Calculus of Variations

Exponential stability of a free boundary problem with spherical symmetry for a gas bubble immersed in a bounded incompressible liquid

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Abstract

This paper is mainly concerned with the free boundary problem for an approximate model (for example, arising from the study of sonoluminescence) of a gas bubble of finite mass enclosed within a bounded incompressible viscous liquid, accounting for surface tensions at both the gas-liquid interface and the external free surface of the entire gas-liquid region. It is found that any regular spherically symmetric steady-state solution is characterized by a positive root of a ninth-degree polynomial for which the existence and uniqueness are proved and a one-to-one correspondence between equilibria and pairs of gas mass and liquid volume is established, by a rescaling argument. We prove that these equilibria exhibit nonlinear and exponential asymptotic stability under small perturbations that conserve gas mass and liquid volume, and an equilibrium solution acts as a local minimizer of the energy functional under perturbations that are allowed to be large, as long as the ratio of perturbations to equilibrium remains small, with the proportionality constant determined by the adiabatic constant. Moreover, we construct a global center manifold to apply the center manifold theory. Our results apply to gases and liquids of all sizes. Furthermore, we derive the optimal exponential decay rate for small liquid volumes by analyzing the spectrum bounds of the associated linear operator and show that decreasing the gas mass or increasing the temperature can accelerate the convergence rate, a behavior not seen in unbounded liquid scenarios.

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1 Introduction

Consider the free boundary problem of a gas bubble immersed in an incompressible viscous liquid with finite gas mass and liquid volume. The external liquid dynamics are governed by the following incompressible Navier–Stokes equations:

$$\partial_t(\rho_l v_l) + \operatorname{div}(\rho_l v_l \otimes v_l) = \operatorname{div}(-p_l I + S_l(v_l)), \qquad \text{in } \Omega_l(t), \qquad (1.1a)$$

$$\operatorname{div} v_l = 0, \qquad \text{in } \Omega_l(t), \qquad (1.1b)$$

$$\rho_l c_l \left[\partial_t T_l + (v_l \cdot \nabla) T_l\right] = \operatorname{div}(\kappa_l \nabla T_l) + S_l(v_l) : \nabla v_l, \quad \text{in } \Omega_l(t), \qquad (1.1c)$$

where $\Omega_l(t) \subset \mathbb{R}^3$ is a bounded connected domain that varies with time. The constant $\rho_l > 0$ represents the density of the liquid. The variables v_l, p_l, and T_l denote the velocity field, pressure, and temperature of the liquid, respectively. The tensor product is denoted by $\otimes$, and I represents the identity matrix. The viscous stress tensor is given by $S_l(v_l) = 2\mu_l D(v_l)$, where $\mu_l > 0$ is the dynamic viscosity and $D(v)$ is the symmetric part of the gradient of the velocity field, defined as $(\nabla v + (\nabla v)^\top)/2$. The constants c_l and κ_l denote the specific heat capacity and thermal conductivity of the liquid, respectively. Additionally, $v_l \cdot \nabla$ represents the directional derivative, and $A : B$ denotes the trace of the matrix product $AB^\top$. In the following, vectors, matrices, and tensors will be represented using bold typefaces.

The internal gas is governed by the following compressible Navier–Stokes equations in a simply connected domain (see, e.g., [9]):

$$\partial_t \rho + \operatorname{div}(\rho v_g) = 0, \qquad \text{in } \Omega_g(t), \qquad (1.2a)$$

$$\partial_t(\rho v_g) + \operatorname{div}(\rho v_g \otimes v_g) = \operatorname{div}(-p_g I + S_g(v_g)), \qquad \text{in } \Omega_g(t), \qquad (1.2b)$$

$$\rho T_g (\partial_t s + v_g \cdot \nabla s) = \operatorname{div}(\kappa \nabla T_g) + S_g(v_g) : \nabla v_g, \quad \text{in } \Omega_g(t), \qquad (1.2c)$$

$$p_g = \mathfrak{R} T_g \rho, \qquad \text{in } \Omega_g(t), \qquad (1.2d)$$

$$s = \mathfrak{c} \log (p_g/\rho^\gamma), \qquad \text{in } \Omega_g(t), \qquad (1.2e)$$

where the variables ρ, v_g, p_g, T_g, and s denote the density, velocity field, pressure, temperature, and entropy per unit of mass of the gas, respectively. The viscous tensor

$$S_g(v_g) = 2\mu \left(D(v_g) - \frac{1}{3}(\operatorname{div} v_g)I \right) + \zeta(\operatorname{div} v_g)I,$$

where μ denotes the dynamic viscosity, and ζ represents the bulk viscosity. The positive constants $\kappa, \mathfrak{c}$ and $\mathfrak{R}$ denote the thermal conductivity, specific heat capacity, and the ratio of the ideal gas constant to the molar mass, respectively. The adiabatic constant $\gamma = 1 + \mathfrak{R}/\mathfrak{c}$, which is 5/3 for monoatomic gases and 7/5 for diatomic gases (see, e.g., [3]), is also included. Moreover, (1.2d) and (1.2e) follow from Boyle's law, Joule's second law, and the second law of thermodynamics for ideal gases.

We take into account the surface tension acting on the gas-liquid interface $\partial\Omega_g(t)$ as well as on the free boundary of the entire gas-liquid region, specifically $\partial\Omega_l(t) \setminus \partial\Omega_g(t)$. The boundary conditions on the gas-liquid interface read

$$v_l \cdot \boldsymbol{\nu} = v_g \cdot \boldsymbol{\nu} = v_-, \qquad \text{on } \partial\Omega_g(t), \qquad (1.3a)$$

$$\boldsymbol{\nu} \cdot [-p_l I + S_l(v_l) - (-p_g I + S_g(v_g))] = -\sigma H \boldsymbol{\nu}, \quad \text{on } \partial\Omega_g(t), \qquad (1.3b)$$

$$T_g = T_l, \qquad \text{on } \partial\Omega_g(t), \qquad (1.3c)$$

where $\boldsymbol{\nu}$ denotes the unit outer normal to $\partial\Omega_g(t)$, and v_- is the normal velocity of the interface. The mean curvature is given by $H = -\overline{\nabla} \cdot \boldsymbol{\nu}$, where $\overline{\nabla}$ represents the tangential

gradient (see, e.g., [8]), and the constant $\sigma > 0$ is the surface tension coefficient. Similarly, on the external free surface, it holds

$$\mathrm{v}_l \cdot \boldsymbol{\nu} = v_+, \qquad \text{on } \partial\Omega_l(t) \setminus \partial\Omega_g(t), \qquad (1.4\mathrm{a})$$
$$\boldsymbol{\nu} \cdot (-p_l\mathrm{I} + \mathrm{S}_l(\mathrm{v}_l)) = \bar{\sigma}H\boldsymbol{\nu}, \quad \text{on } \partial\Omega_l(t) \setminus \partial\Omega_g(t), \qquad (1.4\mathrm{b})$$
$$T_l = T_c, \qquad \text{on } \partial\Omega_l(t) \setminus \partial\Omega_g(t), \qquad (1.4\mathrm{c})$$

where $\boldsymbol{\nu}$ denotes the unit outer normal, and v_+ represents the normal velocity of the free surface. The constant $\bar{\sigma} > 0$ denotes the surface tension coefficient of the external free boundary $\partial\Omega_l(t) \setminus \partial\Omega_g(t)$, with $\sigma \neq \bar{\sigma}$ in general. Moreover, the temperature outside the gas-liquid region remains constant at the temperature T_c, which is positive and provides the continuity condition (1.4c). Finally, we assume that the compatibility conditions hold for the initial data $\mathrm{v}_l(\cdot, 0), T_l(\cdot, 0), \rho(\cdot, 0), \mathrm{v}_g(\cdot, 0)$ and $p_g(\cdot, 0)$ in the initial gas-liquid domain $\overline{\Omega_g(0) \cup \Omega_l(0)}$.

Since we are considering the scenario where the liquid volume is finite, the divergence-free condition (1.1b) ensures that the liquid volume remains constant

$$|\Omega_l(t)| \equiv V \in (0, \infty), \quad t \geqslant 0.$$

Moreover, by the conservation of mass (1.2a), we have

$$\int_{\Omega_g(t)} \rho(x, t)dx = \int_{\Omega_g(0)} \rho_0(x)dx, \quad t > 0, \qquad (1.5)$$

where $\rho_0(\cdot) = \rho(\cdot, 0)$ is the initial gas density.

1.1 The approximate model of the full free boundary problem (1.1)–(1.4)

This paper is primarily concerned with the stability of equilibrium (time-independent) solutions for the following approximate model (1.6)–(1.8), which arises, for instance, in the study of sonoluminescence [1, 2]. This model reads

$$\rho_l (\partial_t \mathrm{v}_l + \mathrm{v}_l \cdot \nabla\mathrm{v}_l) - \mu_l\Delta\mathrm{v}_l + \nabla p_l = 0, \quad \text{in } \Omega_l(t), \qquad (1.6\mathrm{a})$$
$$\mathrm{div}\, \mathrm{v}_l = 0, \qquad \text{in } \Omega_l(t), \qquad (1.6\mathrm{b})$$
$$T_l \equiv T_c, \qquad \text{in } \Omega_l(t), \qquad (1.6\mathrm{c})$$

where T_c is a prescribed temperature consistent with the boundary condition (1.4c),

$$\partial_t\rho + \mathrm{div}(\rho\mathrm{v}_g) = 0, \qquad \text{in } \Omega_g(t), \qquad (1.7\mathrm{a})$$
$$p_g = p_g(t), \qquad \text{in } \Omega_g(t), \qquad (1.7\mathrm{b})$$
$$\rho T_g (\partial_t s + \mathrm{v}_g \cdot \nabla s) = \mathrm{div}(\kappa\nabla T_g), \quad \text{in } \Omega_g(t), \qquad (1.7\mathrm{c})$$
$$p_g = \mathfrak{R}T_g\rho, \qquad \text{in } \Omega_g(t), \qquad (1.7\mathrm{d})$$
$$s = \mathfrak{c}\log\left(p_g/\rho^\gamma\right), \qquad \text{in } \Omega_g(t), \qquad (1.7\mathrm{e})$$

and the boundary conditions

$$v_l \cdot \boldsymbol{\nu} = v_g \cdot \boldsymbol{\nu} = v_-, \qquad \text{on } \partial\Omega_g(t), \qquad (1.8a)$$

$$(p_g - p_l)\boldsymbol{\nu} + 2\mu_l\boldsymbol{\nu} \cdot D(v_l) = -\sigma H\boldsymbol{\nu}, \qquad \text{on } \partial\Omega_g(t), \qquad (1.8b)$$

$$T_g = T_l, \qquad \text{on } \partial\Omega_g(t), \qquad (1.8c)$$

$$v_l \cdot \boldsymbol{\nu} = v_+, \qquad \text{on } \partial\Omega_l(t) \setminus \partial\Omega_g(t), \qquad (1.8d)$$

$$- p_l\boldsymbol{\nu} + 2\mu_l\boldsymbol{\nu} \cdot D(v_l) = \bar{\sigma} H\boldsymbol{\nu}, \qquad \text{on } \partial\Omega_l(t) \setminus \partial\Omega_g(t). \qquad (1.8e)$$

The above approximation gas system (1.7) can be found in [3, 9]. As noted in [3, Section 2], the gas pressure equation (1.7b) is derived under the high sound velocity assumption. This implies that disturbances in the fluid propagate quickly through the gas region, allowing the momentum equation (1.2b) to be simplified to $\nabla p_g \equiv 0$ in $\Omega_g(t)$.

For gas dynamic (1.7), if we substitute (1.7e) into (1.7c) and eliminate the temperature T_g by using (1.7d), it holds

$$(\partial_t + v_g \cdot \nabla)\left[\mathfrak{c}\left(\log p_g - \log \rho^\gamma\right)\right] = \kappa\Delta\left(\rho^{-1}\right), \quad \text{in } \Omega_g(t).$$

Upon simplification by applying (1.7a), we obtain

$$\dot{p}_g/p_g = \kappa\mathfrak{c}^{-1}\Delta\left(\rho^{-1}\right) - \gamma\operatorname{div} v_g, \quad \text{in } \Omega_g(t), \qquad (1.9)$$

where we have denoted $\dot{p}_g = \partial_t p_g$. This implies that the divergence of the gas velocity is determined by both the gas density and the pressure at the gas-liquid interface. Therefore, system (1.7) is equivalent to

$$\partial_t\rho + \operatorname{div}(\rho v_g) = 0, \qquad \text{in } \Omega_g(t), \qquad (1.10a)$$

$$p_g = p_g(t), \qquad \text{in } \Omega_g(t), \qquad (1.10b)$$

$$\dot{p}_g/p_g = \kappa\mathfrak{c}^{-1}\Delta\left(\rho^{-1}\right) - \gamma\operatorname{div} v_g, \quad \text{in } \Omega_g(t). \qquad (1.10c)$$

Furthermore, by eliminating the divergence of the gas velocity, one has

$$\partial_t\rho = \frac{\kappa}{\gamma\mathfrak{c}}\left(\Delta\log\rho - \frac{|\nabla\rho|^2}{\rho^2}\right) - v_g \cdot \nabla\rho + \frac{\dot{p}_g}{\gamma p_g}\rho, \quad \text{in } \Omega_g(t). \qquad (1.11)$$

From the form of the equivalent gas system (1.10), it is evident that without imposing the irrotational condition ($\operatorname{curl} v_g = 0$) or any symmetry assumptions, the approximate model (1.6)–(1.8), in general, may not have a unique solution. Even for steady-state solutions, except for the spherically symmetric equilibria, other solutions involve rigid rotations. Additionally, when considering steady-state solutions, the liquid viscosity and the presence of surface tension require that both the gas bubble and the gas-liquid region exhibit spherical symmetry, as per Alexandrov's theorem. A recent study [10] shows that assuming the flows are irrotational, the shape of any steady-state gas is exclusively spherical by the surface tension alone.

Remark 1 We discuss the steady-state solutions of the full model (1.1)–(1.4). Specifically, surface tension and liquid viscosity alone do not guarantee spherically symmetric steady-state solutions. For instance, we cannot exclude the possibility of a rotational liquid velocity of the form $v_l(x) = (c_1, c_2, c_3)^\top + (w_1, w_2, w_3)^\top \times (x_1, x_2, x_3)^\top$, where $(c_1, c_2, c_3)^\top$

and $(w_1, w_2, w_3)^\top \neq 0$ are constant vectors. However, it remains unknown whether such a liquid velocity can be used to construct a nonspherical steady-state gas state that satisfies the corresponding boundary conditions. Moreover, if the liquid is assumed to be irrotational, surface tension alone ensures a spherical outer boundary. On the other hand, when both surface tension and liquid viscosity are present, and both gas and liquid are irrotational, it remains unclear whether the steady-state configuration must be spherical. Nevertheless, under the additional assumption of an inviscid gas bubble, it can be shown that both the equilibrium gas region and the entire gas-liquid domain must be spherical. Thus, classifying equilibrium solutions to the full model (1.1)–(1.4) presents significant challenges. In particular, the existence of nonspherically symmetric equilibrium solutions remains an open problem, which we intend to investigate in future research.

Therefore, our focus will be on spherically symmetric solutions to the free boundary problem (1.6)–(1.8). For a spherically symmetric solution, the gas domain $\Omega_g(t)$ is assumed to be a ball centered at the origin with radius $R(t)$. Given the density ρ of the gas in the spherical region B_R, we define the corresponding mass as

$$\mathrm{M}[\rho, R] = \int_{B_R} \rho(x)dx.$$

1.2 Main results

We will study the stability of the spherically symmetric equilibria of free boundary problem (1.6)–(1.8) under the assumption of spherical symmetry. Under this assumption, system (1.6)–(1.8) is equivalent to problem (2.1) for the density and radius of the gas bubble, as will be shown in Proposition 1. Based on this, we establish a one-to-one correspondence between the equilibrium gas state $(\rho, R) \equiv (\rho_\dagger, R_\dagger)$ and the mass-volume pair (M, V) for system (1.6)–(1.8).

Theorem 1 *There exists a smooth bijective mapping $(M, V) \in (0, \infty) \times (0, \infty) \mapsto (R_\dagger[M, V], \rho_\dagger[M, V])$, as defined in (2.11) and (2.12), such that any regular spherically symmetric equilibrium solution to system (1.6)–(1.8) is uniquely determined by the mass-volume pair (M, V).*

Remark 2 If no confusion arises, we use subscript $\dagger$ to indicate the steady-state solutions or other related quantities. For instance, $R_\dagger$ represents the equilibrium gas radius and $E_\dagger$ denotes the equilibrium energy.

Remark 3 Based on Theorem 1, we will show in Appendix A that, for the original full free boundary problem (1.1)–(1.4), the mass-volume pairs are the sole determinants of the regular spherically symmetric equilibria provided that the liquid temperature remains constant.

To state our stability result, we introduce the following manifold of equilibria to system (2.1), which is parameterized by the gas mass and liquid volume

$$\Sigma = \{(\rho(x,t), R(t)) \equiv (\rho_\dagger[M,V], R_\dagger[M,V]) \mid 0 < M, V < \infty\}, \tag{1.12}$$

where $\rho_\dagger$ and $R_\dagger$ are smooth functions defined in Theorem 1.

The main result of this paper is to demonstrate the nonlinear and exponential asymptotic stability of spherically symmetric equilibria Σ to small perturbations. This is significant because the spherically symmetric solution to system (1.6)–(1.8) can be reconstructed from the density and radius of the gas bubble. Therefore, we can conclude that the equilibria of the free boundary problem (1.6)–(1.8) are also nonlinearly and exponentially asymptotically stable. It should be noted that in the following theorem, B_1 refers to the unit open ball and the norm $\|\cdot\|_{C_y^{2+2\alpha}(B_1)}$ is defined in Appendix B, where the global existence and the uniqueness of spherically symmetric solutions are also provided.

Theorem 2 *Given any mass of the gas, $M > 0$, and any liquid volume $V > 0$. For free boundary problem (2.1) with the liquid volume V, there exists a constant $\eta_0 > 0$ such that the following holds:*

(i) For any initial data $(\rho_0, R_0, \dot{R}_0)$ such that the mass $\mathrm{M}[\rho_0, R_0] = M$ and

$$\|\rho_0(R_0 y) - \rho_\dagger[M,V]\|_{C_y^{2+2\alpha}(B_1)} + |R_0 - R_\dagger[M,V]| + |\dot{R}_0| \leqslant \eta_0, \tag{1.13}$$

where $\alpha \in (0, 1/2)$ and the equilibrium $(\rho_\dagger[M,V], R_\dagger[M,V]) \in \Sigma$, the global-in-time solution $(\rho(r,t), R(t))$ satisfies

$$\begin{aligned}
&\|\rho(R(t)y,t) - \rho_\dagger[M,V]\|_{C_y^{2+2\alpha}(B_1)} \\
&\quad + |R(t) - R_\dagger[M,V]| + |\dot{R}| + |\ddot{R}| + |\dddot{R}| \to 0, \quad \text{as } t \to \infty.
\end{aligned} \tag{1.14}$$

(ii) The global solution $(\rho(r,t), R(t))$ converges to the equilibrium at an exponential rate. More precisely, there exists a constant $\varpi_\dagger > 0$, such that

$$\begin{aligned}
&\|\rho(R(t)y,t) - \rho_\dagger[M,V]\|_{C_y^{2+2\alpha}(B_1)} \\
&\quad + |R(t) - R_\dagger[M,V]| + |\dot{R}| = O\left(e^{-\varpi_\dagger t}\right), \quad \text{as } t \to \infty.
\end{aligned}$$

Therefore, any spherical equilibrium solution of free boundary problem (1.6)–(1.8) is nonlinearly and exponentially asymptotically stable.

1.3 Background and history

The dynamics of gas bubbles immersed in a liquid with finite volume is a multifaceted subject that intersects fluid dynamics, thermodynamics, and material science. This topic has significant implications in various fields such as industrial engineering [7], environmental science [13], and biomedical applications [5, 21]. Other discussion of bubble phenomena and applications can be found in the review article [16].

Gas bubbles in liquids exhibit complex behavior due to the interplay between pressure, surface tension, and the surrounding fluid's viscosity. When a gas bubble is immersed in a liquid, the dynamics of the bubble are influenced by both the properties of the gas and the

liquid, as well as the interactions at their interface. The gas-liquid interface plays a crucial role in bubble dynamics. The behavior at this interface is influenced by surface tension and the pressure difference between the gas inside the bubble and the surrounding liquid. The movement and deformation of this interface can affect the stability and motion of the bubble. The finite volume constraint adds an additional layer of complexity, as it implies that the bubble dynamics are affected by altering the pressure distribution and potential interactions with surfaces within the liquid.

In the unbounded liquid scenario, where $\Omega_l(t) = \mathbb{R}^3 \setminus \overline{\Omega_g(t)}$, Prosperetti [15] first considered the approximate model (1.6), (1.7), and (1.8a)–(1.8c) for the original problem (1.1)–(1.3) (without external boundary conditions (1.4)). Later, Biro and Velázquez [3] proved the global existence of solutions in Hölder space for initial data near the spherically symmetric equilibria, as well as the Lyapunov stability of the equilibria under small mass-preserving perturbations, assuming that the liquid is inviscid on the gas-liquid free interface (i.e., $p_g \boldsymbol{\nu} - p_l \boldsymbol{\nu} = -\sigma H \boldsymbol{\nu}$ in (1.8b)). They also considered the liquid pressure far away from the gas bubble, represented by an external forcing term $p_\infty(t) = \lim_{x \to \infty} p_l(x, t)$. Subsequently, Lai and Weinstein [9] proved the exponentially asymptotic stability of the manifold of spherically symmetric equilibria, taking into account the liquid viscosity on the gas-liquid interface. They also demonstrated the existence and uniqueness of an exponentially asymptotically stable periodic spherically symmetric pulsating solution [11], given a small-amplitude, time-periodic $p_\infty(t)$.

When considering compressible liquids, Shapiro and Weinstein focused on the linearized problem of a gas bubble immersed in an inviscid compressible liquid. They demonstrated that the system exhibits exponential point-wise decay towards a family of equilibria [18]. In the case of an unbounded liquid governed by compressible Navier–Stokes equations and a homogeneous spherical gas bubble following the polytropic gas law, Zhao and Zou proved the existence of global solutions and the asymptotic stability of the spherical equilibria in this free boundary problem [22].

From the physical point of view, it is natural and of fundamental importance to consider the case when the gas-liquid region is bounded, for which we are not aware of any relevant rigorous theoretical studies, though some numerical investigations are available. Indeed, Siegel conducted simulations in two dimensions, assuming that the gas motion is governed by the Stokes equations with a free boundary [20]. Lozinski and Romerio presented numerical results for the case that both the gas and the liquid are incompressible, and the entire gas-liquid system occupies a bounded, time-independent domain [14]. For the problem considered in this paper, the possibility of collision of the two free surfaces is one of issues we have to resolve, which does not occur in the case when the liquid region is unbounded without an exterior boundary, as most studied in the references mentioned earlier.

1.4 Novelties and structure of the paper

The novelties of this study are as follows.

(i) We consider the scenario of a gas bubble immersed in a liquid with a finite volume. Unlike in an unbounded liquid, the presence of distinct boundaries in a finite liquid volume imposes additional constraints on the bubble's movement and interactions. Specifically, the bubble dynamics are influenced by a well-defined free boundary where

surface tension plays a crucial role. In this bounded gas-liquid system, the steady-state gas-liquid region must assume a spherical shape.

(ii) For general surface tension coefficients $\sigma \neq \bar{\sigma}$, we analyze the unique positive real root of a ninth-order polynomial to establish that spherical equilibrium solutions correspond uniquely to specific pairs of gas mass M and liquid volume V. Indeed, we prove that the steady-state radius of the gas bubble $R_\dagger \in (\sqrt{I}/\sqrt{1+\bar{\sigma}/\sigma}, \sqrt{I})$, where $I = \frac{3\Re T_c M}{8\pi\sigma}$, and $R_\dagger$ is a positive real solution to the following polynomial (see (2.10) in Sect. 2)

$$\mathbb{P}(R) = \left(\frac{\bar{\sigma}^3}{\sigma^3} + 1\right) R^9 - 3IR^7 + \frac{3V}{4\pi}R^6 + 3I^2R^5 - \frac{9V}{4\pi}IR^4 - I^3R^3 + \frac{9V}{4\pi}I^2R^2 - \frac{3V}{4\pi}I^3.$$

As a ninth-order polynomial, it is quite challenging to determine the distribution of the roots or directly analyze its monotonicity. Indeed, the polynomial $\mathbb{P}(R)$ fails to exhibit monotonicity for general coefficients $\bar{\sigma}, \sigma, I$ and V. In order to resolve this issue, we observe that, by the scaling $R = w\sqrt{I}, w \in (1/\sqrt{1+\bar{\sigma}/\sigma}, 1)$, we can separate a distinct positive power factor $I^{\frac{9}{2}}w^9$, i.e.,

$$\mathbb{P}(w\sqrt{I}) = -\left[\left(\frac{1}{\sqrt{I}w^3} - \frac{1}{\sqrt{I}w}\right)^3 \left(I^{\frac{3}{2}}w^3 + \frac{3V}{4\pi}\right) - \frac{\bar{\sigma}^3}{\sigma^3}\right] I^{\frac{9}{2}}w^9 \triangleq -\mathbb{L}(w)I^{\frac{9}{2}}w^9.$$

The advantage of doing so is that we can obtain the monotonicity of $\mathbb{L}(w)$, which leads us to prove the existence and uniqueness of the steady-state gas radius, and determine its upper and lower bounds in terms of given physical quantities $\bar{\sigma}, \sigma, I$ and V, as another benefit of this strategy. In contrast, when the liquid region is unbounded [9], the steady-state gas radius satisfies a cubic equation $4\pi p_\infty R_\dagger^3 + 8\pi\sigma R_\dagger^2 - 3\Re T_c M = 0$ with a constant $p_\infty > 0$, and the existence and uniqueness of its positive root can be readily deduced.

(iii) We analyze the energy dissipation for general solutions (not necessarily symmetric) to the free boundary problem (1.6)–(1.8), which is important to the asymptotic stability analysis in general and extends the results from the unbounded liquid case presented in [9, Proposition 7.4]. Our results demonstrate that the equilibrium solution serves as a local minimizer of the energy functional even when subjected to relatively large perturbations. Specifically, this means that perturbations in the gas density can be accommodated up to a certain proportion of the equilibrium state, with the proportionality constant depending solely on the adiabatic constant γ. This result is notable because it shows that the extent of acceptable disturbance is independent of the liquid volume, making it particularly relevant for scenarios involving small liquid volumes.

(iv) When applying the central manifold theory, we construct general global central manifolds based on the algebraic equations of the steady-state solution parameterized by the mass-volume pair. This approach generalizes the local central manifold identified in [9, Lemma 9.6].

(v) For sufficiently small liquid volumes, we nearly achieve the optimal exponential decay rate. Additionally, we demonstrate that reducing the gas mass or increasing the temperature can accelerate the convergence — an effect not observed with an infinite liquid volume. For spherically symmetric solutions, the convergence rate is solely determined by the spectrum of a linear operator, specifically $\sup\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{sp}(\mathcal{L}) \setminus \{0\}\}$ (see

Sect. 4.2 for details). By analyzing its upper bound and estimating the lower bound of a specific negative eigenvalue, we establish the existence of a constant $\Theta_0 \in (0,1)$ such that

$$-\pi^2\chi < \sup\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{sp}(\mathcal{L}) \setminus \{0\}\} \leqslant -\Theta_0\pi^2\chi,$$

where $\pi^2\chi \sim \sqrt{T_c}/\sqrt{M}$, revealing a clear correlation between the exponential decay index, gas mass M, and external temperature T_c.

The rest of the paper is structured as follows. In Sect. 2, we reduce free boundary problem (1.6)–(1.8) to system (2.1) and specify the equilibria under the spherical symmetry assumption. Section 3 demonstrates the energy dissipation and shows that the steady-state solution acts as a local minimizer of the energy functional under proportional perturbations. In Sect. 4, we establish the nonlinear and exponential asymptotic stability of the equilibria using center manifold theory, as detailed in Appendix C. The calculation of equilibria for problem (1.1)–(1.4) is provided in Appendix A, while Appendix B covers the well-posedness and Lyapunov stability of system (2.1). Appendix D includes necessary verifications for applying the center manifold theory.

2 Spherically symmetric solutions and equilibria

In this section, we will show that, under the assumption of spherical symmetry, the free boundary problem (1.6)–(1.8) can be simplified to a system involving the gas density and the bubble radius, provided that the liquid volume and other parameters are fixed. Furthermore, we will demonstrate that the regular spherically symmetric equilibrium solution is uniquely determined by the gas mass and liquid volume.

We assume that the velocity fields of the gas and the liquid are spherically symmetric (e.g., $\mathrm{v}_g(x,t) = v_g(|x|,t)x/|x|, x \neq 0$), and the other scalar variables are radial (e.g., $\rho(x,t) = \rho(|x|,t)$), to rewrite problem (1.6)–(1.8) as follows.

Proposition 1 *Given the liquid volume V, solving the regular spherically symmetric solution to system (1.6)–(1.8) reduces to an initial boundary value problem for the bubble radius R(t) and the gas density $\rho(r,t) = \rho(x,t)$, where $r = |x| \leqslant R(t)$. More precisely, for $r \leqslant R(t)$ and $t > 0$, we have*

$$\partial_t \rho(r,t) = \frac{\kappa}{\gamma\mathfrak{c}}\Delta_r \log \rho(r,t) + \frac{\dot{p}}{\gamma p(t)}\left(\frac{r\partial_r\rho(r,t)}{3} + \rho(r,t)\right), \tag{2.1a}$$

$$\dot{R} = -\frac{\kappa}{\gamma\mathfrak{c}}\frac{\partial_r\rho(R(t),t)}{\rho(R(t),t)^2} - \frac{\dot{p}}{3\gamma p(t)}R(t), \tag{2.1b}$$

$$p(t) = \mathfrak{R}T_c\rho(R(t),t), \tag{2.1c}$$

$$\rho(R(t),t) = \frac{1}{\mathfrak{R}T_c}\left\{4\mu_l\left(\frac{\dot{R}}{R(t)} - \frac{\dot{\bar{R}}}{\bar{R}(t)}\right) + \frac{2\sigma}{R(t)} + \frac{2\bar{\sigma}}{\bar{R}(t)}\right.$$
$$\left. + \rho_l\left[\left(R(t) - \frac{R(t)^2}{\bar{R}(t)}\right)\ddot{R} + \left(\frac{3}{2} - \frac{2R(t)}{\bar{R}(t)} + \frac{R(t)^4}{2\bar{R}(t)^4}\right)\dot{R}^2\right]\right\}, \tag{2.1d}$$

where $\overline{R} = \sqrt[3]{R^3 + (3V/4\pi)}$ represents the external radius of the entire gas-liquid region. The initial condition reads $(\rho(\cdot, 0), R(0), \dot{R}(0)) = (\rho_0, R_0, \dot{R}_0)$. Above, we omit the subscript for the gas pressure p_g, and denote the operator $\Delta_r(\cdot) = r^{-2}\partial_r(r^2\partial_r(\cdot))$ with $r = |x|$.

Proof Recalling that the gas and gas-liquid regions are spheres $B_{R(t)}$ and $B_{\overline{R}(t)}$ centered at the origin, respectively, we can assume that the spherically symmetric velocities are $v_l(x,t) = v_l(r,t)x/r$ and $v_g(x,t) = v_g(r,t)x/r$ for $x \neq 0$, where v_l and v_g are scalar functions. Additionally, p_l, ρ, T_g and s are radial functions. Direct calculations yield

$$\rho_l\partial_t v_l = \mu_l\left(\Delta_r v_l - 2v_l r^{-2}\right) - \rho_l v_l \partial_r v_l - \partial_r p_l, \qquad R(t) \leqslant r \leqslant \overline{R}(t), \qquad (2.2\text{a})$$

$$r^{-2}\partial_r(r^2 v_l) = 0, \qquad R(t) \leqslant r \leqslant \overline{R}(t), \qquad (2.2\text{b})$$

$$\partial_t\rho + r^{-2}\partial_r(\rho r^2 v_g) = 0, \qquad r \leqslant R(t), \qquad (2.3\text{a})$$

$$\dot{p}p^{-1} = \kappa\mathfrak{c}^{-1}r^{-2}\partial_r\left(r^2\partial_r\left(\rho^{-1}\right)\right) - \gamma r^{-2}\partial_r(r^2 v_g), \qquad r \leqslant R(t), \qquad (2.3\text{b})$$

and the boundary conditions (1.8) become

$$v_l(R(t),t) = v_g(R(t),t) = \dot{R}(t), \quad v_l(\overline{R}(t),t) = \dot{\overline{R}}(t), \qquad (2.4\text{a})$$

$$p(t) - p_l(R(t),t) + 2\mu_l\partial_r v_l(R(t),t) = 2\sigma R(t)^{-1}, \qquad (2.4\text{b})$$

$$-p_l(\overline{R}(t),t) + 2\mu_l\partial_r v_l(\overline{R}(t),t) = -2\overline{\sigma}\overline{R}(t)^{-1}, \qquad (2.4\text{c})$$

$$T(R(t),t) = T_c, \qquad (2.4\text{d})$$

where we have used the fact that the curvature of a sphere B_R is $-2R^{-1}$.

Clearly that (2.1c) follows from (1.6c), (1.7b), (1.7e) and (1.8c). Then, the divergence-free condition (2.2b) and kinematic boundary condition (2.4a) imply that for $t > 0$,

$$v_l(r,t) = R(t)^2\dot{R}(t)r^{-2} = \overline{R}(t)^2\dot{\overline{R}}(t)r^{-2}, \quad R(t) \leqslant r \leqslant \overline{R}(t). \qquad (2.5)$$

We also note that $R^2\dot{R} = \overline{R}^2\dot{\overline{R}}$. Omitting the variable t and substituting (2.5) into (2.2a), we have $(2R\dot{R}^2 + R^2\ddot{R})r^{-2} = 2R^4\dot{R}^2 r^{-5} - \rho_l^{-1}\partial_r p_l$. By integrating, we deduce that for $R(t) \leqslant r \leqslant \overline{R}(t)$ and $t > 0$,

$$p_l(r,t) - p_l(R,t) = \rho_l\left[R^4\dot{R}^2\left(R^{-4} - r^{-4}\right)/2 - \left(2R\dot{R}^2 + R^2\ddot{R}\right)\left(R^{-1} - r^{-1}\right)\right].$$

Moreover, (2.5) implies $\partial_r v_l(r,t) = -2R^2\dot{R}r^{-3} = -2\overline{R}^2\dot{\overline{R}}r^{-3}$. As a result, $\partial_r v_l(R(t),t) = -2\dot{R}R^{-1}$ and $\partial_r v_l(\overline{R}(t),t) = -2\overline{R}^{-1}\dot{\overline{R}}$. Therefore, we obtain $p(t) - p_l(R(t),t) - 4\mu_l\dot{R}R^{-1} = 2\sigma R^{-1}$ and $-p_l(\overline{R}(t),t) - 4\mu_l\overline{R}^{-1}\dot{\overline{R}} = -2\overline{\sigma}\overline{R}^{-1}$. Combining these calculations, it follows that

$$p_l(r,t) = p(t) - 4\mu_l\dot{R}R^{-1} - 2\sigma R^{-1} + \rho_l\left[\frac{1}{2}\dot{R}^2 - \frac{1}{2}R^4\dot{R}^2 r^{-4} - \frac{r-R}{r}\left(2\dot{R}^2 + R\ddot{R}\right)\right],$$

where $R \leqslant r \leqslant \overline{R}$. Setting $r = \overline{R}$, one has

$$p(t) = 4\mu_l \left(\frac{\dot{R}}{R} - \frac{\dot{\overline{R}}}{\overline{R}} \right) + \frac{2\sigma}{R} + \frac{2\bar{\sigma}}{\overline{R}} - \rho_l \left[\frac{\dot{R}^2}{2} - \frac{R^4 \dot{R}^2}{2\overline{R}^4} - \frac{\overline{R} - R}{\overline{R}} \left(2\dot{R}^2 + R\ddot{R} \right) \right]. \quad (2.6)$$

From the gas dynamics in (1.10) and (1.11), for $0 \leqslant r \leqslant R(t)$, we have

$$v_g = \frac{\kappa}{\gamma\mathfrak{c}} \partial_r \left(\frac{1}{\rho} \right) - \frac{r}{3\gamma} \frac{\dot{p}}{p} \text{ and } \partial_t \rho = \frac{\kappa}{\gamma\mathfrak{c}} \Delta_r \log \rho + \frac{\dot{p}}{3\gamma p} r \partial_r \rho + \frac{\dot{p}}{\gamma p} \rho. \quad (2.7)$$

Thus, equation (2.1a) follows. Taking the time derivative of both sides of (2.1c), we obtain $\dot{p} p^{-1} = \partial_t \rho(R,t) \rho(R,t)^{-1} + \dot{R} \partial_r \rho(R,t) \rho(R,t)^{-1}$. Evaluating (2.7) at $r = R(t)$ and using the kinematic boundary condition (2.4a), we derive (2.1b). Finally, combining (2.1c) and (2.6) leads to the boundary condition (2.1d). This completes the proof. $\square$

We note that a regular solution $(\rho(r,t), R(t))$ to system (2.1) corresponds a spherically symmetric solution $(v_l, p_l, \rho, v_g, p_g, T_g, s)$ to system (1.6)–(1.8), where the velocity v_l is given by formula (2.5).

In the following proof and the rest of the paper, the external radius will always be denoted by

$$\overline{R} = \sqrt[3]{R^3 + \overline{V}},$$

where $\overline{V} = 3V/4\pi$ represents the modified liquid volume and R denotes a generic radius of the gas bubble.

Proof (*Proof of Theorem* 1) According to Proposition 1, it suffices to compute the steady solutions to system (2.1). Setting $\partial_t \rho = \dot{R} = 0$, we obtain from (2.1a) and (2.1b) that $R(t) \equiv R_\dagger$ (equilibrium gas radius), $\Delta \log \rho = 0$ in $B_{R_\dagger}$, and $\partial_r \rho(R_\dagger) = 0$. Then it follows that $\rho(r) \equiv \rho_\dagger$ (equilibrium gas density). This, combined with (2.1d) yields $\rho_\dagger = \frac{2}{\mathfrak{R}T_c} \left(\frac{\sigma}{R_\dagger} + \frac{\bar{\sigma}}{\overline{R}_\dagger} \right)$. Moreover, the conservative mass of the gas can be expressed by the pair $(\rho_\dagger, R_\dagger)$, i.e., $M = \frac{4\pi}{3} \rho_\dagger R_\dagger^3$. Therefore, for any mass-volume pair $(M, V) \in (0, \infty)^2$, the equilibrium $(\rho_\dagger, R_\dagger)$ is determined by the following algebraic equations:

$$\frac{4\pi}{3} \rho_\dagger R_\dagger^3 = M \text{ and } \rho_\dagger = \frac{2}{\mathfrak{R}T_c} \left(\frac{\sigma}{R_\dagger} + \frac{\bar{\sigma}}{\overline{R}_\dagger} \right). \quad (2.8)$$

We denote

$$I = \frac{3\mathfrak{R}T_c M}{8\pi\sigma} \text{ and } \sigma_{\text{ratio}} = \frac{\bar{\sigma}}{\sigma} > 0.$$

Then, (2.8) is equivalent to

$$\frac{1}{R_\dagger} + \frac{\sigma_{\text{ratio}}}{\overline{R}_\dagger} = \frac{I}{R_\dagger^3} \text{ or } \frac{\sigma_{\text{ratio}}^3}{R_\dagger^3 + \overline{V}} = \left(\frac{I - R_\dagger^2}{R_\dagger^3} \right)^3, \quad (2.9)$$

where $\overline{V} = 3V/4\pi$. The equilibrium radius $R_\dagger$ (if it exists) is exactly a positive real root to the polynomial

$$\mathbb{P}(x) = (\sigma_{\text{ratio}}^3 + 1)x^9 - 3Ix^7 + \overline{V}x^6 + 3I^2x^5 - 3I\overline{V}x^4 - I^3x^3 + 3I^2\overline{V}x^2 - I^3\overline{V}. \quad (2.10)$$

At the same time, $R_\dagger$ satisfies $R_\dagger \in (\sqrt{I}/\sqrt{1 + \sigma_{\text{ratio}}}, \sqrt{I})$ by using (2.9).

From these observations, we introduce a function $\mathbb{K}(w) = \mathbb{P}(w\sqrt{I})$, where the variable $w \in [1/\sqrt{1 + \sigma_{\text{ratio}}}, 1]$, and a direct calculation shows

$$\mathbb{K}(w) = -\left[\left(I^{-\frac{1}{2}}w^{-3} - I^{-\frac{1}{2}}w^{-1}\right)^3 \left(I^{\frac{3}{2}}w^3 + \overline{V}\right) - \sigma_{\text{ratio}}^3\right] I^{\frac{9}{2}}w^9 \triangleq -\mathbb{L}(w)I^{\frac{9}{2}}w^9.$$

We note that the function $\mathbb{L}(w)$ is strictly decreasing, since for $w \in \left(1/\sqrt{1 + \sigma_{\text{ratio}}}, 1\right)$, it holds

$$\mathbb{L}'(w) = 3I^{-\frac{3}{2}}\left[-2I^{\frac{3}{2}}w^3 - \overline{V}(3 - w^2)\right]\left(w^2 - 1\right)^2 w^{-10} < 0.$$

Furthermore, we have

$$\mathbb{L}(1/\sqrt{1 + \sigma_{\text{ratio}}}) = \overline{V}I^{-\frac{3}{2}}(1 + \sigma_{\text{ratio}})^{\frac{3}{2}}\sigma_{\text{ratio}}^3 > 0 \text{ and } \mathbb{L}(1) = -\sigma_{\text{ratio}}^3 < 0.$$

In the above, we emphasize that $\mathbb{L}(1/\sqrt{1 + \sigma_{\text{ratio}}})$ is positive regardless of the modified volume $\overline{V}$.

We conclude that $\mathbb{L}$ has a unique positive solution in $(1/\sqrt{1 + \sigma_{\text{ratio}}}, 1)$. Therefore, the same result holds for $\mathbb{K}(w)$. Coming back to $\mathbb{P}(x)$, the ninth-degree polynomial, it possesses a unique positive root $R_\dagger$ within $(\sqrt{I}/\sqrt{1 + \sigma_{\text{ratio}}}, \sqrt{I})$, which is uniquely determined by M and V, with the other parameters being constants. Thus, the map $R_\dagger[M, V]$ is well-defined and satisfies

$$\sqrt{\frac{3\Re T_c M}{8\pi\sigma}} \Big/ \sqrt{1 + \frac{\bar{\sigma}}{\sigma}} < R_\dagger[M, V] < \sqrt{\frac{3\Re T_c M}{8\pi\sigma}}. \quad (2.11)$$

The smoothness of the map $R_\dagger[M, V]$ is the consequence of the smooth dependence of a simple root to the polynomial $\mathbb{P}$ on its coefficients.

Once we determine the equilibrium gas radius $R_\dagger[M, V]$, the equilibrium density is expressed as

$$\rho_\dagger[M, V] = \frac{3M}{4\pi R_\dagger[M, V]^3} = \frac{2}{\Re T_c}\left(\frac{\sigma}{R_\dagger[M, V]} + \frac{\bar{\sigma}}{\overline{R}_\dagger[M, V]}\right), \quad (2.12)$$

and the equilibrium gas pressure p follows by using (2.1c). Then, we can recover the corresponding steady state solution to system (1.6)–(1.8). That is, $v_g = v_l \equiv 0$, $T_g \equiv T_c$, $s \equiv \mathfrak{c}\log(p/\rho_\dagger^\gamma)$, and $p_l \equiv 2\bar{\sigma}/R_\dagger$.

In turn, given any equilibrium state $(\rho_\dagger, R_\dagger)$, we can specify the mass of the gas and liquid volume by (2.9) since the coefficients of both $\overline{V}$ and $(R_\dagger^2 - I)^3$ are nonzero. More precisely, there exists a one-to-one correspondence between $(\rho_\dagger, R_\dagger)$ and (M, V)

$$(M, V) = \frac{4\pi R_\dagger[M, V]^3}{3}\left(\rho_\dagger[M, V], \left[\frac{2\sigma^2}{(\Re T_c \rho_\dagger[M, V]R_\dagger[M, V] - 2\sigma)\bar{\sigma}}\right]^3 - 1\right). \quad (2.13)$$

This completes the proof. $\qquad\qquad\square$

Remark 4 If there is no surface tension on the external free boundary (i.e., $\bar{\sigma} = 0$), equations (2.8) reduce to a quadratic function. In this case, the equilibrium radius simplifies to $R_\dagger = \sqrt{\frac{3\Re T_c M}{8\pi\sigma}}$, which is independent of the liquid volume. Consequently, (2.13) no longer holds. Nevertheless, $R_\dagger$ remains the unique positive root of (2.10) and corresponds to the critical case of (2.11).

The local well-posedness, global well-posedness, and the Lyapunov stability for system (2.1) are detailed in Appendix B. In the next section, after introducing the physical energy, we will prove the energy dissipation and show that any equilibrium solution acts as a local minimizer of the energy functional.

3 Energy dissipation and local minimizers of the energy functional

The physical energy of a general solution (without any symmetric assumption) to the approximate model (1.6)–(1.8) is given by $E(t) = E_1(t) + E_2(t)$. The energy $E_1(t)$ denotes the Helmholtz free energy (see, e.g., [12])

$$E_1(t) = \mathfrak{c}\int_{\Omega_g(t)}\rho T_g dx - T_c\int_{\Omega_g(t)}\rho s dx,$$

The energy $E_2(t)$ includes both the kinetic energy of the liquid and the surface energy, encompassing the gas-liquid interface as well as the external free boundary of the gas-liquid region

$$E_2(t) = \frac{1}{2}\int_{\Omega_l(t)}\rho_l|v_l|^2 dx + \sigma\int_{\partial\Omega_g(t)}dS + \bar{\sigma}\int_{\partial\Omega_l(t)\setminus\partial\Omega_g(t)}dS.$$

3.1 Energy dissipation law for general solutions to problem (1.6)–(1.8)

Recalling that the liquid volume is finite and $\overline{\Omega_g(t) \cup \Omega_l(t)}$ is bounded, we compute the associated energy dissipation.

Proposition 2 *For free boundary problem (1.6)–(1.8), we have the following energy dissipation law:*

$$\frac{d}{dt} E(t) = -\frac{\mu_l}{2} \int_{\Omega_l(t)} \left| \nabla \mathbf{v}_l + \nabla \mathbf{v}_l^\top \right|^2 dx - \kappa T_c \int_{\Omega_g(t)} \frac{|\nabla \rho|^2}{\rho^2} dx. \tag{3.1}$$

Proof Given that boundary conditions (1.8a) and (1.8d) are satisfied, we recall the transport formulas for the time-dependent regions

$$\frac{d}{dt} \int_{\Omega_g(t)} f dx = \int_{\Omega_g(t)} (\partial_t + \mathbf{v}_g \cdot \nabla) f + \operatorname{div} \mathbf{v}_g f dx,$$

$$\frac{d}{dt} \int_{\Omega_l(t)} f dx = \int_{\Omega_l(t)} (\partial_t + \mathbf{v}_l \cdot \nabla) f dx,$$

and the analogous formula on the liquid moving surface $\partial\Omega_l(t)$, which encompasses both the gas-liquid interface $\partial\Omega_g(t)$ and the external free boundary $\partial\Omega_l(t) \setminus \partial\Omega_g(t)$ (see, e.g., [19])

$$\frac{d}{dt} \int_{\partial\Omega_l(t)} f dS = \int_{\partial\Omega_l(t)} (\partial_t + \mathbf{v}_l \cdot \nabla) f + \overline{\nabla} \cdot \mathbf{v}_l f dS.$$

These, along with (1.6a), (1.8b), (1.8e), integration by parts formula, and the divergence theorem on the moving surface, yield

$$\dot{E}_2(t)$$
$$= \rho_l \int_{\Omega_l(t)} (\partial_t \mathbf{v}_l + \mathbf{v}_l \cdot \nabla \mathbf{v}_l) \cdot \mathbf{v}_l dx + \sigma \int_{\partial\Omega_g(t)} \overline{\nabla} \cdot \mathbf{v}_l dS + \bar{\sigma} \int_{\partial\Omega_l(t)\backslash\partial\Omega_g(t)} \overline{\nabla} \cdot \mathbf{v}_l dS$$

$$= \mu_l \int_{\Omega_l(t)} \Delta \mathbf{v}_{l,i} \mathbf{v}_l^i dx - \int_{\Omega_l(t)} \partial_i p_l \mathbf{v}_l^i dx + \sigma \int_{\partial\Omega_g(t)} H \left[\mathbf{v}_l \cdot (-\boldsymbol{\nu}) \right] dS$$

$$+ \bar{\sigma} \int_{\partial\Omega_l(t)\backslash\partial\Omega_g(t)} H \left(\mathbf{v}_l \cdot \boldsymbol{\nu} \right) dS$$

$$= - \mu_l \int_{\Omega_l(t)} |\nabla \mathbf{v}_l|^2 dx + \mu_l \int_{\partial\Omega_g(t)} (-\boldsymbol{\nu}^j)\partial_j \mathbf{v}_{l,i} \mathbf{v}_l^i dS$$

$$+ \mu_l \int_{\partial\Omega_l(t)\backslash\partial\Omega_g(t)} \boldsymbol{\nu}^j \partial_j \mathbf{v}_{l,i} \mathbf{v}_l^i dS - \int_{\partial\Omega_g(t)} -\boldsymbol{\nu}_i p_l \mathbf{v}_l^i dS - \int_{\partial\Omega_l(t)\backslash\partial\Omega_g(t)} \boldsymbol{\nu}_i p_l \mathbf{v}_l^i dS$$

$$- \sigma \int_{\partial\Omega_g(t)} H \left(\mathbf{v}_l \cdot \boldsymbol{\nu} \right) dS + \bar{\sigma} \int_{\partial\Omega_l(t)\backslash\partial\Omega_g(t)} H \left(\mathbf{v}_l \cdot \boldsymbol{\nu} \right) dS$$

$$= - \mu_l \int_{\Omega_l(t)} |\nabla \mathbf{v}_l|^2 dx + \int_{\partial\Omega_g(t)} \left(p_l \boldsymbol{\nu} - \mu_l \boldsymbol{\nu}^j \partial_j \mathbf{v}_l - \sigma H \boldsymbol{\nu} \right) \cdot \mathbf{v}_l dS$$

$$- \int_{\partial\Omega_l(t)\backslash\partial\Omega_g(t)} \left(p_l \boldsymbol{\nu} - \mu_l \boldsymbol{\nu}^j \partial_j \mathbf{v}_l + \bar{\sigma} H \right) \cdot \boldsymbol{\nu} dS$$

$$= - \mu_l \int_{\Omega_l(t)} |\nabla \mathbf{v}_l|^2 dx + p_g \int_{\partial\Omega_g(t)} \mathbf{v}_l \cdot \boldsymbol{\nu} dS + \mu_l \int_{\partial\Omega_g(t)} \partial_j \mathbf{v}_l^k \mathbf{v}_l^j \boldsymbol{\nu}_k dS$$

$$- \mu_l \int_{\partial\Omega_l(t)\backslash\partial\Omega_g(t)} \partial_j \mathbf{v}_l^k \mathbf{v}_l^j \boldsymbol{\nu}_k dS$$

$$= - \mu_l \int_{\Omega_l(t)} \nabla \mathbf{v}_l : \left(\nabla \mathbf{v}_l + \nabla \mathbf{v}_l^\top \right) dx + p_g \int_{\partial\Omega_g(t)} \mathbf{v}_l \cdot \boldsymbol{\nu} dS.$$

For the energy $E_1(t)$, we apply the transport formula, gas system (1.7), boundary conditions (1.8a) and (1.8c) to obtain

$$\dot{E}_1(t) = \frac{\mathfrak{c}}{\mathfrak{R}} \dot{p}_g \int_{\Omega_g(t)} 1 dx + \frac{\mathfrak{c}}{\mathfrak{R}} p_g \int_{\Omega_g(t)} \operatorname{div} \mathbf{v}_g dx - T_c \int_{\Omega_g(t)} \rho \left(\partial_t s + \mathbf{v}_g \cdot \nabla s \right) dx$$

$$= \frac{\dot{p}_g}{\gamma - 1} \int_{\Omega_g(t)} 1 dx + \frac{p_g}{\gamma - 1} \int_{\Omega_g(t)} \operatorname{div} \mathbf{v}_g dx - T_c \kappa \int_{\Omega_g(t)} \frac{\Delta T_g}{T_g} dx$$

$$= \frac{\dot{p}_g}{\gamma - 1} \int_{\Omega_g(t)} 1 dx + \frac{p_g}{\gamma - 1} \int_{\partial\Omega_g(t)} \mathbf{v}_g \cdot \boldsymbol{\nu} dS$$

$$- (-1) T_c \kappa \int_{\Omega_g(t)} - \frac{|\nabla T_g|^2}{T_g^2} dx - T_c \kappa \int_{\partial\Omega_g(t)} \frac{\nabla T_g \cdot \boldsymbol{\nu}}{T_g} dS$$

$$= \frac{\dot{p}_g}{\gamma - 1} \int_{\Omega_g(t)} 1 dx - T_c \kappa \int_{\Omega_g(t)} \frac{|\nabla \rho|^2}{\rho^2} dx + \frac{p_g}{\gamma - 1} \int_{\partial\Omega_g(t)} \mathbf{v}_l \cdot \boldsymbol{\nu} dS$$

$$- \kappa \int_{\partial\Omega_g(t)} \nabla T_g \cdot \boldsymbol{\nu} dS,$$

where we have used the relation $\gamma = 1 + \mathfrak{R}/\mathfrak{c}$. Noting that

$$\kappa \int_{\partial \Omega_g(t)} \nabla T_g \cdot \boldsymbol{\nu} dS = \kappa \int_{\Omega_g(t)} \Delta T_g dx = \frac{p_g}{\mathfrak{R}} \int_{\Omega_g(t)} \kappa \Delta \left(\rho^{-1} \right) dx,$$

and recalling (1.9), we deduce

$$\kappa \int_{\partial \Omega_g(t)} \nabla T_g \cdot \boldsymbol{\nu} dS = \frac{\mathfrak{c}}{\mathfrak{R}} \dot{p}_g \int_{\Omega_g(t)} 1 dx + \frac{\mathfrak{c}\gamma}{\mathfrak{R}} p_g \int_{\Omega_g(t)} \operatorname{div} v_g dx$$

$$= \frac{\dot{p}_g}{\gamma - 1} \int_{\Omega_g(t)} 1 dx + \frac{\gamma}{\gamma - 1} p_g \int_{\partial \Omega_g(t)} v_l \cdot \boldsymbol{\nu} dS.$$

Collecting the above calculations, energy dissipation (3.1) follows since the algebraic identity $\nabla v_l : \left(\nabla v_l + \nabla v_l^\top \right) = \left| \nabla v_l + \nabla v_l^\top \right|^2 /2$ holds.   □

3.2 Equilibria as local minimizers of the energy functional relative to proportional perturbations

To establish the nonlinear stability, we will show that any equilibrium density-radius pair $(\rho_\dagger[M,V], R_\dagger[M,V])$, determined by the mass-volume pair (M, V), serves as a local minimizer of the energy functional $E(t)$ with respect to mass-persevering and volume-invariant perturbations.

For this problem, in the case of an infinite liquid volume and an external forcing term $p_\infty(t)$, Biro and Velázquez [3, Lemma 4.2] used Taylor's theorem with Peano's remainder form to expand the total energy at the equilibrium state up to quadratic terms and derived the coercivity energy estimate. Based on this estimate, the Lyapunov stability was established [3, Theorem 4.1]. More recently, Lai and Weinstein [9, Theorem 7.5] extended these results to scenarios with more general far-field pressure conditions. The key coercive energy estimate [9, (7.17)] was obtained under sufficiently small perturbations, which depend on the mass of the gas bubble.

We extend the results to the case where the liquid volume V is finite. This is a significant improvement as we have removed the assumptions, regarding the upper and lower bounds of the gas density and the bubble radius, made in previous studies (such as $C^{-1} \leqslant \rho, R \leqslant C$ in [3, 9]), and now allow for relatively larger perturbations, as long as they are within a certain proportion of the equilibrium density (see (3.2)). It should also be noted that the constant δ_0 in (3.2) is independent of the constants T_c, ρ_l, the equilibrium state $(\rho_\dagger, R_\dagger)$, the mass M, and the volume V. This is a crucial result that enables us to prove the main stability results, specifically (1.14) in Theorem 2.

The strategy we have adopted is as follows. We first consider a specific proportion of disturbance to the equilibrium density. Since the mass of the gas and the volume of the liquid remain constant, the amplitude of the disturbance in the bubble radius will be determined by the change in gas density. Next, we utilize Taylor's theorem to approximate the energy functional near the equilibrium state. We then apply the Lagrangian form of the remainder, which provides the most precise expression for the nonlinear terms. This is because the bulk terms I_7, I_8, and I_9 (in the second-order derivative below) are related to the liquid volume.

Peano's form of the remainder cannot provide the necessary control, as the liquid volume can be arbitrarily small.

Theorem 3 *There exists a constant $\delta_0 > 0$ depending only on γ, such that the following holds: given any equilibrium $(\rho_\dagger, R_\dagger) = (\rho_\dagger[M, V], R_\dagger[M, V])$ with mass M and liquid volume V, for any mass-persevering and volume-invariant spherically symmetric state $(\rho(r, t), R(t), \overline{R}(t) = \left(R(t)^3 + \overline{V}\right)^{1/3})$, such that*

$$\left\| \frac{\rho(\cdot, t) - \rho_\dagger}{\rho_\dagger} \left(1 + |\log \rho_\dagger|\right) \right\|_{L^\infty(B_{R(t)})} \leqslant \delta_0, \tag{3.2}$$

then, we have

$$E(t) \geqslant E_\dagger + \frac{1}{4} \left\{ M\mathfrak{c} T_c \left(\frac{\bar{\rho}(1, t) - \rho_\dagger}{\rho_\dagger} - \frac{3}{4\pi \rho_\dagger} \int_{B_1} \bar{\rho}(y, t) - \rho_\dagger dy \right)^2 \right.$$

$$+ \frac{\rho_l R_\dagger^5}{4\pi \rho_\dagger^2} \left(1 - \frac{R_\dagger}{\overline{R}_\dagger}\right) \left(\int_{B_1} \dot{\bar{\rho}}(y, t) dy \right)^2 + \frac{\mathfrak{R} T_c R_\dagger^3}{3\rho_\dagger} \int_{B_1} |\bar{\rho}(y, t) - \rho_\dagger|^2 \, dy \tag{3.3}$$

$$\left. + \frac{R_\dagger^3}{\pi \rho_\dagger^2} \left[\frac{\sigma}{2R_\dagger} + \frac{\bar{\sigma}}{\overline{R}_\dagger} \left(1 - \frac{R_\dagger^3}{2\overline{R}_\dagger^3}\right) \right] \left(\int_{B_1} \bar{\rho}(y, t) - \rho_\dagger dy \right)^2 \right\},$$

where $E(t) = E[\bar{\rho}(\cdot, t), R(t)]$ with $\bar{\rho}(y, t) = \rho(R(t)|y|, t)$, $E_\dagger = E[\rho_\dagger, R_\dagger]$, and $t > 0$ is arbitrary. Moreover, $(\rho(r, t), R(t))$ is not restricted to be a solution of system (2.1).

Proof Setting $x = R(t)y$ for $y \in \overline{B_1}$, the domain $\overline{B_{R(t)}}$ is fixed to $\overline{B_1}$. Defining $\bar{\rho}(|y|, t) = \rho(R(t)|y|, t)$ and using (1.7d), we have $p = \mathfrak{R} T_c \bar{\rho}(1, t)$. Then, the energy $E_1(t)$ becomes

$$E_1(t) = \frac{4\pi \mathfrak{c}}{3\mathfrak{R}} p_g R^3 - \mathfrak{c} T_c \, \mathrm{M}[\rho, R] \log p_g + \mathfrak{c}\gamma T_c \int_{B_R} \rho \log \rho \, dx$$

$$= \frac{4\pi \mathfrak{c} T_c}{3} \bar{\rho}(1, t) R(t)^3 - \mathfrak{c} T_c M \log \left[\mathfrak{R} T_c \bar{\rho}(1, t)\right] + \mathfrak{c}\gamma T_c R(t)^3 \int_{B_1} \bar{\rho}(x, t) \log \bar{\rho}(x, t) dx.$$

Combined with the energy $E_2(t)$ in the spherically symmetric scenario, i.e.,

$$E_2(t) = 2\pi \rho_l R^3 \dot{R}^2 - 2\pi \rho_l R^4 \dot{R}^2 / \overline{R} + 4\pi \sigma R(t)^2 + 4\pi \bar{\sigma} \overline{R}(t)^2,$$

we obtain that $E(t) = E[\bar{\rho}(x, t), R(t)]$ is a functional of $(\bar{\rho}(x, t), R(t))$ since $\overline{R}(t)$ can be expressed by $R(t)$ and V

$$E(t) = \frac{4\pi \mathfrak{c} T_c}{3} \bar{\rho}(1, t) R^3 - \mathfrak{c} T_c M \log(\mathfrak{R} T_c) - \mathfrak{c} T_c M \log \bar{\rho}(1, t) + \mathfrak{c}\gamma T_c R^3 \int_{B_1} \bar{\rho} \log \bar{\rho}$$

$$+ 2\pi \rho_l R^3 \dot{R}^2 - 2\pi \rho_l R^4 \dot{R}^2 / \overline{R} + 4\pi (\sigma R^2 + \bar{\sigma} \overline{R}^2),$$

where we omit dx and abbreviate the notations $R = R(t), \bar{\rho} = \bar{\rho}(x,t)$, etc.

Fix any $t > 0$, we set $\bar{\rho}_\delta(\cdot, t) = \rho_\dagger + \delta\varrho(\cdot, t)$ where $0 \leqslant \delta \leqslant 1$ and $\|\varrho(\cdot, t)\|_{L^\infty(B_1)} \leqslant \rho_\dagger/2$. Then, there exists a unique $R_\delta = R_\delta(t) > 0$ such that $R_\delta(t)^3 \int_{B_1} \bar{\rho}_\delta(y, t) = M$, where $M = \frac{4\pi}{3}\rho_\dagger R_\dagger^3$. Define $\rho_\delta(x, t) = \bar{\rho}_\delta(x/R_\delta, t)$, and we have

$$\|\rho_\delta(\cdot, t) - \rho_\dagger\|_{L^\infty(B_{R_\delta(t)})} \leqslant \delta\rho_\dagger/2. \tag{3.4}$$

Also, (ρ_δ, R_δ) satisfies $\mathrm{M}[\rho_\delta(\cdot, t), R_\delta(t)] = M$, i.e.,

$$\frac{4\pi\rho_\dagger}{3}R_\dagger^3 = M = R_\delta(t)^3 \int_{B_1} (\rho_\dagger + \delta\varrho(y, t)) = R_\delta(t)^3 \left(\frac{4\pi\rho_\dagger}{3} + \delta\int_{B_1} \varrho(y, t) \right).$$

Then, we have

$$R_\delta(t)^3 = \frac{R_\dagger^3}{1 + \delta\phi(t)} \text{ where } \phi(t) = \frac{3}{4\pi\rho_\dagger}\int_{B_1} \varrho(y, t). \tag{3.5}$$

Also, it holds

$$\dot{R}_\delta(t) = \frac{-\delta R_\dagger \dot{\phi}(t)}{3\left(1 + \delta\phi(t)\right)^{\frac{4}{3}}} \text{ and } \bar{R}_\delta(t) = \left(\frac{R_\dagger^3}{1 + \delta\phi(t)} + \overline{V} \right)^{\frac{1}{3}}.$$

Next, we define $f(\delta) = f(\delta, t) = E[\rho_\dagger + \delta\varrho(\cdot, t), R_\delta(t),]$ for $\delta \in [0, 1]$ and it holds

$$f(\delta) = \frac{4\pi\mathfrak{c}T_c}{3}(\rho_\dagger + \delta\varrho(1, t))\frac{R_\dagger^3}{1 + \delta\phi(t)} - \mathfrak{c}T_c M \log(\mathfrak{R}T_c) - \mathfrak{c}T_c M \log(\rho_\dagger + \delta\varrho(1, t))$$

$$+ \frac{\mathfrak{c}\gamma T_c R_\dagger^3}{1 + \delta\phi(t)}\int_{B_1} (\rho_\dagger + \delta\varrho)\log(\rho_\dagger + \delta\varrho) + 2\pi\rho_l \frac{R_\dagger^3}{1 + \delta\phi(t)}\left(\frac{-\delta R_\dagger \dot{\phi}(t)}{3\left(1 + \delta\phi(t)\right)^{\frac{4}{3}}} \right)^2$$

$$- 2\pi\rho_l \left(\frac{R_\dagger}{(1 + \delta\phi(t))^{\frac{1}{3}}} \right)^4 \left(\frac{-\delta R_\dagger \dot{\phi}(t)}{3\left(1 + \delta\phi(t)\right)^{\frac{4}{3}}} \right)^2 \left(\frac{R_\dagger^3}{1 + \delta\phi(t)} + \overline{V} \right)^{-\frac{1}{3}}$$

$$+ 4\pi\left[\sigma\left(\frac{R_\dagger}{(1 + \delta\phi(t))^{\frac{1}{3}}} \right)^2 + \bar{\sigma}\left(\frac{R_\dagger^3}{1 + \delta\phi(t)} + \overline{V} \right)^{\frac{2}{3}} \right].$$

From Taylor's theorem with Lagrangian remainder, we have $f(\delta) = f(0) + f'(0)\delta + f''(0)\delta^2/2 + f'''(\xi(\delta))\delta^3/6$, where $0 \leqslant \xi(\delta) \leqslant \delta$. Direct calculations show that

$$f'(0) = \frac{4\pi R_\dagger^3}{3} \mathfrak{c} T_c \varrho(1,t) - \frac{4\pi R_\dagger^3}{3} \mathfrak{c} T_c \rho_\dagger \phi - \frac{4\pi R_\dagger^3}{3} \mathfrak{c} T_c \varrho(1,t) - \mathfrak{c}\gamma T_c R_\dagger^3 \phi \int_{B_1} \rho_\dagger \log \rho_\dagger$$

$$+ \mathfrak{c}\gamma T_c R_\dagger^3 \int_{B_1} (\varrho \log(\rho_\dagger) + \varrho) - \frac{8}{3}\pi \left[\sigma R_\dagger^2 \phi + \bar{\sigma} R_\dagger^3 \phi \left(R_\dagger^3 + \overline{V} \right)^{-\frac{1}{3}} \right]$$

$$= -\frac{4\pi R_\dagger^3}{3} \mathfrak{c} T_c \rho_\dagger \frac{3}{4\pi\rho_\dagger} \int_{B_1} \varrho - \mathfrak{c}\gamma T_c \frac{4\pi R_\dagger^3}{3} \frac{3}{4\pi\rho_\dagger} \int_{B_1} \varrho\rho_\dagger \log \rho_\dagger + \mathfrak{c}\gamma T_c R_\dagger^3 \int_{B_1} \varrho$$

$$+ \mathfrak{c}\gamma T_c R_\dagger^3 \log \rho_\dagger \int_{B_1} \varrho - \frac{4\pi}{3} R_\dagger^3 \left(2\sigma/R_\dagger + 2\bar{\sigma}/\overline{R}_\dagger \right) \frac{3}{4\pi\rho_\dagger} \int_{B_1} \varrho$$

$$= (\gamma - 1)\mathfrak{c} T_c R_\dagger^3 \int_{B_1} \varrho - R_\dagger^3 \rho_\dagger^{-1} \left(2\sigma/R_\dagger + 2\bar{\sigma}/\overline{R}_\dagger \right) \int_{B_1} \varrho$$

$$= \left[\mathfrak{R} T_c - \rho_\dagger^{-1} \left(2\sigma/R_\dagger + 2\bar{\sigma}/\overline{R}_\dagger \right) \right] R_\dagger^3 \int_{B_1} \varrho = 0,$$

where we have used $\gamma = 1 + \mathfrak{R}/\mathfrak{c}$ and (2.8). For $f''(0)$, one has

$$f''(\delta) = -\frac{2M\mathfrak{c} T_c \varrho(1,t)}{\rho_\dagger (1+\delta\phi)^2}\phi + \frac{2M\mathfrak{c} T_c}{\rho_\dagger}\frac{\rho_\dagger + \delta\varrho(1,t)}{(1+\delta\phi)^3}\phi^2 + \frac{M\mathfrak{c} T_c \varrho(1,t)^2}{(\rho_\dagger + \delta\varrho(1,t))^2}$$

$$+ \left(\frac{2\mathfrak{c}\gamma T_c R_\dagger^3}{(1+\delta\phi)^3} \int_{B_1} (\rho_\dagger + \delta\varrho) \log(\rho_\dagger + \delta\varrho) \right)\phi^2$$

$$- \left(\frac{2\mathfrak{c}\gamma T_c R_\dagger^3}{(1+\delta\phi)^2} \int_{B_1} (\varrho \log(\rho_\dagger + \delta\varrho) + \varrho) \right)\phi + \frac{\mathfrak{c}\gamma T_c R_\dagger^3}{1+\delta\phi} \int_{B_1} \left(\frac{\varrho^2}{\rho_\dagger + \delta\varrho} \right)$$

$$+ 4\pi\rho_l \left[\frac{R_\dagger^5}{9(1+\delta\phi)^{\frac{11}{3}}} - \frac{R_\dagger^6}{9(1+\delta\phi)^4} \left(\frac{R_\dagger^3}{1+\delta\phi} + \overline{V} \right)^{-\frac{1}{3}} \right]\dot{\phi}^2$$

$$+ 8\pi\rho_l\delta\frac{d}{d\delta} \left[\frac{R_\dagger^5}{9(1+\delta\phi)^{\frac{11}{3}}} - \frac{R_\dagger^6}{9(1+\delta\phi)^4} \left(\frac{R_\dagger^3}{1+\delta\phi} + \overline{V} \right)^{-\frac{1}{3}} \right]\dot{\phi}^2$$

$$+ 2\pi\rho_l\delta^2\frac{d^2}{d^2\delta} \left[\frac{R_\dagger^5}{9(1+\delta\phi)^{\frac{11}{3}}} - \frac{R_\dagger^6}{9(1+\delta\phi)^4} \left(\frac{R_\dagger^3}{1+\delta\phi} + \overline{V} \right)^{-\frac{1}{3}} \right]\dot{\phi}^2$$

$$+ \frac{8\pi}{3} \left[\frac{5\sigma R_\dagger^2}{3(1+\delta\phi)^{\frac{8}{3}}} + \frac{2\bar{\sigma} R_\dagger^3}{(1+\delta\phi)^3} \left(\frac{R_\dagger^3}{1+\delta\phi} + \overline{V} \right)^{-\frac{1}{3}} \right.$$

$$\left. - \frac{\bar{\sigma} R_\dagger^6}{3(1+\delta\phi)^4} \left(\frac{R_\dagger^3}{1+\delta\phi} + \overline{V} \right)^{-\frac{4}{3}} \right]\phi^2 \triangleq \sum_{i=1}^{10} I_i.$$

Therefore, setting $\delta = 0$ yields

$$f''(0) = -2M\mathfrak{c}T_c\frac{\varrho(1,t)}{\rho_\dagger}\phi + 2M\mathfrak{c}T_c\phi^2 + M\mathfrak{c}T_c\frac{\varrho(1,t)^2}{\rho_\dagger^2} + 2\gamma M\mathfrak{c}T_c\log\rho_\dagger\phi^2$$

$$- 2\gamma M\mathfrak{c}T_c\log\rho_\dagger\phi^2 - 2M\mathfrak{c}\gamma T_c\phi^2 + \mathfrak{c}\gamma T_c R_\dagger^3\rho_\dagger^{-1}\int_{B_1}\varrho^2$$

$$+ \frac{4\pi\rho_l R_\dagger^5}{9}\left(1 - \frac{R_\dagger}{\overline{R}_\dagger}\right)\dot\phi^2 + \frac{M}{\rho_\dagger}\left[\frac{10\sigma}{3R_\dagger} + 4\frac{\bar\sigma}{\overline{R}_\dagger} - \frac{2}{3}\frac{\bar\sigma}{\overline{R}_\dagger}\frac{R_\dagger^3}{\overline{R}_\dagger^3}\right]\phi^2$$

$$= M\mathfrak{c}T_c\left(\frac{\varrho(1,t)}{\rho_\dagger} - \phi\right)^2 + \frac{4\pi\rho_l R_\dagger^5}{9}\left(1 - \frac{R_\dagger}{\overline{R}_\dagger}\right)\dot\phi^2$$

$$- M(\mathfrak{c} + 2\mathfrak{R})T_c\phi^2 + \mathfrak{c}\gamma T_c R_\dagger^3\rho_\dagger^{-1}\int_{B_1}\varrho^2 + \frac{M}{\rho_\dagger}\left[\frac{10\sigma}{3R_\dagger} + 4\frac{\bar\sigma}{\overline{R}_\dagger} - \frac{2}{3}\frac{\bar\sigma}{\overline{R}_\dagger}\frac{R_\dagger^3}{\overline{R}_\dagger^3}\right]\phi^2$$

$$\geqslant M\mathfrak{c}T_c\left(\frac{\varrho(1,t)}{\rho_\dagger} - \frac{3}{4\pi\rho_\dagger}\int_{B_1}\varrho\right)^2 + \frac{4\pi\rho_l R_\dagger^5}{9}\left(1 - \frac{R_\dagger}{\overline{R}_\dagger}\right)\dot\phi^2$$

$$- (\mathfrak{c} + \mathfrak{R} - \varepsilon\mathfrak{R})T_c R_\dagger^3\rho_\dagger^{-1}\int_{B_1}\varrho^2 + (\mathfrak{c} + \mathfrak{R})T_c R_\dagger^3\rho_\dagger^{-1}\int_{B_1}\varrho^2$$

$$- M(\mathfrak{R} + \varepsilon\mathfrak{R})T_c\phi^2 + (1+\varepsilon)\frac{M}{\rho_\dagger}\left[\frac{2\sigma}{R_\dagger} + \frac{2\bar\sigma}{\overline{R}_\dagger}\right]\phi^2$$

$$+ \frac{M}{\rho_\dagger}\left[\frac{4\sigma}{3R_\dagger}\left(1 - \frac{3}{2}\varepsilon\right) + \frac{2\bar\sigma}{\overline{R}_\dagger}\left(1 - \frac{R_\dagger^3}{3\overline{R}_\dagger^3} - \varepsilon\right)\right]\phi^2$$

$$\geqslant M\mathfrak{c}T_c\left(\frac{\varrho(1,t)}{\rho_\dagger} - \frac{3}{4\pi\rho_\dagger}\int_{B_1}\varrho\right)^2 + \frac{\rho_l R_\dagger^5}{4\pi\rho_\dagger^2}\left(1 - \frac{R_\dagger}{\overline{R}_\dagger}\right)\left(\int_{B_1}\dot\varrho\right)^2 + \varepsilon\frac{\mathfrak{R}T_c R_\dagger^3}{\rho_\dagger}\int_{B_1}\varrho^2$$

$$+ \frac{R_\dagger^3}{\pi\rho_\dagger^2}\left[\frac{\sigma}{R_\dagger}\left(1 - \frac{3}{2}\varepsilon\right) + \frac{\bar\sigma}{\overline{R}_\dagger}\left(\frac{3}{2} - \frac{R_\dagger^3}{2\overline{R}_\dagger^3} - \frac{3}{2}\varepsilon\right)\right]\left(\int_{B_1}\varrho\right)^2,$$

where $\varepsilon \in (0, 2/3)$ and we have used $\gamma = 1 + \mathfrak{R}/\mathfrak{c}$, (2.8), (3.5), and Cauchy's inequality $\phi^2 \leqslant \frac{3}{4\pi\rho_\dagger^2}\int_{B_1}\varrho^2$. Chosen $\varepsilon = 1/3$, it holds

$$\frac{1}{2}f''(0)\delta^2 \geqslant \frac{M\mathfrak{c}T_c}{2}\left(\frac{\varrho(1,t)}{\rho_\dagger} - \frac{3}{4\pi\rho_\dagger}\int_{B_1}\varrho\right)^2\delta^2 + \frac{\rho_l R_\dagger^5}{8\pi\rho_\dagger^2}\left(1 - \frac{R_\dagger}{\overline{R}_\dagger}\right)\left(\int_{B_1}\dot\varrho\right)^2\delta^2$$

$$+ \frac{\mathfrak{R}T_c R_\dagger^3}{6\rho_\dagger}\int_{B_1}\varrho^2\delta^2 + \frac{R_\dagger^3}{2\pi\rho_\dagger^2}\left[\frac{\sigma}{2R_\dagger} + \frac{\bar\sigma}{\overline{R}_\dagger}\left(1 - \frac{R_\dagger^3}{2\overline{R}_\dagger^3}\right)\right]\left(\int_{B_1}\varrho\right)^2\delta^2$$

$$\triangleq J_1\delta^2 + J_2\delta^2 + J_3\delta^2 + J_4\delta^2.$$

To deal with f''', using $\delta \leqslant 1$ and $\|\varrho(\cdot,t)\|_{L^\infty(B_1)} \leqslant \rho_\dagger/2$, we note that all the denominators of I_i are bounded, i.e., $1/2 \leqslant 1 + \xi(\delta)\phi \leqslant 3/2$, $\rho_\dagger/2 \leqslant \rho_\dagger + \xi(\delta)\varrho(r,t) \leqslant 3\rho_\dagger/2$. With this lower bound, all the terms in $f'''(\xi(\delta))\delta^3/6$ coming from I_1, I_2, I_4, I_5, I_6 and I_{10} can be absorbed into $J_3\delta^2$ and $J_4\delta^2 \geqslant \frac{R_\dagger^3}{4\pi\rho_\dagger^2}\left(\frac{\sigma}{R_\dagger} + \frac{\bar\sigma}{\overline{R}_\dagger}\right)\left(\int_{B_1}\varrho\right)^2\delta^2$ for $\delta \leqslant \delta_0^1/(1 + |\log\rho_\dagger|)$, where $\delta_0^1 > 0$ depends only on $\mathfrak{c}/\mathfrak{R}$. We show the calculation of I_1, I_5, I_{10} and the others are similar or easier. The derivatives of the first two terms can be absorbed by using (2.8)

$$\left| \frac{dI_1}{d\delta}(\xi(\delta))\delta^3 \right| = \left| \frac{\Re T_c \rho_\dagger}{\Re} \frac{\mathfrak{c}\varrho(1,t)}{(1+\xi(\delta)\phi)^3 \rho_\dagger} \frac{3R_\dagger^3}{\pi \rho_\dagger^2} \left(\int_{B_1} \varrho \right)^2 \right| \delta^3 \leqslant \left(\frac{72\mathfrak{c}}{\Re} \delta \right) J_4 \delta^2,$$

$$\left| \frac{dI_5}{d\delta}(\xi(\delta))\delta^3 \right| = \left| -2 \left(\frac{2\mathfrak{c}\gamma T_c R_\dagger^3}{(1+\xi(\delta)\phi)^3} \int_{B_1} (\varrho \log(\rho_\dagger + \xi(\delta)\varrho) + \varrho) \right) \phi^2 \right.$$

$$\left. + \left(\frac{2\mathfrak{c}\gamma T_c R_\dagger^3}{(1+\xi(\delta)\phi)^2} \int_{B_1} \left(\frac{\varrho^2}{\rho_\dagger + \xi(\delta)\varrho} \right) \right) \phi \right| \delta^3$$

$$\leqslant \frac{12\mathfrak{c}\gamma}{\Re} \Re T_c \rho_\dagger \frac{R_\dagger^3}{\pi \rho_\dagger^2} \max \left\{ \left| \log \frac{3\rho_\dagger}{2} \right|, \left| \log \frac{\rho_\dagger}{2} \right| \right\} \left(\int_{B_1} \varrho \right)^2 \delta^3$$

$$+ \frac{12\mathfrak{c}\gamma}{\Re} \Re T_c \rho_\dagger \frac{R_\dagger^3}{\pi \rho_\dagger^2} \left(\int_{B_1} \varrho \right)^2 \delta^3 + \frac{12\mathfrak{c}\gamma}{\Re} \frac{\Re T_c R_\dagger^3}{3\rho_\dagger} \int_{B_1} \varrho^2 \delta^3$$

$$\leqslant \left[\frac{200(\mathfrak{c}+\Re)}{\Re} (1+|\log \rho_\dagger|) \delta \right] J_4 \delta^2 + \left(\frac{24(\mathfrak{c}+\Re)}{\Re} \delta \right) J_4 \delta^2.$$

For the derivative of I_{10}, it is sufficient to notice that $|\phi| \leqslant 1/2$ from (3.5), and

$$\left| \frac{d}{d\delta} \left[\left(\frac{R_\dagger^3}{1+\delta\phi} + \overline{V} \right)^{-\frac{1}{3}} \right] (\xi(\delta)) \right| = \left| \frac{1}{3} \left(\frac{R_\dagger^3}{1+\xi(\delta)\phi} + \overline{V} \right)^{-\frac{4}{3}} \frac{R_\dagger^3 \phi}{(1+\xi(\delta)\phi)^2} \right| \leqslant \frac{100}{R_\dagger},$$

$$\left| \frac{d}{d\delta} \left[\left(\frac{R_\dagger^3}{1+\delta\phi} + \overline{V} \right)^{-\frac{4}{3}} \right] (\xi(\delta)) \right| = \left| \frac{4}{3} \left(\frac{R_\dagger^3}{1+\xi(\delta)\phi} + \overline{V} \right)^{-\frac{7}{3}} \frac{R_\dagger^3 \phi}{(1+\xi(\delta)\phi)^2} \right| \leqslant \frac{100}{R_\dagger^4},$$

where we have used

$$\left(\frac{R_\dagger^3}{1+\xi(\delta)\phi} + \overline{V} \right)^{-\frac{1}{3}} \leqslant \left(\frac{2}{3} R_\dagger^3 + \frac{2}{3} \overline{V} \right)^{-\frac{1}{3}} \leqslant \left(\frac{3}{2} \right)^{\frac{1}{3}} R_\dagger^{-1},$$

which is independent of the liquid volume $V \in (0, \infty)$. Then, $\frac{d}{d\delta} I_{10}(\xi(\delta))\delta^3/6$ can be absorbed into $J_4 \delta^2$.

For the derivative of I_3 which contains the term $\varrho(1,t)^3$, we have

$$\left| \frac{dI_3}{d\delta}(\xi(\delta))\delta^3 \right| = \left| \frac{2M\mathfrak{c} T_c \varrho(1,t)^3}{(\rho_\dagger + \xi(\delta)\varrho(1,t))^3} \right| \delta^3 \leqslant (32\delta) J_1 \delta^2 + \left(\frac{16\mathfrak{c}}{\Re} \delta \right) J_3 \delta^2,$$

where we have used

$$|\varrho(1,t)||\varrho(1,t)|^2 \leqslant \frac{\rho_\dagger}{2} \left[\left| \frac{\varrho(1,t)}{\rho_\dagger} - \frac{1}{|B_1|\rho_\dagger} \int_{B_1} \varrho \right| + |B_1|^{-\frac{1}{2}} \rho_\dagger^{-1} \left(\int_{B_1} \varrho^2 \right)^{\frac{1}{2}} \right]^2 \rho_\dagger^2$$

$$\leqslant \rho_\dagger^3 \left[\left(\frac{\varrho(1,t)}{\rho_\dagger} - \frac{1}{|B_1|\rho_\dagger} \int_{B_1} \varrho \right)^2 + |B_1|^{-1} \rho_\dagger^{-2} \left(\int_{B_1} \varrho^2 \right) \right].$$

Therefore, this term can also be absorbed into $f''(0)\delta^2/2$ for all $0 < \delta < \delta_0^2$, where $\delta_0^2 > 0$ depends on $\mathfrak{c}/\mathfrak{R}$.

To deal with the derivatives of I_7, I_8 and I_9 containing $\left(\int_{B_1} \dot{\varrho}\right)^2$, we need to calculate the first three order derivative of $\rho_l R_\dagger^5 \psi(\delta)\dot{\phi}^2$ where

$$\psi(\delta) = (1 + \delta\phi)^{-\frac{11}{3}} - R_\dagger (1 + \delta\phi)^{-4} \left(R_\dagger^3 (1 + \delta\phi)^{-1} + \overline{V}\right)^{-\frac{1}{3}}.$$

We only control $d\psi/d\delta$ since $d^2\psi/d^2\delta$, $d^3\psi/d^3\delta$ can be calculated similarly. By direct calculation, we obtain

$$\frac{d\psi}{d\delta}(\delta) = \frac{\phi}{3(1 + \delta\phi)^{\frac{14}{3}}} \left[12 \left(1 + \overline{V}R_\dagger^{-3} (1 + \delta\phi)\right)^{-\frac{1}{3}} - \left(1 + \overline{V}R_\dagger^{-3} (1 + \delta\phi)\right)^{-\frac{4}{3}} - 11\right].$$

Therefore, from $|\phi| \leqslant 1/2$ and $\frac{\overline{R}_\dagger - R_\dagger}{R_\dagger} = 1 - \left(1 + \overline{V}R_\dagger^{-3}\right)^{-\frac{1}{3}}$, it follows that

$$\left|\frac{d\psi}{d\delta}(\xi(\delta))\right| \leqslant 6 \left[12 \left|1 - \left(1 + \overline{V}R_\dagger^{-3} (1 + 1/2)\right)^{-\frac{1}{3}}\right| + \left|1 - \left(1 + \overline{V}R_\dagger^{-3} (1 + 1/2)\right)^{-\frac{4}{3}}\right|\right]$$

$$\leqslant 6 \left[19 \left|1 - \left(1 + \overline{V}R_\dagger^{-3}\right)^{-\frac{1}{3}}\right| + 7 \left|1 - \left(1 + \overline{V}R_\dagger^{-3}\right)^{-\frac{1}{3}}\right|\right]$$

$$\leqslant 200 \left(\frac{\overline{R}_\dagger - R_\dagger}{R_\dagger}\right),$$

and $\overline{V}R_\dagger^{-3} \leqslant \delta_0^3$ where $\delta_0^3 > 0$ is a small constant independent of $T_c, \rho_l, \rho_\dagger, R_\dagger, M$ and V. Moreover, for all $\alpha > 0$, we have

$$1 - \left(1 + \overline{V}R_\dagger^{-3} (1 + 1/2)\right)^{-\frac{\alpha}{3}} \leqslant 1 \leqslant \frac{1}{1 - (1 + \delta_0^3)^{-\frac{1}{3}}} \frac{\overline{R}_\dagger - R_\dagger}{R_\dagger}, \quad \overline{V}R_\dagger^{-3} \geqslant \delta_0^3.$$

We conclude that

$$\left|\frac{d\psi}{d\delta}(\xi(\delta))\right| \leqslant 100 \left[\left(1 - \left(1 + \delta_0^3\right)^{-\frac{1}{3}}\right)^{-1} + 1\right] \frac{\overline{R}_\dagger - R_\dagger}{R_\dagger},$$

for all V and $R_\dagger$. Therefore, all the terms $\frac{d}{d\delta}(I_7 + I_8 + I_9)(\xi(\delta))\delta^3$ containing $\dot{\phi}^2$ can be absorbed into $J_2\delta^2$, for all $\delta < \delta_0^4$ with δ_0^4 independent of $T_c, \rho_l, \rho_\dagger, R_\dagger, M$ and V.

Summarizing the above results and recalling that we choose any $(\rho_\delta(\cdot, t), R_\delta(t))$ satisfying (3.4). We conclude that for all $(\rho(\cdot, t), R(t))$ satisfying

$$\|\rho(\cdot, t) - \rho_\dagger\|_{L^\infty(B_{R(t)})} \leqslant \delta_0 \rho_\dagger / (1 + |\log \rho_\dagger|),$$

or equivalently, (3.2), where $\delta_0 > 0$ is a constant that depends on δ_0^i and hence only on $\mathfrak{c}/\mathfrak{R}$, or γ, it holds

$$E[\bar\rho, R] \geqslant E_\dagger + \frac{1}{4}\left\{ M\mathfrak{c}T_c \left(\frac{\bar\rho(1,t) - \rho_\dagger}{\rho_\dagger} - \frac{3}{4\pi\rho_\dagger} \int_{B_1} (\bar\rho(y,t) - \rho_\dagger) \right)^2 \right.$$

$$+ \frac{\rho_l R_\dagger^5}{4\pi\rho_\dagger^2}\left(1 - \frac{R_\dagger}{\overline{R}_\dagger}\right)\left(\int_{B_1}\dot{\bar\rho}\right)^2 + \frac{\mathfrak{R}T_c R_\dagger^3}{3\rho_\dagger}\int_{B_1}(\bar\rho(y,t) - \rho_\dagger)^2$$

$$+ \frac{R_\dagger^3}{\pi\rho_\dagger^2}\left[\frac{\sigma}{2R_\dagger} + \frac{\bar\sigma}{\overline{R}_\dagger}\left(1 - \frac{R_\dagger^3}{2\overline{R}_\dagger^3}\right)\right]\left. \left(\int_{B_1}(\bar\rho(y,t) - \rho_\dagger)\right)^2 \right\},$$

where we have used $\bar\rho(\cdot,t) = \bar\rho_\delta(\cdot,t) = \rho_\dagger + \delta\varrho(\cdot,t)$. This completes the proof. $\qquad\square$

4 Nonlinear and exponential asymptotic stability

4.1 Proof of the nonlinear asymptotic stability

In this subsection, we prove the first part of main Theorem 2, especially the asymptotic stability in (1.14).

Given the equilibrium state $(\rho_\dagger[M,V], R_\dagger[M,V])$ determined by the mass-volume pair (M, V) and any initial data $(\rho_0, R_0, \dot R_0)$, since free boundary problem (1.6)–(1.8) is equivalent to system (2.1) in the spherical case, the energy dissipation (3.1) reduces to

$$\dot E(\tau) = -\kappa T_c \int_{B_{R(\tau)}} \frac{|\nabla_r \rho(|x|,\tau)|^2}{\rho(|x|,\tau)^2}dx - 16\pi\mu_l \frac{\overline{V}R(\tau)}{R(\tau)^3 + \overline{V}}\dot R(\tau)^2, \qquad (4.1)$$

where ∇_r denotes the radial gradient. In the above, we have used the velocity formula (2.5), $\Omega_g(t) = B_{R(t)}$ and $\Omega_l(t) = B_{\overline{R}(t)} \setminus \overline{B_{R(t)}}$.

Integrating from 0 to t and applying the local minimizer (3.3) together with the Lyapunov stability (B.1)–(B.3), we obtain

$$\kappa T_c \int_0^t \int_{B_{R(\tau)}} \frac{|\nabla_r \rho(|x|,\tau)|^2}{\rho(|x|,\tau)^2}dxd\tau + 16\pi\mu_l \int_0^t \frac{\overline{V}R(\tau)\dot R(\tau)^2}{R(\tau)^3 + \overline{V}}d\tau$$
$$\leqslant E_0 - E_\dagger - (E(t) - E_\dagger)$$
$$\leqslant E_0 - E_\dagger,$$

provided the constant $\eta_0 > 0$ is small enough, which is independent of the initial data. Also, we have denoted $E_0 = E[\rho_0, R_0]$ and $E_\dagger = E[\rho_\dagger, R_\dagger]$ in the above. By the regularity of $(\rho(x,t), R(t))$ in (B.3), we deduce that the time-dependent functions

$$\int_{B_{R(\tau)}} \frac{|\nabla_r \rho(|x|,\tau)|^2}{\rho(|x|,\tau)^2}dx, \text{ and } R(\tau)\dot R(\tau)^2 \geqslant 0$$

are uniformly continuous. This, combined with the bound $0 \leqslant \overline{V}/\left(R(\tau)^3 + \overline{V}\right) \leqslant 1$, follows that the function $\overline{V}R(\tau)(\dot{R}(\tau))^2/\left(R(\tau)^3 + \overline{V}\right)$ is also uniformly continuous independent of the liquid volume V.

The remaining proof of (1.14) is similar to that of [9, Proposition 8.1], as we can apply Theorem 1, Barbalat's lemmas in stability theory, and interpolations. This proof is valid because we have shown that the algebraic system (2.8), or equivalently, the equation

$$(\sigma_{\text{ratio}}^3 + 1)R^9 - 3IR^7 + \overline{V}R^6 + 3I^2R^5 - 3I\overline{V}R^4 - I^3R^3 + 3I^2\overline{V}R^2 - I^3\overline{V} = 0,$$

has a unique positive root for any fixed liquid volume.

4.2 Proof of the exponential convergence rate

The exponential convergence in Theorem 2 is established through the utilization of the center manifold theory in Appendix C, where the relevant definitions are provided.

Step 1: We start with transforming free boundary problem (2.1) into an equivalent system within an appropriate Banach space. Specifically, we select $Z = \ell^2$ as specified in (C.1).

Proposition 3 *Under the assumptions of Theorem 2, if we abbreviate the equilibrium state* $(\rho_{\dagger}[M, V], R_{\dagger}[M, V]) = (\rho_{\dagger}, R_{\dagger})$, *and decompose the global-in-time solution* (ρ, R) *as follows*

$$\begin{cases} \rho(R(t)y, t) = \rho_{\dagger} + \varrho(y, t) = \rho_{\dagger} + \varrho_1(y, t) + \varrho_2(t), & 0 \leqslant y \leqslant 1, \quad t > 0, \\ \varrho_2(t) = \rho(R(t), t) - \rho_{\dagger}, & t > 0, \\ R(t) = R_{\dagger} + \mathcal{R}(t), & t > 0, \end{cases} \tag{4.2}$$

then, system (2.1) is converted to the following initial-boundary value problem

$$\partial_t \varrho_1(y, t) = \chi \Delta \varrho_1(y, t) - \left(1 - \frac{1}{\gamma}\right)\dot{\varrho}_2 + \Pi, \ 0 \leqslant y \leqslant 1, \ \varrho_1(1, \cdot) \equiv 0, \quad t > 0, \tag{4.3a}$$

$$\dot{\mathcal{R}} = -\frac{R_{\dagger}}{\rho_{\dagger}}\left(\chi \partial_y \varrho_1(1, t) + \frac{1}{3\gamma}\dot{\varrho}_2\right) + \Phi, \qquad t > 0, \tag{4.3b}$$

$$\varrho_2 = \frac{1}{\Re T_c}\left[-\left(\frac{2\sigma}{R_{\dagger}^2} + \frac{2\bar{\sigma}R_{\dagger}^2}{\overline{R}_{\dagger}^4}\right)\mathcal{R} + \frac{4\mu_l\overline{V}}{R_{\dagger}(R_{\dagger}^3 + \overline{V})}\dot{\mathcal{R}} + \rho_l\tilde{R}_{\dagger}\ddot{\mathcal{R}}\right] + \Psi, \qquad t > 0, \tag{4.3c}$$

with the initial condition

$$(\varrho_1(y, 0), \varrho_2(0), \mathcal{R}(0), \dot{\mathcal{R}}(0)) = (\rho_0(R_0y) - \rho_0(R_0), \rho_0(R_0) - \rho_{\dagger}, R_0 - R_{\dagger}, \dot{R}_0), \tag{4.4}$$

for $0 \leqslant y \leqslant 1$. Moreover, in system (4.3), we denote $\tilde{R}_{\dagger} = R_{\dagger} - R_{\dagger}^2/\overline{R}_{\dagger}, \chi = \kappa/(R_{\dagger}^2\rho_{\dagger}\gamma\mathfrak{c})$, and the following nonlinear terms

$$\Pi = \frac{\kappa}{\gamma \mathfrak{c}} \left[\frac{1}{(R_\dagger + \mathcal{R})^2 (\rho_\dagger + \varrho(y,t))} - \frac{1}{R_\dagger^2 \rho_\dagger} \right] \Delta_y \varrho_1(y,t)$$
$$- \frac{\kappa}{\gamma \mathfrak{c}} \frac{|\nabla_y \varrho_1(y,t)|^2}{(R_\dagger + \mathcal{R})^2 (\rho_\dagger + \varrho(y,t))^2} + \frac{1}{\gamma} \frac{\dot\varrho_2}{\rho_\dagger + \varrho_2} \left(\frac{1}{3} y \partial_y \varrho_1(y,t) + \varrho_1(y,t) \right), \tag{4.5a}$$

$$\Phi = -\frac{\kappa}{\gamma \mathfrak{c}} \left[\frac{1}{(R_\dagger + \mathcal{R})(\rho_\dagger + \varrho_2)^2} - \frac{1}{R_\dagger \rho_\dagger^2} \right] \partial_y \varrho_1(1,t) - \frac{\mathcal{R}\dot\varrho_2}{3\gamma(\rho_\dagger + \varrho_2)}$$
$$+ \frac{R_\dagger}{3\gamma} \frac{\varrho_2 \dot\varrho_2}{\rho_\dagger(\rho_\dagger + \varrho_2)}, \tag{4.5b}$$

$$\Psi = \frac{1}{\mathfrak{R}T_c} \left[4\mu_l \overline{V} \left(\frac{1}{R(R^3 + \overline{V})} - \frac{1}{R_\dagger(R_\dagger^3 + \overline{V})} \right) \dot{\mathcal{R}} + \left(\frac{2\sigma}{R_\dagger^2} \mathcal{R} + \frac{2\sigma}{R_\dagger + \mathcal{R}} - \frac{2\sigma}{R_\dagger} \right) \right.$$
$$\left. + \left(\frac{2\bar\sigma R_\dagger^2}{\overline{R_\dagger^4}} \mathcal{R} + \frac{2\bar\sigma}{\overline{R}} - \frac{2\bar\sigma}{\overline{R_\dagger}} \right) \right] \tag{4.5c}$$
$$+ \frac{\rho_l}{\mathfrak{R}T_c} \left[R\ddot{\mathcal{R}} + \left(\frac{R_\dagger^2}{\overline{R_\dagger}} \ddot{\mathcal{R}} - \frac{R^2}{\overline{R}} \ddot{\mathcal{R}} \right) + \left(\frac{3}{2} - \frac{2R}{\overline{R}} + \frac{R^4}{2\overline{R}^4} \right) \dot{\mathcal{R}}^2 \right].$$

We denote the normalized radial Dirichlet eigenfunctions defined on the unit ball B_1 by $\left\{ \Xi_j(y) = \frac{\sin(j\pi y)}{\sqrt{2\pi} y} \right\}_{j=1}^\infty$, which satisfies $-\Delta_y \Xi_j = \zeta_j \Xi_j$ with $\zeta_j = (j\pi)^2$, $\Xi_j|_{y=1} = 0$, and $\int_{B_1} \Xi_j^2(|x|)\,dx = 1$ for $j \geqslant 1$. We expand ϱ_1 as

$$\varrho_1(y,t) = \sum_{j=1}^\infty \theta_j(t) \Xi_j(y). \tag{4.6}$$

Then, system (4.3) is further equivalent to the following infinite dimensional dynamical system for $z = \begin{bmatrix} \varrho_2 & \mathcal{R} & \dot{\mathcal{R}} & \theta_1 & \theta_2 & \cdots \end{bmatrix}^\top$. That is,

$$\dot{z} = \mathcal{L}z + \mathcal{N}(z,\dot{z}) = \mathcal{L}z + \mathcal{N}^1(z)\dot{z} + \mathcal{N}^0(z), \tag{4.7}$$

where both the linear operator $\mathcal{L}$ and the nonlinear term $\mathcal{N}(z,\dot{z})$ are defined in (4.11). The terms $\mathcal{N}^1$ and $\mathcal{N}^0$ are given in (4.14). Finally, the initial condition of problem (4.7) can be deduced from the original condition (4.4).

Proof To begin with, (4.3a) and (4.3b) are derived by substituting (4.2) into (2.1). We point out that (4.3c) is deduced by applying (2.8), Taylor's theorem $(R_\dagger + x)^{-1} = R_\dagger^{-1} - R_\dagger^{-2} x + \cdots$ and $((R_\dagger + x)^3 + \overline{V})^{-\frac{1}{3}} = (R_\dagger^3 + \overline{V})^{-\frac{1}{3}} - R_\dagger^2 \overline{R}_\dagger^{-4} x + \cdots$. More precisely, we calculate as follows

$$
\begin{aligned}
\varrho_2(t) &= \frac{1}{\Re T_c}\left[4\mu_l \frac{\overline{V}}{R(R^3+\overline{V})}\dot{\mathcal{R}} + \frac{2\sigma}{R_\dagger+\mathcal{R}} - \frac{2\sigma}{R_\dagger} + \frac{2\bar\sigma}{\overline{R}} - \frac{2\bar\sigma}{\overline{R}_\dagger}\right.\\
&\quad \left.+\rho_l\left(R\ddot{R} - \frac{R^2}{\overline{R}}\ddot{R} + \left(\frac{3}{2}-\frac{2R}{\overline{R}}+\frac{R^4}{2\overline{R}^4}\right)\dot{R}^2\right)\right]\\
&= \frac{1}{\Re T_c}\left[4\mu_l \frac{\overline{V}}{R_\dagger(R_\dagger^3+\overline{V})}\dot{\mathcal{R}} - \frac{2\sigma}{R_\dagger^2}\mathcal{R} - \frac{2\bar\sigma R_\dagger^2}{\overline{R}_\dagger^4}\mathcal{R} + \rho_l\tilde{R}_\dagger\ddot{\mathcal{R}}\right]\\
&\quad + \frac{1}{\Re T_c}\left[\left(4\mu_l\frac{\overline{V}}{R(R^3+\overline{V})}\dot{\mathcal{R}} - 4\mu_l\frac{\overline{V}}{R_\dagger(R_\dagger^3+\overline{V})}\dot{\mathcal{R}}\right)\right.\\
&\quad \left.+\left(\frac{2\sigma}{R_\dagger^2}\mathcal{R} + \frac{2\sigma}{R_\dagger+\mathcal{R}} - \frac{2\sigma}{R_\dagger}\right) + \left(\frac{2\bar\sigma R_\dagger^2}{\overline{R}_\dagger^4}\mathcal{R} + \frac{2\bar\sigma}{\overline{R}} - \frac{2\bar\sigma}{\overline{R}_\dagger}\right)\right]\\
&\quad + \frac{\rho_l}{\Re T_c}\left[\mathcal{R}\ddot{\mathcal{R}} + \left(\frac{R_\dagger^2}{\overline{R}_\dagger}\ddot{\mathcal{R}} - \frac{R^2}{\overline{R}}\ddot{\mathcal{R}}\right) + \left(\frac{3}{2} - \frac{2R}{\overline{R}} + \frac{R^4}{2\overline{R}^4}\right)\dot{R}^2\right],
\end{aligned}
$$

where we have denoted $\tilde{R}_\dagger = R_\dagger - R_\dagger^2/\overline{R}_\dagger$. Then, equations (4.3c) and (4.5c) follow.

Next, substituting (4.6) into (4.3a) and computing the inner product in $L^2(B_1)$ with $\Xi_k(y)$ yields the following

$$
\dot\theta_k = -\chi\zeta_k\theta_k - \xi_k\dot\varrho_2 + \Pi_k, \quad \xi_k = \frac{(-1)^{k-1}2^{3/2}(\gamma-1)}{\sqrt{\pi}\gamma k}, \quad \Pi_k = \int_{B_1}\Pi\Xi_k dy. \quad (4.8)
$$

Using $\partial_y\Xi_j(1) = \sqrt{\pi}(-1)^j j/\sqrt{2}$, the second equation (4.3b) becomes

$$
\dot{\mathcal{R}} = \sum_{j=1}^\infty \theta_j\omega_j - \frac{R_\dagger}{3\gamma\rho_\dagger}\dot\varrho_2 + \Phi, \quad \omega_j = -\frac{R_\dagger\chi}{\rho_\dagger}\sqrt{\frac{\pi}{2}}(-1)^j j. \quad (4.9)
$$

Moreover, the third equation (4.3c) implies

$$
\frac{\Re T_c}{\rho_l\tilde{R}_\dagger}\varrho_2 = -\left(\frac{2\sigma}{\rho_l\tilde{R}_\dagger R_\dagger^2} + \frac{2\bar\sigma R_\dagger^2}{\rho_l\tilde{R}_\dagger\overline{R}_\dagger^4}\right)\mathcal{R} + \frac{4\mu_l\overline{V}}{\rho_l\tilde{R}_\dagger R_\dagger(R_\dagger^3+\overline{V})}\dot{\mathcal{R}} + \ddot{\mathcal{R}} + \frac{\Re T_c\Psi}{\rho_l\tilde{R}_\dagger}.
$$

As a consequence, problem (4.3) forms an infinite-dimensional dynamical system

$$
\begin{bmatrix}
\frac{R_\dagger}{3\gamma\rho_\dagger} & 1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
\xi_1 & 0 & 0 & 1 & 0 & \cdots \\
\xi_2 & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
\varrho_2 \\ \mathcal{R} \\ \dot{\mathcal{R}} \\ \theta_1 \\ \theta_2 \\ \vdots
\end{bmatrix}'
=
$$

$$
\begin{bmatrix}
0 & 0 & 0 & \omega_1 & \omega_2 & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
\frac{\mathfrak{R}T_c}{\rho_l\bar{R}_\dagger} & \frac{2\sigma}{\rho_l\bar{R}_\dagger R_\dagger^2}+\frac{2\bar\sigma R_\dagger^2}{\rho_l\bar{R}_\dagger\bar{R}_\dagger^4} & \frac{-4\mu_l\overline{V}}{\rho_l\bar{R}_\dagger R_\dagger(R_\dagger^3+\overline{V})} & 0 & 0 & \cdots \\
0 & 0 & 0 & -\chi\zeta_1 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\chi\zeta_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
\varrho_2 \\ \mathcal{R} \\ \dot{\mathcal{R}} \\ \theta_1 \\ \theta_2 \\ \vdots
\end{bmatrix}
+
\begin{bmatrix}
\Phi \\ 0 \\ -\frac{\mathfrak{R}T_c}{\rho_l\bar{R}_\dagger}\Psi \\ \Pi_1 \\ \Pi_2 \\ \vdots
\end{bmatrix}.
$$

Multiplying both sides of the above equation by the inverse of the infinite-dimensional matrix and denoting $\mathrm{z} = \begin{bmatrix} \varrho_2 & \mathcal{R} & \dot{\mathcal{R}} & \theta_1 & \theta_2 & \cdots \end{bmatrix}^\top$, (4.7) follows, where the linear operator $\mathcal{L}$ equals

$$
\begin{bmatrix}
0 & 0 & -\frac{3\gamma\rho_\dagger}{R_\dagger} & \omega_1\frac{3\gamma\rho_\dagger}{R_\dagger} & \omega_2\frac{3\gamma\rho_\dagger}{R_\dagger} & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
\frac{\mathfrak{R}T_c}{\rho_l\bar{R}_\dagger} & \frac{2\sigma\bar{R}_\dagger^4+2\bar\sigma R_\dagger^4}{\rho_l\bar{R}_\dagger R_\dagger^2\bar{R}_\dagger^4} & \frac{-4\mu_l\overline{V}}{\rho_l\bar{R}_\dagger R_\dagger(R_\dagger^3+\overline{V})} & 0 & 0 & \cdots \\
0 & 0 & \xi_1\frac{3\gamma\rho_\dagger}{R_\dagger} & -\xi_1\omega_1\frac{3\gamma\rho_\dagger}{R_\dagger}-\chi\zeta_1 & -\xi_1\omega_2\frac{3\gamma\rho_\dagger}{R_\dagger} & \cdots \\
0 & 0 & \xi_2\frac{3\gamma\rho_\dagger}{R_\dagger} & -\xi_2\omega_1\frac{3\gamma\rho_\dagger}{R_\dagger} & -\xi_2\omega_2\frac{3\gamma\rho_\dagger}{R_\dagger}-\chi\zeta_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad (4.10)
$$

and the nonlinear term $\mathcal{N} = \mathcal{N}(\mathrm{z},\dot{\mathrm{z}})$ is equal to

$$
\begin{bmatrix}
\frac{3\gamma\rho_\dagger}{R_\dagger}\Phi & 0 & -\frac{\mathfrak{R}T_c}{\rho_l\bar{R}_\dagger}\Psi & -\xi_1\frac{3\gamma\rho_\dagger}{R_\dagger}\Phi+\Pi_1 & -\xi_2\frac{3\gamma\rho_\dagger}{R_\dagger}\Phi+\Pi_2 & \cdots
\end{bmatrix}^\top, \quad (4.11)
$$

where $\Phi = \Phi(\mathrm{z},\dot{\mathrm{z}})$, $\Psi = \Psi(\mathrm{z},\dot{\mathrm{z}})$, $\Pi_k = \Pi_k(\mathrm{z},\dot{\mathrm{z}})$, ξ_k and ω_k are defined in (4.5b), (4.5c), (4.8) and (4.9), respectively. Writing $\mathcal{N}(\mathrm{z},\dot{\mathrm{z}}) = \mathcal{N}(\mathrm{z},\mathrm{w})$ (denoting $\mathrm{w} = \dot{\mathrm{z}}$), we further decompose

$$
\begin{cases}
\Pi(\mathrm{z},\mathrm{w}) = \langle \mathbf{\Pi}^1(\mathrm{z}),\mathrm{w}\rangle + \Pi^0(\mathrm{z}), \\
\Phi(\mathrm{z},\mathrm{w}) = \langle \mathbf{\Phi}^1(\mathrm{z}),\mathrm{w}\rangle + \Phi^0(\mathrm{z}), \\
\Psi(\mathrm{z},\mathrm{w}) = \langle \mathbf{\Psi}^1(\mathrm{z}),\mathrm{w}\rangle + \Psi^0(\mathrm{z}),
\end{cases}
\quad (4.12)
$$

i.e.,

$$\Pi(z, w) = \left\langle \left[\frac{a}{\gamma(\rho_\dagger + \varrho_2)} \sum_{j=1}^{\infty} \theta_j \left(\frac{y}{3} \partial_y \Xi_j + \Xi_j \right) \quad 0 \quad 0 \quad \cdots \right]^\top, w \right\rangle$$

$$+ \frac{|\nabla_y \varrho_1|^2}{(R_\dagger + \mathcal{R})^2 (\rho_\dagger + \varrho)^2} - \frac{\kappa}{\gamma \mathfrak{c}} \left[\frac{1}{(R_\dagger + \mathcal{R})^2 (\rho_\dagger + \varrho)} - \frac{1}{R_\dagger^2 \rho_\dagger} \right] \sum_{j=1}^{\infty} \zeta_j \theta_j \Xi_j,$$

$$\Phi(z, w) = \left\langle \left[-\frac{\mathcal{R}}{3\gamma(\rho_\dagger + \varrho_2)} + \frac{R_\dagger \varrho_2}{3\gamma \rho_\dagger (\rho_\dagger + \varrho_2)} \quad 0 \quad 0 \quad \cdots \right]^\top, w \right\rangle$$

$$- \frac{\kappa}{\gamma \mathfrak{c}} \left[\frac{1}{(R_\dagger + \mathcal{R})(\rho_\dagger + \varrho_2)^2} - \frac{1}{R_\dagger \rho_\dagger^2} \right] \sum_{j=1}^{\infty} \sqrt{\frac{\pi}{2}} (-1)^j j \theta_j,$$

$$\Psi(z, w) = \left\langle \left[0 \quad 0 \quad \frac{\rho_l}{\mathfrak{R} T_c} \left[\mathcal{R} + \left(\frac{R_\dagger^2}{R_\dagger} - \frac{R^2}{R} \right) \right] \quad 0 \quad 0 \quad \cdots \right]^\top, w \right\rangle$$

$$+ \frac{1}{\mathfrak{R} T_c} \left[4\mu_l \overline{V} \left(\frac{1}{R(R^3 + \overline{V})} - \frac{1}{R_\dagger (R_\dagger^3 + \overline{V})} \right) \dot{\mathcal{R}} + \left(\frac{2\sigma}{R_\dagger^2} \mathcal{R} + \frac{2\sigma}{R_\dagger + \mathcal{R}} - \frac{2\sigma}{R_\dagger} \right) \right.$$

$$\left. + \left(\frac{2\bar{\sigma} R_\dagger^2}{\overline{R}_\dagger^4} \mathcal{R} + \frac{2\bar{\sigma}}{\overline{R}} - \frac{2\bar{\sigma}}{\overline{R}_\dagger} \right) + \rho_l \left(\frac{3}{2} - \frac{2R}{\overline{R}} + \frac{R^4}{2\overline{R}^4} \right) \dot{R}^2 \right].$$

Above, $\langle \cdot, \cdot \rangle$ denotes inner product in the Hilbert space ℓ^2. Therefore, the nonlinear term $\mathcal{N}(z, w)$ can be further decomposed as follows

$$\mathcal{N}(z, w) = \mathcal{N}^1(z)w + \mathcal{N}^0(z), \tag{4.13}$$

where

$$\begin{cases} \mathcal{N}^1 = \left[\frac{3\gamma \rho_\dagger}{R_\dagger} \Phi^1 \quad 0 \quad -\frac{\mathfrak{R} T_c}{\rho_l R_\dagger} \Psi^1 \quad \Pi_1^1 - \xi_1 \frac{3\gamma \rho_\dagger}{R_\dagger} \Phi^1 \quad \Pi_2^1 - \xi_2 \frac{3\gamma \rho_\dagger}{R_\dagger} \Phi^1 \quad \cdots \right]^\top, \\ \mathcal{N}^0 = \left[\frac{3\gamma \rho_\dagger}{R_\dagger} \Phi^0 \quad 0 \quad -\frac{\mathfrak{R} T_c}{\rho_l R_\dagger} \Psi^0 \quad \Pi_1^0 - \xi_1 \frac{3\gamma \rho_\dagger}{R_\dagger} \Phi^0 \quad \Pi_2^0 - \xi_2 \frac{3\gamma \rho_\dagger}{R_\dagger} \Phi^0 \quad \cdots \right]^\top. \end{cases} \tag{4.14}$$

Here, $\mathcal{N}^1(z)w$ denotes the point-wise inner product $\left[\left\langle \frac{3\gamma \rho_\dagger}{R_\dagger} \Phi^1(z), w \right\rangle \quad \langle 0, w \rangle \quad \cdots \right]^\top$,

$\Pi_j^0(z) = \int_{B_1} \Pi^0(z) \Xi_j \, dx$, and $\Pi_j^1(z) = \int_{B_1} \Pi^1(z) \Xi_j \, dx$. This completes the proof of the proposition. $\qquad \square$

Step 2: The asymptotic stability result (part (i)) in Theorem 2 can now be expressed for system (4.7) within the Banach space ℓ^2 (cf. Lemma 3). Next, to examine the decay rate, we analyze the spectrum of the linear operator $\mathcal{L} : \ell^2 \to \ell^2$ in the following proposition, with its proof provided in Appendix D.

Proposition 4 *Denote the spectrum of the linear operator $\mathcal{L}$ defined in (4.10) by $\mathrm{sp}(\mathcal{L})$. Then, $\mathrm{sp}(\mathcal{L}) = \{0\} \cup \{\lambda \in \mathbb{C} : \mathbb{M}(\lambda) = 0\}$. The eigenvalue $\lambda = 0$ has multiplicity one, and $\mathbb{M}(\lambda)$ is a meromorphic function defined by*

$$\mathbb{M}(\lambda) = \frac{\pi}{\mathfrak{R} T_c \gamma} \left(\frac{4}{3} + \sum_{k=1}^{\infty} \frac{8(\gamma-1)\chi}{\pi^2 \chi k^2 + \lambda} \right) \left[\rho_l \tilde{R}_\dagger \lambda^2 + \frac{4\mu_l \overline{V}}{R_\dagger(R_\dagger^3 + \overline{V})} \lambda - \left(\frac{2\sigma}{R_\dagger^2} + \frac{2\bar{\sigma} R_\dagger^2}{\overline{R}_\dagger^4} \right) \right] \tag{4.15}$$
$$+ 4\pi \frac{\rho_\dagger}{R_\dagger},$$

where $\tilde{R}_\dagger = R_\dagger - R_\dagger^2/\overline{R}_\dagger$ and $\chi = \kappa/(R_\dagger^2 \rho_\dagger \gamma \mathfrak{c})$ as in Proposition 3.

Moreover, there exists a constant $\varpi > 0$, such that $\sup\{\operatorname{Re}(\lambda) : \lambda \in \operatorname{sp}(\mathcal{L}) \setminus \{0\}\} \leqslant -2\varpi < 0.$

By further analyzing the spectrum bound $\sup\{\operatorname{Re}(\lambda) : \lambda \in \operatorname{sp}(\mathcal{L}) \setminus \{0\}\}$, we discover that the characterization of the operator spectrum is more accurate when the liquid volume is small.

More precisely, given the mass of the gas $M > 0$, we assume that the liquid volume $V > 0$ is sufficiently small. Then, using the notations in Appendix D and applying (2.8), we have

$$\begin{aligned} B^2 - 4KC^2 &= \left(\frac{4\mu_l \overline{V}}{R_\dagger(R_\dagger^3 + \overline{V})} \right)^2 - \frac{8\mathfrak{R} T_c \rho_\dagger \overline{R}_\dagger}{\rho_l R_\dagger^2 (\overline{R}_\dagger - R_\dagger)} \left(\rho_l \frac{R_\dagger \overline{R}_\dagger - R_\dagger^2}{\overline{R}_\dagger} \right)^2 \\ &= \frac{16\mu_l^2}{R_\dagger^2 (R_\dagger^3 + \overline{V})^2} \overline{V}^2 - 16\rho_l \left(\frac{\sigma}{R_\dagger} + \frac{\bar{\sigma}}{\overline{R}_\dagger} \right) \left(1 - \frac{R_\dagger}{\overline{R}_\dagger} \right) \\ &\leqslant \frac{16\mu_l^2}{R_\dagger^8} \overline{V}^2 - \frac{16\rho_l \sigma}{R_\dagger} \left(1 - \left(1 + \overline{V}/R_\dagger^3 \right)^{-1/3} \right) \\ &\leqslant \frac{16\mu_l^2}{R_\dagger^8} \overline{V}^2 - \frac{16\rho_l \sigma}{R_\dagger} \left(\frac{3\overline{V}}{R_\dagger^3} + O\left(\overline{V}^2\right) \right) < 0, \end{aligned}$$

where we have used the lower bound of the equilibrium radius $R_\dagger$ in (2.11). Then, from the results in Lemma 4, direct calculations show that $K = \frac{2\mathfrak{R} T_c \rho_\dagger \overline{R}_\dagger}{\rho_l R_\dagger^2 (\overline{R}_\dagger - R_\dagger)}$ is large, and

$$\varpi = \frac{1}{2} \min \left\{ \Theta_2 \pi^2 \chi, \max \left\{ \frac{6\mu_l}{\rho_l R_\dagger^2 \overline{R}_\dagger^2} \frac{1}{1 + O(V)}, \Theta_1 \pi^2 \chi \right\} \right\} = \frac{1}{2} \Theta_0 \pi^2 \chi,$$

where the constant Θ_0 satisfying $0 < \Theta_1 \leqslant \Theta_0 \leqslant \Theta_2 < 1$. However, by inspecting the form of $\mathbb{M}(\lambda)$, it has a pole (nearest to the origin, i.e., choosing $k = 1$) at $-\pi^2 \chi$. For λ near this pole, since $\overline{V} > 0$ is sufficiently small, it holds that the following term in the bracket of (4.15)

$$\rho_l \tilde{R}_\dagger \lambda^2 + \frac{4\mu_l \overline{V}}{R_\dagger(R_\dagger^3 + \overline{V})} \lambda - \left(\frac{2\sigma}{R_\dagger^2} + \frac{2\bar{\sigma} R_\dagger^2}{\overline{R}_\dagger^4} \right) \quad \text{is close to} \quad -\frac{2\sigma}{R_\dagger^2} - \frac{2\bar{\sigma} R_\dagger^2}{\overline{R}_\dagger^4} < 0.$$

Consequently, $\mathbb{M}(\lambda)$ tends to $-\infty$ as λ approaches the pole $-\pi^2 \chi$ from the right side on the real axis. According to the estimates of **Case 3** in Lemma 4, $\mathbb{M}(\lambda) > 0$ on $\{\lambda \in \mathbb{R} : \lambda \geqslant 0\}$. This observation shows that at least on the interval $(-\pi^2 \chi, 0)$, $\mathbb{M}(\lambda)$ has a real root. Thus, we find both the lower and upper bounds

$$-\pi^2\chi < \sup\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{sp}(\mathcal{L}) \setminus \{0\}\} \leqslant -\Theta_0\pi^2\chi.$$

From the center manifold theory in Appendix C and main Theorem 2, the index $-\varpi_\dagger$ in the convergence rate $e^{-\varpi_\dagger t}$ is determined by this spectrum bound (difference by a constant). Therefore, it is reasonable to anticipate that the convergence rate of the spherically symmetric solution will be determined by the magnitude of $-\pi^2\chi$. Utilizing the radius bound (2.11) again, we see that

$$-\pi^2\chi = -\frac{\kappa\pi^2}{R_\dagger^2\rho_\dagger\gamma\mathfrak{c}} = -\frac{4\kappa\pi^3 R_\dagger}{3M\gamma\mathfrak{c}} \in \frac{\kappa\pi^2}{\gamma\mathfrak{c}}\sqrt{\frac{2\pi\mathfrak{R}}{3\sigma}}\left(-\sqrt{\frac{T_c}{M}}, -\sqrt{\frac{T_c}{M}}\Big/\sqrt{1+\frac{\bar{\sigma}}{\sigma}}\right).$$

We conclude that for small liquid volumes, a reduction in gas mass or an increase in temperature can accelerate convergence. This effect has not been observed when the liquid volume is infinite.

Remark 5 When the liquid volume is sufficiently small, surface tension dominates over liquid viscosity, and the evolution of the free boundary is primarily governed by capillary forces. In this case, even small asymmetric perturbations may disrupt the spherical boundary, generate large curvature variations, and potentially lead to pinch-off or instability. In contrast, small spherically symmetric perturbations preserve the uniform curvature of the free surface, allowing us to establish the nonlinear asymptotic stability of the corresponding equilibrium. However, understanding the stability or instability of the system under asymmetric perturbations, particularly when the liquid volume is small, is a challenging open problem that we plan to consider in future work.

Step 3: As we have demonstrated the asymptotic stability in the space ℓ^2 (cf. Lemma 3), we will decompose the space into the direct sum $\ell^2 = X \oplus Y$ and then derive a corresponding system, (4.17) below, which is equivalent to systems (4.3) and (4.7) in Proposition 3, as well as the original free boundary problem (2.1).

Following the setup in Appendix C, we first observe that according to Proposition 4, the eigenvalue $\lambda = 0$ has multiplicity one. Therefore, the linear operator $\mathcal{L}$ defined in (4.10) has a one-dimensional kernel $X = \ker\mathcal{L} = span(\mathrm{U})$, where

$$\mathrm{U} = \begin{bmatrix} -2\sigma/\left(\mathfrak{R}T_c R_\dagger^2\right) - 2\bar{\sigma}R_\dagger^2/\left(\mathfrak{R}T_c\overline{R}_\dagger^4\right) & 1 & 0 & 0 & 0 & \cdots \end{bmatrix}^\top. \tag{4.16}$$

Moreover, the vector $\Upsilon^\top = [4\pi/3 \quad 4\pi\rho_\dagger/R_\dagger \quad 0 \quad \gamma\xi_1/(\gamma-1) \quad \gamma\xi_2/(\gamma-1) \quad \cdots]$ is the corresponding left eigenvector of $\mathcal{L}$ once we note that $\sum_{j=1}^{\infty}\xi_j^2 = 4(\gamma-1)^2\pi/\left(3\gamma^2\right)$.

Applied the equilibrium algebraic equations (2.8), it follows that

$$\langle\Upsilon^\top, \mathrm{U}\rangle = \frac{4\pi}{3\mathfrak{R}T_c}\left(-\frac{2\sigma}{R_\dagger^2} - \frac{2\bar{\sigma}R_\dagger^2}{\overline{R}_\dagger^4} + \frac{3\mathfrak{R}T_c\rho_\dagger}{R_\dagger}\right) = \frac{4\pi}{3\mathfrak{R}T_c}\left(\frac{4\sigma}{R_\dagger^2} + \left(\frac{6\bar{\sigma}}{\overline{R}_\dagger R_\dagger} - \frac{2\bar{\sigma}R_\dagger^2}{\overline{R}_\dagger^4}\right)\right),$$

where we notice that the term $6\bar{\sigma}/\overline{R}_\dagger R_\dagger - 2\bar{\sigma}R_\dagger^2/\overline{R}_\dagger^4$ is strict positive since $R_\dagger < \overline{R}_\dagger$. Then, we can normalize $\Upsilon^\top$ such that $\langle \Upsilon_0^\top, U \rangle = 1$ by setting $\Upsilon_0^\top = \langle \Upsilon^\top, U \rangle^{-1} \Upsilon^\top$.

Having obtained both the left and right eigenvectors, we decompose $\ell^2 = X \oplus Y$ as follows: $z = x + y$, where $x = \mathcal{Q}_1 z = (\Upsilon_0^\top z)U \in X$ and $y = \mathcal{Q}_2 z = z - \mathcal{Q}_1 z \in Y$.

Since $\mathcal{L}U = \Upsilon^\top \mathcal{L} = 0$, we have $0 = \Upsilon_0^\top \mathcal{L}z = \mathcal{Q}_1 \mathcal{L}z$, $\mathcal{L}\mathcal{Q}_1 z = (\Upsilon_0^\top z)\mathcal{L}U = 0$, and $\mathcal{Q}_2 \mathcal{L}z = \mathcal{L}\mathcal{Q}_2 z = \mathcal{L}y = \mathcal{L}z$. In particular, $\mathcal{L}|_X = 0$, and $\mathcal{Q}_2 \mathcal{L}|_Y = \mathcal{L}\mathcal{Q}_2|_Y$ is the restriction of $\mathcal{L}$ on Y satisfying $\sup\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{sp}(\mathcal{L}|_Y)\} \leqslant -2\varpi < 0$.

Then, we derive a dynamical system of (x, y) from (4.7), which is of the same form as system (C.3). That is,

$$
\begin{cases}
\dot{x} = \mathcal{Q}_1 \mathcal{N}(x + y, \dot{x} + \dot{y}) \\
\quad = \mathcal{Q}_1\left[\mathcal{N}^1(x + y)[\dot{x} + \dot{y}]\right] + \mathcal{Q}_1 \mathcal{N}^0(x + y), & t > 0, \\
\dot{y} = \mathcal{L}y + \mathcal{Q}_2 \mathcal{N}(x + y, \dot{x} + \dot{y}) \\
\quad = \mathcal{L}y + \mathcal{Q}_2\left[\mathcal{N}^1(x + y)[\dot{x} + \dot{y}]\right] + \mathcal{Q}_2 \mathcal{N}^0(x + y), & t > 0.
\end{cases}
\tag{4.17}
$$

The initial condition can be deduced from (4.4).

Step 4: We verify the remaining requirements in (i)–(iii) in Appendix C, especially the decay estimates in (C.2). Since the one-dimensional subspace X is $\mathcal{L}$-invariant ($\mathcal{L}|_X = 0$) and Y is closed, we need to check the following results.

Proposition 5 *The subspace Y is $e^{\mathcal{L}t}$-invariant, and for any $t \geqslant 0$, $\left\|e^{\mathcal{L}t}\mathcal{Q}_2\right\|_{\ell^2} \leqslant ce^{-\varpi t}$, where the constant $c > 0$, and the index $\varpi > 0$ is given in Lemma 4.*

Proof Given any $y_0 \in Y$, one has

$$
e^{\mathcal{L}t}y_0 = \sum_{n=0}^{\infty} \frac{(t\mathcal{L})^n}{n!}y_0 = \mathcal{Q}_2 y_0 + \sum_{n=1}^{\infty} \frac{(t\mathcal{Q}_2\mathcal{L})^n}{n!}y_0 = \mathcal{Q}_2\left(y_0 + t\mathcal{L}\sum_{n=1}^{\infty} \frac{(t\mathcal{Q}_2\mathcal{L})^{n-1}}{n!}y_0\right),
$$

where we have used the fact that for $n \geqslant 1$, $\mathcal{L}^{n-1}\mathcal{L}y_0 = \mathcal{L}^{n-1}\mathcal{Q}_2\mathcal{L}y_0 = \mathcal{L}^{n-2}\mathcal{L}\mathcal{Q}_2\mathcal{L}y_0 = \mathcal{L}^{n-2}\mathcal{Q}_2\mathcal{L}\mathcal{Q}_2\mathcal{L}y_0 = (\mathcal{Q}_2\mathcal{L})^n y_0$. Therefore, we deduce $e^{\mathcal{L}t}y_0 \in Y$, and we conclude that Y is $e^{\mathcal{L}t}$-invariant. The operator estimate $\left\|e^{\mathcal{L}t}\mathcal{Q}_2\right\|$ follows from the boundedness of $\mathcal{Q}_2$ and the spectrum analysis in Proposition 4, together with the Gearhart-Prüss theorem [6, 17] for C_0 semigroups. $\qquad\square$

Step 5: In the following, we show the existence of a global center manifold for (4.17), and verify the Lyapunov stability of the zero solution to the equation on the center manifold, as required in Lemma 2.

Compared to the local center manifold constructed in [9, Lemma 9.6], it is worth mentioning that the manifold of equilibria Σ given in (1.12) is actually a global center manifold since we do not utilize the smallness assumption in this part.

Proposition 6 *Given any $(\rho_\dagger, R_\dagger) = (\rho_\dagger[M, V], R_\dagger[M, V]) \in \Sigma$, for $\alpha \in \mathbb{R}$, we define $\hat{\rho}(\alpha)$ by*

$$\hat{\rho}(\alpha) = \frac{1}{\Re T_c} \left(2\sigma / \hat{R}(\alpha) + 2\bar{\sigma} \Big/ \sqrt[3]{\hat{R}(\alpha)^3 + \overline{V}} \right), \qquad (4.18)$$

where $\hat{R}$ is an arbitrary positive C^1 function such that $\hat{R}(0) = R_\dagger$ and $\hat{R}'(0) \neq 1$.

Then, it holds $\hat{\rho}(0) = \rho_\dagger$. Denote $\mathrm{x} \in X$ by $\mathrm{x} = \alpha \mathrm{U}, \alpha \in \mathbb{R}$, and define the curve $\mathrm{y} = h(\mathrm{x}) = h(\alpha \mathrm{U})$ by

$$h(\mathrm{x}) = \left[\hat{\rho}(\alpha) + \left(\frac{2\sigma}{\Re T_c R_\dagger^2} + \frac{2\bar{\sigma} R_\dagger^2}{\Re T_c \overline{R}_\dagger^4} \right) \alpha - \rho_\dagger \quad \hat{R}(\alpha) - \alpha - R_\dagger \quad 0 \quad 0 \quad \cdots \right]^\top.$$

It follows that the curve h is a global center manifold for system (4.17).

Moreover, for $\mathrm{x}(t) = \alpha(t)\mathrm{U}$ with $|\alpha(t)|$ small enough, the equation on the center manifold given by

$$\dot{\mathrm{x}} = \mathcal{Q}_1 \left[\mathcal{N}^1(\mathrm{x} + h(\mathrm{x}))[\dot{\mathrm{x}} + h'(\mathrm{x})\dot{\mathrm{x}}] \right] + \mathcal{Q}_1 \mathcal{N}^0(\mathrm{x} + h(\mathrm{x})), \qquad (4.19)$$

is trivial. That is, (4.19) is equivalent to $\dot{\alpha} \equiv 0$. Therefore, the zero solution to (4.19) is Lyapunov stable.

Proof Without loss of generality, for any $(\mathrm{x}(0), h(\mathrm{x}(0)))$, we consider the solution in the form of $(\mathrm{x}(t) = \alpha(t)\mathrm{U}, \mathrm{y}(t))$ to system (4.17). Note that the initial data yield the initial condition $\mathrm{z}(0) = \mathrm{x}(0) + h(\mathrm{x}(0)) = \left[\hat{\rho}(\alpha) - \rho_\dagger \quad \hat{R}(\alpha) - R_\dagger \quad 0 \quad 0 \quad \cdots \right]^\top.$

Recalling in Proposition 3, $\mathrm{z} = \left[\varrho_2 \quad \mathcal{R} \quad \dot{\mathcal{R}} \quad \theta_1 \quad \theta_2 \quad \cdots \right]^\top$, combining with the changing variables (4.2) and the decomposition (4.6), we deduce that $\theta_i(0) = 0$ and therefore $\varrho_1(\cdot, 0) \equiv 0$. Also, it holds $R(0) = \hat{R}(\alpha(0)), \varrho_2(0) = \hat{\rho}(\alpha(0)) - \rho_\dagger$, and $\rho(R(0)y, 0) = \rho_\dagger + \varrho_2(0) = \hat{\rho}(\alpha(0))$ for any $y \leqslant 1$. That is, $\rho(\cdot, 0) \equiv \hat{\rho}(\alpha(0))$.

From the trajectory defined in (4.18), these initial data are exactly the equilibrium of the gas-liquid system with the gas mass

$$M = \frac{4\pi}{3} \hat{\rho}(\alpha(0))\hat{R}(\alpha(0))^3$$

and the liquid volume V. Therefore, the global-in-time solution is

$$(\mathrm{x}(t), \mathrm{y}(t)) \equiv (\mathrm{x}(0), \mathrm{y}(0)) = (\mathrm{x}(0), h(\mathrm{x}(0))).$$

In other words, it follows that

$$\rho(x, t) \equiv \hat{\rho}(\alpha(0)), x \in B_{\hat{R}(\alpha(0))}, R(t) \equiv \hat{R}(\alpha(0)), \text{ and } \alpha(t) \equiv \alpha(0).$$

Next, we verify that the curve $\mathrm{y} = h(\mathrm{x})$ is tangent to the subspace X at the origin by differentiating (4.18)

$$\hat{\rho}'(\alpha) = -\frac{1}{\Re T_c}\left[\frac{2\sigma}{\hat{R}(\alpha)^2} + \frac{2\bar{\sigma}\hat{R}(\alpha)^2}{\left(\hat{R}(\alpha)^3 + \overline{V}\right)^{4/3}}\right]\hat{R}'(\alpha). \tag{4.20}$$

It follows that

$$\hat{\rho}'(0) = -\left(\frac{2\sigma}{\Re T_c R_\dagger^2} + \frac{2\bar{\sigma}R_\dagger^2}{\Re T_c \overline{R}_\dagger^4}\right)\hat{R}'(0),$$

by setting $\alpha = 0$ and using the assumption $\hat{R}(0) = R_\dagger$. Recalling the vector U defined in (4.16), we conclude that

$$\left.\frac{dh(\alpha \mathrm{U})}{d\alpha}\right|_{\alpha=0} = \left[-\left(\frac{2\sigma}{\Re T_c R_\dagger^2} + \frac{2\bar{\sigma}R_\dagger^2}{\Re T_c \overline{R}_\dagger^4}\right)\left(\hat{R}'(0) - 1\right) \quad \hat{R}'(0) - 1 \quad 0 \quad \cdots\right]^\top \in X.$$

Finally, we check that equation (4.19) on the center manifold is trivial, provided $\mathrm{x}(t) = \alpha(t)\mathrm{U}$ with $|\alpha(t)|$ sufficiently small. Since $\mathrm{y}(t) = h(\mathrm{x}(t))$ on the center manifold, we have

$$\begin{cases} \mathrm{z}(t) = \mathrm{x}(t) + h(\mathrm{x}(t)) = \left[\hat{\rho}(\alpha(t)) - \rho_\dagger \quad \hat{R}(\alpha(t)) - R_\dagger \quad 0 \quad 0 \quad \cdots\right]^\top, \\ \dot{\mathrm{z}}(t) = \dot{\mathrm{x}}(t) + h'(\mathrm{x}(t))\dot{\mathrm{x}}(t) = \dot{\alpha}(t)\left[\hat{\rho}'(\alpha(t)) \quad \hat{R}'(\alpha(t)) \quad 0 \quad 0 \quad \cdots\right]^\top. \end{cases} \tag{4.21}$$

Thus, for the nonlinear terms $\mathbf{\Pi}^0, \mathbf{\Pi}^1, \Phi^0, \mathbf{\Phi}^1, \Psi^0$, and $\mathbf{\Psi}^1$ defined in (4.12), from $\dot{\mathcal{R}} \equiv 0, \theta_i \equiv 0$, and $\varrho_1 \equiv 0$, we deduce that

$$\begin{cases} \Pi^0(\mathrm{x}(t) + h(\mathrm{x}(t))) = 0, \quad \mathbf{\Pi}^1(\mathrm{x}(t) + h(\mathrm{x}(t))) = 0, \\ \Phi^0(\mathrm{x}(t) + h(\mathrm{x}(t))) = 0, \quad \mathbf{\Phi}^1(\mathrm{x}(t) + h(\mathrm{x}(t))) = [g(\alpha(t)) \quad 0 \quad 0 \quad \cdots]^\top, \\ \Psi^0(\mathrm{x}(t) + h(\mathrm{x}(t))) = \frac{1}{\Re T_c}\left[\frac{2\sigma}{R_\dagger^2}\left(\hat{R}(\alpha(t)) - R_\dagger\right) + \frac{2\sigma}{\hat{R}(\alpha(t))} - \frac{2\sigma}{R_\dagger}\right. \\ \qquad\qquad\qquad\qquad \left. + \frac{2\bar{\sigma}R_\dagger^2}{\overline{R}_\dagger^4}\left(\hat{R}(\alpha(t)) - R_\dagger\right) + \frac{2\bar{\sigma}}{\left(\hat{R}(\alpha(t))^3 + \overline{V}\right)^{\frac{1}{3}}} - \frac{2\bar{\sigma}}{\overline{R}_\dagger}\right] \\ \mathbf{\Psi}^1(\mathrm{x}(t) + h(\mathrm{x}(t))) = \left[0 \quad 0 \quad \frac{\rho_l}{\Re T_c}\left(\hat{R}(\alpha(t)) - R_\dagger + \frac{R_\dagger^2}{\overline{R}_\dagger} - \frac{\hat{R}(\alpha(t))^2}{\left(\hat{R}(\alpha(t))^3 + \overline{V}\right)^{\frac{1}{3}}}\right) \quad 0 \quad \cdots\right]^\top, \end{cases}$$

where

$$g(\alpha(t)) = \frac{R_\dagger}{3\gamma\rho_\dagger}\left(1 - \frac{\rho_\dagger\hat{R}(\alpha(t))}{\hat{\rho}(\alpha(t))R_\dagger}\right). \tag{4.22}$$

By (4.14), the nonlinear term $\mathcal{N}^1(\mathrm{x}(t) + h(\mathrm{x}(t)))\left[\dot{\mathrm{x}}(t) + h'(\mathrm{x}(t))\dot{\mathrm{x}}(t)\right]$ equals

$$3\gamma\rho_\dagger R_\dagger^{-1}\hat{\rho}'(\alpha(t))g(\alpha(t))\dot{\alpha}(t)\begin{bmatrix}1 & 0 & 0 & -\xi_1 & -\xi_2 & \cdots\end{bmatrix}^\top,$$

and $\mathcal{N}^0(\mathrm{x}(t) + h(\mathrm{x}(t))) = \begin{bmatrix} 0 & 0 & -\frac{\mathfrak{R}T_c}{\rho_l \bar{R}_\dagger} \Psi^0(\mathrm{x}(t) + h(\mathrm{x}(t))) & 0 & 0 & \cdots \end{bmatrix}^\top$. Then, noting that $\left\langle \Upsilon_0^\top, \mathcal{N}^0(\mathrm{x}(t) + h(\mathrm{x}(t))) \right\rangle = 0$, equation (4.19) is further equivalent to

$$\dot{\mathrm{x}} = \left\langle \Upsilon_0^\top, \mathcal{N}^1(\mathrm{x}(t) + h(\mathrm{x}(t)))[\dot{\mathrm{x}} + h'(\mathrm{x}(t))\dot{\mathrm{x}}] \right\rangle \mathrm{U}$$
$$= 4\pi \rho_\dagger R_\dagger^{-1} \left\langle \Upsilon^\top, \mathrm{U} \right\rangle^{-1} \hat{\rho}'(\alpha(t)) g(\alpha(t)) \dot{\alpha} \mathrm{U},$$

where we have used the identity $\sum_{j=1}^{\infty} \xi_j^2 = \frac{4(\gamma-1)^2\pi}{3\gamma^2}$ and the left eigenvector $\Upsilon^\top$. Combining (4.22) and (4.20), we conclude that

$$\dot{\alpha}\left[1 - 4\pi\rho_\dagger R_\dagger^{-1} \left\langle \Upsilon^\top, \mathrm{U} \right\rangle^{-1} \hat{\rho}'(\alpha(t)) g(\alpha(t)) \right]$$
$$= \dot{\alpha}\left[1 + \frac{4\pi \hat{R}'(\alpha(t))}{3\left\langle \Upsilon^\top, \mathrm{U} \right\rangle \mathfrak{R}T_c \gamma} \left(\frac{2\sigma}{\hat{R}(\alpha(t))^2} + \frac{2\bar{\sigma}\hat{R}(\alpha(t))^2}{(\hat{R}(\alpha(t))^3 + \overline{V})^{\frac{4}{3}}}\right)\left(1 - \frac{\rho_\dagger \hat{R}(\alpha(t))}{\hat{\rho}(\alpha(t))R_\dagger}\right)\right]$$
$$\triangleq \dot{\alpha}\left\{1 + \left[1 - \rho_\dagger \hat{R}(\alpha(t))/(\hat{\rho}(\alpha(t))R_\dagger)\right] K(\alpha(t))\right\}$$

vanishes, since $\dot{\mathrm{x}}(t) = \dot{\alpha}(t)\mathrm{U}$ and $\mathrm{U} \neq 0$. Note that for $|\alpha(t)|$ small enough, $K(\alpha(t))$ is bounded and the factor $1 - \rho_\dagger \hat{R}(\alpha(t))/(\hat{\rho}(\alpha(t))R_\dagger)$ is sufficiently small. This yields that $\dot{\alpha}(t) = 0$ for all $|\alpha(t)|$ sufficiently small. In other words, the dynamic on the center manifold is trivial, and the proof is completed. $\qquad\square$

Step 6: The assumptions for the nonlinear terms in system (4.7) are verified in the following lemma.

Lemma 1 *For the nonlinear term $\mathcal{N}(\mathrm{z}, \mathrm{w})$ defined in (4.11), where z is computed by (4.2) and (4.6) from the solution $(\bar{\rho}, R)$ of problem (B.1). Then, we have $\mathcal{N}(\mathrm{z}, \mathrm{w}) \in \ell^2$, $\mathcal{N}(0, \cdot) \equiv 0$, $\partial_\mathrm{w}\mathcal{N}(0, 0) = 0$, and $\partial_\mathrm{z}\mathcal{N}(0, 0) = 0$.*

Proof The fact that $\mathcal{N}(\mathrm{z}, \mathrm{w}) \in \ell^2$ follows from the same arguments in [9, Proposition 9.7]. Also, it is clear that $\mathcal{N}(0, \mathrm{w}) \equiv 0$ for all w by using the definition in (4.12). To compute the partial derivatives, we utilize the decomposition $\mathcal{N}(\mathrm{z}, \mathrm{w}) = \mathcal{N}^1(\mathrm{z})\mathrm{w} + \mathcal{N}^0(\mathrm{z})$ from (4.13). Applying (4.12) again, we have $|\Pi^1(\mathrm{z})| + |\Phi^1(\mathrm{z})| + |\Psi^1(\mathrm{z})| \leqslant O(\|\mathrm{z}\|)$ and $|F^0(\mathrm{z})| + |\Phi^0(\mathrm{z})| + |\Psi^0(\mathrm{z})| \leqslant O(\|\mathrm{z}\|^2)$. To see this, for terms $\Psi^1(\mathrm{z})$ and $\Psi^0(\mathrm{z})$, using Taylor's theorem at the equilibrium radius $R_\dagger$ (e.g., $\overline{R}^{-1} = \overline{R}_\dagger^{-1} - R_\dagger^2\overline{R}_\dagger^{-4}\mathcal{R} + O(\mathcal{R}^2)$), one has

$$\frac{\rho_l}{\mathfrak{R}T_c}\left|\mathcal{R} + \frac{R_\dagger^2}{\overline{R}_\dagger} - \frac{R^2}{\overline{R}}\right| = O(\|\mathrm{z}\|), \quad \text{as } \|\mathrm{z}\| \to 0.$$

Therefore, $|\Psi^1(\mathrm{z})| \leqslant O(\|\mathrm{z}\|)$, as $\|\mathrm{z}\| \to 0$. Also, from the ratio $R/\overline{R} < 1$, it holds

$$\left| \Psi^0(z) \right| = \left| \frac{1}{\Re T_c} \left[4\mu_l \overline{V} \left(\frac{1}{R(R^3 + \overline{V})} - \frac{1}{R_\dagger(R_\dagger^3 + \overline{V})} \right) \dot{\mathcal{R}} + \frac{2\sigma \mathcal{R}^2}{R_\dagger^2(R_\dagger + \mathcal{R})} \right. \right.$$

$$\left. \left. + 2\bar{\sigma} \left(\frac{R_\dagger^2}{R_\dagger^4} \mathcal{R} + \frac{1}{R} - \frac{1}{R_\dagger} \right) \right] + \frac{\rho_l}{\Re T_c} \left(\frac{3}{2} - \frac{2R}{\overline{R}} + \frac{R^4}{2\overline{R}^4} \right) \dot{\mathcal{R}}^2 \right|$$

$$\leqslant O\left(\|\mathcal{R}\dot{\mathcal{R}}\| + \|\mathcal{R}\|^2 + \|\dot{\mathcal{R}}\|^2 \right) \leqslant O(\|z\|^2), \quad \text{as } \|z\| \to 0.$$

We conclude that $\mathcal{N}^1(0) = \mathcal{N}^0(0) = 0$, and $\partial_z \mathcal{N}^0(0) = 0$ by applying (4.14). This completes the proof. $\qquad\square$

Step 7: By incorporating the aforementioned preparations, we close the proof of the nonlinear exponential decay rate.

We recall that free boundary problem (2.1) is equivalent to system (B.1) defined in B_1, and is also equivalent to dynamical system (4.7) by Proposition 3.

Applying part (i) in Theorem 2 and Lemma 3, we obtain $|\dddot{R}(t)| + |\ddot{R}(t)| \to 0$ and $z(t) \to 0$ as $t \to \infty$. We also deduce that $\partial_t \bar{\rho} \to 0$ uniformly from (B.1a). Combining these results, $\|\dot{z}(t)\|_{\ell^2} \to 0$ follows. Without loss of generality, we may assume that $\|\dot{z}(t)\|$ is small enough for all time. Applying Lemma 2 to system (4.7), there exists $\alpha \in \mathbb{R}$ with $|\alpha|$ small, such that

$$\|x(t) - \alpha U + y(t) - h(\alpha U)\|_{\ell^2} = O(e^{-\varpi_0 t}), \quad \text{as } t \to \infty.$$

In other words, from the first equation in (4.21), we obtain

$$\left\| \begin{bmatrix} \varrho_2 & \mathcal{R} & \dot{\mathcal{R}} & \theta_1 & \theta_2 & \cdots \end{bmatrix} - \begin{bmatrix} \hat{\rho}(\alpha) - \rho_\dagger & \hat{R}(\alpha) - R_\dagger & 0 & 0 & \cdots \end{bmatrix} \right\|_{\ell^2} = O(e^{-\varpi_0 t}),$$

as $t \to \infty$. Clearly, $(\hat{\rho}(\alpha), \hat{R}(\alpha))$ coincides with the equilibrium $(\rho_\dagger[M, V], R_\dagger[M, V])$, since $z(t) \to 0$. This yields the exponential convergence of $(\varrho, \varrho_1, \mathcal{R}, \dot{\mathcal{R}})$

$$\|\varrho_1\|_{L^2(B_1)}^2 + (\rho(R(t), t) - \rho_\dagger[M, V])^2 + (R(t) - R_\dagger[M, V])^2 + \dot{R}(t)^2 = O\left(e^{-2\varpi_0 t} \right),$$

as $t \to \infty$, where we have used the changing variables in (4.2). Again from (4.2), it holds $\rho(R(t)y, t) - \rho_\dagger[M, V] = \varrho_1(y, t) + \rho(R(t), t) - \rho_\dagger[M, V]$, and we deduce that

$$\|\rho(R(t)y, t) - \rho_\dagger[M, V]\|_{L_y^2(B_1)}^2 = O\left(e^{-2\varpi_0 t} \right), \quad \text{as } t \to \infty,$$

by triangle inequality. The remaining proof is similar to the $W^{1,\infty}$ estimates and the $C^{2+2\alpha}$ decay estimates in [9, Lemma 9.8 and Proposition 9.9]. This completes the proof of Theorem 2.

Regular spherical equilibria to system (1.1)–(1.4)

In this appendix, we show that the regular spherically symmetric equilibrium solutions to the original full free boundary problem (1.1)–(1.4) are determined by the mass-volume pairs, provided that the liquid temperature remains constant.

Proposition 7 Any regular spherical equilibrium solution to system (1.1)–(1.4) with mass (of the gas) M and liquid volume V satisfying $T_l \equiv T_c$ is given by

$$v_l \equiv 0, \quad v_g \equiv 0, \quad p_l \equiv \frac{2\bar{\sigma}}{\overline{R}_\dagger[M,V]}, \quad \Omega_g = B_{R_\dagger[M,V]}, \quad \overline{\Omega_g} \cup \Omega_l \equiv B_{\overline{R}_\dagger[M,V]},$$

$$\rho_\dagger[M,V] \equiv \frac{2}{\mathfrak{R}T_c}\left(\frac{\sigma}{R_\dagger[M,V]} + \frac{\bar{\sigma}}{\overline{R}_\dagger[M,V]}\right), \quad p_g \equiv 2\left(\frac{\sigma}{R_\dagger[M,V]} + \frac{\bar{\sigma}}{\overline{R}_\dagger[M,V]}\right),$$

$$T_g \equiv T_c, \quad T_l \equiv T_c, \quad s \equiv \mathfrak{c}\log\left((\mathfrak{R}T_c)^\gamma\left(\frac{2\sigma}{R_\dagger[M,V]} + \frac{2\bar{\sigma}}{\overline{R}_\dagger[M,V]}\right)^{1-\gamma}\right),$$

where $R_\dagger \in C^\infty((0,\infty)^2;(0,\infty))$ is the smooth map denoting the equilibrium radius of the gas bubble as defined in Theorem 1, $\overline{R}_\dagger[M,V] = \sqrt[3]{R_\dagger[M,V]^3 + \overline{V}}$ represents the external radius of the entire gas-liquid region and $\overline{V} = 3V/4\pi$ is the modified liquid volume.

Proof Assume that $v_l(x) = v_l(r)x/r$ with $r = |x|$. The divergence-free condition reads $\partial_r v_l(r) + (2/r)v_l(r) = 0$, or $\partial_r(r^2 v_l(r)) = 0, R_\dagger \leqslant r \leqslant \overline{R}_\dagger$. Therefore, $v_l(r) = a/r^2, R_\dagger \leqslant r \leqslant \overline{R}_\dagger$ for some constant a. However, the boundary condition (1.3a) implies $v_l(R_\dagger) = 0$. Thus, $v_l \equiv 0$, and (1.1a) becomes $\nabla p_l = 0$. From (1.4b), we conclude that the pressure $p_l \equiv 2\bar{\sigma}/\overline{R}_\dagger$.

For the gas velocity v_g, (1.2a) implies $\partial_r(r^2 \rho v_g) = 0$ for $r \leqslant R_\dagger$. Therefore, $r^2 \rho v_g$ is a constant. Again by (1.3a), $v_g(R_\dagger) = 0$ follows and $\rho v_g \equiv 0$. Since we consider the regular solution, $\rho \neq 0$ by (1.2e). Therefore, $v_g \equiv 0$ and p_g is a constant from (1.2b). Now that $v_l = v_g \equiv 0$, (1.2b) yields $p_g \equiv 2\sigma/R_\dagger + p_l = 2\sigma/R_\dagger + 2\bar{\sigma}/\overline{R}_\dagger$. Moreover, (1.2c) becomes $\Delta T_g = 0$ in $B_{R_\dagger}$. Since T_g is regular, we have $T_g \equiv T_g(R_\dagger) = T_c$ by (1.3c) and the maximum principle.

Finally, by (1.2d), $\rho \equiv \left(2\sigma/R_\dagger + 2\bar{\sigma}/\overline{R}_\dagger\right)/(\mathfrak{R}T_c)$. Due to the conservation of mass (1.5), $M = \frac{4\pi}{3}\rho R_\dagger^3$. These imply that the spherically symmetric equilibrium $(\rho, R_\dagger)$ can be obtained by solving the algebraic equation (2.8). Therefore, the proposition follows from the same arguments as in the proof of Theorem 1. $\qquad\square$

Well-posedness and Lyapunov stability of problem (2.1)

To state the well-posedness and the Lyapunov stability results, given any gas density and bubble radius $(\rho(\cdot,t), R(t))$, we define $\bar{\rho}(y,t) = \rho(R(t)y,t), y \in \overline{B}_1$ for $t > 0$, and introduce the norm $\|\cdot\|_{C^{2+2\alpha}(B_A)}$ as follows. For a function $f(r), r < A$, where $A > 0$ is a constant, we define a radial function $\tilde{f}(x) = f(|x|)$ for $x \in B_A$ and

$$\|f\|_{C_r^{2+2\alpha}} = \left\|\tilde{f}\right\|_{C_x^{2+2\alpha}} = \max_{|\beta|\leqslant 2}\sup_{x\in B_A}\left|D^\beta\tilde{f}(x)\right| + \sup_{x_1\neq x_2\in B_A}\frac{\left|D^2\tilde{f}(x_1) - D^2\tilde{f}(x_2)\right|}{|x_1 - x_2|^{2\alpha}}.$$

We state the local well-posedness by adapting the arguments in [3, Theorem 3.1]

Proposition 8 Fix any liquid volume $V > 0$, for problem (2.1) with initial data $(\rho_0, R_0, \dot{R}_0)$, where the initial gas radius $R_0 > 0$ and density $\rho_0 \in C^{2+2\alpha}([0, R_0]; (\eta, \infty))$ for some $\alpha \in (0, 1/2)$ and $\eta > 0$, there exists a unique solution $R \in C_t^{3+\alpha}([0, \delta])$ and $\rho \in C_t^{1+\alpha}([0, \delta]; C_r^{2+2\alpha}([0, R(t))))$, where $\delta = \delta(\|\rho_0\|_{C^{2+2\alpha}}) > 0$.

Proof By changing the variable $x = R(t)y$, we reduce problem (2.1) on $B_{\overline{R}(t)}$ to a problem on the fixed domain

$$\partial_t \bar{\rho}(y, t) = \frac{\kappa \Delta_y \log \bar{\rho}(y, t)}{\gamma \mathfrak{c} R^2} + \frac{\dot{p}}{\gamma p}\left(\frac{y \partial_y \bar{\rho}(y, t)}{3} + \bar{\rho}(y, t)\right), \quad |y| \leqslant 1, \quad t > 0, \qquad \text{(B.1a)}$$

$$\dot{R} = -\frac{\kappa \partial_y \bar{\rho}(1, t)}{\gamma \mathfrak{c} R \bar{\rho}(1, t)^2} - \frac{R \dot{p}}{3 \gamma p}, \quad p(t) = \Re T_c \bar{\rho}(1, t), \qquad t > 0, \qquad \text{(B.1b)}$$

$$\bar{\rho}(1, t) = \frac{1}{\Re T_c}\left[4\mu_l\left(\frac{\dot{R}}{R} - \frac{\dot{\overline{R}}}{\overline{R}}\right) + \frac{2\sigma}{R} + \frac{2\bar{\sigma}}{\overline{R}} \right.$$
$$\left. + \rho_l\left(\frac{\overline{R} - R}{\overline{R}} R \ddot{R} + \left(\frac{3}{2} - \frac{2R}{\overline{R}} + \frac{R^4}{2\overline{R}^4}\right) \dot{R}^2\right)\right], \qquad t > 0, \qquad \text{(B.1c)}$$

where $\bar{\rho}(y, t) = \rho(R(t)y, t)$. Compared to the proof in [3, Theorem 3.1], the extra terms in (B.1c) are analytic in $R > 0$ and $\dot{R}$, since $\overline{R} = (R^3 + \overline{V})^{\frac{1}{3}} > \overline{V}^{\frac{1}{3}}$ and $\dot{\overline{R}} = R^2 \dot{R}/\overline{R}^2$. Thus, (B.1c) can be written in the same form as equation (3.18) in [3, Theorem 3.1], i.e., an analytic function of $R, \dot{R}, \ddot{R}$ and t. Then, (2.1) can be treated as that in [3, Theorem 3.1], and one can follow the same procedure to complete the proof. $\qquad \square$

The global well-posedness and stability results are derived following the proof in [3] (see also [9, Section 6]), where the global well-posedness of the free boundary problem (2.1) is established under conditions where the liquid volume is infinite, there are no viscous terms at the gas-liquid interface, and the external far-field pressure is constant. Additionally, they showed the Lyapunov stability of the problem when the initial data are sufficiently close to a spherically symmetric equilibrium.

For the problem we are considering here for which the liquid has a finite volume V, the presence of viscous terms on the gas-liquid interface (see (1.8c)) and the external liquid free surface (see (1.8e)) lead to the boundary condition (2.1d). Consequently, $-16\pi\mu_l \overline{V}(\dot{R}(t))^2 R(t)/(R(t)^3 + \overline{V})$ appears on the right-hand side of the energy dissipation law (4.1) and it is a negative term. As a result, the arguments used in the proof of [3, Theorem 4.1], particularly those involving key energy dissipation estimates (4.16) and (4.40) in [3], remain applicable.

More precisely, fix the gas mass M and the liquid volume V, given $\varepsilon_0 > 0$, there exists $\eta_0 = \eta_0(\varepsilon_0) > 0$ such that the following holds: for any mass-preserving ($M = \mathrm{M}[\rho_0, R_0]$) initial data $(\rho_0, R_0, \dot{R}_0)$ with liquid volume V satisfying

$$\|\rho_0(R_0 y) - \rho_\dagger[M, V]\|_{C_y^{2+2\alpha}(B_1)} + |R_0 - R_\dagger[M, V]| + |\dot{R}_0| \leqslant \eta_0, \qquad \text{(B.1)}$$

the global-in-time solution satisfies

$$\|\rho(R(t)y, t) - \rho_\dagger[M, V]\|_{C_y^{2+2\alpha}(B_1)} + |R(t) - R_\dagger[M, V]| + |\dot{R}| \leqslant \varepsilon_0, \qquad \text{(B.2)}$$

$$\|p_g\|_{C_t^{1+\alpha}(\mathbb{R}_+)} \leqslant \varepsilon_0, \text{ and } \|R(t) - R_\dagger[M, V]\|_{C_t^{3+\alpha}(\mathbb{R}_+)} \leqslant \varepsilon_0, \ \forall t > 0. \qquad \text{(B.3)}$$

Therefore, given the gas mass M and the liquid volume V, we have the spherical equilibrium Lyapunov stability for mass-preserving and volume-invariant perturbations to the equilibrium $(\rho_\dagger[M, V], R_\dagger[M, V])$.

Center manifold theory

In this short appendix, we recall the center manifold theory developed in [9], and we refer to [4] for more information on this topic. Consider the following equation on a Banach space $(Z, \|\cdot\|)$

$$\dot{z} = \mathcal{L}z + \mathcal{N}(z, \dot{z}), \quad z(0) \in Z, \qquad \text{(C.1)}$$

where $\mathcal{N}(z, w) : Z \times Z \to Z$ has a uniformly continuous second order derivative satisfying $\mathcal{N}(0, w) = 0 = \partial_{(z,w)}\mathcal{N}(0, 0) = 0$. We further assume that

(i) $Z = X \oplus Y$, where X is a finite-dimensional $\mathcal{L}$-invariant subspace and Y is a closed $e^{\mathcal{L}t}$-invariant subspace.
(ii) All the eigenvalues of $\mathcal{L}|_X$ have zero real parts.
(iii) Let $\mathcal{Q}_1 : Z \to X$ be a projection, and $\mathcal{Q}_2 = I_X - \mathcal{Q}_1$. There exist positive constants ϖ and c, such that

$$\left\| e^{\mathcal{L}t}\mathcal{Q}_2 \right\|_{Y \to Y} \leqslant ce^{-\varpi t}, \quad t \geqslant 0. \quad \text{(C.2)}$$

Decompose a solution to (C.1) as $z = x + y = \mathcal{Q}_1 z + \mathcal{Q}_2 z$. Then, (C.1) becomes

$$\begin{cases} \dot{x} = \mathcal{L}|_X x + f(x, y, \dot{x}, \dot{y}), & \text{where } f(x, y, \dot{x}, \dot{y}) = \mathcal{Q}_1 \mathcal{N}(x + y, \dot{x} + \dot{y}), \\ \dot{y} = \mathcal{Q}_2 \mathcal{L}y + g(x, y, \dot{x}, \dot{y}), & \text{where } g(x, y, \dot{x}, \dot{y}) = \mathcal{Q}_2 \mathcal{N}(x + y, \dot{x} + \dot{y}). \end{cases} \qquad \text{(C.3)}$$

Recall that an invariant manifold for (C.3) is a curve $y = h(x)$, defined for $|x|$ small, such that the solution $(x(t), y(t))$ passing through $(x(0), h(x(0)))$ satisfies $y(t) = h(x(t))$. A center manifold is an invariant manifold that is tangent to the subspace X at the origin. Let $\mathcal{M}$ be a center manifold for (C.3) given by $y = h(x)$. The equation on the center manifold is given by

$$\dot{x} = \mathcal{L}|_X x + f(x, h(x), \dot{x}, h'(x)\dot{x}). \qquad \text{(C.4)}$$

Assume that $z(t)$ converges to some point in $\mathcal{M}$ as $t \to \infty$, and that $\sup_{t \geqslant 0} \|\dot{z}(t)\|$ is sufficiently small. Then, the following lemma holds:

Lemma 2 *Denote by $(x(t), y(t))$ a solution of (C.3). Assume that there exists $\varepsilon > 0$ such that if $\|(x(0), y(0))\| < \varepsilon$, then $\|(\dot{x}(t), \dot{y}(t))\| < \varepsilon$ for any $t \geqslant 0$. Then there exist constants $C_1, \varpi_1 > 0$ such that $\|y(t) - h(x(t))\| \leqslant C_1 e^{-\varpi_1 t} \|y(0) - h(x(0))\|$ for $t \geqslant 0$.*

If we further assume that the zero solution of (C.4) is Lyapunov stable, then there exists a solution $\hat{x}(t)$ of (C.4) such that $x(t) = \hat{x}(t) + O(e^{-\varpi_0 t})$ and $y(t) = h(\hat{x}(t)) + O(e^{-\varpi_0 t})$, as $t \to \infty$, where $\varpi_0 = \min(\varpi, \varpi_1)$ and ϖ is given in (C.2).

Asymptotic stability in ℓ^2 and the spectrum analysis

We first reformulate the asymptotic stability result in the first part of Theorem 2 for system (1.6)–(1.8) by using the variable z to express it in terms of the equivalent system (4.7) within Banach space ℓ^2. The proof involves adapting [9, Proposition 9.2], as it primarily addresses the results related to (2.1a) and (2.1b).

Lemma 3 *Fix any mass-volume pair (M, V). For any initial data $(\rho_0, R_0, \dot{R}_0) \in C_r^{2+2\alpha}(B_{R_0}) \times \mathbb{R}_+ \times \mathbb{R}$ with $\alpha \in (0, 1/2)$ such that $M[\rho_0, R_0] = M$, denote $\varrho_2(0) = \rho_0(R_0) - \rho_\dagger[M, V], \mathcal{R}(0) = R_0 - R_\dagger[M, V]$, and $\theta_k(0) = \int_{B_1} (\rho_0(R_0 y) - \rho_0(R_0)) \Xi_k(y) \, dy$, where Ξ_k is defined in Proposition 3. Then, we have $z(0) = (\varrho_2(0), \mathcal{R}(0), \dot{R}_0, \theta_1(0), \theta_2(0), \cdots)^\top \in \ell^2$, and the sequence $\{j^2 \theta_j(0)\}_{j=1}^\infty \in \ell^2$.*

Furthermore, assume that (1.13) holds. Let $(\rho, R) \in C_{r,t}^{2+2\alpha, 1+\alpha}(B_{R(t)} \times [0, \infty)) \times C_t^{3+\alpha}$ be the global solution of (2.1) with initial data $(\rho_0, R_0, \dot{R}_0)$ and liquid volume V as in Theorem 2. Let z be the corresponding solution to system (4.7). Then, it follows that $\{j^2 \theta_j\}_{j=1}^\infty \in \ell^2$ and $\|\dot{z}\|_{\ell^2} + \|z(t)\|_{\ell^2} \to 0$, as $t \to \infty$.

Then, we analyze the spectrum of the operator $\mathcal{L}$ on the space ℓ^2 using the Laplace transform. Given a function $f(t)$ defined for $t \geqslant 0$, we denote the Laplace transform by $\tilde{f}(\tau) = \int_0^\infty e^{-t\tau} f(t) dt$.

Proof of Proposition 4 Note that the linear system $\dot{z} = \mathcal{L}z$ is equivalent to the linear part of system (4.3)

$$\partial_t \varrho_1 = \chi \Delta_y \varrho_1 - \left(1 - \gamma^{-1}\right) \dot{\varrho}_2, \quad 0 \leqslant y \leqslant 1, \quad \varrho_1(1, t) = 0, \quad t > 0, \tag{D.1}$$

$$\dot{\mathcal{R}} = -\chi R_\dagger \rho_\dagger^{-1} \partial_y \varrho_1(1, t) - R_\dagger \gamma^{-1} \rho_\dagger^{-1} \dot{\varrho}_2 / 3, \qquad t > 0, \tag{D.2}$$

$$\varrho_2 = (\mathfrak{R} T_c)^{-1} \left(-A\mathcal{R} + B\dot{\mathcal{R}} + \rho_l \tilde{R}_\dagger \ddot{\mathcal{R}}\right), \qquad t > 0, \tag{D.3}$$

where

$$A = 2\sigma/R_\dagger^2 + 2\bar{\sigma} R_\dagger^2/\overline{R}_\dagger^4 \text{ and } B = 4\mu_l \overline{V}/R_\dagger(R_\dagger^3 + \overline{V}).$$

Similar to the proof as in Proposition 3, substituting the decomposition (4.6) into (D.1) and testing by Ξ_k, it holds $\dot{\theta}_k(t) = -\chi \zeta_k \theta_k(t) - \xi_k \dot{\varrho}_2(t)$. Taking the Laplace transform yields

$$\tilde{\theta}_k(\tau) = \frac{\theta_k(0) + \xi_k \varrho_2(0)}{\chi \zeta_k + \tau} - \frac{\xi_k \tau}{\chi \zeta_k + \tau} \tilde{\varrho}_2(\tau). \tag{3.4}$$

Again by the Laplace transform, (D.2) and (D.3) become

$$- \mathcal{R}(0) + \tau \widetilde{\mathcal{R}}(\tau) = \sum_{j=1}^{\infty} \widetilde{\theta}_j(\tau) \omega_j - 3^{-1} R_\dagger \gamma^{-1} \rho_\dagger^{-1}(-\varrho_2(0) + \tau \widetilde{\varrho}_2(\tau)), \tag{D.5}$$

$$(\rho_l \tilde{R}_\dagger \tau^2 + B\tau - A) \widetilde{\mathcal{R}}(\tau) - \mathfrak{R} T_c \widetilde{\varrho}_2(\tau) = \rho_l \tilde{R}_\dagger (\dot{\mathcal{R}}(0) + \tau \mathcal{R}(0)). \tag{D.6}$$

Substituting (3.4) and (D.6) into (D.5) and using $\xi_j \omega_j = \frac{2(\gamma-1)\chi R_\dagger}{\gamma \rho_\dagger}$, we obtain

$$-\mathcal{R}(0) + \tau \widetilde{\mathcal{R}}(\tau) = -(\mathfrak{R}T_c)^{-1} \left[(\rho_l \tilde{R}_\dagger \tau^2 + B\tau - A)\tau \widetilde{\mathcal{R}}(\tau) - \rho_l \tilde{R}_\dagger \tau (\dot{\mathcal{R}}(0) + \tau \mathcal{R}(0)) \right]$$
$$\cdot \left(\sum_{k=1}^{\infty} \frac{\xi_k \omega_k}{\chi \zeta_k + \tau} + \frac{R_\dagger}{3\gamma \rho_\dagger} \right) + \left[\sum_{k=1}^{\infty} \frac{\theta_k(0) + \xi_k \varrho_2(0)}{\chi \zeta_k + \tau} \omega_k + \frac{R_\dagger}{3\gamma \rho_\dagger} \varrho_2(0) \right].$$

Therefore, we obtain $\tau \widetilde{\mathcal{R}}(\tau) \mathbb{M}(\tau) = \mathbb{S}(\tau)$, where $\mathbb{M}(\tau)$ is defined in (4.15) and

$$\mathbb{S}(\tau) = 4\pi \frac{\rho_\dagger}{R_\dagger} \left[\frac{\rho_l \tilde{R}_\dagger \tau}{\mathfrak{R}T_c} \left(\sum_{k=1}^{\infty} \frac{\xi_k \omega_k}{\chi \zeta_k + \tau} + \frac{R_\dagger}{3\gamma \rho_\dagger} \right) (\dot{\mathcal{R}}(0) + \tau \mathcal{R}(0)) \right.$$
$$\left. + \sum_{k=1}^{\infty} \frac{\theta_k(0) + \xi_k \varrho_2(0)}{\chi \zeta_k + \tau} \omega_k + \frac{R_\dagger}{3\gamma \rho_\dagger} \varrho_2(0) + \mathcal{R}(0) \right]$$

is analytic for all $\tau \in \mathbb{C} \setminus \{-\pi^2 \chi j^2 : j = 1, 2, \cdots\}$ since $-\chi \zeta_j = -\pi^2 \chi j^2$. Then, (3.4) and (D.6) yield that $\widetilde{z} = (\widetilde{\varrho}_2(\tau), \widetilde{\mathcal{R}}(\tau), \widetilde{\dot{\mathcal{R}}}(\tau), \widetilde{\theta}_1(\tau), \widetilde{\theta}_2(\tau), \cdots)^\top = (\mathcal{L} - \tau I)^{-1} \widetilde{z}(0)$ for $\tau \in \mathbb{C} \setminus \{-\pi^2 \chi j^2 : j = 1, 2, \cdots\}$ satisfying $\tau \mathbb{M}(\tau) \neq 0$.

To estimate the upper bound of $\sup\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{sp}(\mathcal{L}) \setminus \{0\}\}$, we rewrite

$$\mathbb{M}(\lambda) = 4\pi \frac{\rho_\dagger}{R_\dagger} + \frac{1}{\mathfrak{R}T_c} \left(\sum_{k=1}^{\infty} \frac{1}{\gamma} \frac{8(\gamma-1)\pi\chi}{\chi \pi^2 k^2 + \lambda} + \frac{4\pi}{3\gamma} \right) (C\lambda^2 + B\lambda - A),$$

where $C = \rho_l(R_\dagger - R_\dagger^2 / \overline{R}_\dagger)$. The remaining proof of the lemma is a consequence of Lemma 4 below. □

Lemma 4 *There exists a constant $\varpi > 0$ such that $x < -2\varpi$ for all complex roots $\lambda = x + iy$ to the function $\mathbb{M}$ in (4.15). The constant ϖ can be chosen as*

$$\varpi = \frac{1}{2} \min \left\{ \Theta_2 \pi^2 \chi, \max \left\{ \frac{B}{2C}, \min \left\{ \Theta_1 \pi^2 \chi, \sqrt{\frac{K}{2}} \right\} \right\} \right\}, \quad B^2 \leqslant 4KC^2,$$

$$\frac{1}{2} \min \left\{ \Theta_2 \pi^2 \chi, \frac{B - \sqrt{B^2 - 4KC^2}}{2C} \right\}, \quad B^2 > 4KC^2,$$

where the constants Θ_1 and Θ_2 are defined in (D.15) and (D.21), respectively. Moreover, $K = 2\mathfrak{R}T_c \rho_\dagger \overline{R}_\dagger / [\rho_l R_\dagger^2 (\overline{R}_\dagger - R_\dagger)]$.

Proof Let $\lambda = x + iy$ be a complex root of $\mathbb{M}$. Substituting $\lambda = x + iy$ into (4.15) and splitting the real and imaginary parts of $\mathbb{M}$, we obtain

$$\frac{1}{\Re T_c}\left\{\left[\frac{4\pi}{3\gamma}+\sum_{j=1}^{\infty}\frac{1}{\gamma}\frac{8(\gamma-1)\pi^3\chi^2 j^2}{(\pi^2\chi j^2+x)^2+y^2}\right]\left[C\left(x^2-y^2\right)+Bx-A\right]\right.$$

$$\left.+\sum_{j=1}^{\infty}\frac{1}{\gamma}\frac{8(\gamma-1)\pi\chi}{(\pi^2\chi j^2+x)^2+y^2}\left[(Cx+B)(x^2+y^2)-Ax\right]\right\}+4\pi\frac{\rho_\dagger}{R_\dagger}=0, \tag{D.8a}$$

$$\frac{i}{\Re T_c}\left\{\left[\frac{4\pi}{3\gamma}+\sum_{j=1}^{\infty}\frac{1}{\gamma}\frac{8(\gamma-1)\pi\chi\left(\pi^2\chi j^2+x\right)}{(\pi^2\chi j^2+x)^2+y^2}\right](2Cx+B)\,y\right.$$

$$\left.-\sum_{j=1}^{\infty}\frac{1}{\gamma}\frac{8(\gamma-1)\pi\chi}{(\pi^2\chi j^2+x)^2+y^2}\left[C\left(x^2-y^2\right)+Bx-A\right]y\right\}=0. \tag{D.8b}$$

We will denote $D=\frac{4\pi}{3\gamma}+\sum_{j=1}^{\infty}\frac{1}{\gamma}\frac{8(\gamma-1)\pi\chi\left(\pi^2\chi j^2+x\right)}{(\pi^2\chi j^2+x)^2+y^2}$, $E=\sum_{j=1}^{\infty}\frac{1}{\gamma}\frac{8(\gamma-1)\pi^3\chi^2 j^2}{(\pi^2\chi j^2+x)^2+y^2}$, and $F=\sum_{j=1}^{\infty}\frac{1}{\gamma}\frac{8(\gamma-1)\pi\chi}{(\pi^2\chi j^2+x)^2+y^2}$ to simplify the above formulas.

Case 1: $0<y^2\leqslant K/2$. If $y\neq 0$, eliminating y for the imaginary part (D.8b) yields

$$0=\left(\frac{4\pi}{3\gamma}+E\right)(2Cx+B)+F\left[C(x^2+y^2)+A\right]. \tag{D.9}$$

From $E>0$ and $F\left[C(x^2+y^2)+A\right]>0$, it holds

$$x<-\frac{B}{2C}<0, \tag{D.10}$$

provided $y\neq 0$. Splitting F in (D.9) and substituting into the real part (D.8a), we derive

$$\left(\frac{4\pi}{3\gamma}+E\right)\left\{C(x^2-y^2)+Bx-A-\frac{(2Cx+B)\left[(Cx+B)(x^2+y^2)-Ax\right]}{C(x^2+y^2)+A}\right\}$$
$$=-4\pi\Re T_c\frac{\rho_\dagger}{R_\dagger}. \tag{D.11}$$

Denote the term in the big brace by ω, and a straightforward calculation shows that

$$\left[C(x^2+y^2)+A\right]\omega$$
$$=-C^2(x^2+y^2)^2-B\left(2Cx+B\right)(x^2+y^2)+A\left(2Cx^2+2Bx-2Cy^2\right)-A^2$$
$$>-C^2(x^2+y^2)^2-2AC(x^2+y^2)-A^2=-\left[C(x^2+y^2)+A\right]^2.$$

In the above, we have used (D.10) to deduce $2Cx^2+2Bx=x\left(2Cx+2B\right)>-x(2Cx)$. Then, one has $\omega>-C(x^2+y^2)-A$, and (D.11) yields

$$\frac{1}{\Re T_c}\left(\frac{4\pi}{3\gamma}+E\right)\left[C(x^2+y^2)+A\right]>4\pi\frac{\rho_\dagger}{R_\dagger}. \tag{D.12}$$

Assume that $x\geqslant-\Theta\pi^2\chi$, where $\Theta\in(0,1)$ will be chosen later. It is clear that $x\geqslant-\Theta\pi^2\chi j^2$ for all $j\geqslant 1$. By equilibrium equation (2.8) and recalling $R_\dagger<\overline{R}_\dagger$, it holds

$$\frac{2\Re T_c \rho_\dagger}{R_\dagger} < KC + A = \frac{2\Re T_c \rho_\dagger}{R_\dagger} + \frac{2\sigma}{R_\dagger^2} + \frac{2\bar\sigma R_\dagger^2}{\overline{R}_\dagger^4} < \frac{3\Re T_c \rho_\dagger}{R_\dagger}. \tag{D.13}$$

We further assume that

$$x \geqslant -\sqrt{K - y^2}, \tag{D.14}$$

if $0 < y^2 \leqslant K$. Since (D.10) implies $x < 0$, we have $C(x^2 + y^2) + A \leqslant KC + A$, if $0 < y^2 \leqslant K$. Then (D.12) yields

$$0 > -\frac{1}{\Re T_c}\left(\frac{4\pi}{3\gamma} + \frac{8(\gamma - 1)}{\pi\gamma}\frac{1}{(1 - \Theta)^2}\sum_{j=1}^{\infty} j^{-2}\right)(KC + A) + 4\pi\frac{\rho_\dagger}{R_\dagger},$$

where we have substituted $x \geqslant -\Theta\pi^2\chi j^2$. Upon simplification the above inequality, one has

$$\Theta > \Theta_1 \triangleq 1 - \sqrt{(\gamma - 1)\bigg/\left(\frac{3\Re T_c\rho_\dagger\gamma}{R_\dagger(KC + A)} - 1\right)} \in (0, 1). \tag{D.15}$$

We simply choose $\Theta = \Theta_1$ to reach a contradiction to (D.14), since we will deduce $\Theta > \Theta_1$. Thus, we have for $0 < y^2 \leqslant K$ that $x < -\min\left\{\Theta_1\pi^2\chi, \sqrt{K - y^2}\right\}$. Combining (D.10), we have

$$x < -\max\left\{\frac{B}{2C}, \min\left\{\Theta_1\pi^2\chi, \sqrt{\frac{K}{2}}\right\}\right\}, \text{ if } 0 < y^2 \leqslant \frac{K}{2}. \tag{D.16}$$

Case 2: $y^2 > K/2$. We eliminate D in the imaginary part (D.8b) and substitute it into the real part (D.8a)

$$4\pi\rho_\dagger R_\dagger^{-1}\Re T_c(2Cx + B) = -F\left|C\lambda^2 + B\lambda - A\right|^2. \tag{D.17}$$

Note that $A, B, C > 0$ and therefore $\left|C\lambda^2 + B\lambda - A\right|^2 = C^2\left|\lambda - \lambda_1\right|^2\left|\lambda - \lambda_2\right|^2$ has real roots $\lambda_i \in \mathbb{R}, i = 1, 2$. Then, we have $\left|C\lambda^2 + B\lambda - A\right|^2 > C^2 y^4$ since $|\lambda - \lambda_i| = |x + iy - \lambda_i| > |iy|$. Using $y^2 > K/2$, we further derive

$$4\pi\rho_\dagger R_\dagger^{-1}\Re T_c(2Cx + B) < -C^2\left(\frac{K}{2}\right)^2\sum_{j=1}^{\infty}\frac{1}{\gamma}\frac{8(\gamma - 1)\pi\chi}{\pi^4\chi^2 j^4 + K/2}. \tag{D.18}$$

Then, recalling the definitions of C, K and χ, we have from (D.18) that

$$x < -\frac{B}{2C} - \frac{(\gamma-1)\mathfrak{R}T_c}{\pi^4 \rho_l \kappa} \frac{\rho_\dagger^2}{1 - R_\dagger/\overline{R}_\dagger} \left(\sum_{j=1}^{\infty} \frac{1}{j^4 + \frac{K}{2\pi^4 \chi^2}} \right) < -\frac{B}{2C}, \text{ if } y^2 > \frac{K}{2}. \quad \text{(D.19)}$$

Case 3: $y = 0$. We claim that $x < 0$. Suppose $x \geqslant 0$, and we rewrite $\mathbb{M}(x)$ as

$$\frac{\pi x(Cx+B)}{\mathfrak{R}T_c\gamma} \left(\frac{4}{3} + \sum_{j=1}^{\infty} \frac{8(\gamma-1)\chi}{\pi^2 \chi j^2 + x} \right) + \left[\frac{4\pi \rho_\dagger}{R_\dagger} - \frac{\pi A}{\mathfrak{R}T_c\gamma} \left(\frac{4}{3} + \sum_{j=1}^{\infty} \frac{8(\gamma-1)\chi}{\pi^2 \chi j^2 + x} \right) \right].$$

From $x \geqslant 0$, the first term is non-negative, and we apply (2.8) to deduce that the term in the bracket is greater than

$$\frac{8\pi}{\mathfrak{R}T_c} \left(\frac{\sigma}{R_\dagger^2} + \frac{\bar{\sigma}}{R_\dagger \overline{R}_\dagger} \right) - \frac{\pi}{\mathfrak{R}T_c\gamma} \left(\frac{2\sigma}{R_\dagger^2} + \frac{2\bar{\sigma}R_\dagger^2}{\overline{R}_\dagger^4} \right) \left(\frac{4}{3} + \frac{8(\gamma-1)\chi}{\pi^2\chi} \sum_{j=1}^{\infty} j^{-2} \right)$$

$$\geqslant \frac{8\pi}{\mathfrak{R}T_c} \left(\frac{\sigma}{R_\dagger^2} + \frac{\bar{\sigma}}{R_\dagger \overline{R}_\dagger} \right) - \frac{1}{3} \frac{8\pi}{\mathfrak{R}T_c} \left(\frac{\sigma}{R_\dagger^2} + \frac{\bar{\sigma}R_\dagger^2}{\overline{R}_\dagger^4} \right) > 0.$$

This contradicts $\mathbb{M}(x) = 0$. To search for a negative upper bound, we assume that $x \geqslant -\Theta'\pi^2\chi$, where $0 < \Theta' < 1$ will be chosen. We further assume that

$$x > \frac{-B + \sqrt{B^2 - 4KC^2}}{2C}, \quad \text{(D.20)}$$

if $B^2 > 4KC^2$. In this case, $Cx^2 + Bx - A \geqslant -KC - A$, and this inequality also holds when $B^2 \leqslant 4KC^2$. Then, we have

$$0 = \mathbb{M}(x) > -\frac{1}{\mathfrak{R}T_c} \left(\frac{4\pi}{3\gamma} + \frac{8(\gamma-1)}{\pi\gamma(1-\Theta')} \sum_{j=1}^{\infty} j^{-2} \right) (KC+A) + 4\pi \frac{\rho_\dagger}{R_\dagger}$$

$$= -\frac{1}{\mathfrak{R}T_c} \left(\frac{4\pi}{3\gamma} + \frac{4\pi}{3} \left(1 - \frac{1}{\gamma} \right) \frac{1}{1-\Theta'} \right) (KC+A) + 4\pi \frac{\rho_\dagger}{R_\dagger}.$$

Recalling (D.13), it follows that $3\mathfrak{R}T_c\rho_\dagger/R_\dagger < [\gamma^{-1} + (1-\gamma^{-1})(1-\Theta')^{-1}](KC+A)$, or equivalently,

$$\Theta' > \Theta_2 \triangleq 1 - (\gamma-1) \Big/ \left(\frac{3\mathfrak{R}T_c\gamma\rho_\dagger}{R_\dagger(KC+A)} - 1 \right) \in (0,1). \quad \text{(D.21)}$$

Therefore, we reach a contradiction to (D.20) by choosing $\Theta' = \Theta_2$.

Summing up, in the case of $y = 0$, $x < -\min\left\{ \Theta_2\pi^2\chi, \frac{B-\sqrt{B^2-4KC^2}}{2C} \right\}$, if $B^2 > 4KC^2$. Otherwise, $x < -\Theta_2\pi^2\chi$. This, combined with the upper bounds (D.16), (D.19) and the fact $\Theta_2 > \Theta_1$ gives the upper bound. $\qquad \square$

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that there is no Conflict of interest.

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