right\|_{L^{3}(\Omega_{t})} \left\| \mathfrak{D}_{t} v \right\|_{L^{6}(\Omega_{t})} \right) \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})} \\ &\leq C \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} \right) \left( \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})}^{2} + \left\| \mathfrak{D}_{t} v \right\|_{H^{\frac{3}{2}}(\Omega_{t})}^{2} \right) \\ &\leq C \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} \right) E(t). \end{split}$$

Combining the above three lemmas, we obtain

$$\frac{d}{dt}\mathscr{E}(t) \leqslant C(\mathscr{C}_{\dagger}) \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} + 1 \right) E(t) + I(t),$$
(3.19)

where

$$I(t) = \int_{\Omega_t} \mathfrak{D}_t^2 p \nabla \cdot \mathfrak{D}_t^2 v dx.$$
(3.20)

To estimate I(t), we make use of the following result.

Lemma 3.10. Under (3.8), we have

$$\left\|\nabla \cdot \mathfrak{D}_{t}^{2} v\right\|_{H^{\frac{1}{2}}(\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger}) \left(\left\|\nabla v\right\|_{L^{\infty}(\Omega_{t})} + \left\|\nabla^{2} p\right\|_{L^{2}(\Omega_{t})}\right) \sqrt{E(t)}, \quad t \in (0, T^{\dagger}).$$
(3.21)

*Proof.* By Lemma 2.3, it follows that

$$\left\|\nabla \cdot \mathfrak{D}_{t}^{2} v\right\|_{H^{\frac{1}{2}}(\Omega_{t})} \leqslant C\left(\left\|\partial_{i} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i}\right\|_{H^{\frac{1}{2}}(\Omega_{t})} + \left\|\partial_{i} v^{j} \partial_{j} v^{k} \partial_{k} v^{i}\right\|_{H^{\frac{1}{2}}(\Omega_{t})}\right)$$

For the first term, by applying Lemma 3.5, we obtain

$$\begin{split} \left\| \partial_{i} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} \right\|_{H^{\frac{1}{2}}(\Omega_{t})} &\leqslant C \left( \| \nabla v \|_{L^{\infty}(\Omega_{t})} \left\| \nabla \mathfrak{D}_{t} v \right\|_{H^{\frac{1}{2}}(\Omega_{t})} + \left\| \nabla \mathfrak{D}_{t} v \right\|_{L^{\frac{12}{5}}(\Omega_{t})} \left\| \nabla v \right\|_{W^{\frac{1}{2},12}(\Omega_{t})} \right) \\ &\leqslant C \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} \left\| \mathfrak{D}_{t} v \right\|_{H^{\frac{3}{2}}(\Omega_{t})} + \left\| \nabla^{2} p \right\|_{L^{2}(\Omega_{t})}^{\frac{1}{2}} \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})}^{\frac{1}{2}} \sqrt{E(t)} \right) \\ &\leqslant C \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla^{2} p \right\|_{L^{2}(\Omega_{t})} \right) \sqrt{E(t)}, \end{split}$$

where we have used the following estimates

$$\begin{aligned} \|\nabla\mathfrak{D}_{t}v\|_{L^{\frac{12}{5}}(\Omega_{t})} &\leqslant C \left\|\nabla\mathfrak{D}_{t}v\right\|_{H^{\frac{1}{4}}(\Omega_{t})} \leqslant C \left\|\mathfrak{D}_{t}v\right\|_{H^{\frac{5}{4}}(\Omega_{t})} \leqslant C \left\|\nabla p\right\|_{H^{1}(\Omega_{t})}^{\frac{1}{2}} \left\|\mathfrak{D}_{t}v\right\|_{H^{\frac{3}{2}}(\Omega_{t})}^{\frac{1}{2}} \\ \|\nabla v\|_{W^{\frac{1}{2},12}(\Omega_{t})} &\leqslant C \left\|\nabla v\right\|_{W^{1,6}(\Omega_{t})}^{\frac{1}{2}} \left\|\nabla v\right\|_{L^{\infty}(\Omega_{t})}^{\frac{1}{2}} \leqslant C \left\|\nabla v\right\|_{L^{\infty}(\Omega_{t})}^{\frac{1}{2}} \|v\|_{H^{3}(\Omega_{t})}^{\frac{1}{2}}, \end{aligned}$$

which can be derived by applying 3.4, (1.1a), and the Sobolev embedding theorem.

To handle the second term, recalling from (3.8) that the  $L^2$ -norms of the velocity and vorticity are bounded by  $C(\mathscr{C}_{\dagger})$ , and combining these with the divergence-free condition (1.1b) and the lower-order div-curl estimate in (2.9) for k = 1, we obtain the following bound

$$\|v\|_{H^{1}(\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger}) \left( \|\nabla \cdot v\|_{L^{2}(\Omega_{t})} + \|\nabla \times v\|_{L^{2}(\Omega_{t})} + \|v_{n}\|_{H^{\frac{1}{2}}(\partial\Omega_{t})} + \|v\|_{L^{2}(\Omega_{t})} \right) \leqslant C(\mathscr{C}_{\dagger}).$$
(3.22)

We then apply Lemmas 3.4 and 3.5, and the Sobolev embedding theorem to deduce

$$\left\| |\nabla v|^{2} \right\|_{H^{\frac{1}{2}}(\Omega_{t})} \leqslant C \left\| \nabla v \right\|_{W^{\frac{1}{2},4}(\Omega_{t})}^{2} \leqslant C \left\| v \right\|_{W^{\frac{3}{2},4}(\Omega_{t})}^{2}$$
$$\leqslant C \left\| v \right\|_{L^{6}(\Omega_{t})} \left\| v \right\|_{H^{3}(\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger}) \left\| v \right\|_{H^{3}(\Omega_{t})},$$
(3.23)

since, by the Sobolev embedding theorem and (3.22), it holds that

$$\|v\|_{L^{6}(\Omega_{t})} \leqslant C \, \|v\|_{H^{1}(\Omega_{t})} \leqslant C(\mathscr{C}_{\dagger})$$

Therefore, from Lemma 3.5, the Sobolev embedding theorem, and (3.23), it follows that

$$\begin{split} \left\| \partial_{i} v^{j} \partial_{j} v^{k} \partial_{k} v^{i} \right\|_{H^{\frac{1}{2}}(\Omega_{t})} &\leqslant C(\mathscr{C}_{\dagger}) \left( \| \nabla v \|_{L^{\infty}(\Omega_{t})} \left\| | \nabla v |^{2} \right\|_{H^{\frac{1}{2}}(\Omega_{t})} + \| \nabla v \|_{W^{\frac{1}{2},\frac{1}{\delta}}(\Omega_{t})} \left\| | \nabla v |^{2} \right\|_{L^{\frac{2}{1-2\delta}}(\Omega_{t})} \right) \\ &\leqslant C(\mathscr{C}_{\dagger}) \left( \| \nabla v \|_{L^{\infty}(\Omega_{t})} \| v \|_{H^{3}(\Omega_{t})} + \| \nabla v \|_{L^{\infty}(\Omega_{t})} \| v \|_{H^{3-3\delta}(\Omega_{t})} \| \nabla v \|_{L^{\frac{2}{1-2\delta}}(\Omega_{t})} \right) \\ &\leqslant C(\mathscr{C}_{\dagger}) \left( \| \nabla v \|_{L^{\infty}(\Omega_{t})} \sqrt{E(t)} + \| \nabla v \|_{L^{\infty}(\Omega_{t})} \| v \|_{H^{3}(\Omega_{t})}^{1-\frac{3\delta}{2}} \| v \|_{H^{3}(\Omega_{t})}^{\frac{3\delta}{2}} \right) \\ &\leqslant C(\mathscr{C}_{\dagger}) \| \nabla v \|_{L^{\infty}(\Omega_{t})} \sqrt{E(t)}, \end{split}$$

where  $\delta > 0$  is sufficiently small, and we have used the following results, which are obtained by applying (3.22), Sobolev's embedding and interpolation

$$\begin{aligned} \|v\|_{H^{3-3\delta}(\Omega_{t})} &\leqslant C(\mathscr{C}_{\dagger}) \, \|v\|_{H^{3}(\Omega_{t})}^{1-\frac{3\delta}{2}} \, \|v\|_{H^{1}(\Omega_{t})}^{\frac{3\delta}{2}} \leqslant C(\mathscr{C}_{\dagger}) \, \|v\|_{H^{3}(\Omega_{t})}^{1-\frac{3\delta}{2}}, \\ \|\nabla v\|_{L^{\frac{2}{1-2\delta}}(\Omega_{t})} &\leqslant C(\mathscr{C}_{\dagger}) \, \|v\|_{H^{1+3\delta}(\Omega_{t})}^{1-\frac{3\delta}{2}} \leqslant C(\mathscr{C}_{\dagger}) \, \|v\|_{H^{3}(\Omega_{t})}^{1-\frac{3\delta}{2}} \, \|v\|_{H^{1}(\Omega_{t})}^{1-\frac{3\delta}{2}} \leqslant C(\mathscr{C}_{\dagger}) \, \|v\|_{H^{3}(\Omega_{t})}^{\frac{3\delta}{2}}. \end{aligned}$$
e, (3.21) follows.

Therefore, (3.21) follows.

**Proposition 3.11.** Under (3.8), it holds that

$$|I(t)| \leq C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial\Omega_{t})} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} + 1 \right) E(t), \quad t \in (0, T^{\dagger}),$$

$$(3.24)$$

where I(t) is defined in (3.20), and the energy functional (3.11) satisfies:

$$\frac{d\mathscr{E}}{dt} \leqslant C(\mathscr{C}_{\dagger}) \left( \|\nabla v\|_{L^{\infty}(\Omega_{t})} + \|\overline{\nabla}v\|_{L^{\infty}(\partial\Omega_{t})} + 1 \right) E(t), \quad t \in (0, T^{\dagger}).$$
(3.25)

*Proof.* Consider the following elliptic equation

. .

$$\begin{cases} -\Delta u = \nabla \cdot \mathfrak{D}_t^2 v, & \text{in } \Omega_t, \\ u = 0, & \text{on } \partial \Omega_t. \end{cases}$$
(3.26)

We begin by applying the elliptic estimates in Lemma 2.9

$$\begin{aligned} \|\partial_n u\|_{H^1(\partial\Omega_t)} + \|\nabla u\|_{H^{\frac{3}{2}}(\Omega_t)} &\leq C \left\|\nabla \cdot \mathfrak{D}_t^2 v\right\|_{H^{\frac{1}{2}}(\Omega_t)}, \\ \|u\|_{H^1(\Omega_t)} &\leq C \left\|\mathfrak{D}_t^2 v\right\|_{L^2(\Omega_t)}. \end{aligned}$$
(3.27)

We then integrate by parts to obtain

$$I(t) = -\int_{\Omega_t} \Delta \mathfrak{D}_t^2 p u dx - \int_{\partial \Omega_t} \mathfrak{D}_t^2 p \partial_n u dS \eqqcolon I_1(t) + I_2(t).$$

By applying Lemma 2.5 and integrating by parts once more, we have

$$\begin{split} -I_2(t) &= \int_{\partial\Omega_t} \left( -\Delta_{\mathrm{II}}(\mathfrak{D}_t v \cdot n) + \mathfrak{R} \right) \partial_n u dS \\ &= \int_{\partial\Omega_t} \overline{\nabla}(\mathfrak{D}_t v \cdot n) \cdot \overline{\nabla} \partial_n u dS + \int_{\partial\Omega_t} \mathfrak{R} \partial_n u dS. \end{split}$$

By the normal trace theorem, (3.26), (3.18), and the second elliptic estimate in (3.27), we obtain

$$\begin{aligned} \|\partial_n u\|_{H^{-\frac{1}{2}}(\partial\Omega_t)} &\leq C(\|\nabla u\|_{L^2(\Omega_t)} + \|\Delta u\|_{H^{-1}(\Omega_t)}) \\ &\leq C(\|\mathfrak{D}_t^2 v\|_{L^2(\Omega_t)} + \|\nabla \cdot \mathfrak{D}_t^2 v\|_{H^{-1}(\Omega_t)}) \\ &\leq C \|\mathfrak{D}_t^2 v\|_{L^2(\Omega_t)} \,. \end{aligned}$$

Then, we use the first elliptic estimate in (3.27), (3.21), and (3.15) to deduce

$$|I_2(t)| \leq C(\left\|\overline{\nabla}(\mathfrak{D}_t v \cdot n)\right\|_{L^2(\partial\Omega_t)} \|\partial_n u\|_{H^1(\partial\Omega_t)} + \|\mathfrak{R}\|_{H^{\frac{1}{2}}(\partial\Omega_t)} \|\partial_n u\|_{H^{-\frac{1}{2}}(\partial\Omega_t)})$$

$$\begin{split} &\leqslant C\sqrt{E(t)} \left\| \nabla \cdot \mathfrak{D}_{t}^{2} v \right\|_{H^{\frac{1}{2}}(\Omega_{t})} \\ &+ C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial\Omega_{t})} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} + 1 \right) \sqrt{E(t)} \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})} \\ &\leqslant C(\mathscr{C}_{\dagger}) \left( \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla^{2} p \right\|_{L^{2}(\Omega_{t})} \right) E(t) \\ &+ C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial\Omega_{t})} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} + 1 \right) E(t) \\ &\leqslant C(\mathscr{C}_{\dagger}) \left( \left\| \overline{\nabla} v \right\|_{L^{\infty}(\partial\Omega_{t})} + \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} + \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} + 1 \right) E(t). \end{split}$$

For  $I_1(t)$ , by Lemma 2.4, (1.1a), and the divergence-free condition (1.1b), we integrate by parts to obtain

$$\begin{split} I_{1}(t) &= \int_{\Omega_{t}} \left[ 4\partial_{i}v^{j}\partial_{j}\mathfrak{D}_{t}^{2}v^{i} + 3\partial_{i}\mathfrak{D}_{t}v^{j}\partial_{j}\mathfrak{D}_{t}v^{i} - 12\partial_{i}v^{l}\partial_{l}v^{j}\partial_{j}\mathfrak{D}_{t}v^{i} \\ &+ 6\partial_{i}v^{l}\partial_{l}v^{j}\partial_{j}v^{k}\partial_{k}v^{i} + \sum_{j}\partial_{j}\left(2\partial_{j}v_{i}\mathfrak{D}_{t}^{2}v^{i} + \partial_{j}\mathfrak{D}_{t}v_{i}\mathfrak{D}_{t}v^{i}\right) \right] udx \\ &= \int_{\Omega_{t}} \left[ -4\partial_{i}v^{j}\mathfrak{D}_{t}^{2}v^{i}\partial_{j}u - 2\sum_{j}\partial_{j}v_{i}\mathfrak{D}_{t}^{2}v^{i}\partial_{j}u + \sum_{i,j}\partial_{j}\mathfrak{D}_{t}v_{i}\partial_{i}p\partial_{j}u \\ &+ 3\partial_{i}\mathfrak{D}_{t}v^{j}\partial_{j}\mathfrak{D}_{t}v^{i}u - 12\partial_{i}v^{l}\partial_{l}v^{j}\partial_{j}\mathfrak{D}_{t}v^{i}u + 6\partial_{i}v^{l}\partial_{l}v^{j}\partial_{j}v^{k}\partial_{k}v^{i}u \right] dx. \end{split}$$

We use (3.27) and the Sobolev embedding theorem to estimate as follows

$$\begin{split} \left\| \partial_{i} v^{j} \mathfrak{D}_{t}^{2} v^{i} \partial_{j} u \right\|_{L^{1}(\Omega_{t})} &\leq C \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})} \left\| \nabla u \right\|_{L^{2}(\Omega_{t})} \\ &\leq C \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} E(t), \end{split} \\ \begin{split} \sum_{j} \left\| \partial_{j} v_{i} \mathfrak{D}_{t}^{2} v^{i} \partial_{j} u \right\|_{L^{1}(\Omega_{t})} &\leq C \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} E(t), \end{aligned} \\ \begin{split} \sum_{i,j} \left\| \partial_{j} \mathfrak{D}_{t} v_{i} \partial_{i} p \partial_{j} u \right\|_{L^{1}(\Omega_{t})} &\leq C \left\| \nabla p \right\|_{L^{6}(\Omega_{t})} \left\| \nabla \mathfrak{D}_{t} v \right\|_{L^{3}(\Omega_{t})} \left\| \mathfrak{D}_{t}^{2} v \right\|_{L^{2}(\Omega_{t})} \\ &\leq C \left\| \nabla p \right\|_{H^{1}(\Omega_{t})} E(t). \end{split}$$

For the fourth term, we apply (1.1a), (2.2), and integrate by parts to deduce

$$\begin{split} \int_{\Omega_t} \partial_i \mathfrak{D}_t v^j \partial_j \mathfrak{D}_t v^i u dx &= -\int_{\Omega_t} \mathfrak{D}_t v^j \partial_j \nabla \cdot \mathfrak{D}_t v u dx - \int_{\Omega_t} \mathfrak{D}_t v^j \partial_j \mathfrak{D}_t v^i \partial_i u dx \\ &= \int_{\Omega_t} |\nabla \cdot \mathfrak{D}_t v|^2 \, u dx + \int_{\Omega_t} \mathfrak{D}_t v^j \nabla \cdot \mathfrak{D}_t v \partial_j u dx \\ &- \int_{\Omega_t} \mathfrak{D}_t v^j \partial_j \mathfrak{D}_t v^i \partial_i u dx \\ &= \int_{\Omega_t} \partial_i v^j \partial_j v^i \partial_k v^l \partial_l v^k u dx - \sum_j \int_{\Omega_t} \partial_j p \partial_i v^k \partial_k v^i \partial_j u dx \\ &+ \sum_j \int_{\Omega_t} \partial_j p \partial_j \mathfrak{D}_t v^i \partial_i u dx. \end{split}$$

Then, it follows that

$$\begin{split} \left| \int_{\Omega_t} \partial_i \mathfrak{D}_t v^j \partial_j \mathfrak{D}_t v^i u dx \right| &\leq C \left\| \nabla v \right\|_{L^{\infty}(\Omega_t)} \left\| |\nabla v|^3 \right\|_{L^{\frac{6}{5}}(\Omega_t)} \|u\|_{L^{6}(\Omega_t)} \\ &+ C \left\| \nabla p \right\|_{L^{6}(\Omega_t)} \left\| |\nabla v|^2 \right\|_{L^{3}(\Omega_t)} \|\nabla u\|_{L^{2}(\Omega_t)} \\ &+ C \left\| \nabla p \right\|_{L^{6}(\Omega_t)} \left\| \nabla \mathfrak{D}_t v \right\|_{L^{3}(\Omega_t)} \|\nabla u\|_{L^{2}(\Omega_t)} \,. \end{split}$$

Note that by applying Lemma 3.4 and (3.22), one obtains

$$\left\| |\nabla v|^{2} \right\|_{L^{3}(\Omega_{t})} \leq \|v\|_{W^{1,6}(\Omega_{t})}^{2} \leq C \|v\|_{H^{1}(\Omega_{t})} \|v\|_{H^{3}(\Omega_{t})} \leq C(\mathscr{C}_{\dagger}) \|v\|_{H^{3}(\Omega_{t})},$$

$$\left\| |\nabla v|^{3} \right\|_{L^{\frac{6}{5}}(\Omega_{t})} \leq \|v\|_{W^{1,\frac{18}{5}}(\Omega_{t})} \leq C \|v\|_{H^{3}(\Omega_{t})} \|v\|_{L^{6}(\Omega_{t})}^{2} \leq C(\mathscr{C}_{\dagger}) \|v\|_{H^{3}(\Omega_{t})}.$$

$$(3.28)$$

These, combined with the second elliptic estimate in (3.27) and the Sobolev embedding theorem, yield that

$$\left|\int_{\Omega_t} \partial_i \mathfrak{D}_t v^j \partial_j \mathfrak{D}_t v^i u dx\right| \leqslant C(\mathscr{C}_{\dagger}) \left( \|\nabla v\|_{L^{\infty}(\Omega_t)} + \|\nabla p\|_{H^1(\Omega_t)} \right) E(t).$$

Similarly, for the fifth term, from (3.28) and the second elliptic estimate in (3.27), it follows that

$$\begin{split} \left\| \partial_{i} v^{l} \partial_{l} v^{j} \partial_{j} \mathfrak{D}_{t} v^{i} u \right\|_{L^{1}(\Omega_{t})} &\leqslant \left\| |\nabla v|^{2} \right\|_{L^{3}(\Omega_{t})} \left\| \nabla^{2} p \right\|_{L^{2}(\Omega_{t})} \|u\|_{L^{6}(\Omega_{t})} \\ &\leqslant C(\mathscr{C}_{\dagger}) \left\| \nabla^{2} p \right\|_{L^{2}(\Omega_{t})} E(t), \end{split}$$

For the last term, note that by applying integration by parts, one has

$$\int_{\Omega_t} \partial_i v^l \partial_l v^j \partial_j v^k \partial_k v^i u dx = -\int_{\Omega_t} v^l \partial_l v^j \partial_j v^k \partial_k v^i \partial_i u dx - \int_{\Omega_t} v^l \partial_i \left( \partial_l v^j \partial_j v^k \right) \partial_k v^i u dx.$$

Then, we apply (3.28), (3.27), the Sobolev embedding theorem, and (3.22) to obtain

$$\begin{split} \left| \int_{\Omega_{t}} v^{l} \partial_{l} v^{j} \partial_{j} v^{k} \partial_{k} v^{i} \partial_{i} u dx \right| &\leq C \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} \left\| v \right\|_{L^{6}(\Omega_{t})} \left\| |\nabla v|^{2} \right\|_{L^{3}(\Omega_{t})} \left\| \nabla u \right\|_{L^{2}(\Omega_{t})} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} \left\| v \right\|_{H^{3}(\Omega_{t})} \left\| \nabla u \right\|_{L^{2}(\Omega_{t})} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} E(t). \end{split}$$

Similarly, one has

$$\begin{split} \left| \int_{\Omega_{t}} v^{l} \partial_{i} \left( \partial_{l} v^{j} \partial_{j} v^{k} \right) \partial_{k} v^{i} u dx \right| &\leq C \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} \left\| v \right\|_{L^{6}(\Omega_{t})} \left\| \nabla^{2} v \right\|_{L^{6}(\Omega_{t})} \left\| \nabla v \right\|_{L^{2}(\Omega_{t})} \left\| u \right\|_{L^{6}(\Omega_{t})} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} \left\| v \right\|_{H^{3}(\Omega_{t})} \left\| u \right\|_{H^{1}(\Omega_{t})} \\ &\leq C(\mathscr{C}_{\dagger}) \left\| \nabla v \right\|_{L^{\infty}(\Omega_{t})} E(t). \end{split}$$

We conclude that

$$I_1(t) \leqslant C(\mathscr{C}_{\dagger}) \left( \|\nabla v\|_{L^{\infty}(\Omega_t)} + \|\nabla p\|_{H^1(\Omega_t)} \right) E(t),$$

and (3.24) follows.

Finally, recalling (2.3), we estimate the pressure by considering the following elliptic equation

$$\begin{cases} -\Delta p = \partial_i v^j \partial_j v^i, & \text{in } \Omega_t, \\ p = \mathscr{H}_{\partial \Omega_t}, & \text{on } \partial \Omega_t. \end{cases}$$

By standard elliptic estimates, and using the curvature bound in (3.8) and (3.22), we obtain

$$\begin{aligned} \|p\|_{H^{2}(\Omega)} &\leq C\left(\|\Delta p\|_{L^{2}(\Omega)} + \|\mathscr{H}_{\partial\Omega_{t}}\|_{H^{\frac{3}{2}}(\partial\Omega)}\right) \\ &\leq C\left(\|\partial_{i}v^{j}\partial_{j}v^{i}\|_{L^{2}(\Omega)} + C(\mathscr{C}_{\dagger})\right) \\ &\leq C\left(\|\nabla v\|_{L^{2}(\Omega_{t})}\|\nabla v\|_{L^{\infty}(\Omega)} + C(\mathscr{C}_{\dagger})\right) \\ &\leq C(\mathscr{C}_{\dagger})\left(\|\nabla v\|_{L^{\infty}(\Omega_{t})} + 1\right). \end{aligned}$$
(3.29)

Combining the above estimate with (3.19) and (3.24), (3.25) follows, and we have completed the proof of the proposition.

In the following proposition, we establish the equivalence of two energy functionals.

**Proposition 3.12.** Under (3.8), for any time  $t \in (0, T^{\dagger})$ , we have

$$E(t) \leqslant C(\mathscr{C}_{\dagger})(1 + \mathscr{E}(t)). \tag{3.30}$$

*Proof.* Recalling that the  $L^2$ -norm of the vorticity is bounded by  $C(\mathscr{C}_{\dagger})$  as stated in (3.8), by interpolation, we obtain

$$\begin{aligned} \|\nabla \times v\|_{H^{2}(\Omega_{t})}^{2} &\leq C\left(\|\nabla \times v\|_{L^{2}(\Omega_{t})}^{2} + \|\nabla^{2}\left(\nabla \times v\right)\|_{L^{2}(\Omega_{t})}^{2}\right) \\ &\leq C(\mathscr{C}_{\dagger})\left(\left\|\nabla^{2}\left(\nabla \times v\right)\right\|_{L^{2}(\Omega_{t})}^{2} + 1\right) \end{aligned}$$

$$\leq C(\mathscr{C}_{\dagger})\left(\mathscr{E}(t)+1\right). \tag{3.31}$$

We aim to control

$$\|\mathfrak{D}_t v \cdot n\|_{L^2(\partial\Omega_t)}^2, \|\mathfrak{D}_t v\|_{H^{\frac{3}{2}}(\Omega_t)}^2, \text{ and } \|v\|_{H^3(\Omega_t)}$$

Using the divergence theorem, (3.9), (1.1a), (3.8), and (3.29), we have

$$\begin{aligned} \|\mathfrak{D}_{t}v\cdot n\|_{L^{2}(\partial\Omega_{t})}^{2} &= \int_{\partial\Omega_{t}} [(\mathfrak{D}_{t}v\cdot n)\mathfrak{D}_{t}v]\cdot ndS \\ &\leqslant \int_{\Omega_{t}} |(\mathfrak{D}_{t}v\cdot n)\nabla\cdot\mathfrak{D}_{t}v|\,dx + \int_{\Omega_{t}} |\nabla\mathfrak{D}_{t}v\ast\mathfrak{D}_{t}v|\,dx + \int_{\Omega_{t}} |\mathfrak{D}_{t}v\ast\nabla n\ast\mathfrak{D}_{t}v|\,dx \\ &\leqslant C(\mathscr{C}_{\dagger})\left(\|\mathfrak{D}_{t}v\|_{L^{2}(\Omega_{t})}^{2} + \|\nabla\cdot\mathfrak{D}_{t}v\|_{L^{2}(\Omega_{t})}^{2} + \|\nabla\mathfrak{D}_{t}v\|_{L^{2}(\Omega_{t})}^{2} \|\mathfrak{D}_{t}v\|_{L^{2}(\Omega_{t})}\right) \\ &\leqslant C(\mathscr{C}_{\dagger})\left(\|\nabla p\|_{H^{1}(\Omega_{t})}^{2} + \|\nabla\cdot\mathfrak{D}_{t}v\|_{L^{2}(\Omega_{t})}^{2} + 1\right) \\ &\leqslant C(\mathscr{C}_{\dagger})\left(\|\nabla v\|_{L^{\infty}(\Omega_{t})}^{2} + \|\nabla\cdot\mathfrak{D}_{t}v\|_{L^{2}(\Omega_{t})}^{2} + 1\right). \end{aligned}$$
(3.32)

In the last step, we have utilized the interpolation inequality

$$\|\nabla v\|_{L^{\infty}(\Omega_{t})}^{2} \leq C\left(\|v\|_{H^{3}(\Omega_{t})}^{2} + \|v\|_{L^{2}(\Omega_{t})}^{2}\right),$$

along with the  $L^2$ -bound of the velocity given in (3.8).

Noting that  $\nabla \times \mathfrak{D}_t v$  vanishes, (3.32), and applying (2.9) for  $k = \frac{3}{2}$ , it follows that

$$\begin{split} \|\mathfrak{D}_{t}v\|_{H^{\frac{3}{2}}(\Omega_{t})}^{2} &\leqslant C(\mathscr{C}_{\dagger})\left(\|\mathfrak{D}_{t}v\cdot n\|_{H^{1}(\partial\Omega_{t})}^{2} + \|\mathfrak{D}_{t}v\|_{L^{2}(\Omega_{t})}^{2} + \|\nabla\cdot\mathfrak{D}_{t}v\|_{H^{\frac{1}{2}}(\Omega_{t})}^{2}\right) \\ &\leqslant C(\mathscr{C}_{\dagger})\left(\|v\|_{H^{3}(\Omega_{t})}^{2} + \|\nabla\cdot\mathfrak{D}_{t}v\|_{H^{\frac{1}{2}}(\Omega_{t})}^{2} + \mathscr{E}(t) + 1\right). \end{split}$$

By (2.2) and (3.23), we are able to control

$$\left\|\nabla \cdot \mathfrak{D}_{t} v\right\|_{H^{\frac{1}{2}}(\Omega_{t})}^{2} \leqslant C \left\|\left|\nabla v\right|^{2}\right\|_{H^{\frac{1}{2}}(\Omega_{t})}^{2} \leqslant C(\mathscr{C}_{\dagger}) \left\|v\right\|_{H^{3}(\Omega_{t})}^{2}.$$

By combining the above estimates, we obtain

$$\left\|\mathfrak{D}_{t}v\cdot n\right\|_{L^{2}(\partial\Omega_{t})}^{2}+\left\|\mathfrak{D}_{t}v\right\|_{H^{\frac{3}{2}}(\Omega_{t})}^{2}\leqslant C(\mathscr{C}_{\dagger})\left(\left\|v\right\|_{H^{3}(\Omega_{t})}^{2}+\mathscr{E}(t)+1\right)$$

Therefore, it suffices to bound  $||v||^2_{H^3(\Omega_t)}$ . By applying (2.9) for k = 3, the divergence-free condition (1.1b), (3.8) and (3.31), it follows that

$$\begin{split} \|v\|_{H^{3}(\Omega_{t})}^{2} &\leqslant C(\mathscr{C}_{\dagger}) \left( \|v_{n}\|_{H^{\frac{5}{2}}(\partial\Omega_{t})} + \|v\|_{L^{2}(\Omega_{t})} + \|\nabla \cdot v\|_{H^{2}(\Omega_{t})} + \|\nabla \times v\|_{H^{2}(\Omega_{t})} \right) \\ &\leqslant C(\mathscr{C}_{\dagger}) \left( \|\nabla \times v\|_{H^{2}(\Omega_{t})} + C(\mathscr{C}_{\dagger}) \right) \\ &\leqslant C(\mathscr{C}_{\dagger})(\mathscr{E}(t) + 1). \end{split}$$

We conclude that

$$\|\mathfrak{D}_t v \cdot n\|_{L^2(\partial\Omega_t)}^2 + \|\mathfrak{D}_t v\|_{H^{\frac{3}{2}}(\Omega_t)}^2 + \|v\|_{H^3(\Omega_t)}^2 \leqslant C(\mathscr{C}_{\dagger})(\mathscr{E}(t)+1)$$

and (3.30) follows. This completes the proof.

Finally, we complete the classification result stated in Theorem 1.1.

Proof of Theorem 1.1. By applying (3.25) and (3.30), it follows that

$$\frac{d\mathscr{E}}{dt} \leq C(\mathscr{C}_{\dagger}) \left( \|\nabla v\|_{L^{\infty}(\Omega_{t})} + \|\overline{\nabla}v\|_{L^{\infty}(\partial\Omega_{t})} + 1 \right) (1 + \mathscr{E}(t)), \quad 0 < t < T^{\dagger}.$$
(3.33)

Integrating the above, and using (3.3), (3.8), and again (3.30), we have

$$\sup_{0 \le t < T^{\dagger}} \mathscr{E}(t) \le C(\mathscr{C}_{\dagger}, T^{\dagger}) \, (\mathscr{E}(0) + 1) \text{ and } \sup_{0 \le t < T^{\dagger}} E(t) \le C(\mathscr{C}_{\dagger}, T^{\dagger}) \, (\mathscr{E}(0) + 1) \,.$$
(3.34)

Then, from the definition in (3.12) and the fact that  $T^{\dagger} < \infty$ , we have  $v(\cdot, T^{\dagger}) \in H^3(\Omega_{T^{\dagger}})$  and  $\nabla p \in H^{\frac{3}{2}}(\Omega_{T^{\dagger}})$ . By the trace theorem, it holds

$$\left\| \mathscr{H}_{\partial\Omega_{T^{\dagger}}} \right\|_{H^{2}(\partial\Omega_{T^{\dagger}})} < \infty$$

This gives  $\partial \Omega_{T^{\dagger}} \in H^4$  by Lemma 2.7. In other words, the solution  $(v(\cdot, t), \Omega_t)$  does not develop singularities at time  $T^{\dagger}$  and can be extended for some time. This leads to a contradiction, and the proof is complete.

# 4. Proof of Theorem 1.5

In this section, we impose the assumption of simple connectivity to prove Theorem 1.5. Leveraging the boundary regularity in (3.8), we apply [26, Proposition 1 and Corollary 1] to derive the following result, where the constant depends only on  $\mathscr{C}_{\dagger}$ .

**Lemma 4.1.** Let  $\Omega_t$  be a bounded, simply connected domain with  $\partial \Omega_t \in C^{2,\alpha}$ , where  $\alpha > 0$  is sufficiently small, and let the vector field u(x,t) be an  $H^3(\Omega_t)$  solution on the time interval  $[0,T^{\dagger})$  to the following equation

$$\int \nabla \cdot u = 0, \quad in \ \Omega_t, \tag{4.1a}$$

$$\bigcup_{n = 0, \quad on \ \partial\Omega_t.$$
(4.1b)

Then, the following estimate holds

$$\|u\|_{W^{1,\infty}(\Omega_t)} \leqslant C(\mathscr{C}_{\dagger}) \left\{ \left[ 1 + \log^+ \left( \|\nabla \times u\|_{H^2(\Omega_t)} \right) \right] \|\nabla \times u\|_{L^{\infty}(\Omega_t)} + 1 \right\}, \quad 0 < t < T^{\dagger},$$

$$(4.2)$$

where  $\log^+(\cdot) = \max(0, \log(\cdot))$ .

*Proof.* Recalling the uniform interior and exterior ball condition of the free boundary throughout the time interval  $[0, T^{\dagger})$  given in (3.8), i.e.,

$$\inf_{0 \leqslant t < T^{\dagger}} \mathscr{R}(\Omega_t) > \mathscr{C}_{\dagger}^{-1} > 0.$$

This implies that the free boundary will not become sufficiently close or form a self-interaction at the time of existence. Moreover, the uniform  $H^{\frac{7}{2}}$ -regularity of the free boundary in (3.8) ensures that  $\partial \Omega_t$  is uniformly  $C^{2,\alpha}$ -regular. Consequently, the logarithmic estimate constant in (4.2) depends only on the constant  $\mathscr{C}_{\dagger}$ . This concludes the proof.

We are now able to improve the blow-up criterion (5) given in Theorem 1.1.

Proof of Theorem 1.5. Arguing by contradiction, let us assume that the maximal time  $T^{\dagger} < \infty$ . In this case, either the velocity field  $v(\cdot, T^{\dagger}) \notin H^3(\Omega_{T^{\dagger}})$ , or the free boundary  $\partial \Omega_{T^{\dagger}} \notin H^4$ . Assume that none of the first four scenarios in Theorem 1.1 occur. As in the proof of Theorem 1.1 in Section 3, assumptions (3.1), (3.2), and (3.4) hold. However, it is crucial to emphasize that we do not adopt the assumption (3.3). Instead, we make the following assumption

$$\sup_{0 \leqslant t < T^{\dagger}} \|\nabla \times v\|_{L^{2}(\Omega_{t})} + \int_{0}^{T^{\dagger}} \|\nabla \times v\|_{L^{\infty}(\Omega_{t})} dt \leqslant C(\mathscr{C}_{\dagger}).$$

$$(4.3)$$

As a result, the vorticity bound (3.6) is valid, and all the assumptions in (3.8) are satisfied. This enables us to apply the estimates (3.25) and (3.30). Consequently, we are able to derive the estimate (3.33) (prior to applying assumption (3.3)), and obtain

$$\frac{d\log\left(\mathscr{E}+1\right)}{dt} \leqslant C(\mathscr{C}_{\dagger}) \left( \|\nabla v\|_{L^{\infty}(\Omega_{t})} + \left\|\overline{\nabla}v\right\|_{L^{\infty}(\partial\Omega_{t})} + 1 \right), \quad 0 < t < T^{\dagger}.$$

Therefore, from assumption (3.2), it follows that

$$\log\left(\mathscr{E}(t)+1\right) - \log\left(\mathscr{E}(0)+1\right) \leqslant C(\mathscr{C}_{\dagger}) \int_{0}^{t} \left( \|\nabla v\|_{L^{\infty}(\Omega_{s})} + \|\overline{\nabla}v\|_{L^{\infty}(\partial\Omega_{s})} + 1 \right) ds$$
$$\leqslant C(\mathscr{C}_{\dagger}) \int_{0}^{t} \left( \|\nabla v\|_{L^{\infty}(\Omega_{s})} + 1 \right) ds. \tag{4.4}$$

Let w(x,t) be the solution of the following boundary value problem

$$\begin{cases} \Delta w = 0, & \text{in } \Omega_t, \\ \partial_n w = v_n, & \text{on } \partial \Omega_t, \end{cases}$$

where  $0 < t < T^{\dagger}$ . Owing to the regularity of the free boundary in (3.8), we obtain the following Schauder estimates (cf. [42, Lemma 4.3]),

$$\|w\|_{C^{2,\alpha}(\Omega_t)} \leq C(\mathscr{C}_{\dagger}) \|g\|_{C^{1,\alpha}(\partial\Omega_t)}, \quad \alpha \in (0,1).$$

$$(4.5)$$

We define

$$u = v - \nabla w$$

and observe that the divergence-free condition (1.1a) yields

$$\begin{cases} \nabla \cdot u = \nabla \cdot v - \Delta w = 0, & \text{in } \Omega_t, \\ u_n = v_n - \partial_n w = 0, & \text{on } \partial \Omega_t \end{cases}$$

That is, u solves equation (4.1). We also note that

$$\nabla \times u = \nabla \times v, \quad \text{ in } \Omega_t$$

since  $\nabla \times (\nabla w)$  vanishes. Invoking the log-type estimate (4.2), assumptions (3.1), (3.2), the Schauder estimate (4.5), the energy equivalence (3.30), and the Sobolev embedding theorem, we obtain

$$\begin{split} \|v\|_{W^{1,\infty}(\Omega_t)} &\leqslant \|u\|_{W^{1,\infty}(\Omega_t)} + \|\nabla w\|_{W^{1,\infty}(\Omega_t)} \\ &\leqslant C(\mathscr{C}_{\dagger}) \left\{ \left[ 1 + \log^+ \left( \|\nabla \times v\|_{H^2(\Omega_t)} \right) \right] \|\nabla \times v\|_{L^{\infty}(\Omega_t)} + 1 \right\} + C \|w\|_{W^{2,\alpha}(\Omega_t)} \\ &\leqslant C(\mathscr{C}_{\dagger}) \left\{ \left[ \log e + \log^+ \left( \|v\|_{H^3(\Omega_t)} \right) \right] \|\nabla \times v\|_{L^{\infty}(\Omega_t)} + 1 \right\} + C(\mathscr{C}_{\dagger}) \|v_n\|_{C^{1,\alpha}(\partial\Omega_t)} \\ &\leqslant C(\mathscr{C}_{\dagger}) \left( \log E(t) \|\nabla \times v\|_{L^{\infty}(\Omega_t)} + 1 \right) + C(\mathscr{C}_{\dagger}) \\ &\leqslant C(\mathscr{C}_{\dagger}) \left( \log (\mathscr{E}(t) + 1) \|\nabla \times v\|_{L^{\infty}(\Omega_t)} + 1 \right), \end{split}$$

where  $t \in (0, T^{\dagger})$  and the constant  $\alpha \in (0, 1)$  is chosen to be sufficiently small.

Invoking (4.4), it follows that

$$\begin{split} \log\left(\mathscr{E}(t)+1\right) &\leqslant \ \log\left(\mathscr{E}(0)+1\right) + C(\mathscr{C}_{\dagger}) \int_{0}^{t} \left(\|\nabla v\|_{L^{\infty}(\Omega_{s})}+1\right) ds \\ &\leqslant \ \log\left(\mathscr{E}(0)+1\right) + C(\mathscr{C}_{\dagger}) \int_{0}^{t} \left(\log\left(\mathscr{E}(s)+1\right) \|\nabla \times v\|_{L^{\infty}(\Omega_{s})}+1\right) ds, \end{split}$$

where  $t \in (0, T^{\dagger})$ . By applying Grönwall's inequality and assumption (4.3), we have

$$\begin{split} \log\left(\mathscr{E}(t)+1\right) &\leqslant \log\left(\mathscr{E}(0)+1\right) \exp\left(C(\mathscr{C}_{\dagger}) \int_{0}^{T^{\dagger}} \left\|\nabla \times v\right\|_{L^{\infty}(\Omega_{s})} + 1 ds\right) \\ &\leqslant \exp\left[C(\mathscr{C}_{\dagger}) \left(T^{\dagger}+1\right)\right] \log\left(\mathscr{E}(0)+1\right). \end{split}$$

Invoking (3.30) once more, we conclude that

$$\sup_{0 \leqslant t < T^{\dagger}} \log E(t) \leqslant C(\mathscr{C}_{\dagger}) \sup_{0 \leqslant t < T^{\dagger}} \log \left(\mathscr{E}(t) + 1\right) \leqslant C(\mathscr{C}_{\dagger}) \exp\left(T^{\dagger} + 1\right) \log \left(\mathscr{E}(0) + 1\right),$$

and (3.34) follows. As discussed at the end of the proof of Theorem 1.1, this leads to a contradiction. Therefore, assumption (4.3) is violated, and we have

$$\sup_{0 \le t < T^{\dagger}} \|\nabla \times v\|_{L^{2}(\Omega_{t})} + \int_{0}^{T^{\dagger}} \|\nabla \times v\|_{L^{\infty}(\Omega_{t})} dt = \infty.$$

$$(4.6)$$

We claim that (4.6) is equivalent to (1.8). Without loss of generality, we consider the case where

$$\sup_{0 \leqslant t < T^{\dagger}} \| \nabla \times v \|_{L^{2}(\Omega_{t})} = \infty \text{ and } \int_{0}^{T^{\dagger}} \| \nabla \times v \|_{L^{\infty}(\Omega_{t})} dt < \infty$$

**m**+

Recalling that on the time interval  $[0, T^{\dagger})$ ,

$$v \in C([0, T^{\dagger}); H^3(\Omega_t))$$
 and  $\partial \Omega_t \in C([0, T^{\dagger}); H^4)$ ,

it follows that

$$\nabla \times v \in C([0, T^{\dagger}); L^2(\Omega_t))$$

since

$$\| (\nabla \times v) (\cdot, t) - (\nabla \times v) (\cdot, s) \|_{L^2} \leqslant C \| v(\cdot, t) - v(\cdot, s) \|_{H^3}, \quad 0 \leqslant t, s < T^{\dagger}.$$

As a consequence,  $\|\nabla \times v\|_{L^2(\Omega_t)}$  is a continuous function of the time t. Recalling that  $T^{\dagger} < \infty$ , we deduce that

 $\sup_{0 \le t < T^{\dagger}} \|\nabla \times v\|_{L^{2}(\Omega_{t})} = \infty \text{ is equivalent to } \limsup_{t \to T^{\dagger}} \|\nabla \times v\|_{L^{2}(\Omega_{t})} = \infty.$ 

This concludes the proof.

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# Declarations

**Conflict of interest** The authors declare that there is no conflict of interest.

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